On the Testability of Identification in Some Nonparametric Models with Endogeneity

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Three Nonparametric Models

Conditional Mean IV: Let \((Y, X, Z) \in \mathbb{R} \times \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}\) have distribution \(P:\)

\[ Y = \theta_0(X) + \epsilon \quad \quad E_P[\epsilon|Z] = 0 \]
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\]

Non-separable IV: Let \((Y, X, Z) \in \mathbb{R} \times \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}\) have distribution \(P\):

\[
Y = \theta_0(X, \epsilon) \quad \quad \quad \quad \quad P(\theta_0(X, \epsilon) \leq \theta_0(X, \tau)|Z) = \tau
\]

where in addition \(\tau \mapsto \theta_0(X, \tau)\) is assumed strictly monotonic almost surely.
In conditional mean IV, identification requires a unique solution (in $\theta$) to:

$$E_P[Y|Z] = E_P[\theta(X)|Z]$$

Since Newey & Powell (2003), identification through completeness condition

$$E_P[\theta(X)|Z] = 0 \quad P - a.s. \quad \Rightarrow \quad \theta(X) = 0 \quad P - a.s.$$  

**Comments**

- More general: bounded completeness or $L^q(P)$ completeness.
- Sometimes referred to as nonparametric rank condition.
- Also used in identification of quantile and nonseparable models.
Testability

Problems

- Completeness conditions are difficult to interpret.
- Hard to motivate from economic theory.

Questions

- Are completeness assumptions testable under reasonable conditions?
- More generally: is point identification testable in these three models?

Answers

- We show no nontrivial tests for completeness exist.
- We show no nontrivial tests for identification exist in these three models.
Linear Model Intuition

Linear IV: Suppose \((Y, X, Z) \in \mathbb{R}^3\) with distribution \(P \in \mathbb{P}\), and satisfy:

\[
Y = X\theta_0 + \epsilon \quad \quad \quad \quad E_P[Z\epsilon] = 0
\]

\(\Rightarrow \theta_0\) is identified if and only if \(E_P[XZ] \neq 0\) – i.e. \(\theta_0 = E_P[XY]/E_P[XZ]\).

Testing Rank Condition

\[
H_0 : E_P[XZ] = 0 \quad \quad \quad \quad H_1 : E_P[XZ] \neq 0
\]

Bahadur and Savage (1956)

- **Negative**: If \(P\) is rich enough, only test is the trivial test.
- **Positive**: Learn how to restrict \(P\) for tests to exist (example bounded).

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General Setup

\[ H_0 : P \in P_0 \quad \text{and} \quad H_1 : P \in P_1 \]

where \( P_1 \equiv P \setminus P_0 = \{\text{distributions that are complete (or model identified)}\} \).

Main Result

Any test \( \phi_n \) that controls asymptotic size at level \( \alpha \in (0, 1) \), in the sense:

\[
\limsup_{n \to \infty} \sup_{P \in P_0} E_{P^n}[\phi_n] \leq \alpha ,
\]

(for \( P^n \equiv \bigotimes_{i=1}^{n} P \)) will have no power against any alternative, in the sense:

\[
\limsup_{n \to \infty} \sup_{P \in P_1} E_{P^n}[\phi_n] \leq \alpha .
\]

Conclusion holds for all three models, under common assumptions on \( P \).
Nonparametric IV

Quantile/Nonseparable IV

Uniformly Valid Inference
Bahadur & Savage (1956), Romano (2004), and many others ...
General Outline

Setup
- Notation and Assumptions.
- Useful Lemma.

Testing Completeness
- The null and alternative hypothesis.
- Main result and proof strategy.

