# Nonparametric IV and Partial Identification

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# Instrumental Variables

$$Y = \theta_0(X) + \epsilon \ ,$$

with $E[\epsilon|X] \neq 0$, $E[\epsilon|Z] = 0$ and $\theta \in \Theta$ for some smooth set of functions $\Theta$.

Defining $\Upsilon : L^2(X) \rightarrow L^2(Z)$ by $\Upsilon(\theta) = E[\theta(X)|Z]$ we obtain the equation:

$$E[Y|Z] = \Upsilon(\theta) \ .$$

**Problem is ill posed if:**

- $\Upsilon^{-1}$ exists but is not continuous (*instability*).
- $E[Y|Z]$ is not in the image of $\Upsilon$ (*nonexistence*).
- $\Upsilon$ is not injective (*nonuniqueness*).

# Set of Solutions

Assuming existence and for $\mathcal{N}(\Upsilon)$ the null space of $\Upsilon$, the set of solutions is:

$$\mathbf{V}_0 \equiv \theta_0 + \mathcal{N}(\Upsilon) \ .$$

**Note:** Identified set is an affine vector space, potentially infinite dimensional.

**Restrict domain of $\Upsilon$ to $\Theta$**

- The relevant set of solutions becomes $\mathbf{V}_0 \cap \Theta$.
- Regularize through compactness of $\Theta \Rightarrow$ nonuniqueness with stability.
- Inverse correspondence $\Upsilon^{-1}$ is upper hemicontinuous.

# Parameter Space

**Norms** $X \in \mathbf{R}^{d_x}$, $m, m_0$ integers, $\delta_0 > \delta > 0$ scalars with $m_0 > \frac{d_x}{2}$ and $\frac{d_x}{m} + \frac{d_x}{\delta} < 2$

$$||\theta||_s^2 \equiv \sum_{|\lambda| \leq m+m_0} \int [D^\lambda \theta(x)]^2 (1 + x'x)^{\delta_0} dx \qquad ||\theta||_{c\delta} \equiv \max_{|\lambda| \leq m} \sup_x |D^\lambda \theta(x)| (1 + x'x)^{\frac{\delta}{2}}$$

**Vector Spaces** For $\mathcal{X} \subseteq \mathbf{R}^{d_x}$ the support of $X$ define the metric vector spaces:

$$W^s(\mathcal{X}) \equiv \{\theta : \mathcal{X} \to \mathbf{R} \text{ s.t. } ||\theta||_s < \infty\} \qquad W^{c\delta}(\mathcal{X}) \equiv \{\theta : \mathcal{X} \to \mathbf{R} \text{ s.t. } ||\theta||_{c\delta} < \infty\}$$

**Parameter Space** $\Theta$ is the closure under $|| \cdot ||_{c\delta}$ of a sphere in $W^s(\mathcal{X})$:

$$\Theta \equiv \text{cl}\{\theta \in W^s(\mathcal{X}) : ||\theta||_s \leq B\} \ .$$

**Key:** $\Theta$ is bounded in $W^s(\mathcal{X})$ and compact in $W^{c\delta}(\mathcal{X})$.

# Identification

**Question:** Does restriction $\theta_0 \in \Theta$ identify $\theta_0$?

**Answer:** If $\theta_0 \in W^s(\mathcal{X})$, then unlikely **unless** $\mathcal{N}(\Upsilon) \cap W^s(\mathcal{X}) = \emptyset$, since:

$$\mathbf{V}_0 \cap \Theta \supseteq \underbrace{\{\theta_0 + \mathcal{N}(\Upsilon) \cap W^s(\mathcal{X})\}}_{\text{affine vector space}} \cap \underbrace{\{\theta \in W^s(\mathcal{X}) : \|\theta\|_s \leq B\}}_{\text{sphere}} .$$

**Three cases**

- Sphere does not intersect affine space *(misspecification)*
- Sphere intersects affine space at many points *(partial identification)*
- Sphere is tangent to affine space *(identification)*

**Basic Insight:** $\mathcal{N}(\Upsilon) \cap W^s(\mathcal{X}) \neq \emptyset$, then identification fails for $B$ large.

