Aggregate Welfare and Output with Heterogeneous Agents

David R. Baqaee
UCLA

Ariel Burstein
UCLA

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Abstract

We characterize how societal welfare responds to shocks in economies with heterogeneous agents, in which preferences are non-homothetic and potentially changing over time. We provide a characterization in both partial equilibrium, taking changes in prices and incomes as given, and in general equilibrium, taking technologies as given. We generalize Hulten's theorem, the basis for constructing aggregate quantity indices, to this context. Our results generalize the representative agent results in Baqaee and Burstein (2021). We apply these results to characterize the gains from international trade in economies with heterogeneous agents and non-homothetic preferences.

1 Introduction

Baqaee and Burstein (2021) study how welfare responds to changes in technologies when preferences are non-homothetic and or changing over time. For simplicity, for their general equilibrium results, Baqaee and Burstein (2021) assume that there exists a representative agent. This assumption implies that the definition of social welfare is unambiguous. In this short companion paper, we discuss how the results in that paper can be generalized to environments where there is no representative agent. We propose a notion of societal welfare and show that using this notion of welfare, the results in Baqaee and Burstein (2021) readily generalize.

To measure the change in social welfare from some initial situation t_0 to some terminal situation t_1, we ask: "*what is the minimum amount the aggregate endowment in t_0 must change so that it is possible to make every consumer indifferent between t_0 and t_1?*" In other words, from a social perspective, t_1 is preferred to t_0 if, and only if, the only way to make everyone as well off in the t_0 economy as in the t_1 economy requires that we increase the aggregate endowment. The increase in the aggregate endowment necessary is our measure of the change in social welfare. This welfare measure is related to the Kaldor-Hicks compensation principle, commonly used in welfare analysis, which deems a change socially desirable if the winners can hypothetically compensate the losers. This notion of social welfare simply asks if a Pareto-improvement is feasible. As such, this welfare notion is silent on the impact of the change on inequality, which requires making interpersonal utility comparisons and taking a stand on the extent to which transfers between agents take place in practice as well as the potential costs of implementing such transfers (see Antras et al. (2017) for a recent discussion of alternative social welfare criteria).

We study the answer to this question in both partial equilibrium, where prices are taken as given, and in general equilibrium, where technologies are taken as given. For general equilibrium economies, we provide a generalization of Hulten (1978) that captures societal welfare in economies with heterogenous agents, non-homothetic preferences, and taste shocks. As in Baqaee and Burstein (2021), this notion of social welfare is useful since it can be computed using only information about expenditure shares and elasticities of substitution in t_1, and does not require direct knowledge of income elasticities, the taste shocks, or the evolution of the distribution of income. Whereas our baseline economy features perfect competition, we show how our results generalize to economies with distortions. Using our general equilibrium results, we characterize gains from trade in economies with heterogeneous agents and non-homothetic preferences.

If the economy has a representative agent, then the welfare notion we introduce, and our characterizations, are the same as the ones in Baqaee and Burstein (2021). Furthermore, if this representative agent has homothetic and stable preferences, then this welfare notion also coincides with a Divisia index for real consumption, and Hulten (1978) can be used to measure changes in welfare.

The outline of this paper is as follows. In Section 2, we introduce our notion of social welfare and characterize its properties in partial equilibrium, taking prices as given. In Section 3, we extend these definitions and results to general equilibrium economies where prices are endogenously determined. In Section 4, we show how to compute changes in general equilibrium welfare in economies where production and consumption functions are nested (potentially non-homothetic) CES aggregators with taste shifters. In Section 5, we briefly discuss how to extend our results to economies with distortions. In Section 6, we characterize the aggregate gains from trade in economies with heterogeneous agents and non-homothetic preferences.

2 Aggregate welfare in partial equilibrium

The economy is populated by households indexed by $h \in \{1, ..., H\}$. Each h has a set of preference relations over bundles of goods, $\{\succeq_{x_h}\}$. The index x_h represents anything that affects h's preference rankings over bundles of goods. For every $x_h \in X$, we represent the preference relation $\succeq_{x_h}$ by a utility function $u_h(c_h; x_h)$, where $c_h \in \mathbb{R}^N$ and N is the number of goods in the consumption bundle.

Given preferences encapsulated in u_h, the expenditure function for any x_h is

$$e_h(p, u; x_h) = \min_{c_h} \{p \cdot c_h : u_h(c_h; x_h) = u\},$$

where $p \in \mathbb{R}^N$ is a price vector over all relevant goods in the preference relation. The budget share for agent h on good i (given prices, preferences, and a level of utility) is

$$b_{hi}(p, u; x_h) \equiv \frac{p_i c_{hi}(p, u; x_h)}{e_h(p, u; x_h)} = \frac{\partial \log e_h(p, u; x_h)}{\partial \log p_i}. \tag{1}$$

Budget shares can differ across agents due to differences in preferences or in incomes.

There are three properties of preferences that are analytically convenient benchmarks throughout the rest of the analysis. Preferences over goods are *stable* if $\succeq_{x_h}$ is the same as $\succeq_{x'_h}$ for every x_h and x'_h. Preferences are *homothetic* if whenever $c_h \sim_{x_h} c'_h$ then $ac_h \sim_{x_h} ac'_h$ for every $a > 0$, which implies that we can write $u_h(ac_h; x_h) = au_h(c_h; x_h)$. Preferences are *aggregable* and homothetic if $b_{hi}(p, u, x_h) = b_i(p, x)$ for every $h \in H$. In words, all households must have the same budget shares, and these budget shares depend on prices and some aggregate taste shifter.