Quantile/Nonseparable IV
- Quantile IV: Main result and proof strategy.
- Nonseparable IV: Main result.
1 Setup

2 Completeness

3 Quantile IV

4 Nonseparable IV
Notation

Let $\mathcal{M}$ be the set of all probability measures on $\mathbb{R} \times \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$, and define:

$$\mathcal{M}(\nu) \equiv \{ P \in \mathcal{M} : P \ll \nu \}$$

We will require $P \subseteq \mathcal{M}(\nu)$ for some measure $\nu$ satisfying the following:
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We will require $\mathcal{P} \subseteq \mathcal{M}(\nu)$ for some measure $\nu$ satisfying the following:

Main Assumption (A)

- $\nu$ is a $\sigma$-finite Borel measure on $\mathbb{R} \times \mathbb{R}^{d_x} \times \mathbb{R}^{d_z}$.
- $\nu = \nu_y \times \nu_x \times \nu_z$ for $\nu_y$, $\nu_x$ and $\nu_z$ Borel measures on $\mathbb{R}$, $\mathbb{R}^{d_x}$ and $\mathbb{R}^{d_z}$.
- The measure $\nu_x$ is atomless on $\mathbb{R}^{d_x}$.
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- The measure $\nu_x$ is atomless on $\mathbb{R}^{d_x}$.

**Comments**
- Support restrictions imposed through $\nu$ (example $(X, Z) \in [0, 1]^{d_x+d_z}$).
- $\nu$ product measure does not require $P \in \mathcal{P}$ to be product measure.
Discussion

νᵢ atomless

- May be relaxed, but νᵢ cannot be purely discrete.
- If dᵢ > 1, then sufficient for one coordinate to be atomless.

Example

- Suppose νᵢ and νᵦ have discrete support \{x₁, \ldots, xₙ\} and \{z₁, \ldots, zₜ\}.

\[ \Pi(P) \equiv \{ s \times t \text{ matrix with } \Pi(P)_{j,k} = P(X = x_j | Z = z_k) \} \]

- Newey & Powell (2003) showed P is complete iff \text{rank}(\Pi(P)) = s.
- Test can be constructed through uniform confidence region for Π(P).
Useful Lemma

\[ \|P_1 - P_2\|_{TV} \equiv \sup_{g:|g|\leq 1} \frac{1}{2} \left| \int gdP_1 - \int gdP_2 \right| \]

Lemma If for all \( P \in \mathcal{P}_1 \), there is \( \{P_k\} \) with \( P_k \in \mathcal{P}_0 \) and \( \|P - P_k\|_{TV} = o(1) \)

\[ \sup_{P \in \mathcal{P}_1} E_{P_n}[\phi_n] \leq \sup_{P \in \mathcal{P}_0} E_{P_n}[\phi_n] \]

for every sequence of test functions \( \{\phi_n\} \) and for every \( n \).

Comments

- Small modification of Theorem 1 in Romano (2004).
- Result implies its asymptotic analogue.
- Intuition: If every \( P \in \mathcal{P}_1 \) is in the boundary of \( \mathcal{P}_0 \), then we conclude

\[ \text{Size Control} \Rightarrow \text{No Power} \]
Useful Lemma

Key Idea: Since \( |\phi_n| \leq 1 \) for any test function, \( \|P - P_k\|_{TV} = o(1) \) implies:

\[
\left| \int \phi_n dP_k^n - \int \phi_n dP^n \right| \leq \sup_{g:|g| \leq 1} \frac{1}{2} \left| \int gdP_k^n - \int gdP^n \right| = o(1)
\]

Comments

- Total Variation distance plays no role in the definition of \( P_0 \) and \( P_1 \).
- Metrics compatible with weak topology may be too weak for result.
- Stronger metric, implies harder to show \( P_k \to P \).

Goal: Show in problems we study, lack of identification \((P_0)\) is “dense”.

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1 Setup

2 Completeness

3 Quantile IV

4 Nonseparable IV
Completeness

\[ \begin{align*}
H_0 : P &\in \mathcal{P}_0 & H_1 : P &\in \mathcal{P}_1
\end{align*} \]

where \( \mathcal{P} = \mathcal{M}(\nu) \) for some \( \nu \in \mathcal{M} \), \( \mathcal{P}_0 \equiv \mathcal{P} \setminus \mathcal{P}_1 \) and we additionally define:

\[ \mathcal{P}_1 \equiv \{ P \in \mathcal{P} : E_P[\theta(X)|Z] = 0 \text{ for } \theta \in L^\infty(P) \Rightarrow \theta(X) = 0 \ P - a.s. \} \]