# Counterexamples

**Goal:** Find $f_{X|Z}(x|z)$ such that $\int \psi(x) f_{X|Z}(x|z) dx = 0$ for some $\psi \in W^s(\mathcal{X})$.

**Basic Approach:**

- Suppose $\{\psi_k(x)\}_{k=1}^{\infty}$ and $\{\phi_j(z)\}_{j=1}^{\infty}$ are basis of $L^2(X)$ and $L^2(Z)$.
- Let $f_{X|Z}(x|z)$ be square integrable and admitting for an expansion (in $\|\cdot\|_{L^2}$)

$$f_{X|Z}(x|z) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} \psi_k(x) \phi_j(z) \ .$$

- If for some $k^*$ we have $\psi_{k^*} \in W^s(\mathcal{X})$ and $a_{k^* j} = 0$ for all $j$ then:

$$\theta_0 + \psi_{k^*} \in \mathbf{V}_0 \cap W^s(\mathcal{X}) \ .$$

**But!** Most basis functions are in $W^s(\mathcal{X})$ and it is easy to construct $f_{X|Z}(x|z)$.

# Counterexamples

**Example 1:** Suppose $(X, Z) \in [-1, 1]^2$ and $f_{X|Z}(x|z)$ is polynomial of finite order.

**Example 2:** $(X, Z)$ scalars, correlation arbitrarily close to one, but not identified.

In many instances, the set of densities for which identification fails is dense.

$$\mathbf{D}(K) \equiv \{ f : K \to \mathbf{R} : f \geq 0, \int_K f(x,z)dxdz = 1, \ f \text{ is continuous } \} .$$

Further define the subset of $\mathbf{D}(K)$ for which identification fails by:

$$\mathbf{D}_{\emptyset}(K) \equiv \{ f \in \mathbf{D}(K) : 0 \neq \theta \in W^s(\mathcal{X}), \text{ such that } \int_K \theta(x) f(x,z)dx = 0 \ \ \forall z \} .$$

**Lemma** If $K$ is compact, then $\mathbf{D}_{\emptyset}(K)$ is dense in $\mathbf{D}(K)$ under $\| \cdot \|_{\infty}$.

# Literature Review

**Nonparametric/Semiparametric IV:** Ai & Chen (2003), Darolles, Florens & Renault (2003), Newey & Powell (2003), Blundell, Chen & Kristensen (2004), Hall & Horowitz (2005), Horowitz (2006, 2007), Chen & Pouzo (2008, 2010).

**Causality/Triangular systems:** Newey, Powell & Vella (1999), Chesher (2003, 2005, 2007), Imbens & Newey (2006).

**Partial Identification:** Manski (2003), Chernozhukov, Hong & Tamer (2004), Severini & Tripathi (2006, 2007), Romano & Shaikh (2008, 2009), Chernozhukov, Lee & Rosen (2009).

**Specification Testing:** Anderson & Rubin (1949), Bierens (1990), Bierens & Ploberger (1997), Stinchcombe & White (1998).

# Talk Outline

1. Testing Framework.
2. Test Statistic and Asymptotic Distribution.
3. Almost Sure Consistent Bootstrap.
4. Monte Carlo Evidence.

# Testing Framework

# Basic Setup

**Identified Set:** Models consistent with the exogeneity assumption on $Z$

$$\Theta_0 \equiv \{\theta \in \Theta : E[Y - \theta(X)|Z] = 0\}$$

**Hypothesis Tests:** Does at least one element of $\Theta_0$ satisfy a restriction $R$?

$$H_0 : \Theta_0 \cap R \neq \emptyset \qquad H_1 : \Theta_0 \cap R = \emptyset$$

**Under Identification:** If $\Theta_0$ is a singleton, so that $\Theta_0 = \{\theta_0\}$, then we have:

$$H_0 : \Theta_0 \cap R \neq \emptyset \Leftrightarrow \theta_0 \in R \qquad H_1 : \Theta_0 \cap R = \emptyset \Leftrightarrow \theta_0 \notin R$$

**Hence:** Under identification, analysis simplifies to inference on true parameter.

# The Set R

Functions in the set $R$ are assume to satisfy a linear equality restriction:

$$R \equiv \{\theta \in W^{c\delta}(\mathcal{X}) : L(\theta) = l\ \}$$

**Assumption:** $L : (W^{c\delta}(\mathcal{X}), \|\cdot\|_{c\delta}) \to (\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ is linear, continuous operator.