We denote aggregate income by I, and household h's share of aggregate income by ω_h, so that h's income is $I_h = \omega_h I$. A consumption allocation across households, $\{c_h\}$, is feasible if it satisfies the aggregate budget constraint, $p \cdot \sum_h c_h = I$, where the i'th element of $\sum_h c_h$ is aggregate consumption of good i.

In Baqaee and Burstein (2021), when we study the microeconomic problem we consider shifts in the budget set of individual (or identical) agents. We now consider shifts in budget sets in an economy with heterogeneous agents. In this section we consider exogenous price changes, and in the next section we endogenize them.

Specifically, consider a shift in budget constraint as prices and aggregate incomes change from p_{t_0} and I_{t_0} to p_{t_1} and I_{t_1}, and the income distribution changes from $\{\omega_{ht_0}\}$ to $\{\omega_{ht_1}\}$. Here, t_0 and t_1 simply index the vector of prices and income being compared. For concreteness, we refer to this index as time, but it could equally refer to space. This change in the budget sets is accompanied by changes in tastes from $\{x_{ht_0}\}$ to $\{x_{ht_1}\}$.

Our baseline measure of aggregate welfare in partial equilibrium is defined as follows.

Definition 1 (Welfare in partial equilibrium)**.** The change in welfare measured using the *equivalent variation* with *final preferences* is $EV^{PE} = \phi^*$ where ϕ^* solves

$$\min_{\phi \in \mathbb{R}} \{\exists \{c_h\} \text{ s.t. } p_{t_0} \cdot \sum_h c_h = \exp(\phi) I_{t_0} \text{ and } u_h(c_h; x_{ht_1}) \geq u_h(c_{ht_1}; x_{ht_1}) \forall h\}, \tag{2}$$

where c_{ht_1} is the consumption bundle of agent h in t_1.

In words, EV^{PE} is the minimum (log) percentage increase in aggregate income under p_{t_0} such that there exists a Pareto improvement compared to t_1, according to $\{\succeq_{hx_{t_1}}\}$. The definition of Pareto improvement is standard: there exists a feasible consumption bundle such that all agents, with preferences $\{\succeq_{hx_{t_1}}\}$, are (weakly) better off than with their budget constraints at t_1.[1] From a social perspective, t_1 is preferred to t_0 if, and only if, EV^{PE} is positive. Our welfare measure is related to the Kaldor-Hicks compensation principle commonly used in welfare analysis, whereby a change is deemed socially desirable if the winners can hypothetically compensate the losers.[2]

[1] EV^{PE} is a function of prices and aggregate income at t_0, p_{t_0}, I_{t_0}, prices and aggregate income at t_1, p_{t_1}, I_{t_1}, the income distribution across households at t_1, $\{\omega_{ht_1}\}$, and tastes by household at t_1, $\{x_{ht_1}\}$. Consumption by household h at t_1, c_{ht_1} is the solution to utility maximization given prices p_{t_1} and income $I_{ht_1} = \omega_{ht_1} I_{t_1}$. Since (hypothetical) income redistributions are allowed to evaluate Pareto improvements, the distribution of income at t_0, $\{\omega_{ht_0}\}$, is irrelevant for EV^{PE}.

[2] In principle, we could also measure changes in welfare using compensating (instead of equivalent) variation, or by using initial (rather than final) preferences. Combining EV with final preferences (CV with

The following proposition characterizes changes in the partial equilibrium welfare measure.

Proposition 1 (Partial Equilibrium Welfare)**.** *For any smooth path of prices, aggregate income, income distribution, and tastes that unfold as a function of t between t_0 and t_1, the welfare change is given by*

$$EV^{PE} = \log \frac{I_{t_1}}{I_{t_0}} - \int_{t_0}^{t_1} \sum_{i \in N} b_i^{ev}(p) d \log p_i, \tag{3}$$

where

$$b_i^{ev}(p) \equiv \sum_h \frac{I_h^{ev}(p)}{\sum_{h'} I_{h'}^{ev}(p)} b_{hi}^{ev}(p), \tag{4}$$

with

$$\log I_h^{ev}(p(t)) = \log I_{ht_1} - \int_t^{t_1} \sum_i b_{hi}^{ev}(p) \, d \log p_i, \tag{5}$$

and $b_{hi}^{ev}(p) \equiv b_{hi}(p, u_h(c_{ht_1}; x_{ht_1}); x_{ht_1})$ denotes agent h budget shares on good i at prices p, fixing final preferences x_{ht_1} and final utility $u_h(c_{ht_1}; x_{ht_1})$.

In an economy in which all agents are identical, the expression for EV^{PE} collapses to the expression for EV^m in Lemma 1 of Baqaee and Burstein (2021), which can also be used to examine the welfare changes of individual agents.[3] With heterogeneous agents, the weight assigned to individual households, $\frac{I_h^{ev}(p)}{\sum_{h'} I_{h'}^{ev}(p)}$, when calculating aggregate budget shares is obtained from expression (5), which indicates how individual households must be compensated so that they can attain $u_h(c_{ht_1}; x_{ht_1})$ as prices change.

In contrast to equation (3), real consumption weights price changes by observed budget shares:

$$\Delta \log Y = \log \frac{I_{t_1}}{I_{t_0}} - \int_{t_0}^{t_1} \sum_{i \in N} b_{it} \frac{d \log p_{it}}{dt} dt, \tag{6}$$

where $b_{it} \equiv \sum_h \omega_{ht} b_{hi}(p, \omega_{ht} I_t; x_{ht})$. Hence, there are two reasons why gaps may appear between real consumption and welfare EV^{PE}. First, welfare-relevant consumption shares by individual households b_{hi}^{ev} are not the same as observed consumption shares b_{hi} (unless preferences are homothetic and stable). Second, welfare-relevant income shares at t

initial preferences) is natural since this requires preserving the shape of the indifference curve at the final (initial) allocation. We focus on EV using final preferences to streamline the presentation. See Baqaee and Burstein (2021) for a discussion and characterization of these other welfare measures.