Comments

- Using \( L^\infty(P) \) \( \Rightarrow \) test for bounded completeness.
- Replacing \( L^\infty(P) \) with \( L^q(P) \) for \( 1 \leq q < \infty \) just enlarges \( \mathcal{P}_0 \).
- No power in this setting \( \Rightarrow \) no power in test of \( L^q(P) \) completeness.
Completeness

**Theorem** Let $\mathbb{P} = \mathcal{M}(\nu)$ and Assumption (A) hold. Then if \( \{\phi_n\} \) satisfies:

$$
\limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} E_{P^n}[\phi_n] \leq \alpha,
$$

for \( P^n \equiv \bigotimes_{i=1}^{n} P \) and level \( \alpha \in (0, 1) \), then it follows that it also satisfies:

$$
\limsup_{n \to \infty} \sup_{P \in \mathcal{P}_1} E_{P^n}[\phi_n] \leq \alpha.
$$

**Comments**

- If $\nu$ has compact support, then support of $P \in \mathbb{P}$ uniformly bounded.
- In contrast, $\nu$ with compact support suffices in linear IV model.
Proof Outline

**Step 1** Fix $P \in P_1$, let $f \equiv dP/d\nu$, show $\sup_{g:|g|\leq 1} |\int g(f_k - f)d\nu| = o(1)$:

$$f_k(x, z) = \sum_{i=1}^{K_k} \pi_{ik} 1\{(x, z) \in S_{ik}\} \quad f_k \geq 0 \quad \int f_k d\nu = 1$$

**Step 2** $\{S_{ik}\}_{i=1}^{K_k}$ can be chosen to be the product of two collections of sets:

- $\{U_{ik}\}$ a partition of the set $[-M_k, M_k]^d_x$ some $M_k \in (0, \infty)$.
- $\{V_{ik}\}$ a partition of the set $[-M_k, M_k]^d_z$ same $M_k \in (0, \infty)$.

**Step 3** Since $\nu_x$ is atomless, we can partition each $U_{ik}$ into $(U_{ik}^{(1)}, U_{ik}^{(2)})$:

$$\nu_x(U_{ik}^{(1)}) = \nu_x(U_{ik}^{(2)}) = \frac{1}{2} \nu_x(U_{ik})$$
Proof Outline

Step 4 Let $P_k$ be measure with $dP_k/d\nu = f_k$, and define the function:

$$\theta_k(x) \equiv \sum_{i=1}^{D_k} (1\{x \in U_{ik}^{(1)}\} - 1\{x \in U_{ik}^{(2)}\})$$

Step 5 Then: (i) $\theta_k$ is bounded, (ii) $\theta_k(X) \neq 0$ $P_k - a.s.$, and (iii):

$$\int_{V_{nk}} \int_{U_{tk}} \psi(z) \theta_k(x) \nu_x(dx) \nu_z(dz)$$

$$= \int_{V_{nk}} \psi(z) \int_{U_{tk}} (1\{x \in U_{tk}^{(1)}\} - 1\{x \in U_{tk}^{(2)}\}) \nu_x(dx) \nu_z(dz)$$

$$= 0$$

However, recall $dP_k/d\nu = \sum_{i=1}^{K_k} \pi_{ik} 1\{(x, z) \in S_{ik}\}$ with $S_{ik} = V_{nk} \times U_{tk}$ ...
Step 6 Therefore, \( E_{P_k}[\psi(Z)\theta_k(X)] = 0 \) for all \( P_k \)-integrable \( \psi \), and hence:

\[
E_{P_k}[\theta_k(X)|Z] = 0 \quad P_k - a.s.
\]

Step 7 Therefore, \( P_k \in \mathbf{P}_0 \) for all \( k \), and \( \|P_k - P\|_{TV} = o(1) \). By Lemma,

\[
\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_1} E_{P^n}[\phi_n] \leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n] \leq \alpha
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Proof Outline

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\limsup_{n \to \infty} \sup_{P \in P_1} E_{P_n}[\phi_n] \leq \limsup_{n \to \infty} \sup_{P \in P_0} E_{P_n}[\phi_n] \leq \alpha
\]

Comments

- The sequence \( \{\theta_k\} \) developed in the proof is not differentiable.
- Proof may be modified so \( \{\theta_k\} \) is infinitely differentiable.
- \( \Rightarrow L^\infty(P) \) may be replaced by Sobolev space or Ball.
- Similarly, we may also impose smoothness restrictions on \( dP/d\nu \).
Comments

Two Important Features

- Completeness may be testable under alternative specifications of \( P \). However, standard “nonparametric” approaches do not seem to apply.
- Assumptions routinely employed that are non testable but “reasonable”.

