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ON THE TESTABILITY OF IDENTIFICATION IN SOME NONPARAMETRIC MODELS WITH ENDOGENEITY

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NOTES AND COMMENTS

ON THE TESTABILITY OF IDENTIFICATION IN SOME
NONPARAMETRIC MODELS WITH ENDOGENEITY

BY IVAN A. CANAY, ANDRES SANTOS, AND AZEEM M. SHAIKH¹

This paper examines three distinct hypothesis testing problems that arise in the context of identification of some nonparametric models with endogeneity. The first hypothesis testing problem we study concerns testing necessary conditions for identification in some nonparametric models with endogeneity involving mean independence restrictions. These conditions are typically referred to as completeness conditions. The second and third hypothesis testing problems we examine concern testing for identification directly in some nonparametric models with endogeneity involving quantile independence restrictions. For each of these hypothesis testing problems, we provide conditions under which *any* test will have power no greater than size against *any* alternative. In this sense, we conclude that no nontrivial tests for these hypothesis testing problems exist.

KEYWORDS: Instrumental variables, identification, completeness, bounded completeness.

1. INTRODUCTION

INSTRUMENTAL VARIABLES (IV) METHODS have a prominent role in econometrics due to their ability to uncover causal effects in observational studies. Though traditionally parametric in nature, an important literature has extended IV methods to a variety of nonparametric settings. Among these extensions, of particular prominence is the additively separable specification in which, for an outcome of interest Y , a regressor W , and an instrument Z , it is assumed that

$$(1) \quad Y = \theta(W) + \varepsilon,$$

with ε mean independent of Z . Under the maintained assumption that the model is correct, Newey and Powell (2003) showed identification of θ to be equivalent to the joint distribution of W and Z satisfying a completeness condition. Complementing their prevalent use in statistics (Lehmann and Scheffé (1950, 1955)), completeness conditions have since then been widely used in econometrics—see Hall and Horowitz (2005), Blundell, Chen, and Kristensen

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(2007), Hu and Schennach (2008), Cunha, Heckman, and Schennach (2010), Berry and Haile (2010a), Darolles, Fan, Florens, and Renault (2011), and the references therein.

Despite the widespread use of completeness conditions in econometrics, little evidence has been provided about their reasonableness in applications of interest to economists. We note, however, that since completeness conditions impose restrictions on the distribution of the observed data, it is potentially possible to provide such evidence by testing the validity of these assumptions. This paper explores precisely this possibility. Specifically, we study whether it is possible to test the null hypothesis that a completeness condition does not hold against the alternative that it does hold. Such a hypothesis testing problem is consistent with a setting in which a researcher wishes to assert that the model is identified and hopes to find evidence in favor of this claim in the data. This setup is also analogous to tests of rank conditions in linear models with endogeneity, where the null hypothesis is that of rank-deficiency.

In this paper, we show that, under commonly imposed restrictions on the distribution of the data, the null hypothesis that the completeness condition does not hold is in fact untestable. Formally, we establish that *any* test will have power no greater than size against *any* alternative. It is therefore not possible to provide empirical evidence in favor of the completeness condition by means of such a test. This conclusion is in contrast to the testability of a failure of the rank condition in linear specifications of θ , for which nontrivial tests do exist under reasonable assumptions. Thus, while completeness conditions provide an intuitive generalization of the rank condition in a linear specification, the empirical implications of these assumptions are substantially different in this sense.

We additionally derive analogous results in two other prominent nonparametric models with endogeneity. The first such model follows the specification in (1) with a prespecified conditional quantile of ε assumed independent of Z —see Chernozhukov and Hansen (2005), Horowitz and Lee (2007), Gagliardini and Scaillet (2012), and Chen and Pouzo (2012). The second such model follows a specification in which θ is allowed to depend nonseparably on both W and ε , with the dependence on ε being monotonic, and all conditional quantiles of ε assumed independent of Z —see Chernozhukov and Hansen (2005), Imbens and Newey (2009), Torgovitsky (2012), d’Haultfoeuille and Février (2012), and Berry and Haile (2009, 2010b). Due to the nonlinear nature of such models, simple, global rank conditions such as completeness conditions are unavailable. For this reason, we instead directly consider the testability of the null hypothesis that identification fails against the alternative hypothesis that it holds. Analogously to our results concerning the testability of completeness conditions, we obtain conditions under which no nontrivial tests exist for these hypothesis testing problems either.

We emphasize that our results should not be interpreted as an indictment against nonparametric methods in models with endogeneity. The nontestability of completeness conditions or identification does not imply that these

assumptions are false. The impossibility of providing supporting empirical evidence by means of such a test, however, does suggest that it is prudent to justify their use with alternative arguments in favor of their validity. Indeed, much research relies on assumptions that are untestable, but are nonetheless deemed reasonable on different grounds. Important progress in this regard has been recently made by [Andrews \(2011\)](#) and [Chen, Chernozhukov, Lee, and Newey \(2012\)](#), who established different genericity results for distributions satisfying completeness restrictions. Each of these results provides a sense under which completeness conditions are satisfied by “most” distributions in a large class of distributions. Thus, while completeness conditions may be untestable, one may argue that they are “commonly satisfied” by appealing to such results.

We further stress that completeness or identification assumptions are not necessary for conducting nonparametric inference in models with endogeneity. As is well understood in semiparametric models, certain inferential methods can be robust to whether identification is “strong,” “weak,” or partial—see, among others, [Anderson and Rubin \(1949\)](#), [Kleibergen \(2002\)](#), and [Moreira \(2003\)](#) for results in the classical, linear IV setting, and [Chernozhukov, Hansen, and Jansson \(2007, 2009\)](#) for extensions to settings with quantile restrictions. Insights from these papers have also been employed in the development of inferential procedures for general partially identified models—see [Chernozhukov, Hong, and Tamer \(2007\)](#) and [Romano and Shaikh \(2008\)](#). Our impossibility results emphasize the value of extending these approaches to nonparametric settings. Recent work toward this end includes [Freyberger and Horowitz \(2012\)](#) and [Santos \(2012\)](#), who, in the context of the first model we consider, developed inferential procedures that are robust to the possibility that the completeness condition fails and the model is therefore only partially identified. See also [Chen, Tamer, and Torgovitsky \(2011\)](#) for inferential procedures that are robust to partial identification in semiparametric likelihood models.

Finally, it is important to note that our results do not preclude the testability of completeness conditions or identification under alternative restrictions on θ or the distribution of the observed data. For instance, our arguments may not extend easily to settings where θ is restricted to be a density ([Hoderlein, Nesheim, and Simoni \(2012\)](#)) or is semiparametrically specified ([Ai and Chen \(2003\)](#)). We further discuss the possible implications of alternative restrictions on θ or the distribution of the observed data in the text—see [Remarks 3.1–3.2](#) and the discussion following those remarks.

This paper contributes to an important literature on impossibility results in econometrics—see [Leeb and Pötscher \(2008\)](#) and [Müller \(2008\)](#) for recent examples, and [Dufour \(2003\)](#) for an excellent overview. First among such results is [Bahadur and Savage \(1956\)](#), who documented the impossibility of conducting nontrivial inference on the mean without appropriate restrictions on the distribution of the observed data. [Romano \(2004\)](#) later showed that this result is the consequence of the set of distributions satisfying the alternative hypothesis lying in the closure, under the Total Variation distance, of the set of

distributions satisfying the null hypothesis. Intuitively, if, for every distribution satisfying the alternative hypothesis, there exists an arbitrarily “close” distribution satisfying the null hypothesis, then it will be impossible to discriminate between them from data—see Pötscher (2002) for related ideas. Our results share this intuition, but require novel arguments for showing that distributions for which a completeness condition or identification fails can approximate distributions for which it holds arbitrarily well in the Total Variation distance.

The remainder of the paper is organized as follows. Section 2 elaborates on the nature of our impossibility results, and reviews a general framework for deriving them. The main results are developed in Section 3. Section 4 briefly concludes. All proofs are contained in the Appendix.

2. SETUP

Before formally stating the three distinct hypothesis problems we consider, it will be useful to introduce some notation and elaborate on the nature of our impossibility results. Toward this end, we let $\{V_i\}_{i=1}^n$ be an independent and identically distributed (i.i.d.) sequence of random variables with distribution $P \in \mathbf{P}$, and denote by P^n the n -fold product $\otimes_{i=1}^n P$. The hypothesis testing problems we study may then be expressed as

$$(2) \quad H_0 : P \in \mathbf{P}_0 \quad \text{versus} \quad H_1 : P \in \mathbf{P}_1,$$

where \mathbf{P}_0 is the subset of \mathbf{P} for which the null hypothesis holds and $\mathbf{P}_1 = \mathbf{P} \setminus \mathbf{P}_0$ is the subset of \mathbf{P} for which the alternative hypothesis holds. For a sequence of (possibly randomized) tests $\{\phi_n\}_{n=1}^\infty$, the corresponding size at sample size n is given by

$$\sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n].$$

In our analysis, we will show that under commonly imposed restrictions on the set of distributions \mathbf{P} , the three hypothesis testing problems we examine share the property that

$$(3) \quad \sup_{P \in \mathbf{P}_1} E_{P^n}[\phi_n] \leq \sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n]$$

for any sequence of (possibly randomized) tests $\{\phi_n\}_{n=1}^\infty$ and any sample size n . Equivalently, result (3) establishes that, for *all* tests, the power against *any* alternative $P \in \mathbf{P}_1$ is always bounded above by the size of the test. It also follows from such an assertion that any sequence of (possibly randomized) tests $\{\phi_n\}_{n=1}^\infty$ that controls asymptotic size at level $\alpha \in (0, 1)$ will have asymptotic power no larger than α against any alternative. Formally, (3) immediately yields that

$$(4) \quad \limsup_{n \rightarrow \infty} \sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n] \leq \alpha \quad \implies \quad \limsup_{n \rightarrow \infty} \sup_{P \in \mathbf{P}_1} E_{P^n}[\phi_n] \leq \alpha.$$

We therefore conclude that no nontrivial tests exist for hypothesis testing problems satisfying property (3). In other words, in such settings, no test can outperform a procedure that simply ignores the data and rejects the null hypothesis with probability α .