[3]The partial equilibrium welfare measure in this paper is referred to in Baqaee and Burstein (2021) as microeconomic welfare, and the general equilibrium measure in this paper as macroeconomic welfare in Baqaee and Burstein (2021). Whereas the welfare notions are the same in both papers, we change the labels to make it clear that the distinction is about exogenous vs. endogenous prices and not about a single agent versus a collection of agents.

$I_h^{ev}(p(t))/\sum_{h'} I_{h'}^{ev}(p(t))$ are not equal to observed income shares ω_{ht}, and this matters in (4) if preferences are non-aggregable — that is, if $b_{hi} \neq b_{h'i}$ for some households h and h'.

Implementation. According to Proposition 1, we can calculate changes in aggregate welfare given changes in prices and income using terminal total expenditures by household and budget shares by product and household as a function of prices, $b_{hi}^{ev}(p)$. To compute $b_h^{ev}(p)$ given prices, we need to know terminal budget shares by product and household as well as terminal elasticities of substitution. Conditional on knowledge of these statistics, we do not need to know income elasticities, taste shocks, or changes in the income distribution. Intuitively, income elasticities or the income distribution in t_0 do not matter because EV^{PE} adjusts the level of income of each household in t_0 to make every consumer as well off as they are in t_1. Tastes in t_0 do not matter because EV^{PE} uses fixed preferences in t_1.

As discussed in the proof of Proposition 1, budget shares $b^{ev}(p)$ are those of a hypothetical representative agent with homothetic and stable preferences with expenditure function $e^{ev}(p,u) = \sum_h e_h(p, u_h(c_{ht_1}; x_{ht_1}); x_{ht_1})\, u$.

For example, suppose that preferences are non-homothetic CES with taste shocks. The expenditure function for household h is

$$e_h(p_t, u_t; x_{ht}) = \left(\sum_i \bar{x}_{hi} x_{hit} p_{it}^{1-\theta_h} u_{ht}^{\xi_{hi}}\right)^{\frac{1}{1-\theta_h}}. \tag{7}$$

The parameter ξ_{hi} is the utility elasticity of good i, θ_h is the (constant utility) elasticity of substitution across goods, and x_{hit} is a demand shifter (i.e. a taste shock) that generates changes in expenditure shares not attributable to changes in income or prices. When ξ_{hi} is equal for every i, final demand of agent h is homothetic, and when x_{hit} is constant for all i, final demand is stable.

Using Proposition 1 we obtain[4]

$$\begin{aligned} EV^{PE} &= \log \frac{I_{t_1}}{I_{t_0}} + \log \frac{\sum_h e_h(p_{t_0}, u_{ht_1}; x_{ht_1})}{\sum_h e_h(p_{t_1}, u_{ht_1}; x_{ht_1})} \\ &= \log \frac{I_{t_1}}{I_{t_0}} + \log \sum_h \omega_{ht_1} \left[\sum_i b_{hit_1} \left(\frac{p_{it_0}}{p_{it_1}}\right)^{1-\theta_h}\right]^{\frac{1}{1-\theta_h}}. \end{aligned} \tag{8}$$

[4]Recall that EV^{PE} can be derived from the the expenditure function $e^{ev}(p,u)$, which in this example is the expenditure function of a homothetic and stable nested CES aggregator. The outer nest has an elasticity of substitution 0 across households and the inner nest for household h has an elasticity of substitution θ_h across goods.

Note that income elasticities and taste shocks are not directly required.

3 Aggregate welfare in general equilibrium

Consider a neoclassical closed economy with heterogeneous agents. Each good $i \in N$ has a production function

$$y_i = A_i G_i \left(\{m_{ij}\}_{j \in N}, \{l_{if}\}_{f \in F} \right),$$

where m_{ij} are intermediate inputs used by i and produced by j, and l_{if} denotes primary factor inputs used by i for each factor $f \in F$. The exogenous scalar A_i is a Hicks-neutral productivity shifter. Without loss of generality, we assume that G_i has constant returns to scale since decreasing returns to scale can be captured by adding producer-specific factors.[5]

Let A be the $N \times 1$ vector of technology shifters and L be the $F \times 1$ vector of primary (exogenously given) factor endowments. We denote by $\mathcal{P}(A, L)$ the set of feasible consumption allocations $\{c_h\}$.

We consider perfectly competitive equilibria. The vector $\omega_h \in \mathbb{R}^F$ denotes the endowment share of agent h over each primary factor, with $\sum_h \omega_{hf} = 1$ for each f. For each A, L, $\{\omega_h\}$, and $\{x_h\}$, we denote equilibrium prices and aggregate income by $p(A, L, \{\omega_h\}, \{x_h\})$ and $I(A, L, \{\omega_h\}, \{x_h\})$. These equilibrium prices and incomes are unique up to the choice of a numeraire.[6]

Define the sales shares relative to GDP of each good or factor i to be

$$\lambda_i = \frac{p_i y_i}{I} \mathbf{1}(i \in N) + \frac{w_i L_i}{I} \mathbf{1}(i \in F),$$

where w_i and L_i are the price and quantity of factor i.[7]

We consider a change in technologies and factor endowments from A_{t_0}, L_{t_0} to A_{t_1}, L_{t_1}, a change in factor endowment shares from $\{\omega_{ht_0}\}$ to $\{\omega_{ht_1}\}$, and a change in tastes from $\{x_{ht_0}\}$ to $\{x_{ht_1}\}$.