**Comments**

- Restriction of $L$ linear compensated by flexibility in choosing $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$.
- Strength of norm $\|\cdot\|_{c\delta}$ makes continuity easy to verify.
- Assumption can be relaxed to $R$ closed subset of $W^{c\delta}(\mathcal{X})$ ...
  ... **but** inference becomes potentially conservative.

# Identifiable Functionals

Often we are interested in a functional $f : \Theta \to \mathbf{R}^k$, and the identified set:

$$\mathcal{F}_0 \equiv \{ f(\theta) : \theta \in \Theta_0 \} .$$

**Goal:** Construct a confidence region $C_n(1-\alpha)$ satisfying the requirement:

$$\inf_{\theta \in \Theta_0} \liminf_{n \to \infty} P(f(\theta) \in C_n(1-\alpha)) \geq 1 - \alpha .$$

**Solution:** Proceed by test inversion of the family of null hypotheses:

$$H_0(\gamma) : \Theta_0 \cap R(\gamma) \neq \emptyset \qquad R(\gamma) \equiv \{\theta \in W^{c\delta}(\mathcal{X}) : f(\theta) = \gamma\} .$$

**If** size can be controlled for each $H_0(\gamma)$, **then** coverage requirement is satisfied.

# Identifiable Functionals

**Example:** Suppose we want to know the value of $\theta_0$ at a point $x_0$, then:

$$R(\gamma) = \{\theta \in W^{c\delta}(\mathcal{X}) : \theta(x_0) = \gamma\} \ .$$

Just let $(\mathcal{L}, \|\cdot\|_{\mathcal{L}}) = (\mathbf{R}, \|\cdot\|)$ and $L(\theta) \equiv \theta(x_0)$. Also applies to derivatives.

**Example:** Let $\theta(p, x)$ denote a demand function. For elasticity at a point $(p_0, x_0)$:

$$R(\gamma) = \{\theta \in W^{c\delta}(\mathcal{X}) : -p_0 \frac{\partial \theta(p_0, x_0)}{\partial p} \frac{1}{\theta(p_0, x_0)} = \gamma\} \ .$$

Now let $(\mathcal{L}, \|\cdot\|_{\mathcal{L}}) = (\mathbf{R}, \|\cdot\|)$ and $L(\theta) = -p_0 \frac{\partial \theta(p_0, x_0)}{\partial p} - \gamma \theta(p_0, x_0)$ and set:

$$R = \{\theta \in W^{c\delta}(\mathcal{X}) : L(\theta) = 0\} \ .$$

# Specification Testing

Let $\mathcal{X}$ be compact, $\{\psi_k\}_{k=1}^K \in W^{c\delta}(\mathcal{X})$ and define the parametric family:

$$\mathcal{P} \equiv \{\theta \in W^{c\delta}(\mathcal{X}) : \theta(x) = \sum_{k=1}^{K} \beta_k \psi_k(x)\} .$$

Suppose we wish to test whether $\Theta_0$ intersects with the parametric model,

$$H_0 : \Theta_0 \cap \mathcal{P} \neq \emptyset \qquad H_1 : \Theta_0 \cap \mathcal{P} = \emptyset .$$

Let $(\mathcal{L}, \|\cdot\|_{\mathcal{L}}) = (L^2(\mathcal{X}), \|\cdot\|_{\mathcal{L}})$ and $P_{\mathcal{P}}(\theta)$ be the projection of $\theta \in L^2(\mathcal{X})$ onto $\mathcal{P}$

$$\Theta_0 \cap \mathcal{P} = \Theta_0 \cap R \qquad R = \{\theta \in W^{c\delta}(\mathcal{X}) : L(\theta) = 0\} \qquad L(\theta) = P_{\mathcal{P}}(\theta) - \theta .$$

**Note:** Key property is $\mathcal{P}$ be a vector subspace. Also for semiparametric models.

# Homogeneity

Suppose we wish to test for homogeneous production functions of degree $\alpha$.