2.1. *A Useful Lemma*

Underlying our arguments is a powerful result originally found in Romano (2004), which we restate due to its importance in our derivations. In the statement of the lemma, $\|P - Q\|_{TV}$ denotes the Total Variation distance between probability measures P and Q ; for example, for $\lambda \equiv (P + Q)/2$,

$$\|P - Q\|_{TV} \equiv \frac{1}{2} \int \left| \frac{dQ}{d\lambda} - \frac{dP}{d\lambda} \right| d\lambda.$$

LEMMA 2.1: *Let \mathbf{M} denote the space of Borel probability measures on a separable metric space \mathbf{V} . Suppose $\mathbf{P} \subseteq \mathbf{M}$ and that $\mathbf{P} = \mathbf{P}_0 \cup \mathbf{P}_1$. If, for each $P \in \mathbf{P}_1$, there exists a sequence $\{P_k\}_{k=1}^\infty$ in \mathbf{P}_0 with $\|P - P_k\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$, then every sequence of test functions $\{\phi_n\}_{n=1}^\infty$ satisfies*

$$(5) \quad \sup_{P \in \mathbf{P}_1} E_{P^n}[\phi_n] \leq \sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n] \quad \text{for all } n \geq 1.$$

Heuristically, Lemma 2.1 states that if each $P \in \mathbf{P}_1$ is on the boundary of the set of distributions satisfying the null hypothesis, then, by continuity, the probability of rejection under any $P \in \mathbf{P}_1$ must be no larger than the size of the test. Theorem 1 in Romano (2004) establishes that the appropriate topology for this purpose is that induced by the Total Variation distance. A metric compatible with weak convergence, such as the Lévy–Prokhorov metric, would be insufficient, as it would not guarantee convergence of integrals defining rejection probabilities. By contrast, the Total Variation distance between two measures P and Q is intimately related to the statistical properties of the best test for distinguishing between P and Q (Le Cam (1986)). For this reason, some authors have referred to the Total Variation distance as the “testing metric” (Donoho (1988)).

In each of the three hypothesis testing problems we consider, we establish nonexistence of nontrivial tests by constructing, for each $P \in \mathbf{P}_1$, a sequence $\{P_k\}_{k=1}^\infty$ in \mathbf{P}_0 with $\|P - P_k\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$ and applying Lemma 2.1. It is worth emphasizing that the Total Variation metric plays no role in determining \mathbf{P} nor the null and alternative hypotheses. Indeed, \mathbf{P} may possess other natural topologies, such as that induced by the Euclidean metric in parametric models, or that induced by a Hölder norm in sets of smooth densities. However, Lemma 2.1 reveals that, regardless of what topology \mathbf{P} originally has, for our purposes we must examine \mathbf{P} under the topology induced by the Total Variation metric.

It would be natural to interpret Lemma 2.1 as a negative result, but we believe it is also instructive to emphasize its constructive nature. In particular, the converse of Lemma 2.1 implies that a necessary condition for the existence of nontrivial tests is that there is a $P \in \mathbf{P}_1$ that is not in the closure of \mathbf{P}_0 with respect to $\|\cdot\|_{\text{TV}}$. Thus, an often fruitful approach for restoring testability has been to restrict \mathbf{P} in a manner that ensures that this is the case. For example, Romano (2004) showed that the original impossibility result of Bahadur and Savage (1956) can be circumvented by restricting \mathbf{P} to be a set of distributions satisfying a uniform integrability condition. This restriction on \mathbf{P} , and closely related ones, have been used widely in the recent literature on inference in partially identified models—see, for example, Romano and Shaikh (2008), Andrews and Guggenberger (2009), and Andrews and Soares (2010). Similarly, uniformly valid confidence regions for infinite-dimensional parameters have been obtained by restricting \mathbf{P} through smoothness assumptions (Robins and van der Vaart (2006), Giné and Nickl (2010)) or shape restrictions (Hengartner and Stark (1995)). See also Romano and Shaikh (2012) for restrictions on \mathbf{P} that allow for uniformly valid inference on the empirical process. Unfortunately, finding analogous restrictions on \mathbf{P} for the problems we consider in this paper is very challenging, and we therefore leave it for future work—see Remark 3.1.

3. MAIN RESULTS

In this section, we show, for each of the three hypothesis testing problems we consider, that the power of any test against any alternative is always bounded above by the size of the test; that is, we show that (3) holds. As mentioned previously, these results further imply that any test that controls asymptotic size will have trivial asymptotic power against any alternative; that is, they imply (4).

3.1. Testing Completeness

We begin by establishing the nonexistence of nontrivial tests for completeness conditions. Toward this end, we first need to introduce additional notation and formally define L^q -completeness. Let $V_i = (X_i, Z_i) \in \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$ be random variables distributed according to $P \in \mathbf{P}$. For $Z_i = (Z_i^{(1)}, Z_i^{(2)})$, with the sub-vector $Z_i^{(1)}$ possibly empty, let $W_i = (X_i, Z_i^{(1)}) \in \mathbf{R}^{d_w}$, and for $\Theta(P)$ a set of measurable functions from \mathbf{R}^{d_w} to \mathbf{R} , consider the condition on P given by

$$(6) \quad E_P[\theta(W_i)|Z_i] = 0 \text{ } P\text{-a.s. for } \theta \in \Theta(P) \implies \theta(W_i) = 0 \text{ } P\text{-a.s.}$$

For $1 \leq q \leq \infty$, the distribution P is said to be L^q -complete with respect to W given Z if condition (6) holds with $\Theta(P) = L^q(P_W)$. Here, P_W denotes the marginal distribution of W under P and $L^q(P_W)$ denotes the set of measurable functions from \mathbf{R}^{d_w} to \mathbf{R} with finite $\|\cdot\|_{L^q(P_W)}$ semi-norm—see the Appendix for definitions. For the special cases in which $q = 1$ or $q = \infty$, P is sometimes simply said to be complete with respect to W given Z or bounded complete with

respect to W given Z , respectively. See d’Haultfoeuille (2011) and Andrews (2011) for further discussion on these conditions.

As emphasized by Lemma 2.1, the definition of the set of possible distributions \mathbf{P} plays a fundamental role in setting up the hypothesis testing problem. In our analysis, we restrict attention to sets of measures \mathbf{P} that have a common dominating measure. Specifically, letting $\mathbf{M}_{x,z}$ denote the set of all Borel probability measures on $\mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$, and defining the set

$$(7) \quad \mathbf{M}_{x,z}(\nu) \equiv \{P \in \mathbf{M}_{x,z} : P \ll \nu\},$$

for some Borel measure ν on $\mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$, we let $\mathbf{P} = \mathbf{M}_{x,z}(\nu)$. In this setting, we examine the testability of the null hypothesis that the completeness condition fails, and hence, for a given choice of $\Theta(P)$, we let

$$(8) \quad \mathbf{P}_1 = \mathbf{P} \setminus \mathbf{P}_0 = \{P \in \mathbf{P} : (6) \text{ holds under } P\}.$$

Defining the null hypothesis in this manner is analogous to testing the null hypothesis of a failure of the rank condition in a linear specification—see Remark 3.3. For instance, in a simple model where $(Y_i, X_i, Z_i) \in \mathbf{R}^3$ are distributed according to P , and for some $\theta \in \mathbf{R}$ we have

$$(9) \quad E_P[Z_i(Y_i - X_i\theta)] = 0,$$

the rank condition required for identification reduces to $E_P[X_i Z_i] \neq 0$. Thus, our setup is analogous to testing the null hypothesis $E_P[X_i Z_i] = 0$ (rank deficiency) against the alternative hypothesis $E_P[X_i Z_i] \neq 0$ (rank condition holds). We note that, in this simple example, the null hypothesis that the rank condition holds ($E_P[X_i Z_i] \neq 0$) is untestable under typical assumptions.

We will make use of the following assumptions in establishing our result:

ASSUMPTION 3.1: ν is a positive σ -finite Borel measure on $\mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$.

ASSUMPTION 3.2: $\nu = \nu_x \times \nu_z$, where ν_x and ν_z are Borel measures on \mathbf{R}^{d_x} and \mathbf{R}^{d_z} , respectively.

ASSUMPTION 3.3: The measure ν_x is atomless on \mathbf{R}^{d_x} .