Our baseline measure of aggregate welfare in general equilibrium is defined as follows

[5] A_i is Hicks-neutral without loss of generality. This is because we can capture non-neutral (biased) productivity shocks to input j for producer i by introducing a fictitious producer that buys from j and sells to i with a linear technology. A Hicks-neutral shock to this fictitious producer is equivalent to a non-neutral technology shock to i.

[6] Technically, if the economy does not have a unique equilibrium, then prices and aggregate income are not just a function of the primitives (A, L, $\{\omega_h\}$, and $\{x_h\}$). In this case we require an additional parameter which selects the equilibrium. This kind of multiplicity poses no issues for our analysis, so we suppress dependence on this equilibrium-selection mechanism (if it is needed) to simplify notation.

[7] Whereas $\sum_{i \in N} \lambda_i > 1$ whenever there are intermediate inputs, $\sum_{i \in F} \lambda_i = 1$.

Definition 2 (Welfare in general equilibrium)**.** The change in welfare measured using the *equivalent variation* with *final preferences* is $EV^{GE} = \phi^*$ where ϕ^* solves

$$\min_{\phi \in \mathbb{R}} \{\exists \{c_h\} \in \mathcal{P}(A_{t_0}, \exp(\phi) L_{t_0}) \text{ and } u_h(c_h, x_{t_1}) \geq u_h(c_{ht_1}; x_{ht_1}) \forall h\}, \tag{9}$$

where c_{ht_1} is the consumption bundle of agent h in t_1.

In words, EV^{GE} is the minimum proportional increase in factor endowments at t_0 such that there exists a Pareto improvement compared to t_1, according to $\{\succeq_{hx_{t_1}}\}$.[8] Expressing EV^{GE} in terms of changes in factor endowments rather than aggregate income is convenient in general equilibrium since it can be stated without reference to (endogenous) prices.

For an alternative but equivalent definition of welfare in general equilibrium, denote by $\mathcal{C}_t$ the set of feasible consumption allocations $\{c_h\}$ given A_t and L_t. Then welfare in general equilibrium is given by ϕ^*, which solves

$$\min_{\phi \in \mathbb{R}} \{\exists \{c_h\} \in \exp(\phi)\mathcal{C}_{t_0} \text{ and } u_h(c_h, x_{ht_1}) \geq u_h(c_{ht_1}; x_{ht_1}) \forall h\}, \tag{10}$$

In words, welfare measures the proportional shift in $\mathcal{C}_{t_0}$ necessary to make every household h indifferent between $\exp(\phi)\mathcal{C}_{t_0}$ and $\mathcal{C}_{t_1}$. This is because scaling all factor endowments by a constant under autarky shifts out the PPF by the same constant under our assumptions of a neoclassical economy with constant returns to scale.

The following proposition provides conditions under which partial equilibrium and general equilibrium welfare are equal.

Proposition 2 (Partial Equilibrium vs. General Equilibrium Welfare vs. Real GDP)**.** *General and partial equilibrium welfare changes are equal ($EV^{PE} = EV^{GE} = \Delta \log Y$) if preferences are aggregable, homothetic, and stable. If there is only one primary factor of production (so that prices do not depend on the distribution of income or on taste shocks), then $EV^{PE} = EV^{GE}$, but these are not generically equal to $\Delta \log Y$.*

The next lemma provides a characterization of real GDP in terms of primitive shocks and sales shares.

[8] EV^{GE} is a function of t_0 technologies and factor endowments, A_{t_0}, L_{t_0}, t_1 technologies and factor endowments, A_{t_1}, L_{t_1}, the distribution of factor endowments at t_1, $\{\omega_{ht_1}\}$, and tastes at t_1, $\{x_{ht_1}\}$. Consumption by household h at t_1, c_{ht_1} is the solution to utility maximization given by household h's factor endowment at t_1, $\omega_{ht_1} \cdot L_{ft_1}$, and prices of goods and factors at t_1, which are a function of $A_{t_1}, L_{t_1}, \{\omega_{ht_1}\}, \{x_{ht_1}\}$.

Lemma 1 (Real GDP). *Given a smooth path of technologies, factor quantities, factor endowment shares, and tastes that unfold as a function of time t, the change in real GDP is*

$$\Delta \log Y = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i d \log A_i + \int_{t_0}^{t_1} \sum_{i \in F} \lambda_i d \log L_i, \tag{11}$$

where λ are sales shares which are functions of A, L, $\{\omega_h\}$ and $\{x_h\}$ and can change as a function of time inside the integral.

The proof of Lemma 1 is identical to that of Proposition 4 in Baqaee and Burstein (2021). Lemma 1 is a slight generalization of Hulten (1978) to environments with heterogeneous agents and final demand that is unstable and/or non-homothetic (see Baqaee and Farhi, 2019a for a similar result when preferences are heterogeneous, but stable and homothetic).

Next, we show that a Hulten-style result also exists for changes in welfare. Define $\lambda^{ev}(A, L)$ to be sales shares in a fictional economy with the PPF (A, L) but with a representative consumer whose homothetic and stable expenditure function is $e^{ev}(p, u) = \sum_h e_h \left(p, u_h(c_{ht_1}; x_{ht_1}); x_{ht_1}\right) u$ that owns all factors. We call λ^{ev} the *welfare-relevant* sales share.

Proposition 3 (Macro Welfare). *For any smooth path of technologies, factor quantities, and tastes that unfold as a function of time t, changes in macro welfare are*

$$EV^{GE} = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i^{ev}(A, L) d \log A_i + \int_{t_0}^{t_1} \sum_{i \in F} \lambda_i^{ev}(A, L) d \log L_i. \tag{12}$$

In an economy in which all agents are identical, the expression for EV^{GE} collapses to the expression for EV^M in Proposition 3 of Baqaee and Burstein (2021).