Genericity Arguments

- Alternative justification in favor of completeness assumptions.
- Andrews (2011) shows set of distributions for which it fails is “shy”.
- Chen et al. (2012) show certain measures (over conditional expectation operators) assign zero probability to completeness failure.
1 Setup

2 Completeness

3 Quantile IV

4 Nonseparable IV
Quantile IV

\[ H_0 : P \in P_0 \quad H_1 : P \in P_1 \]

for \( P \) the subset of \( M(\nu) \) consisting of \( P \in M(\nu) \) such that \( \exists \theta_0 \in L^\infty(P) : \)

\[
Y = \theta_0(X) + \epsilon \quad P(\epsilon \leq 0|Z) = \tau P - a.s.
\]

Comments
- Uniqueness of \( \theta \in L^\infty(P) \) understood up to sets of \( P \)-measure zero.
- No easy necessary conditions for identification from completeness: \( \Rightarrow \) We test for identification directly.
\[ H_0 : P \in P_0 \quad \text{and} \quad H_1 : P \in P_1 \]

for \( P \) the subset of \( M(\nu) \) consisting of \( P \in M(\nu) \) such that \( \exists \theta_0 \in L^\infty(P) : \]
\[ Y = \theta_0(X) + \epsilon \quad \quad P(\epsilon \leq 0|Z) = \tau P \text{ a.s.} \]

As before, \( P_0 \equiv P \setminus P_1 \), where now \( P_1 \subset P \) is given by the set of measures:
\[ P_1 \equiv \{ P \in P : \exists! \theta \in L^\infty(P) \text{ s.t. } P(Y \leq \theta(X)|Z) = \tau P \text{ a.s.} \} \]
Quantile IV

\[ H_0 : P \in \mathcal{P}_0 \quad H_1 : P \in \mathcal{P}_1 \]

for \( P \) the subset of \( \mathcal{M}(\nu) \) consisting of \( P \in \mathcal{M}(\nu) \) such that \( \exists \theta_0 \in L^\infty(P) : \)

\[ Y = \theta_0(X) + \epsilon \quad P(\epsilon \leq 0|Z) = \tau P \text{ a.s.} \]

As before, \( \mathcal{P}_0 \equiv P \setminus \mathcal{P}_1 \), where now \( \mathcal{P}_1 \subset \mathcal{P} \) is given by the set of measures:

\[ \mathcal{P}_1 \equiv \{ P \in \mathcal{P} : \exists! \theta \in L^\infty(P) \text{ s.t. } P(Y \leq \theta(X)|Z) = \tau P \text{ a.s.} \} \]

Comments

- Uniqueness of \( \theta \in L^\infty(P) \) understood up to sets of \( P \)-measure zero.
- No easy necessary conditions for identification from completeness:

\[ \Rightarrow \text{ We test for identification directly} \]
Theorem Let $\mathbb{P}$ be as defined, and Assumption (A) hold. If $\{\phi_n\}$ satisfies:

$$\limsup_{n \to \infty} \sup_{P \in \mathbb{P}_0} E_{P^n} [\phi_n] \leq \alpha,$$

for $P^n \equiv \bigotimes_{i=1}^n P$ and level $\alpha \in (0, 1)$, then it follows that it also satisfies:

$$\limsup_{n \to \infty} \sup_{P \in \mathbb{P}_1} E_{P^n} [\phi_n] \leq \alpha.$$

Comments

- We show $\mathbb{P}_0$ is dense in $\mathcal{M}(\nu)$ (not just $\mathbb{P}_1$) w.r.t Total Variation.
- Theorem holds for $L^q(P)$ in place of $L^\infty(P)$ as well.
Proof Outline