Note that since we impose the requirement that $\mathbf{P} = \mathbf{M}_{x,z}(\nu)$ for some ν satisfying Assumptions 3.1–3.3, properties of ν translate into restrictions on \mathbf{P} . For instance, if ν has bounded support, then $\mathbf{P} = \mathbf{M}_{x,z}(\nu)$ implies that the support of (X_i, Z_i) under P is uniformly bounded in $P \in \mathbf{P}$. In particular, by choosing ν_x and ν_z to be the Lebesgue measure on $[0, 1]^{d_x}$ and $[0, 1]^{d_z}$, respectively, we may impose the requirement that the support of (X_i, Z_i) under P be contained in $[0, 1]^{d_x} \times [0, 1]^{d_z}$ for all $P \in \mathbf{P}$. See Hall and Horowitz (2005) and Horowitz and Lee (2007) for examples of the use of such an assumption. It is also worth emphasizing that while Assumption 3.2 imposes that ν be a product measure, the

requirement that $\mathbf{P} = \mathbf{M}_{x,z}(\nu)$ for some such ν does not imply that each $P \in \mathbf{P}$ is itself of such form. On the other hand, the requirement that $\mathbf{P} = \mathbf{M}_{x,z}(\nu)$ for some ν satisfying Assumptions 3.2 and 3.3 does imply that $P\{X_i \neq Z_i\} > 0$ for all $P \in \mathbf{P}$. Thus, our assumptions rule out $X_i = Z_i$ P -a.s. for some $P \in \mathbf{P}$, in which case condition (6) would be trivially satisfied. Finally, we point out that if $d_x > 1$, then Assumption 3.3 may be weakened to instead requiring that at least one component of X_i has an atomless marginal measure. For ease of exposition, however, we impose the stronger than necessary requirement in Assumption 3.3.

Under Assumptions 3.1–3.3, we can now establish the nontestability of bounded completeness.

THEOREM 3.1: *Suppose Assumptions 3.1, 3.2, and 3.3 hold. If $\mathbf{P} = \mathbf{M}_{x,z}(\nu)$, for $\mathbf{M}_{x,z}(\nu)$ as in (7), and \mathbf{P}_0 and \mathbf{P}_1 are as in (8) with $\Theta(P) = L^\infty(P_W)$, then, for any sequence of tests $\{\phi_n\}_{n=1}^\infty$,*

$$(10) \quad \sup_{P \in \mathbf{P}_1} E_{P^n}[\phi_n] \leq \sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n] \quad \text{for all } n \geq 1.$$

Theorem 3.1 establishes the nonexistence of nontrivial tests for bounded completeness. The result is proven by showing that the set of distributions for which bounded completeness holds (\mathbf{P}_1) lies in the closure of the set of distributions for which bounded completeness fails (\mathbf{P}_0) with respect to the Total Variation distance and then appealing to Lemma 2.1. It therefore follows that the conclusion of Theorem 3.1 continues to hold if $\Theta(P)$ instead satisfies $L^\infty(P_W) \subseteq \Theta(P)$. Any such modification only enlarges \mathbf{P}_0 , and hence \mathbf{P}_1 continues to lie in the closure of \mathbf{P}_0 with respect to the Total Variation distance. In particular, by setting $\Theta(P) = L^q(P_W)$ for any $1 \leq q < \infty$, we are able to conclude from Theorem 3.1 that there exist no nontrivial tests of L^q -completeness conditions either. It is worth emphasizing, however, that these results do rely on setting $\mathbf{P} = \mathbf{M}_{x,z}(\nu)$ and having $L^\infty(P_W) \subseteq \Theta(P)$. Thus, Theorem 3.1 does not preclude the existence of nontrivial tests for completeness conditions under more demanding restrictions on $\Theta(P)$ or \mathbf{P} ; see Remarks 3.1–3.3 and the discussion thereafter.

While we focus on the testability of L^q -completeness conditions due to their importance in the literature, it is worth noting that they are only necessary conditions for identification. Indeed, if we consider P as a measure on $\mathbf{R} \times \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$ instead, then $P_{X,Z}$ being L^q -complete with respect to W given Z does not guarantee existence of a solution (in $\theta \in L^q(P_W)$) to the equation

$$(11) \quad E_P[Y_i - \theta(W_i)|Z_i] = 0 \quad P\text{-a.s.}$$

Rather, $P_{X,Z}$ being L^q -complete with respect to W given Z just ensures that if a solution to (11) does exist, then it must be unique—see Proposition 2.1 in Newey and Powell (2003). Thus, $P_{X,Z}$ being L^q -complete with respect to

W given Z is only equivalent to identification under the additional assumption that (11) holds for some $\theta \in L^q(P_W)$. In this sense, completeness requirements are analogous to the rank condition in a linear specification, which is necessary, but not sufficient, for ensuring identification; for example, if $(Y_i, X_i, Z_i) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^2$, then $E_P[X_i|Z_i]$ being full rank does not guarantee the existence of solution (in $\theta \in \mathbf{R}$) to equation (9).

REMARK 3.1: In establishing Theorem 3.1, we construct, for each $P \in \mathbf{P}_1$, a sequence $\{P_k\}_{k=1}^\infty$ in \mathbf{P}_0 such that $\|P - P_k\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$, and then apply Lemma 2.1 to obtain the desired conclusion. This approach requires us to exhibit, for each P_k , a corresponding function $\theta_k \in \Theta(P_k)$ such that

$$P_k\{\theta_k(W_i) \neq 0\} > 0 \quad \text{and} \quad E_{P_k}[\theta_k(W_i)|Z_i] = 0 \quad P_k\text{-a.s.}$$

While the θ_k that appear in the proof are not differentiable everywhere, it is worth emphasizing that this is not an essential feature of the argument. For instance, if ν equals the restriction of the Lebesgue measure on $\mathbf{R}^{d_x+d_z}$ to a compact set $K \subset \mathbf{R}^{d_x+d_z}$ and $\nu\{K\} > 0$, then the construction in Lemma 2.1 of Santos (2012) implies that P_k may be chosen so that both $dP_k/d\nu$ and θ_k are polynomials of finite order. Therefore, no nontrivial tests exist even if $\Theta(P)$ is further restricted to be a smooth class of functions, such as a Hölder space. By rescaling θ_k appropriately, we may, in fact, even restrict $\Theta(P)$ to be a Hölder ball. Analogously, no nontrivial tests exist if, for some finite integer p , we restrict every $P \in \mathbf{P}$ to be such that $dP/d\nu$ is p -times continuously differentiable, nor if we impose explicit bounds on the supremum of the first p derivatives of $dP/d\nu$ on K .

REMARK 3.2: Under additional restrictions, the requirement that ν_x be atomless in Assumption 3.3 may be relaxed to it being a mixture of an atomless and a discrete measure. However, the conclusion of Theorem 3.1 may not apply if ν_x is a purely discrete measure. For example, suppose that ν_x and ν_z have finite support $\{x_1, \dots, x_s\}$ and $\{z_1, \dots, z_t\}$. Let $\Pi(P)$ be the $s \times t$ matrix with entry $\Pi(P)_{jk} = P\{X_i = x_j|Z_i = z_k\}$. Theorem 2.4 in Newey and Powell (2003) fully characterizes L^1 -completeness of P with respect to W given Z in terms of rank conditions on submatrices of $\Pi(P)$. In this setting, nontrivial tests for L^1 -completeness of P with respect to W given Z can therefore be constructed using, for example, uniformly valid confidence regions for $\Pi(P)$. See Anderson (1967) and Romano and Wolf (2000) for relevant results about confidence regions for a univariate mean.

REMARK 3.3: In the context of identification of some linear, semiparametric models with endogeneity, full rank requirements on matrices arise instead of completeness conditions. Specifically,

$$(12) \quad \mathbf{P}_1 = \mathbf{P} \setminus \mathbf{P}_0 = \{P \in \mathbf{P} : E_P[Z_i W_i'] \text{ has full rank}\}.$$

Tests for this purpose have been proposed, among others, by [Anderson \(1951\)](#), [Gill and Lewbel \(1992\)](#), [Cragg and Donald \(1993, 1997\)](#), [Robin and Smith \(2000\)](#), and [Kleibergen and Paap \(2006\)](#). Contrary to the implications of [Theorem 3.1](#), nontrivial tests that control asymptotic size do exist, for example, if the support of (X_i, Z_i) under P is bounded uniformly in $P \in \mathbf{P}$.

[Remarks 3.2](#) and [3.3](#) illustrate two important features of the problem we consider. First, they highlight that, despite their conceptual similarities, completeness conditions and rank conditions in linear specifications differ with regard to their testability. Second, they show that nontrivial tests of completeness may exist under alternative specifications of \mathbf{P} . For example, when \mathbf{P} is given by certain exponential families of distributions, such as the multivariate normal, the sets \mathbf{P}_0 and \mathbf{P}_1 in [\(8\)](#) can be characterized by the rank of appropriate matrices in a manner similar to [\(12\)](#); see [Newey and Powell \(2003, Theorem 2.3\)](#). Thus, completeness conditions are also testable under suitable parametric restrictions on \mathbf{P} . Such restrictions, however, are at odds with the nonparametric nature of the problems in which completeness assumptions are often invoked.

Of course, an assumption being testable should not be considered a necessary condition for its use. Many assumptions are routinely imposed in empirical analysis that are untestable, yet deemed reasonable on an alternative basis. As an example, consider the commonly used assumption that P has bounded support. This assumption is, in fact, untestable when $\mathbf{P} = \mathbf{M}_{x,z}$. Nonetheless, assuming that P has bounded support can often be a natural requirement in a number of economic applications, and thus be deemed reasonable despite it being untestable.