The following corollary provides simple examples in which general equilibrium welfare differs from partial equilibrium welfare.

Corollary 1 (Demand Shocks Only). *In response to changes in preferences, $\{x_h\}$, and factor endowment shares, $\{\omega_h\}$, that keep the PPF, A and L, unchanged between t_0 and t_1,*

$$\Delta \log Y = EV^{GE} = 0.$$

However, partial equilibrium welfare, EV^{PE}, may be nonzero.

Hence, movements along a stable Pareto frontier have no effect on real GDP or general equilibrium welfare, but they can affect partial equilibrium welfare.

According to Proposition 3, growth accounting for welfare should be based on hypothetical sales shares evaluated at current technology but using demand of the fictional representative agent with homothetic and stable preferences, $e^{ev}(p,u)$.

We use the following simple example to show that, when preferences are non-aggregable, real GDP is different from welfare and is path-dependent, even if preferences for each agent are stable and homothetic.

Example 1. Consider an economy with two Cobb-Douglas consumers and two goods. Each good is produced using a fixed, good-specific primary factor. Each household owns the same fraction of both factors, and denote this fraction by ω_h for household h. Let b_{hi} be consumer h's budget share on good i. Let $b_i = \omega_1 b_{1i} + \omega_2 b_{2i}$ be the aggregate budget share on good i.

Consider technology shocks to each good and shocks to the distribution of income between $t_0 = 0$ and $t_1 = 1$. Specifically, for $t \in (0, 1/2)$ set $\omega_1 = 2t$, $\omega_2 = 1-2t$, $\log(A_1, A_2) = (1-(2t)^2, 1-2t)$. For $t \in (1/2, 1)$, set $\omega_1 = 2-2t$, $\omega_2 = 2t-1$ and $\log(A_1, A_2) = (2t-1, 2t-1)$. Hence, the economy starts and ends at the same point. For this reason, $EV^{PE} = EV^{GE} = 0$. However, real GDP is not equal to zero.

$$\Delta \log Y = \int_{t_0}^{t_1} b_1 d\log c_1 + b_2 d\log c_2 = \int_{t_0}^{t_1} b_1 d\log A_1 + b_2 d\log A_2 = \frac{b_{21}-b_{11}}{6}.$$

Hence, as long as $b_{11} \neq b_{21}$, i.e. preferences are non-aggregable, real GDP is non-zero.

4 Computing Aggregate GE Welfare in a CES Model

In this section, we sketch out how to compute changes in aggregate welfare, following Proposition 3, in a model where production and consumption functions are nested (potentially non-homothetic and subject to taste shocks) CES aggregators. Suppose that for each $i \in H+N$, the elasticity of substitution between goods (final demand for consumers and inputs for producers) is θ_i.

We define three endogenous statistics for the economy: the distribution of income χ, the input-output matrix Ω, and the Leontief inverse matrix Ψ.

Distribution of income. The distribution of income is an $(H+N+F)\times 1$ vector. The first H elements are equal to each household's share of aggregate nominal income, and the remaining $N+F$ elements are all zeros.

Input-output matrix. We stack the expenditure shares of the representative household, all producers, and all factors into the $(H+N+F) \times (H+N+F)$ input-output matrix Ω. The first H rows correspond to the households consumption baskets. The next N rows correspond to the expenditure shares of each producer on every other producer and factor. The last F rows correspond to the expenditure shares of the primary factors (which are all zeros, since primary factors do not require any inputs). With some abuse of notation, the heterogeneous agent input-output matrix can be written as

$$\Omega = \left[\begin{array}{ccc|ccc|ccc}
0 & \cdots & 0 & b_{11} & \cdots & b_{1N} & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & & \cdots & & & \cdots & \\
0 & \cdots & 0 & b_{H1} & \cdots & b_{HN} & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & \Omega_{11} & \cdots & \Omega_{1N} & \Omega_{1N+1} & \cdots & \Omega_{1N+F} \\
\vdots & \cdots & \vdots & & \ddots & & & & \\
0 & \cdots & 0 & \Omega_{N1} & & \Omega_{NN} & \Omega_{NN+1} & \cdots & \Omega_{NN+F} \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right]$$

Leontief Inverse. The Leontief inverse matrix is the $(H+N+F) \times (H+N+F)$ matrix defined as

$$\Psi \equiv (I-\Omega)^{-1} = I + \Omega + \Omega^2 + \dots,$$

where I is the identity matrix. The Leontief inverse matrix $\Psi \geq I$ records the *direct and indirect* exposures through the supply chains in the production network. Define the $(H+N+F) \times F$ matrix Ψ^F as the submatrix consisting of the right F columns of Ψ, representing the network-adjusted factor intensities of each good. The sum of network-adjusted factor intensities for every good i is equal to one, $\sum_{f \in F} \Psi_{if} = 1$ because the factor content of every good is equal to one.

Welfare-relevant variables. Solving for changes in welfare in an heterogeneous agent economy is equivalent to solving for changes in welfare in a representative agent economy with Leontieff preferences over the utility of fictional households with t_1 homothetic preferences. Changes in welfare-relevant variables are pinned down by the following system of differential equations. The $(H+N+F) \times 1$ price vector p^{ev} contains H household-specific (homothetic and stable preference) price indices, N good prices, and

F factor prices. For each $i \in H + N + F$,

$$d \log p_i^{ev} = -\sum_j \Psi_{ij}^{ev} d \log A_j + \sum_{f \in F} \Psi_{if}^{ev} d \log \lambda_f^{ev}.$$