**Step 1** Fix $P \in P_1$, let $f \equiv dP/d\nu$, show $\sup_{g:|g| \leq 1} |\int g(f_k - f) d\nu| = o(1)$:

\[
f_k(y, x, z) \equiv \sum_{i=1}^{K_k} \pi_{ik} 1\{(y, x, z) \in S_{ik}\} \quad f_k \geq 0 \quad \int f_k d\nu = 1
\]

**Step 2** \(\{S_{ik}\}_{i=1}^{K_k}\) can be chosen to be the product of three collections of sets

- \(\{U_{ik}\}\) a partition of the set \([-M_k, M_k] d_x\) some \(M_k \in (0, \infty)\).
- \(\{V_{ik}\}\) a partition of the set \([-M_k, M_k] d_z\) same \(M_k \in (0, \infty)\).
- \(\{L_{ik}\}\) a partition of the set \([-M_k, M_k]\) same \(M_k \in (0, \infty)\).

**Step 3** Since \(\nu_x\) is atomless, we can pick \(U_{ik}^{(1)}(\tau) \subset U_{ik}\), and \(U_{ik}^{(2)}(\tau) \subset U_{ik}\):

\[
\nu_x(U_{ik}^{(1)}(\tau)) = \nu_x(U_{ik}^{(2)}(\tau)) = \tau \nu_x(U_{ik}) \quad \nu_x(U_{ik}^{(1)}(\tau) \triangle U_{ik}^{(2)}(\tau)) > 0
\]
Proof Outline

Step 4

Under $P_k$, $Y$ has support contained in $[-M_k, M_k]$. Hence, letting:

$$\theta(l)_k(x, \tau) = D_k \sum_{i=1}^{\{x \in U_l(\tau)\}} \{-2M_k 1_{\{x \in U_{ik}\}}\}$$

we get that

$$1_{\{Y \leq \theta(l)_k(X, \tau)\}} = \sum_{i} 1_{\{X \in U_{ik}\}},$$

almost surely under $P_k$. 

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Proof Outline

Step 4
Under $P_k$, $Y$ has support contained in $[-M_k, M_k]$. Hence, letting:

$$\theta(l) k(x, \tau) = D_k \sum_{i=1}^{\{2M_k\}} \left\{ \begin{array}{l} x \in U_{ik}(\tau) \\ x \not\in U_{ik}(\tau) \end{array} \right\} - 2M_k \sum_{i=1}^{\{2M_k\}} \left\{ x \in U_{ik}(\tau) \right\},$$

we get that

$$\{Y \leq \theta(l) k(x, \tau)\} = \sum_{i=1}^{\{2M_k\}} \left\{ X \in U_{ik}(\tau) \right\},$$
a almost surely under $P_k$. 

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Proof Outline

Step 4
Under $P_k$, $Y$ has support contained in $[-M_k, M_k]$. Hence, letting:

$\theta(l)(x, \tau) = D_k \sum_{i=1}^{\mathbb{1}_{\{x \in U(l)\}}(\tau)} - 2M_k \sum_{i=1}^{\mathbb{1}_{\{x \in U(l)\}}(\tau)}$

we get that $\mathbb{1}_{\{Y \leq \theta(l)(x, \tau)\}} = \sum_{i=1}^{\mathbb{1}_{\{X \in U(l)\}}(\tau)}$, almost surely under $P_k$. 

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Proof Outline

\[ f_x \]

\[ \tau \nu_x \{ U_{4k} \} \]

Under \( P_k \), \( Y \) has support contained in \([ -M_k, M_k ]\).

Hence, letting:

\[ \theta(l; x, \tau) = D_k \sum_{i=1}^{2M_k} \left\{ x \in U_{ik}(l; \tau) \right\} - 2M_k \left\{ x \notin U_{ik}(l; \tau) \right\} \]

we get that:

\[ \{ Y \leq \theta(l; x, \tau) \} = \sum_{i} \{ X \in U_{ik}(l; \tau) \}, \text{ almost surely under } P_k. \]
Proof Outline