The nonexistence of nontrivial tests for completeness, however, does emphasize the value of alternative justifications for its use. Examples of such arguments include [Andrews \(2011\)](#), who showed that the set of distributions for which L^2 -completeness fails has a property analogous to having zero Lebesgue measure on \mathbf{R}^d . Formally, [Andrews \(2011\)](#) showed that the set of distributions for which L^2 -completeness fails is “shy” within a certain set of distributions. See [Hunt, Sauer, and Yorke \(1992\)](#) and [Anderson and Zame \(2001\)](#) for a formal definition of shyness. In related work, [Chen et al. \(2012\)](#) showed that a certain class of probability measures over conditional expectation operators assign zero probability to the set of operators for which completeness fails.

3.2. Testing Identification

In this section, we establish analogous results to [Theorem 3.1](#) for two widely studied nonparametric models with endogeneity that impose conditional quantile independence restrictions. Due to their nonlinear nature, simple, global rank conditions such as completeness conditions are unavailable for these models, and it is for this reason that we directly examine tests for identification instead.

Throughout this section, we let $V_i = (Y_i, X_i, Z_i) \in \mathbf{R} \times \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$ be random variables distributed according to $P \in \mathbf{P}$. As before, we let $Z_i = (Z_i^{(1)}, Z_i^{(2)})$, with the subvector $Z_i^{(1)}$ possibly empty, and let $W_i = (X_i, Z_i^{(1)})$. We will once again focus on sets of distributions \mathbf{P} that are dominated by a common measure ν . Thus, by analogy with the notation used in the preceding section, we let $\mathbf{M}_{y,x,z}$ be the set of all Borel probability measures on $\mathbf{R} \times \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$ and define

$$(13) \quad \mathbf{M}_{y,x,z}(\nu) \equiv \{P \in \mathbf{M}_{y,x,z} : P \ll \nu\}.$$

We will impose the following requirements on the dominating measure ν :

ASSUMPTION 3.4: ν is a positive σ -finite Borel measure on $\mathbf{R} \times \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$.

ASSUMPTION 3.5: $\nu = \nu_y \times \nu_x \times \nu_z$, where ν_y, ν_x , and ν_z are Borel measures on $\mathbf{R}, \mathbf{R}^{d_x}$, and \mathbf{R}^{d_z} , respectively.

Assumptions 3.4 and 3.5 are modifications of Assumptions 3.1 and 3.2 from the previous section to account for the fact that here the random variables take values in $\mathbf{R} \times \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$ rather than just $\mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$. In the results of this section, we will impose the requirement that $\mathbf{P} \subseteq \mathbf{M}_{y,x,z}(\nu)$ for some ν satisfying Assumptions 3.3, 3.4, and 3.5. Properties of ν , such as bounded support, will therefore still translate into restrictions on \mathbf{P} —see the discussion preceding Theorem 3.1. Finally, we note that if $d_x > 1$, then, in each of the following two theorems, Assumption 3.3 may be weakened to requiring that at least one component of X_i has an atomless marginal measure.

3.2.1. Single Quantile Restriction Model

The first model we consider is one where, for an outcome of interest Y_i , an endogenous variable X_i and an instrumental variable Z_i , and each $P \in \mathbf{P}$, there is some $\theta \in \Theta(P)$ such that

$$(14) \quad P\{Y_i - \theta(W_i) \leq 0 | Z_i\} = \tau \quad P\text{-a.s.}$$

for some prespecified $\tau \in (0, 1)$. Here, $\Theta(P)$ is a set of measurable functions from \mathbf{R}^{d_w} to \mathbf{R} , often set to equal $L^q(P_W)$ for some $1 \leq q \leq \infty$. We examine the testability of identification under the maintained assumption that P indeed satisfies (14) for some $\theta \in \Theta(P)$. Thus, we define \mathbf{P} to be

$$(15) \quad \mathbf{P} = \{P \in \mathbf{M}_{y,x,z}(\nu) : \exists \theta \in \Theta(P) \text{ s.t. (14) holds under } P\}.$$

By analogy with our analysis of the testability of completeness conditions, we let the null hypothesis be that identification fails in (14), and the alternative hypothesis be that it holds. Thus, we let

$$(16) \quad \mathbf{P}_1 = \mathbf{P} \setminus \mathbf{P}_0 = \{P \in \mathbf{P} : \exists ! \theta \in \Theta(P) \text{ s.t. (14) holds under } P\},$$

where uniqueness of $\theta \in \Theta(P)$ is understood to be up to sets of measure zero under P .

The next result shows that no nontrivial tests of identification exist in models defined by (14).

THEOREM 3.2: *Let Assumptions 3.3, 3.4, and 3.5 hold. Let $\mathbf{M}_{y,x,z}(\nu)$ be as in (13) and \mathbf{P} be as in (15) with $\Theta(P) = L^\infty(P_W)$. If \mathbf{P}_0 and \mathbf{P}_1 are as in (16), then, for any sequence of tests $\{\phi_n\}_{n=1}^\infty$,*

$$(17) \quad \sup_{P \in \mathbf{P}_1} E_{P^n}[\phi_n] \leq \sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n] \quad \text{for all } n \geq 1.$$

In establishing Theorem 3.2, we show that, for every $P \in \mathbf{P}_1$, there exists a sequence $\{P_k\}_{k=1}^\infty$ in \mathbf{P}_0 such that $\|P - P_k\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$. Our construction does not exploit the fact that $P \in \mathbf{P}_1$, but rather just the fact that $P \in \mathbf{M}_{y,x,z}(\nu)$. It follows that not only \mathbf{P}_1 lies in the closure of \mathbf{P}_0 with respect to the Total Variation metric, but that $\mathbf{M}_{y,x,z}(\nu)$ does as well. As a result, the conclusion of Theorem 3.2 continues to hold if we instead set $\Theta(P) = L^q(P_W)$ for any $1 \leq q < \infty$. It is worth noting that, in contrast to the setting of Theorem 3.1, here letting $\Theta(P) = L^q(P_W)$ for $1 \leq q < \infty$ enlarges \mathbf{P} itself, and so potentially enlarges not only \mathbf{P}_0 , but also \mathbf{P}_1 . As in Theorem 3.1, however, it is important to emphasize that Theorem 3.2 does not preclude the testability of identification under alternative specifications of \mathbf{P} . For instance, Theorem 3.2 does not apply when X_i has discrete support under ν_x —a setting prominently studied in Chernozhukov and Hansen (2005).

REMARK 3.4: In establishing that \mathbf{P}_1 lies in the closure of \mathbf{P}_0 with respect to the Total Variation distance, we construct, for each $P \in \mathbf{P}_1$, a sequence $\{P_k\}_{k=1}^\infty$ in \mathbf{P}_0 such that $\|P - P_k\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$, and for each k , there exist $\theta_k^{(1)}$ and $\theta_k^{(2)}$ in $\Theta(P_k)$ that differ not only on a set with positive probability under P_k , but in the stronger sense of

$$E_{P_k}[(1\{Y_i \leq \theta_k^{(1)}(W_i)\} - 1\{Y_i \leq \theta_k^{(2)}(W_i)\})^2] > 0,$$

while still satisfying

$$P_k\{Y_i \leq \theta_k^{(1)}(W_i) | Z_i\} = P_k\{Y_i \leq \theta_k^{(2)}(W_i) | Z_i\} = \tau \quad P_k\text{-a.s.}$$

This feature of the proof is noteworthy because it may still be the case that $1\{Y_i \leq \theta_k^{(1)}(W_i)\} = 1\{Y_i \leq \theta_k^{(2)}(W_i)\}$ P_k -a.s. for functions $\theta_k^{(1)}$ and $\theta_k^{(2)}$ that differ with positive probability under P_k .

3.2.2. *Nonseparable Model*

The final model we consider is closely related to the single quantile independence model in (14). Specifically, for an outcome of interest Y_i , an endogenous variable X_i , and an instrumental variable Z_i , we now consider a setting in which, for each $P \in \mathbf{P}$, there is some $\theta \in \Theta(P)$ such that

$$(18) \quad P\{Y_i - \theta(W_i, \tau) \leq 0 | Z_i\} = \tau \quad P\text{-a.s. for all } \tau \in (0, 1).$$

Here, $\Theta(P)$ denotes a set of measurable functions $\theta: \mathbf{R}^{d_w} \times [0, 1] \rightarrow \mathbf{R}$ such that P -a.s. the function $\tau \rightarrow \theta(W_i, \tau)$ is strictly increasing. Often, boundedness restrictions are imposed on θ , so, for \mathbf{T} the set of all measurable functions $\theta: \mathbf{R}^{d_w} \times [0, 1] \rightarrow \mathbf{R}$, $\Theta(P)$ is set to equal

$$\mathbf{T}(P) \equiv \{ \theta \in \mathbf{T} : \tau \mapsto \theta(W_i, \tau) \text{ is strictly increasing } P\text{-a.s. and} \\ \sup_{0 \leq \tau \leq 1} \|\theta(\cdot, \tau)\|_{L^\infty(P)} < \infty \}.$$

We consider the problem of testing for identification under the maintained assumption that P indeed satisfies (18) for some $\theta \in \Theta(P)$. We therefore define \mathbf{P} to be given by

$$(19) \quad \mathbf{P} = \{ P \in \mathbf{M}_{y,x,z}(\nu) : \exists \theta \in \Theta(P) \text{ s.t. (18) holds under } P \}.$$

Following the hypothesis problem studied in Theorem 3.2, we examine the testability of the null hypothesis that $P \in \mathbf{P}$ is such that identification fails in (18). Formally, we define

$$(20) \quad \mathbf{P}_1 = \mathbf{P} \setminus \mathbf{P}_0 = \{ P \in \mathbf{P} : \exists! \theta \in \Theta(P) \text{ s.t. (18) holds under } P \},$$

where uniqueness of $\theta \in \Theta(P)$ is understood to mean that if $\theta_1 \in \Theta(P)$ and $\theta_2 \in \Theta(P)$ both satisfy (18), then $P\{\theta_1(W_i, \tau) = \theta_2(W_i, \tau)\} = 1$ for all $\tau \in (0, 1)$.