This equation pins down changes in prices as a function of changes in productivity shocks, changes in factor income shares, and the Leontief inverse. The $(H + N + F) \times 1$ sales share vector λ^{ev} contains H household income shares, N sales shares, and F factor income shares. For each $l \in N + F$,

$$d\lambda_l^{ev} = \sum_{j \in N} \lambda_j^{ev} (\theta_j - 1) Cov_{\Omega_{(j,:)}^{ev}} \left(-d \log p^{ev}, \Psi_{(:,l)}^{ev} \right) + Cov_{\chi^{ev}} \left(d \log p^{ev}, \Psi_{(:,l)}^{ev} \right). \tag{13}$$

This equation pins down sales shares (including factor income shares) as a function of prices, the Leontief inverse, and elasticities of substitution. For each $h \in H$, equation (13) can be simplified to obtain changes in compensating income shares as follows:

$$d\chi_h^{ev} = d\lambda_h^{ev} = Cov_{\chi^{ev}} (d \log p^{ev}, I_{(h)}).$$

For each $i, l \in H + N + F$, changes in the Leontieff inverse are given by

$$d\Psi_{il}^{ev} = \sum_{j \in N} \Psi_{ij}^{ev} (\theta_j - 1) Cov_{\Omega_{(j,:)}^{ev}} \left(-d \log p^{ev}, \Psi_{(:,l)}^{ev} \right).$$

This equation pins down changes in the Leontief inverse as a function of prices, the Leontief inverse, and elasticities of substitution.

Together, these equations form a system of differential equations which pin down the nonlinear path of welfare-relevant Domar weights λ^{ev} for use in Proposition 3. The boundary conditions are that $p^{ev}(t_1) = 1$, $\lambda^{ev}(t_1) = \lambda_{t_1}$, $\Psi^{ev}(t_1) = \Psi_{t_1}$, and $\chi^{ev}(t_1) = \chi_{t_1}$.

As in the partial equilibrium problem, solving these differential equations does not require direct knowledge of income elasticities, taste shocks, or changes in the income distribution. Hence, none of these are required in order to compute EV^{GE}.

5 Distorted economies

In this section, we briefly discuss how to extend the results in Baqaee and Burstein (2021) to economies with inefficient equilibria building on the results of Baqaee and Farhi (2019b) for economies with homothetic and stable preferences. For simplicity in the exposition, we consider a representative agent economy. Consider again the environment in Section

3, but suppose that there are some arbitrary pattern of distorting wedges at point μ, which are implicit or explicit taxes. Without loss of generality, we can assume that μ take the form of output wedges (i.e. a tax wedge between price and marginal cost).[9]

For each A, x and μ, we denote equilibrium prices and aggregate income by $p(A, x, \mu)$ and $I(A, x, \mu)$. These equilibrium prices and incomes are unique up to the choice of a numeraire. Define the *macro indirect utility* function $V(A, L, \mu; x)$ to be the utility achieved by the agent with preferences x under the Walrasian equilibrium with wedges.

Consider shifts in technologies from A_{t_0} to A_{t_1}, along with changes in preferences from x_{t_0} to x_{t_1} and output wedges from μ_{t_0} to μ_{t_1}. We use the same definition of welfare as in Section 3, but we no longer require that the first welfare theorem hold. Our welfare measure EV^{GE} is the proportional change in initial factor endowments so that the representative consumer with preferences $\succeq_{x_{t_1}}$ is indifferent between the economy with initial productivities and wedges and the economy with final productivities, endowments, and wedges.

We characterize changes in real GDP and welfare. For simplicity, we abstract from changes in factor quantities, L. To study this problem we index the path of technologies, preferences, and wedges by time t. The definition of $\Delta \log Y$ is the same as before: $\Delta \log Y = \int_{t_0}^{t_1} \sum_{i \in N} b_{it} d \log c_{it}$. Define $\tilde{\lambda}$ to be the cost-based Domar weight of i, as in Baqaee and Farhi (2019b). That is,

$$\tilde{\lambda}' = b'(I - \mu\Omega)^{-1},$$

where b, μ, and Ω are all functions of A, u, x, and μ.

Proposition 4 (Real GDP)**.** *Given a path of technologies, tastes, and wedges that unfold as a function of time t, the change in real GDP is*

$$\Delta \log Y = \int_{t_0}^{t_1} \sum_{i \in N} \tilde{\lambda}_i(A_t, x_t, \mu_t) \frac{d \log A_{it}/\mu_{it}}{dt} - \int_{t_0}^{t_1} \sum_{i \in F} \tilde{\lambda}_i(A_t, x_t, \mu_t) \frac{d \log \lambda_{it}}{dt} dt. \tag{14}$$

Define $\lambda^{ev}(A, \mu)$ to be sales shares in a fictional economy with productivities A and wedges μ, but where consumers have stable homothetic preferences represented by the expenditure function $e^{ev}(p, u) = e(p, u_{t_1}, x_{t_1}) \frac{u}{u_{t_1}}$ where $u_{t_1} = v(p_{t_1}, I_{t_1}; x_{t_1})$. Let $\tilde{\lambda}^{ev}$ be the equivalent cost-based Domar weights.

Proposition 5 (Macro Welfare)**.** *For any smooth path of technologies, tastes, and wedges that*

[9]This is without loss of generality because we can always introduce a wedge on i's purchases of inputs from j by adding a fictitious middle-man that buys from j on behalf of i. An output wedge on this fictitious middleman is isomorphic to an input-specific wedge in the original economy.

unfold as a function of time t, changes in macro welfare are

$$EV^{GE} = \int_{t_0}^{t_1} \sum_{i \in N} \tilde{\lambda}_i^{ev}(A_t, \mu_t) \frac{d \log A_{it}/\mu_{it}}{dt} dt - \int_{t_0}^{t_1} \sum_{i \in F} \tilde{\lambda}_i^{ev}(A_t, \mu_t) \frac{d \log \lambda_{it}^{ev}}{dt} dt. \quad (15)$$

6 Aggregate gains from international trade

Consider a neoclassical world economy with heterogeneous agents. Let $\mathcal{H}_n$ be the set of households from country n. Let $\mathcal{F}_n$ be the set of primary factors owned by these households (and, in order to ensure balanced trade, that only these households can own), which are used to produce in country n. Denote by $\mathcal{P}_n(A_n, L_n)$ the set of feasible consumption allocations $\{c_h\}_{h \in \mathcal{H}_n}$ if country n is in autarky from the rest of the world, given technologies A_n and factor endowments $L_n \in \mathcal{F}_n$. The world economy is at t_1 in a trade equilibrium with world prices p_{t_1}. Country n is in autarky if costs to import and export goods to other countries are infinite.