$\tau \nu_x \{ U_{4k} \}$

$U_{1k}$  $U_{2k}$  $U_{3k}$  $U_{4k}$  $U_{5k}$  $U_{6k}$  $U_{7k}$

$f_x$
Proof Outline

Under $P_k$, $Y$ has support contained in $[-M_k, M_k]$. Hence, letting:

$$\theta(l)(x, \tau)(\tau) = D_k \sum_{i=1}^{2M_k} \{x \in U^{(i)}(l, \tau)\} - 2M_k \{x \not\in U^{(i)}(l, \tau)\}$$

we get that

$$1\{Y \leq \theta(l)(X, \tau)\} = \sum_{i} 1\{X \in U^{(i)}(l, \tau)\},$$

almost surely under $P_k$.

$$\nu_x \{U^{(1)}(\tau) \Delta U^{(2)}(\tau)\} > 0$$
Proof Outline

\[ \tau \nu_x \{ U_{4k} \} \]

\[ \nu_x \{ U_{4k}^{(1)} (\tau) \Delta U_{4k}^{(2)} (\tau) \} > 0 \]

Step 4
Under \( P_k \), \( Y \) has support contained in \([-M_k, M_k]\). Hence, letting:
\[
\theta(l) = \sum_{i=1}^{2M_k} \mathbb{1}_{\{ x \in U_i(l) \}} - \sum_{i=1}^{2M_k} \mathbb{1}_{\{ x \notin U_i(l) \}}
\]
we get that:
\[
\mathbb{1}_{\{ Y \leq \theta(l) \}} = \sum_{i=1}^{2M_k} \mathbb{1}_{\{ X \in U_i(l) \}}, \text{ almost surely under } P_k.
\]
**Step 4** Under $P_k$, $Y$ has support contained in $[-M_k, M_k]$. Hence, letting:

$$
\theta_{k}^{(l)}(x, \tau) = \sum_{i=1}^{D_k} \left\{ 2M_k 1\{ x \in U_{ik}^{(l)}(\tau) \} - 2M_k 1\{ x \in U_{ik} \setminus U_{ik}^{(l)}(\tau) \} \right\}
$$

we get that $1\{ Y \leq \theta_{k}^{(l)}(X, \tau) \} = \sum_{i} 1\{ X \in U_{ik}^{(l)} \}$, almost surely under $P_k$. 
Proof Outline

Step 5 Then: (i) $\theta_k^{(1)}$ and $\theta_k^{(2)}$ are bounded, (ii) for any $L_{jk} \times V_{nk} \times U_{tk}$:

$$
\int_{L_{jk}} \int_{V_{nk}} \int_{U_{tk}} \psi(z)(1 \{y \leq \theta_k^{(l)}(x, \tau)\} - \tau) \nu_x(dx) \nu_z(dz) \nu_y(dy)
$$

$$
= \int_{L_{jk}} \int_{V_{nk}} \psi(z) \int_{U_{tk}} (1 \{x \in U_{tk}^{(l)}(\tau)\} - \tau) \nu_x(dx) \nu_z(dz) \nu_y(dy)
$$

$$
= 0
$$

However, $dP_k/d\nu = \sum_{i=1}^{K_k} \pi_{ik} 1\{(x, z) \in S_{ik}\}$ with $S_{ik} = L_{jk} \times V_{nk} \times U_{tk} ...$

Step 6 Hence, $E_{P_k}[\psi(Z)(1 \{Y \leq \theta_k^{(l)}(X, \tau)\} - \tau)] = 0$ for $\psi \in L^1(P_k)$, and:

$$
E_{P_k}[1 \{Y \leq \theta_k^{(l)}(X, \tau)\} - \tau|Z] = 0 \quad P_k - a.s.
$$
Step 7 Argue that $P_k(\theta_k^{(1)}(X, \tau) \neq \theta_k^{(2)}(X, \tau)) > 0$ for all $k$.

Step 8 Hence, $P_k \in \mathbf{P}_0$ for all $k$, and $\|P_k - P\|_{TV} = o(1)$. By Lemma,

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}_1} E_{P^n}[\phi_n] \leq \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_0} E_{P^n}[\phi_n] \leq \alpha$$
Proof Outline

Step 7 Argue that \( P_k(\theta_k^{(1)}(X, \tau) \neq \theta_k^{(2)}(X, \tau)) > 0 \) for all \( k \).