It is worth noting that if (18) holds for a single, fixed $\tau \in (0, 1)$, then this model is equivalent to the model in (14) with $\theta(W_i) = \theta(W_i, \tau)$. As a result of this connection between models (14) and (18), it is perhaps to be expected that the conclusion of Theorem 3.2 extends to the present setting. The following result establishes this point, showing that under commonly used restrictions for \mathbf{P} , no nontrivial tests of identification exist in models defined by (18) even if we impose that the function $\tau \mapsto \theta(W_i, \tau)$ be strictly increasing P -a.s.

THEOREM 3.3: *Let Assumptions 3.3, 3.4, and 3.5 hold. Let $\mathbf{M}_{y,x,z}(\nu)$ be as in (13) and \mathbf{P} be as in (19) with $\Theta(P) = \mathbf{T}(P)$. If \mathbf{P}_0 and \mathbf{P}_1 are as in (20), then, for any sequence of tests $\{\phi_n\}_{n=1}^\infty$,*

$$(21) \quad \sup_{P \in \mathbf{P}_1} E_{P^n}[\phi_n] \leq \sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n] \quad \text{for all } n \geq 1.$$

As in Theorem 3.2, our proof implies that not only \mathbf{P}_1 lies in the closure of \mathbf{P}_0 with respect to the Total Variation metric, but that $\mathbf{M}_{y,x,z}(\nu)$ does as well. It therefore follows that the conclusion of Theorem 3.3 continues to hold if we instead set $\Theta(P)$ to be given by

$$\Theta(P) = \left\{ \theta \in \mathbf{T} : \tau \mapsto \theta(W_i, \tau) \text{ is strictly increasing } P\text{-a.s. and} \right. \\ \left. \sup_{0 \leq \tau \leq 1} \|\theta(\cdot, \tau)\|_{L^q(P)} < \infty \right\}$$

for some $1 \leq q < \infty$. It is also worth noting that the fact that \mathbf{P}_1 lies in the closure of \mathbf{P}_0 with respect to the Total Variation metric is established by constructing sequences $\{P_k\}_{k=1}^\infty$ in \mathbf{P}_0 such that, for each k , there exist functions $\theta_k^{(1)}$ and $\theta_k^{(2)}$ in $\Theta(P_k)$ satisfying

$$E_{P_k} \left[\left(1\{Y_i \leq \theta_k^{(1)}(W_i, \tau)\} - 1\{Y_i \leq \theta_k^{(2)}(W_i, \tau)\} \right)^2 \right] > 0$$

for all $\tau \in (0, 1)$. Thus, the functions $\theta_k^{(1)}(\cdot, \tau)$ and $\theta_k^{(2)}(\cdot, \tau)$ differ for every τ not just on a set with positive probability under P_k , but in the stronger sense of Remark 3.4.

4. CONCLUSION

This paper shows that, under commonly imposed restrictions on the distribution of the data, no nontrivial tests for completeness or identification conditions exist in some prominent nonparametric models with endogeneity. Whether testability can be restored under appealing restrictions on either $\Theta(P)$ or \mathbf{P} remains an important and challenging open question.

Two constructive points follow from our results, which we believe have potentially important implications for future research. First, having established the impossibility of producing empirical evidence in favor of completeness or identification conditions, our results emphasize the value of alternative arguments for their plausibility. Recent work that addresses this problem includes Andrews (2011) and Chen, Chernozhukov, Lee, and Newey (2012), who argued in favor of completeness conditions on the basis of genericity arguments. Second, our analysis highlights the significance of both developing inferential methods that are robust to partial identification and comparing their performance to nonrobust inferential approaches. For instance, it would be important to understand which inferential method is preferable when identification is believed to hold. We hope the results and arguments in this paper provide motivation for addressing these challenges in future research.

APPENDIX

Throughout the appendix, we employ the following notation, not necessarily introduced in the text:

$A\Delta B$ For two sets A and B , $A\Delta B \equiv (A \setminus B) \cup (B \setminus A)$.

$\mathcal{B}(A)$ For a Borel set A , the σ -algebra generated by all open subsets of A .

$\|\cdot\|_{L^q(\lambda)}$ For $1 \leq q < \infty$, a measure λ , and measurable function f , $\|f\|_{L^q(\lambda)}^q \equiv \int |f(u)|^q \lambda(du)$.

$\|\cdot\|_{L^\infty(\lambda)}$ For a measure λ and measurable function f , $\|f\|_{L^\infty(\lambda)} \equiv \text{esssup} |f(u)|$, where the essential supremum is defined as $\text{esssup} |f(u)| \equiv \inf\{M \in \mathbf{R} : \lambda\{u : |f(u)| > M\} = 0\}$.

$L^q(\lambda)$ For $1 \leq q \leq \infty$ and a measure λ , $L^q(\lambda)$ is the space of measurable functions f with finite semi-norm $\|f\|_{L^q(\lambda)}$.

LEMMA A.1: *Let $A \subseteq \mathbf{R}^d$ be a Borel set, $\mathcal{B}(A)$ the σ -algebra generated by all open subsets of A , and λ an atomless positive Borel measure on \mathbf{R}^d satisfying $0 < \lambda\{A\} < \infty$. Then, there exists a map $\bar{B} : [0, 1] \rightarrow \mathcal{B}(A)$ such that: (i) $\bar{B}(0) = \emptyset$ and $\bar{B}(1) = A$, (ii) $\bar{B}(\tau) \subseteq \bar{B}(\tau')$ for all $0 \leq \tau \leq \tau' \leq 1$, and (iii) $\lambda\{\bar{B}(\tau)\} = \tau\lambda\{A\}$ for all $\tau \in [0, 1]$. Additionally, there is $\tilde{B} : [0, 1] \rightarrow \mathcal{B}(A)$ satisfying properties (i)–(iii) and such that $\lambda\{\tilde{B}(\tau)\Delta\bar{B}(\tau)\} > 0$ for all $\tau \in (0, 1)$.*

PROOF: Without loss of generality, we assume $\lambda\{A\} = 1$; otherwise, we may just renormalize. Let μ denote the Lebesgue measure on \mathbf{R} , and $\mathcal{B}([0, 1])$ the σ -algebra generated by all open subsets of $[0, 1]$. For any $U_1, U_2 \in \mathcal{B}(A)$, define the equivalence relation $U_1 \sim U_2$ if $\lambda\{U_1\Delta U_2\} = 0$, and denote the set of resulting equivalence classes by E_λ . Similarly, denote by E_μ the equivalence classes on $\mathcal{B}([0, 1])$ generated by μ . Next, observe that since λ is a Borel measure on \mathbf{R}^d , Theorem 7.1.7 in Bogachev (2007b) implies λ is Radon, and hence also separable by Proposition 7.14.12(ii) in Bogachev (2007b). It therefore follows from Theorem 9.3.4 in Bogachev (2007b) that (E_λ, λ) is isomorphic to (E_μ, μ) , that is, there exists a one-to-one mapping $\Gamma : E_\lambda \rightarrow E_\mu$ such that

$$(22) \quad \mu\{\Gamma(V_1)\} = \lambda\{V_1\},$$

$$\Gamma(V_1 \setminus V_2) = \Gamma(V_1) \setminus \Gamma(V_2), \quad \Gamma(V_1 \cup V_2) = \Gamma(V_1) \cup \Gamma(V_2)$$

for any $V_1, V_2 \in E_\lambda$, and where the operations on elements of E_μ and E_λ are defined as the respective operations on arbitrary representatives of the equivalence classes. Next, define a map $B : (0, 1) \rightarrow \mathcal{B}(A)$ satisfying $B(\tau) \in \Gamma^{-1}([0, \tau])$ for any $\tau \in (0, 1)$, and note that, by (22), B satisfies $\lambda\{B(\tau)\} = \tau$ for all $\tau \in (0, 1)$. The map B , however, may not be monotonic, and for this reason we modify it by defining $\bar{B} : [0, 1] \rightarrow \mathcal{B}(A)$ to be given by $\bar{B}(0) = \emptyset$, $\bar{B}(1) = A$ and for any $\tau \in (0, 1)$ by

$$(23) \quad \bar{B}(\tau) \equiv \left[B(\tau) \cup \left\{ \bigcup_{0 < \sigma < \tau, \sigma \in \mathbf{Q}} B(\sigma) \right\} \right] \cap \left\{ \bigcap_{1 > \sigma > \tau, \sigma \in \mathbf{Q}} B(\sigma) \right\},$$

where \mathbf{Q} are the rational numbers. It follows by construction that \bar{B} then satisfies properties (i) and (ii). Moreover, for any $\tau \in (0, 1)$, let $\{a_i(\tau)\}_{i=1}^\infty = \mathbf{Q} \cap (0, \tau)$, and note that, by (22), we have

$$\begin{aligned}
 (24) \quad \lambda\{\bar{B}(\tau) \setminus B(\tau)\} &\leq \lim_{n \rightarrow \infty} \lambda \left\{ \left\{ \bigcup_{i=1}^n B(a_i(\tau)) \right\} \setminus B(\tau) \right\} \\
 &= \lim_{n \rightarrow \infty} \mu \left\{ \left\{ \bigcup_{i=1}^n [0, a_i(\tau)] \right\} \setminus [0, \tau] \right\} = 0.
 \end{aligned}$$