Definition 3 (Gains from trade)**.** The general equilibrium gains for country n from autarky to trade at t_1 are given by ϕ^*, which solves

$$\min_{\phi \in \mathbb{R}} \{\exists \{c_h\}_{h \in \mathcal{H}_n} \in \mathcal{P}_n(A_n, \exp(\phi) L_n) \text{ and } u_h(c_h, x_{ht_1}) \geq u_h(c_{ht_1}; x_{ht_1}) \forall h \in \mathcal{H}_n\}, \quad (16)$$

where c_{ht_1} is the consumption bundle of agent h in t_1.

The gains from trade are measured by the minimum proportional increase in autarky factor endowments such that there exists a Pareto improvement compared to the trade equilibrium. The gains from trade can equivalently be defined by proportional shifts in the consumption possibility frontier, as discussed following Definition 2.[10]

We now discuss how to calculate aggregate gains from trade. Consider a fictional economy in country n with a representative agent with homothetic and stable preferences with expenditure function $e_n^{ev}(p, u) = \sum_{h \in \mathcal{H}_n} e_h \left(p, u_h(c_{ht_1}; x_{ht_1}); x_{ht_1}\right) u$ that owns all factors. This economy takes prices of exports and imports as given, as in a small open economy, at their t_1 trade equilibrium level. Import prices are subject to uniform trade costs τ in addition to those in the primitive trade equilibrium. In the trade equilibrium, $\tau = 1$, and in autarky, $\tau = \infty$. Balanced trade holds for any τ, so it is not necessary to introduce trade costs for exports to model autarky. In an abuse of notation, we denote

[10]Denote by $\mathcal{C}_n$ the set of feasible consumption allocations $\{c_h\}_{h \in \mathcal{H}_n}$ under autarky given A_n and L_n. Then the general equilibrium gains from autarky to trade at t_1 are given by ϕ^*, which solves $\min_{\phi \in \mathbb{R}} \{\exists \{c_h\}_{h \in \mathcal{H}_n} \in e^{\phi} \mathcal{C}_n \text{ and } u_h(c_h, x_{ht_1}) \geq u_h(c_{ht_1}; x_{ht_1}) \forall h \in \mathcal{H}_n\}$.

by $\mathcal{P}_n(A_n, L_n, \tau)$ the set of feasible consumption allocations in this fictional economy. We treat consumption of imports as domestically produced by adding a domestic intermediary that buys these goods and sells them to the consumer. Equilibrium prices of goods produced and consumed domestically in this fictional economy (including consumption of imports and exports) are given by $p_n^{GE}(A_n, L_n, \tau)$.To solve the equilibrium in this fictional economy, we do not need to compute allocations and prices in the rest of the world because export and import prices are fixed and exogenously given.

To solve for the gains from trade, we calculate welfare changes for this fictional country n economy as τ rises from $\tau = 1$ to $\tau = \infty$. Since consumers in this fictional economy have homothetic preferences, we do not need to know income elasticities in country n (or in any other country since world prices are fixed), conditional on knowing expenditure shares and elasticities of substitution in country n at t_1.

If preferences are non-homothetic CES (elasticities are constant along any indifference curve, but not necessarily constant across indifference curves) and agents are homogeneous, the welfare gains in response to a change in τ from $\tau = 1$ to $\tau = \infty$ are equal to those if preferences were homothetic CES using elasticities of substitution and trade shares in the primitive economy evaluated at u_{ht_1}. The sufficient statistics of Arkolakis et al. (2012) apply, where trade elasticities are evaluated at t_1 indifference curves.

To prove this implementation result, define the aggregate indirect utility function (for simplicity in the exposition, we omit taste shocks)

$$V(A_n, L_n, \tau) = \max_{\{c_h\}} W(\{c_h\}) \text{ subject to } \{c_h\} \in \mathcal{P}_n(A_n, L_n, \tau)$$

where

$$W(\{c_h\}) = \min_h \left\{ \frac{u_h(c_h; x_h)}{\bar{u}_h} \right\},$$

with $\bar{u}_h = u_h\left(c_{ht_1}\right)$. A consumption allocation Pareto dominates another one if it entails a higher value of V. The consumption allocation that solves the maximization problem $V(A_n, L_n, 1)$ is $\left\{c_{ht_1}\right\}$. We can re-express the definition of gains of trade as the value of ϕ that solves

$$V(A_n, L_n, 1) = V(A_n, e^{\phi} L_n, \infty).$$

Armed with this aggregate indirect utility function, we can solve for the gains from trade as in Proposition 5 in Baqaee and Burstein (2021).