$$\mathcal{P} \equiv \{\theta \in W^{c\delta}(\mathcal{X}) : \theta(\lambda k, \lambda l) = \lambda^{\alpha}\theta(k,l)\} \ .$$

To characterize homogeneity as a linear restriction, use Euler's Theorem:

$$\theta(\lambda k, \lambda l) = \lambda^{\alpha}\theta(k,l) \quad \Leftrightarrow \quad k\frac{\partial\theta(k,l)}{\partial k} + l\frac{\partial\theta(k,l)}{\partial l} = \alpha\theta(k,l) \ .$$

Let $(\mathcal{L}, \|\cdot\|_{\mathcal{L}}) = (L^{\infty}(\mathcal{X}), \|\cdot\|_{\infty})$ and $L(\theta) = k\frac{\partial\theta(k,l)}{\partial k} + l\frac{\partial\theta(k,l)}{\partial l} - \alpha\theta(k,l)$ to obtain

$$\Theta_0 \cap \mathcal{P} = \Theta_0 \cap R \qquad\qquad R = \{\theta \in W^{c\delta}(\mathcal{X}) : L(\theta) = 0\} \ .$$

and test whether the identified set $\Theta_0$ contains homogenous functions.

# Test Statistic

# Testing Outline

Since $\Theta_0 \subseteq \Theta$, we can rephrase the null hypothesis $H_0 : \Theta_0 \cap R \neq \emptyset$ as:

Is there a $\theta \in \Theta \cap R$ such that $E[Y - \theta(X)|Z] = 0$?

**Goal:** Use this observation for a simple characterization of the null hypothesis.

**Strategy**

- Suppose you had a positive functional $F : \Theta \to \mathbf{R}$ such that:

$$F(\theta) = 0 \quad iff \quad \theta \in \Theta_0 \; .$$

- If $F$ is continuous under $\|\cdot\|_{c\delta}$ and $R$ is closed under $\|\cdot\|_{c\delta}$, then:

$$\Theta_0 \cap R \neq \emptyset \quad iff \quad \min_{\theta \in \Theta \cap R} F(\theta) = 0 \; .$$

... because $\Theta$ is compact under $\|\cdot\|_{c\delta}$.

# Conditional to Unconditional

## Revealing Functions

- Let $\mathcal{Z}$ be the support of $Z$, let $T \subset \mathbf{R}^{d_t}$ and $w : T \times \mathcal{Z} \to \mathbf{R}$ be such that:

$$E[V|Z] = 0 \quad \text{if and only if} \quad E[Vw(t,Z)] = 0 \;\; \forall t \in T \quad (GCR)$$

- **Examples:** Bierens (1990), Stinchcombe & White (1998).

**Lemma:** Under regularity conditions, it follows that if $GCR$ holds:

$$\theta \in \Theta_0 \quad iff \quad \max_{t \in T} \; (E[(Y - \theta(X))w(t,Z)])^2 = 0 \; .$$

Moreover, exploiting compactness and continuity we also get:

$$\Theta_0 \cap R \neq \emptyset \quad iff \quad \min_{\theta \in \Theta \cap R} \max_{t \in T} \; (E[(Y - \theta(X))w(t,Z)])^2 = 0 \; .$$

# Test Statistic

$$I_n(R) \equiv \min_{\theta \in \Theta_n \cap R} \max_{t \in T_n} \Big( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Y_i - \theta(X_i)) w(t, Z_i) \Big)^2$$

## Sieve Details

- $T_n$ is a grid for $T$. The whole set $T$ may be employed as well.
- For the sieve of $\Theta \cap R$ we consider:

$$\Theta_n \cap R \equiv \{\theta \in W^s(\mathcal{X}) : \theta(x) = p^{k_n'}(x)h \text{ for } h \in \mathbf{R}^{k_n},\ L(\theta) = l,\ \ \|\theta\|_s \leq B\}\ .$$

  where $p^{k_n}(x) = (p_1(x), \ldots, p_{k_n}(x))'$ and $p_i \in W^s(\mathcal{X})$ for all $i$.

- Constraints $L(\theta) = l$ is linear in $h$ and $\|\theta\|^2 \leq B^2$ quadratic in $h$.