Similarly, letting $\{b_i(\tau)\}_{i=1}^\infty = \mathbf{Q} \cap (\tau, 1)$, we also obtain by monotone convergence that

$$\begin{aligned}
 (25) \quad \lambda\{B(\tau) \setminus \bar{B}(\tau)\} &= \lim_{n \rightarrow \infty} \lambda \left\{ B(\tau) \setminus \left\{ \bigcap_{i=1}^n B(b_i(\tau)) \right\} \right\} \\
 &= \lim_{n \rightarrow \infty} \mu \left\{ [0, \tau] \setminus \left\{ \bigcap_{i=1}^n [0, b_i(\tau)] \right\} \right\} = 0.
 \end{aligned}$$

We conclude from (24) and (25) that $\lambda\{\bar{B}(\tau)\} = \lambda\{B(\tau)\}$ for all $\tau \in (0, 1)$. Since $\lambda\{B(\tau)\} = \mu\{[0, \tau]\} = \tau$ for all $\tau \in (0, 1)$, $\bar{B}(0) = \emptyset$ and $\bar{B}(1) = A$, it follows that \bar{B} satisfies property (iii) as well.²

To establish the second claim of the lemma, pointwise define $\tilde{B}: [0, 1] \rightarrow \mathcal{B}(A)$ by

$$(26) \quad \tilde{B}(\tau) \equiv A \setminus \bar{B}(1 - \tau).$$

It is then immediate that $\tilde{B}(0) = \emptyset$ and $\tilde{B}(1) = A$, while $\lambda\{\tilde{B}(\tau)\} = \tau$ for all $\tau \in [0, 1]$ additionally yields that

$$(27) \quad \lambda\{\tilde{B}(\tau)\} = \lambda\{A \setminus \bar{B}(1 - \tau)\} = 1 - (1 - \tau) = \tau$$

for all $\tau \in [0, 1]$. Furthermore, for any $0 \leq \tau \leq \tau' \leq 1$, note that $\tau \leq \tau'$ implies $\bar{B}(1 - \tau') \subseteq \bar{B}(1 - \tau)$, and therefore

$$(28) \quad \tilde{B}(\tau) = A \setminus \bar{B}(1 - \tau) \subseteq A \setminus \bar{B}(1 - \tau') = \tilde{B}(\tau').$$

Thus, from (27) and (28), we obtain that $\tilde{B}: [0, 1] \rightarrow \mathcal{B}(A)$ indeed satisfies properties (i)–(iii). To conclude, note that monotonicity of \bar{B} implies

²We would like to thank an anonymous referee for suggesting this method of proof.

$(A \setminus \bar{B}(1 - \tau)) \setminus \bar{B}(\tau) = A \setminus \bar{B}(\max\{\tau, 1 - \tau\})$, and hence

$$(29) \quad \lambda\{\tilde{B}(\tau)\Delta\bar{B}(\tau)\} \geq \lambda\{\tilde{B}(\tau) \setminus \bar{B}(\tau)\} \\ = \lambda\{A \setminus \bar{B}(\max\{\tau, 1 - \tau\})\} = (1 - \max\{\tau, 1 - \tau\}).$$

Therefore, it follows from (29) that $\lambda\{\tilde{B}(\tau)\Delta\bar{B}(\tau)\} > 0$ for all $\tau \in (0, 1)$. *Q.E.D.*

LEMMA A.2: Let Q be a Borel probability measure on $\mathbf{R} \times \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$ satisfying $Q \ll \lambda$ for λ a σ -finite positive Borel measure on $\mathbf{R} \times \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$, and let $f \equiv dQ/d\lambda$. Then, for each $k \in \mathbb{N}$, there exists a positive real number M_k , a natural number K_k , collections $\{A_{ik}\}_{i=1}^{K_k}$, $\{B_{jk}\}_{j=1}^{K_k}$, $\{C_{lk}\}_{l=1}^{K_k}$ that are partitions of $[-M_k, M_k]$, $[-M_k, M_k]^{d_x}$, and $[-M_k, M_k]^{d_z}$, respectively, and a set of positive real numbers $\{\pi_{ijlk}\}_{1 \leq i, j, l \leq K_k}$ such that the sequence of functions $\{f_k\}_{k=1}^\infty$ defined by

$$(30) \quad f_k(y, x, z) = \sum_{1 \leq i, j, l \leq K_k} \pi_{ijlk} 1\{(y, x, z) \in A_{ik} \times B_{jk} \times C_{lk}\}$$

satisfies: (i) $f_k \geq 0$ for all k , (ii) $\int f_k d\lambda = 1$ for all k , and (iii) $\|f_k - f\|_{L^1(\lambda)} \rightarrow 0$ as $k \rightarrow \infty$.

PROOF: Note $f \geq 0$ and $f \in L^1(\lambda)$. Since λ is a Borel measure, it is also regular by Theorem 7.1.7 in Bogachev (2007b), and hence Theorem 13.9 in Aliprantis and Border (2006) implies there is a sequence $\{f_k^*\}_{k=1}^\infty$ of continuous, compactly supported functions such that $f_k^* \geq 0$ for all k and

$$(31) \quad \|f_k^* - f\|_{L^1(\lambda)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Let $\Omega_k \subset \mathbf{R} \times \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$ be the compact support of f_k^* and select $M_k > 0$ sufficiently large so that $\Omega_k \subseteq [-M_k, M_k]^{1+d_x+d_z}$. Additionally, select $0 < \xi_k \downarrow 0$ so that $\lim_{k \rightarrow \infty} \xi_k \lambda\{[-M_k, M_k]^{1+d_x+d_z}\} = 0$, and notice that f_k^* being uniformly continuous on $[-M_k, M_k]^{1+d_x+d_z}$ implies that there exist partitions $\{A_{ik}\}_{i=1}^{K_k}$, $\{B_{jk}\}_{j=1}^{K_k}$, and $\{C_{lk}\}_{l=1}^{K_k}$ of $[-M_k, M_k]$, $[-M_k, M_k]^{d_x}$, and $[-M_k, M_k]^{d_z}$, such that

$$(32) \quad \max_{1 \leq i, j, l \leq K_k} \sup_{(y, x, z) \in A_{ik} \times B_{jk} \times C_{lk}} \sup_{(\tilde{y}, \tilde{x}, \tilde{z}) \in A_{ik} \times B_{jk} \times C_{lk}} |f_k^*(y, x, z) - f_k^*(\tilde{y}, \tilde{x}, \tilde{z})| \leq \xi_k.$$

Letting $S_{ijlk} \equiv A_{ik} \times B_{jk} \times C_{lk}$ for $1 \leq i, j, l \leq K_k$, we then pointwise define the functions

$$(33) \quad \tilde{f}_k(y, x, z) \equiv \sum_{1 \leq i, j, l \leq K_k} \tilde{\pi}_{ijlk} 1\{(y, x, z) \in S_{ijlk}\}, \\ \tilde{\pi}_{ijlk} \equiv \sup_{(y, x, z) \in S_{ijlk}} f_k^*(y, x, z).$$

By construction, $\tilde{\pi}_{ijkl} \geq 0$ for all $1 \leq i, j, l \leq K_k$ and all k , and hence $\tilde{f}_k \geq 0$. Moreover, since f_k^*, \tilde{f}_k vanish outside $[-M_k, M_k]^{1+d_x+d_z}$, and $\{S_{ijkl}\}_{1 \leq i, j, l \leq K_k}$ is a partition of $[-M_k, M_k]^{1+d_x+d_z}$, equations (32) and (33) imply

$$\sup_{(y,x,z) \in \mathbf{R}^{1+d_x+d_z}} |f_k^*(y, x, z) - \tilde{f}_k(y, x, z)| \leq \xi_k.$$