Step 8 Hence, \( P_k \in P_0 \) for all \( k \), and \( \|P_k - P\|_{TV} = o(1) \). By Lemma,

\[
\limsup_{n \to \infty} \sup_{P \in P_1} E_{P^n}[\phi_n] \leq \limsup_{n \to \infty} \sup_{P \in P_0} E_{P^n}[\phi_n] \leq \alpha
\]

Comments

• In the proof, we actually establish the stronger inequality:

\[
E_{P_k}[(1\{Y \leq \theta_k^{(1)}(X, \tau)\} - 1\{Y \leq \theta_k^{(2)}(X, \tau)\})^2] > 0.
\]

• Results holds if identification is up to \( P \) equivalence of \( 1\{Y \leq \theta(X)\} \).
1 Setup

2 Completeness

3 Quantile IV

4 Nonseparable IV
Nonseparable IV

\[ H_0 : P \in P_0 \quad H_1 : P \in P_1 \]

for \( P \) the maximal subset of \( M(\nu) \) s.t. for each \( P \in P \), \( \exists \theta_0 \in L^\infty(P) \), with:

\[
Y = \theta_0(X, \epsilon) \quad P(\theta_0(X, \epsilon) \leq \theta_0(X, \tau)|Z) = \tau \quad P - a.s.
\]
Nonseparable IV

\[ H_0 : P \in P_0 \quad \text{and} \quad H_1 : P \in P_1 \]

for \( P \) the maximal subset of \( \mathcal{M}(\nu) \) s.t. for each \( P \in \mathcal{P} \), \( \exists \theta_0 \in L^\infty(P) \), with:

\[ Y = \theta_0(X, \epsilon) \quad P(\theta_0(X, \epsilon) \leq \theta_0(X, \tau)|Z) = \tau \ P - \text{a.s.} \]

As before, \( P_0 \equiv P \setminus P_1 \), where now \( P_1 \subset P \) is given by the set of measures:

\[ P_1 \equiv \{ P \in \mathcal{P} : \exists! \theta \in L^\infty(P) \text{ s.t. } P(Y \leq \theta(X, \tau)|Z) = \tau \ \forall \tau \ P - \text{a.s.} \} \]
Nonseparable IV

\[ H_0 : P \in P_0 \quad \quad H_1 : P \in P_1 \]

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\[ Y = \theta_0(X, \epsilon) \quad \quad P(\theta_0(X, \epsilon) \leq \theta_0(X, \tau)|Z) = \tau \quad P - a.s. \]

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\[ P_1 \equiv \{ P \in P : \exists! \theta \in L^\infty(P) \text{ s.t. } P(Y \leq \theta(X, \tau)|Z) = \tau \quad \forall \tau \quad P - a.s. \} \]

Comments

- For each \( \tau \in (0, 1) \) model is equivalent to previous one.
- \( \tau \mapsto \theta(X, \tau) \) additionally strictly increasing \( P - a.s. \).
Theorem For $\mathcal{P}$ be as defined, and under Assumption (A), if $\{\phi_n\}$ satisfies:

$$\limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} E_{P^n}[\phi_n] \leq \alpha,$$

for $P^n \equiv \bigotimes_{i=1}^{n} P$ and level $\alpha \in (0, 1)$, then it follows that it also satisfies:

$$\limsup_{n \to \infty} \sup_{P \in \mathcal{P}_1} E_{P^n}[\phi_n] \leq \alpha.$$

Comments

- We show $\mathcal{P}_0$ is dense in $\mathcal{M}(\nu)$ not just $\mathcal{P}_1$ w.r.t. Total Variation.
- Theorem holds for $L^q(P)$ in place of $L^\infty(P)$ completeness as well.
- Essentially same steps, but add monotonicity in $\tau$ to construction.
Conclusion

Testability

- No nontrivial tests for identification exist in three IV models.
- P requirements are satisfied by usual assumptions in the literature.

However ...

- Valid tests may exist under more restrictive assumptions on P.
- Valid tests may also exist under shape restrictions on $\theta_0$.
- Results can aid develop nontrivial tests under additional requirements.

Two Constructive Points

- Highlight importance of alternative justifications – e.g. genericity.
- Emphasize value of procedures that are robust to partial identification.