# Local Parameter Space

- For each $\theta_0 \in \Theta_0 \cap R$, let $\Pi_n \theta_0$ be its projection onto $\Theta_n \cap R$ (under $\|\cdot\|_{L^2}$).

$$\underbrace{\theta_n(x)}_{\text{in } R} \equiv \underbrace{\Pi_n \theta_0(x)}_{\text{in } R} + p^{k_n'}(x)\frac{h}{\sqrt{n}} \ .$$

Since we must have $L(p^{k_n'}h) = 0$, the local values of $h$ are contained in the set

$$\mathcal{H}_{k_n} \equiv \{h \in \mathbf{R}^{k_n} : L(p^{k_n'}h) = 0\} \ .$$

- Distribution depends on effect of local parameters on criterion function. Let,

$$V_{k_n}(T) \equiv \{v : T \to \mathbf{R} \text{ s.t. } v(t) = E[w(t, Z)p^{k_n'}(X)h], \ h \in \mathcal{H}_{k_n}\}$$

Define the function space $V_\infty(T)$ the closure of $\bigcup V_{k_n}(T)$ under $\|\cdot\|_\infty$.

# Some Intuition

$$\min_{\Theta_n \cap R} \max_{T_n} \Big( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Y_i - \theta_n(X_i)) w(t, Z_i) \Big)^2$$

$$\approx \min_{\Theta_n \cap R} \max_{T_n} \Big( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Y_i - \theta_0(X_i)) w(t, Z_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\Pi_n \theta_0(X_i) - \theta_n(X_i)) w(t, Z_i) \Big)^2$$

$$\approx \min_{h \in \mathcal{H}_{k_n}} \max_{T_n} \Big( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Y_i - \theta_0(X_i)) w(t, Z_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{p^{k_n'}(X_i) h}{\sqrt{n}} w(t, Z_i) \Big)^2$$

$$\approx \min_{h \in \mathcal{H}_{k_n}} \max_{T_n} \Big( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underbrace{(Y_i - \theta_0(X_i)) w(t, Z_i)}_{\text{Gaussian Process}} + \underbrace{E[w(t, Z_i) p^{k_n'}(X_i) h]}_{\text{in } V_{k_n}(T)} \Big)^2$$

# Asymptotic Distribution

**Theorem:** Under appropriate regularity conditions, if $\Theta_0 \cap R \neq \emptyset$, then:

$$I_n(R) \xrightarrow{L} \inf_{\theta_0 \in \Theta_0 \cap R} \inf_{v \in V_\infty(T)} \|G(t,\theta_0) - v(t)\|_\infty^2$$

where $G(t,\theta_0)$ is a tight Gaussian process on $L^\infty(T \times \Theta_0)$. If $\Theta_0 \cap R = \emptyset$, then:

$$n^{-1} I_n(R) \xrightarrow{a.s.} \min_{\theta \in \Theta \cap R} \|E[(Y - \theta(X))w(t,Z)]\|_\infty^2$$

.

## Comments

- Statistic has a proper limit distribution under the null hypothesis.
- Statistic diverges to infinity under the alternative hypothesis.

# Alternative Result

**Remark:** If $c_{1-\alpha}$ is the $1-\alpha$ quantile of $\inf_{\Theta_0 \cap R} \|G(t,\theta_0)\|_\infty^2$ ...

... Then it is possible to show that under the null hypothesis:

$$\liminf_{n\to\infty} P(I_n(R) \geq c_{1-\alpha}) \geq 1-\alpha \ .$$

... While under the alternative hypothesis we have:

$$\lim_{n\to\infty} P(I_n(R) \geq c_{1-\alpha}) = 1 \ .$$

**Comments**

- Only requires that $R$ be closed under $\|\cdot\|_{c\delta}$.
- Potentially conservative.

# Bootstrap Procedure

# The Unknowns

Limiting distribution under the null hypothesis:

$$I_n(R) \xrightarrow{L} \inf_{\theta_0 \in \Theta_0 \cap R} \inf_{v \in V_\infty(T)} \|G(t, \theta_0) - v(t)\|_\infty^2 .$$

## Three Unknowns

- Distribution of the Gaussian process $G(t, \theta_0)$.
- Identified set $\Theta_0 \cap R$.
- The function space $V_\infty(T)$.