Thus, we obtain

$$\begin{aligned} (34) \quad \lim_{k \rightarrow \infty} \|f_k^* - \tilde{f}_k\|_{L^1(\lambda)} &= \lim_{k \rightarrow \infty} \int_{[-M_k, M_k]^{1+d_x+d_z}} |f_k^* - \tilde{f}_k| d\lambda \\ &\leq \lim_{k \rightarrow \infty} \xi_k \lambda\{[-M_k, M_k]^{1+d_x+d_z}\} = 0, \end{aligned}$$

since $\xi_k \downarrow 0$ satisfies $\lim_{k \rightarrow \infty} \xi_k \lambda\{[-M_k, M_k]^{1+d_x+d_z}\} = 0$. Finally, let $f_k \equiv \tilde{f}_k / \|\tilde{f}_k\|_{L^1(\lambda)}$ and note that properties (i) and (ii) follow, while (30) holds with $\pi_{ijkl} = \tilde{\pi}_{ijkl} / \|\tilde{f}_k\|_{L^1(\lambda)}$. Moreover, since $\|\tilde{f}_k - f\|_{L^1(\lambda)} \rightarrow 0$ as $k \rightarrow \infty$ by (31) and (34), it follows that $\|\tilde{f}_k\|_{L^1(\lambda)} \rightarrow 1$, and thus

$$\begin{aligned} (35) \quad \|f_k - f\|_{L^1(\lambda)} &\leq \frac{\|\tilde{f}_k - f\|_{L^1(\lambda)}}{\|\tilde{f}_k\|_{L^1(\lambda)}} + \left| 1 - \frac{1}{\|\tilde{f}_k\|_{L^1(\lambda)}} \right| \times \|f\|_{L^1(\lambda)} \\ &= o(1) \quad \text{as } k \rightarrow \infty, \end{aligned}$$

which verifies property (iii), and hence the claim of the lemma follows. *Q.E.D.*

PROOF OF LEMMA 2.1: Fix $P \in \mathbf{P}_1$ and let $\{P_k\}_{k=1}^\infty$ satisfy $\|P_k - P\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$ and $P_k \in \mathbf{P}_0$ for all k . Since $\|P_k - P\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$ implies $\|P_k^n - P^n\|_{TV} \rightarrow \infty$ for any fixed $n \in \mathbb{N}$ (see (4.5) in [Hoeffding and Wolfowitz \(1958\)](#)), the result immediately follows from Theorem 1 in [Romano \(2004\)](#) applied to the sets $\mathbf{P}_{0,n} \equiv \{Q \in \otimes_{i=1}^n \mathbf{M} : Q = P^n \text{ for some } P \in \mathbf{P}_0\}$ and $\mathbf{P}_{1,n} \equiv \{Q \in \otimes_{i=1}^n \mathbf{M} : Q = P^n \text{ for some } P \in \mathbf{P}_1\}$. *Q.E.D.*

PROOF OF THEOREM 3.1: Fix $P \in \mathbf{P}_1$ and let $f \equiv dP/d\nu$. By Assumption 3.1 and Lemma A.2 applied to $Q \equiv \delta_0 \times P$ and $\lambda \equiv \delta_0 \times \nu$ for δ_0 a degenerate measure at 0 on \mathbf{R} , there is $\{f_k\}_{k=1}^\infty$ with

$$(36) \quad \|f_k - f\|_{L^1(\nu)} = o(1) \quad \text{as } k \rightarrow \infty,$$

and in addition, for all k , each f_k satisfies $f_k \geq 0$, $\int f_k d\nu = 1$, and is a simple function of the form

$$(37) \quad f_k(x, z) = \sum_{1 \leq j, l \leq K_k} \pi_{jlk} 1\{(x, z) \in S_{jlk}\}.$$

Here, $S_{jlk} = B_{jk} \times C_{lk}$, and for some $M_k > 0$, the collections $\{B_{jk}\}_{j=1}^{K_k}$ and $\{C_{lk}\}_{l=1}^{K_k}$ are partitions of $[-M_k, M_k]^{d_x}$ and $[-M_k, M_k]^{d_z}$, respectively. Therefore, defining

$$(38) \quad P_k\{E\} \equiv \int_E f_k d\nu$$

for all Borel measurable $E \subseteq \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$, it follows that P_k is a probability measure with $P_k \ll \nu$, and hence $P_k \in \mathbf{P} = \mathbf{M}_{x,z}(\nu)$ for all k . Moreover, by (36), $\|P_k - P\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$.

In what follows, we aim to show that in fact $P_k \in \mathbf{P}_0$ for all k . Assumption 3.3 and Corollary 1.12.10 in Bogachev (2007a) imply that, for each $1 \leq j \leq K_k$, there exist Borel measurable sets $B_{jk}^{(1)}, B_{jk}^{(2)}$ such that $B_{jk} = B_{jk}^{(1)} \cup B_{jk}^{(2)}$, $B_{jk}^{(1)} \cap B_{jk}^{(2)} = \emptyset$ and in addition satisfy

$$(39) \quad \nu_x\{B_{jk}^{(1)}\} = \nu_x\{B_{jk}^{(2)}\} = \frac{1}{2}\nu_x\{B_{jk}\}.$$

Since $\{B_{jk}\}_{j=1}^{K_k}$ is a partition of $[-M_k, M_k]^{d_x}$ by Lemma A.2, we may define a function θ_k by

$$(40) \quad \theta_k(x) \equiv \sum_{j=1}^{K_k} (1\{x \in B_{jk}^{(1)}\} - 1\{x \in B_{jk}^{(2)}\}).$$

Note that $\theta_k \in L^\infty(P_k)$, and $\theta_k \neq 0$ P_k -a.s. due to (37), (40) and $B_{jk}^{(1)} \cap B_{jk}^{(2)} = \emptyset$ for all $1 \leq j \leq K_k$. Moreover, for any bounded $z \mapsto \psi(z)$, we obtain from (37), $f_k = dP_k/d\nu$, and Assumption 3.2:

$$(41) \quad \begin{aligned} E_{P_k}[\psi(Z_i)\theta_k(X_i)] &= \sum_{1 \leq j,l \leq K_k} \pi_{jlk} \int_{C_{lk}} \int_{B_{jk}} \psi(z)\theta_k(x)\nu_x(dx)\nu_z(dz) \\ &= \sum_{1 \leq j,l \leq K_k} \pi_{jlk} (\nu_x\{B_{jk}^{(1)}\} - \nu_x\{B_{jk}^{(2)}\}) \int_{C_{lk}} \psi(z)\nu_z(dz) \\ &= 0, \end{aligned}$$

where the final equality exploited (39). In particular, (41) must hold for $\psi(\cdot) = E_{P_k}[\theta_k(X_i)|Z_i = \cdot]$, and hence we obtain, by the law of iterated expectations, that $E_{P_k}[\theta_k(X_i)|Z_i] = 0$ P_k -a.s. Note that for each k , θ_k can be viewed as a function of $w = (x, z^{(1)})$ that only depends on x , which together with (41) suffices for concluding that $P_k \in \mathbf{P}_0$ for all k . Hence, since $P \in \mathbf{P}_1$ was arbitrary,

trary and $\|P_k - P\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$, the conclusion of the theorem follows by Lemma 2.1. Q.E.D.

PROOF OF THEOREM 3.2: Fix $P \in \mathbf{P}_1$ and let $f \equiv dP/d\nu$. As in the proof of Theorem 3.1, we begin by noting that Lemma A.2 and Assumption 3.4 imply that there is a sequence $\{f_k\}_{k=1}^\infty$ such that

$$(42) \quad \|f_k - f\|_{L^1(\nu)} = o(1) \quad \text{as } k \rightarrow \infty,$$

and in addition, for all k , each f_k satisfies $f_k \geq 0$, $\int f_k d\nu = 1$, and is a simple function of the form

$$(43) \quad f_k(y, x, z) = \sum_{1 \leq i, j, l \leq K_k} \pi_{ijlk} 1\{(y, x, z) \in S_{ijlk}\}.$$

Here, $S_{ijlk} = A_{ik} \times B_{jk} \times C_{lk}$, and for some $M_k > 0$, $\{A_{ik}\}_{i=1}^{K_k}$, $\{B_{jk}\}_{j=1}^{K_k}$, and $\{C_{lk}\}_{l=1}^{K_k}$ form partitions of $[-M_k, M_k]$, $[-M_k, M_k]^{d_x}$, and $[-M_k, M_k]^{d_z}$, respectively. Hence, defining

$$(44) \quad P_k\{E\} \equiv \int_E f_k d\nu$$

for all Borel measurable $E \subseteq \mathbf{R} \times \mathbf{R}^{d_x} \times \mathbf{R}^{d_z}$, we obtain a sequence of probability measure $\{P_k\}_{k=1}^\infty$ satisfying $P_k \in \mathbf{M}_{y,x,z}(\nu)$ for all k , and $\|P_k - P\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$ due to result (42).