# Estimating $V_\infty(T)$

Recall that $V_\infty(T)$ is the closure of $\bigcup V_{k_n}(T)$ under $\|\cdot\|_\infty$, where:

$$V_{k_n}(T) \equiv \{v : T \to \mathbf{R} : \text{ s.t. } v(t) = E[w(t,Z)p^{k'_n}(X)h],\ L(p^{k'_n}h) = 0\ \}\ .$$

For some $b_n \nearrow \infty$ and $B_n \nearrow \infty$, define the sample analogue:

$$\hat{V}_{b_n}(T) \equiv \left\{v : T \to \mathbf{R} : \text{ s.t. } v(t) = \frac{1}{n}\sum_{i=1}^{n} w(t,Z_i)p^{b'_n}(X_i)h,\ L(p^{b'_n}h) = 0,\ \|h\| \leq B_n\ \right\}.$$

## Comments

- $b_n$ plays role of $k_n$ but they need not be equal.
- The norm bound $B_n$ is imposed to obtain a uniform law of large numbers.

# Estimating $G(t, \theta_0)$

Recall that the Gaussian process $G(t, \theta_0)$ on $L^\infty(T \times \Theta_0)$ is the limit of:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Y_i - \theta_0(X_i)) w(t, Z_i) \xrightarrow{L} G(t, \theta_0) \; .$$

For $(Y_i^*, X_i^*, Z_i^*)$ distributed according to the empirical distribution, we define:

$$G_n^*(t, \theta) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{(Y_i^* - \theta(X_i^*)) w(t, Z_i^*) - E^*[(Y_i - \theta(X_i)) w(t, Z_i)] \; .\}$$

**Problem**

- $G_n^*(t, \theta)$ is properly centered for all $(t, \theta) \in T \times \Theta$.
- $G_n^*(t, \theta)$ convergence is in $L^\infty(T \times \Theta)$ not $L^\infty(T \times \Theta_0)$.
- Need to evaluate restriction of $G_n^*(t, \theta)$ to proper domain.

# Estimating $\Theta_0 \cap R$

**Goal:** Use penalty function to evaluate $G_n^*(t,\theta)$ on proper set. Define:

$$P_n^*(t,\theta) \equiv \Big(\frac{1}{n}\sum_{i=1}^{n}(Y_i - \theta(X_i))w(t,Z_i)\Big)^2$$

**Indicator for Identified Set**

If $\lambda_n \nearrow \infty$ at an appropriate rate, then the penalty function:

$$\lambda_n \max_{t\in T} P_n^*(t,\theta)$$

... converges a.s. to zero for all $\theta \in \Theta_0$ ... but diverges a.s. to $+\infty$ for all $\theta \notin \Theta_0$.

# Consistency

$$I_n^*(R) \equiv \inf_{\theta \in \Theta_n \cap R} \inf_{v \in \hat{V}_{b_n}(T)} \max_{t \in T_n} \{(G_n^*(t,\theta) - v(t))^2 + \lambda_n P_n^*(t,\theta)\}$$

**Theorem** Under appropriate regularity conditions, if $\Theta_0 \cap R \neq \emptyset$, then:

$$I_n^*(R) \xrightarrow{L^*} \inf_{\theta_0 \in \Theta_0 \cap R} \inf_{v \in V_\infty(T)} \|G(t,\theta_0) - v(t)\|_\infty^2 \qquad a.s.$$

On the other hand, if $\Theta_0 \cap R = \emptyset$ then we obtain:

$$\lambda_n^{-1} I_n^*(R) \xrightarrow{p^*} \min_{\theta \in \Theta \cap R} \|E[(Y - \theta(X))w(t,Z)]\|_\infty^2 \qquad a.s.$$