We next aim to show that $P_k \in \mathbf{P}_0$ for all k . Toward this end, note that Assumption 3.3 and Lemma A.1 imply that there exist collections of sets $\{B_{jk}^{(1)}(\tau), B_{jk}^{(2)}(\tau)\}_{j=1}^{K_k}$ such that, for all $1 \leq j \leq K_k$,

$$(45) \quad \nu_x\{B_{jk}^{(1)}(\tau)\} = \tau\nu_x\{B_{jk}\}, \quad \nu_x\{B_{jk}^{(2)}(\tau)\} = \tau\nu_x\{B_{jk}\},$$

with $B_{jk}^{(m)}(\tau) \subseteq B_{jk}$ for $m \in \{1, 2\}$, and $\nu_x\{B_{jk}^{(1)}(\tau) \Delta B_{jk}^{(2)}(\tau)\} > 0$ for all $1 \leq j \leq K_k$ such that $\nu_x\{B_{jk}\} > 0$. For $m \in \{1, 2\}$, we may then define functions $\theta_k^{(m)}(\cdot, \tau)$ pointwise in x by

$$(46) \quad \theta_k^{(m)}(x, \tau) \equiv \sum_{j=1}^{K_k} (2M_k 1\{x \in B_{jk}^{(m)}(\tau)\} - 2M_k 1\{x \in B_{jk} \setminus B_{jk}^{(m)}(\tau)\}),$$

and note that $\theta_k^{(m)}(\cdot, \tau) \in L^\infty(P_k)$ for $m \in \{1, 2\}$. Moreover, $Y_i \in [-M_k, M_k]$ P_k -a.s., together with Assumption 3.5 and $f_k = dP_k/d\nu$, with f_k as in (43), allows us to conclude that

$$(47) \quad \begin{aligned} E_{P_k}[(1\{Y_i \leq \theta_k^{(1)}(X_i, \tau)\} - 1\{Y_i \leq \theta_k^{(2)}(X_i, \tau)\})^2] \\ = E_{P_k}[(1\{\theta_k^{(1)}(X_i, \tau) = 2M_k\} - 1\{\theta_k^{(2)}(X_i, \tau) = 2M_k\})^2] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{1 \leq i, j, l \leq K_k} \pi_{ijlk} \nu_y \{A_{ik}\} \nu_z \{C_{lk}\} \\
 &\quad \times \int_{B_{jk}} (1\{\theta_k^{(1)}(x, \tau) = 2M_k\} - 1\{\theta_k^{(2)}(x, \tau) = 2M_k\})^2 \nu_x(dx) \\
 &> 0,
 \end{aligned}$$

where we exploited $(1\{\theta_k^{(1)}(x, \tau) = 2M_k\} - 1\{\theta_k^{(2)}(x, \tau) = 2M_k\})^2 = 1\{x \in B_{jk}^{(1)}(\tau) \Delta B_{jk}^{(2)}(\tau)\}$ for every $x \in B_{jk}$ and $\nu_x\{B_{jk}^{(1)}(\tau) \Delta B_{jk}^{(2)}(\tau)\} > 0$ for some $1 \leq j \leq K_k$ due to P_k having support $[-M_k, M_k]^{1+d_x+d_z}$ implying $P_k\{S_{ijlk} = \pi_{ijlk} \nu\{S_{ijlk}\} > 0$ for some $1 \leq i, j, l \leq K_k$.

We conclude from (47) that $\theta_k^{(1)}(\cdot, \tau)$ and $\theta_k^{(2)}(\cdot, \tau)$ are distinct under $\|\cdot\|_{L^\infty(P_k)}$. Additionally, for $m \in \{1, 2\}$, we have $1\{\theta_k^{(m)}(x, \tau) = 2M_k\} = 1\{x \in B_{jk}^{(m)}(\tau)\}$ for every $x \in B_{jk}$ by (46), and hence $\nu_x\{\{\theta_k^{(m)}(x, \tau) = 2M_k\} \cap B_{jk}\} = \nu_x\{B_{jk}^{(m)}(\tau)\} = \tau \nu_x\{B_{jk}\}$. It follows that, for any bounded $z \mapsto \psi(z)$,

$$\begin{aligned}
 (48) \quad &E_{P_k}[\psi(Z_i)(1\{Y_i \leq \theta_k^{(m)}(X_i, \tau)\} - \tau)] \\
 &= \sum_{1 \leq i, j, l \leq K_k} \pi_{ijlk} \int_{A_{ik}} \int_{C_{lk}} \int_{B_{jk}} \psi(z)(1\{\theta_k^{(m)}(x, \tau) = 2M_k\} - \tau) \\
 &\quad \times \nu_x(dx) \nu_z(dz) \nu_y(dy) \\
 &= \sum_{1 \leq i, j, l \leq K_k} \pi_{ijlk} (\tau \nu_x\{B_{jk}\} - \tau \nu_x\{B_{jk}\}) \int_{A_{ik}} \int_{C_{lk}} \psi(z) \nu_z(dz) \nu_y(dy) \\
 &= 0.
 \end{aligned}$$

In particular, setting $\psi(\cdot) = P_k\{Y_i \leq \theta^{(m)}(X_i, \tau) | Z_i = \cdot\} - \tau$ in (48) implies, by the law of iterated expectations, that $P_k\{Y_i \leq \theta_k^{(m)}(X_i, \tau) | Z_i\} = \tau$ P_k -a.s. for $m \in \{1, 2\}$. Note that for each k , $\theta_k^{(1)}$ and $\theta_k^{(2)}$ can be viewed as functions of $w = (x, z^{(1)})$ that only depend on x , which together with (47) and (48) imply that $P_k \in \mathbf{P}_0$ for all k . Hence, since $P \in \mathbf{P}_1$ was arbitrary and $\|P_k - P\|_{TV} \rightarrow 0$ as $k \rightarrow \infty$, the conclusion of the theorem follows by Lemma 2.1. Q.E.D.

PROOF OF THEOREM 3.3: The proof is very similar to that of Theorem 3.2, and we therefore provide only an outline, emphasizing the differences in the arguments. Fixing $P \in \mathbf{P}_1$, we may obtain a sequence $\{P_k\}_{k=1}^\infty$ such that, for all k , $P_k \in \mathbf{M}_{y,x,z}(\nu)$, $dP_k/d\nu = f_k$ for f_k as defined in (43), and such that (42) holds. To show $P_k \in \mathbf{P}_0$ for all k , let $\mathcal{B}(B_{jk})$ denote the σ -algebra generated by all the open subsets of B_{jk} . By Assumption 3.3 and Lemma A.1, there then exist $B_{jk}^{(1)} : [0, 1] \rightarrow \mathcal{B}(B_{jk})$ and $B_{jk}^{(2)} : [0, 1] \rightarrow \mathcal{B}(B_{jk})$ such that, for all $1 \leq j \leq K_k$ and $m \in \{1, 2\}$, (i) $\nu_x\{B_{jk}^{(m)}(\tau)\} = \tau \nu_x\{B_{jk}\}$, (ii) $B_{jk}^{(m)}(\tau) \subseteq B_{jk}^{(m)}(\tau')$ for all $0 \leq \tau \leq$

$\tau' \leq 1$, and (iii) $\nu_x\{B_{jk}^{(1)}(\tau) \Delta B_{jk}^{(2)}(\tau)\} > 0$ for all $\tau \in (0, 1)$ and $1 \leq j \leq K_k$ such that $\nu_x\{B_{jk}\} > 0$. Following (46), we then define functions $\theta_k^{(m)}$ pointwise by

$$(49) \quad \theta_k^{(m)}(x, \tau) \equiv \sum_{j=1}^{K_k} ((2 + \tau)M_k 1\{x \in B_{jk}^{(m)}(\tau)\} - (3 - \tau)M_k 1\{x \in B_{jk} \setminus B_{jk}^{(m)}(\tau)\}).$$

Observe that $|\theta_k^{(m)}(x, \tau)| \leq 3M_k$ for all $(x, \tau) \in \mathbf{R}^{d_x} \times [0, 1]$, and hence $\theta_k(X_i, \tau)$ is bounded P_k -a.s. uniformly in $\tau \in [0, 1]$. Moreover, since $B_{jk}^{(m)}(\tau) \subseteq B_{jk}^{(m)}(\tau')$ for $m \in \{1, 2\}$ and all $0 \leq \tau \leq \tau' \leq 1$ and $1 \leq j \leq K_k$, it follows from (49) and the support of X_i under P_k being contained in $[-M_k, M_k]^{d_x} = \bigcup_{j=1}^{K_k} B_{jk}$ that $\theta_k^{(m)}(X_i, \tau)$ is strictly monotonic in τ P_k -a.s. In turn, notice that since $\tau \in [0, 1]$ and the support of Y_i is contained in $[-M_k, M_k]$ under P_k , we obtain from (49) that, for all $x \in B_{jk}$, $1\{Y_i \leq \theta_k^{(m)}(x, \tau)\} = 1\{x \in B_{jk}^{(m)}(\tau)\}$ P_k -a.s. Therefore, arguing as in (47) yields that

$$(50) \quad E_{P_k}[(1\{Y_i \leq \theta_k^{(1)}(X_i, \tau)\} - 1\{Y_i \leq \theta_k^{(2)}(X_i, \tau)\})^2] > 0,$$

for all k and $\tau \in (0, 1)$. Thus, we may conclude from (50) that $\theta_k^{(1)}(\cdot, \tau)$ is distinct from $\theta_k^{(2)}(\cdot, \tau)$ under $\|\cdot\|_{L^\infty(P_k)}$ for all $\tau \in (0, 1)$. In turn, arguing as in (48) further implies that, for all k ,

$$(51) \quad P_k\{Y_i \leq \theta_k^{(m)}(X_i, \tau) | Z_i\} = \tau$$

P_k -a.s. for $m \in \{1, 2\}$ and all $\tau \in (0, 1)$. Since, for each k and $\tau \in [0, 1]$, $\theta_k^{(1)}(\cdot, \tau)$ and $\theta_k^{(2)}(\cdot, \tau)$ can be viewed as functions of $w = (x, z^{(1)})$ that only depend on x , results (50) and (51) imply that $P_k \in \mathbf{P}_0$ for all k . The argument can then be finished as in the proof of Theorem 3.2. *Q.E.D.*

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