**Note:** Under the null, bootstrap equivalent to plug-in estimator for $\hat{\Theta}_0$.

# Inference

$$\hat{c}_{1-\alpha} \equiv \inf\{u : P^*(I_n^*(R) \leq u) \geq 1 - \alpha\}$$

**Corollary** Under $H_0$, if limit distribution of $I_n(R)$ is continuous, strictly increasing,

$$\lim_{n\to\infty} P(I_n(R) \leq \hat{c}_{1-\alpha}) = 1 - \alpha \ .$$

On the other hand, if $\Theta_0 \cap R = \emptyset$, then we have:

$$\lim_{n\to\infty} P(I_n(R) > \hat{c}_{1-\alpha}) = 1 \ .$$

**Note:** Consistency is due to $I_n^*(R)$ diverging to infinity at slower rate than $I_n(R)$.

# Monte Carlo

# Monte Carlo

## Distribution Design

- $(X, Z, \epsilon)$ transformed from multivariate normal $(X^*, Z^*, \epsilon^*)$.
- $\rho(X^*, Z^*) = 0.5$ and $\rho(X^*, \epsilon^*) = 0.3$, $(X, Z)$ have compact support.
- True model: $Y = 2\sin(X\pi) + \epsilon$

## Implementation Details

- B-Splines used for sieve $\Theta_n$.
- Weight function $w(t, z) = \phi((t_1 - z)/t_2)$, where $\phi(u)$ is normal pdf.
- 500 replications, sample size of 500.

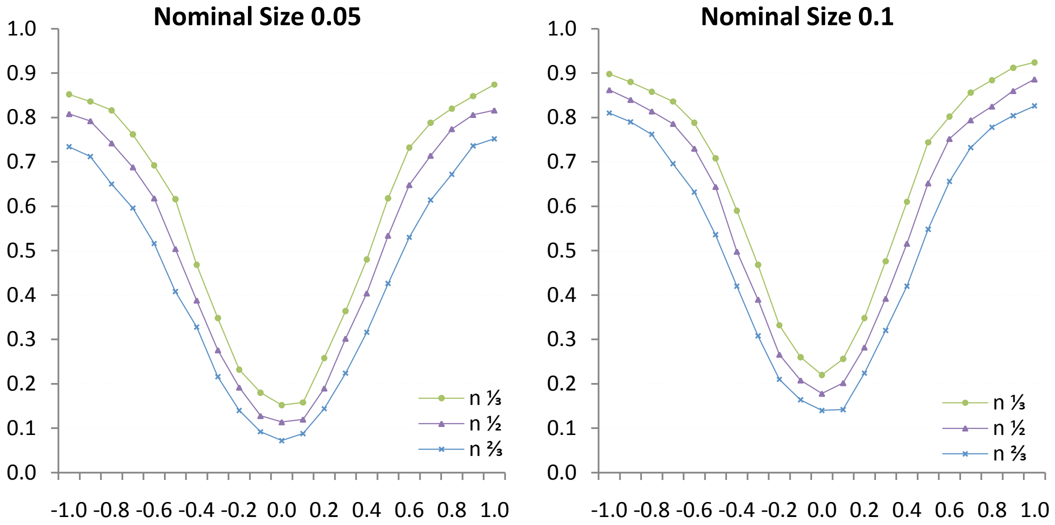
# Empirical Size

**Null hypothesis:** Does $\theta_0(0) = 0$?

| $\alpha/\lambda_n$ | $\lambda_n = 0$ | $\lambda_n = n^{\frac{1}{3}}$ | $\lambda_n = n^{\frac{1}{2}}$ | $\lambda_n = n^{\frac{2}{3}}$ |
|---|---|---|---|---|
| $\alpha = 0.1$ | 0.508 | 0.220 | 0.178 | 0.140 |
| $\alpha = 0.05$ | 0.378 | 0.152 | 0.114 | 0.072 |
| $\alpha = 0.01$ | 0.198 | 0.050 | 0.028 | 0.014 |

**Comments**

- $\lambda_n = 0$ not warranted by theory. Should over-reject.
- $(n^{\frac{1}{3}}, n^{\frac{1}{2}}, n^{\frac{2}{3}}) \approx (7.9, 22.4, 63)$ ... broad range for choices.
- $n^{\frac{1}{3}}$ seems to be too small, controls size poorly.



Nominal Size 0.05
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
n ⅓
n ½
n ⅔
Nominal Size 0.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
n ⅓
n ½
n ⅔

# Final Remarks

## Partial Identification

- Smoothness restriction aid in identification but do not guarantee it.
- Straightforward to construct examples where identification fails.

## Methods for Inference

- Robust to partial identification.
- Identifiable functionals through test inversion.
- Bootstrap procedure for obtaining critical values.