Appendix A Proofs

Proof of Proposition 1. Consider the following social welfare function across households

$$W = \min_h \left\{ \frac{u_h\left(c_h; x_h\right)}{\bar{u}_h} \right\}.$$

For any fixed $\{x_h\}$, a consumption allocation $\{c_h\}$ Pareto dominates another one if and only if it entails a higher value of W. The expenditure function corresponding to this social welfare function is

$$e\left(p, u; \left\{x_{ht_1}\right\}\right) = \sum_h e_h\left(p, \bar{u}_h u; \left\{x_{ht_1}\right\}\right) \tag{17}$$

For each h, we set $\bar{u}_h = u_h\left(c_{ht_1}; x_{ht_1}\right)$, where c_{ht_1} is the consumption bundle at t_1. With this parameter choice, $e_h\left(p_{t_1}, \bar{u}_h; x_{ht_1}\right) = I_{ht_1}$ and $u_{t_1} = 1$. Denote u_{t_0} such that $e\left(p_{t_0}, u_{t_0}; \left\{x_{ht_1}\right\}\right) = I_{t_0}$. We can write EV^{PE} as,

$$\begin{aligned}
EV^{PE} &= \log \frac{e\left(p_{t_0}, u_{t_1}; \left\{x_{ht_1}\right\}\right)}{e\left(p_{t_0}, u_{t_0}; \left\{x_{ht_1}\right\}\right)} \\
&= \log \frac{\sum_h e_h\left(p_{t_0}, \bar{u}_h u_{t_1}; x_{ht_1}\right)}{\sum_h e_h\left(p_{t_0}, \bar{u}_h u_{t_0}; x_{ht_1}\right)} \\
&= \log \frac{I_{t_1}}{I_{t_0}} \frac{\sum_h e_h\left(p_{t_0}, \bar{u}_h; x_{ht_1}\right)}{\sum_h e_h\left(p_{t_1}, \bar{u}_h; x_{ht_1}\right)}
\end{aligned}$$

where the last equality uses $e\left(p_{t_0}, u_{t_0}; \left\{x_{ht_1}\right\}\right) = I_{t_0}$, $e\left(p_{t_1}, u_{t_1}; \left\{x_{ht_1}\right\}\right) = I_{t_1}$, and $u_{t_1} = 1$. The second term in the last line can be written as

$$\log \sum_h e_h\left(p_{t_0}, \bar{u}_h; x_{ht_1}\right) - \log \sum_h e_h\left(p_{t_1}, \bar{u}_h; x_{ht_1}\right) =$$

$$-\int_{t_0}^{t_1} \sum_i \frac{\partial \log \sum_h e_h\left(p, \bar{u}_h; x_{ht_1}\right)}{\partial \log p_i} d\text{log}p_i = -\int_{t_0}^{t_1} \sum_i \sum_h \frac{e_h\left(p, \bar{u}_h; x_{ht_1}\right)}{\sum_h e_h\left(p, \bar{u}_h; x_{ht_1}\right)} b_{hi}\left(p, \bar{u}_h; x_{ht_1}\right) d\log p_i$$

where

$$\begin{aligned}
\log e_h\left(p\left(t\right), \bar{u}_h; x_{ht_1}\right) &= \log e_h\left(p_{t_1}, \bar{u}_h; x_{ht_1}\right) - \int_t^{t_1} \sum_i b_{hi}\left(p, \bar{u}_h; x_{ht_1}\right) d\log p_i \\
&= I_{ht_1} - \int_t^{t_1} \sum_i b_{hi}\left(p, \bar{u}_h; x_{ht_1}\right) d\log p_i
\end{aligned}$$

The expression for EV^{PE} can also be derived using the expenditure function of a representative agent with homothetic and stable preferences

$$e^{ev}(p,u) = \sum_h e_h\left(p, u_h(c_{ht_1}; x_{ht_1}); x_{ht_1}\right) u,$$

with corresponding budget shares $b^{ev}(p)$.

$\square$

Proof of Proposition 2. This proof follows the same steps of Proposition 4 in Baqaee and Burstein (2021). Partial and general equilibrium welfare are equal if and only if $p^{ev}(A,L) = p(A,L,\{\omega_h\},\{x_h\})$ for all $A, L, \{\omega_h\}, \{x_h\}$, where $p^{ev}(A,L)$ denotes prices in the hypothetical economy with a representative consumer with expenditure function $e^{ev}(p,u)$ defined above. This condition is satisfied if either demand is aggregable, homothetic and stable (in which case aggregate budget shares by good, and hence prices, do not depend on the income distribution $\{\omega_h\}$ and taste shifters $\{x_h\}$)), or if prices are pinned down by A and L only, as in a one factor model.

$\square$

Proof of Proposition 3. Define the aggregate indirect utility function

$$V(A,L;\{x_h\}) = \max_{\{c_h\}} W\left(\{c_h\};\{x_h\}\right) \text{ subject to } \{c_h\} \in \mathcal{P}(A,L)$$

where

$$W\left(\{c_h\};\{x_h\}\right) = \min_h \left\{\frac{u_h\left(c_h; x_h\right)}{\bar{u}_h}\right\},$$

with $\bar{u}_h = u_h\left(c_{ht_1}; x_{ht_1}\right)$. With this choice of parameters, the consumption allocation that solves the maximization problem $V(A_{t_1}, L_{t_1}; \{x_{ht_1}\})$ is $\{c_{ht_1}\}$. For any fixed $\{x_h\}$, a consumption allocation Pareto dominates another one if it entails a higher value of V. We can re-express the definition of EV^{GE} as the value of ϕ that solves

$$V(A_{t_1}, L_{t_1}; \{x_{ht_1}\}) = V(A_{t_0}, e^{\phi} L_{t_0}; \{x_{ht_1}\}).$$

Armed with this aggregate indirect utility function, we can solve for EV^{GE} as in Proposition 5 in Baqaee and Burstein (2021). To apply the proof of Proposition 5 in Baqaee and Burstein (2021), we expand the set of goods from N in the primitive economy to $N \times H$ in the expanded economy, indexing goods by hi. All goods with a common i share the same production function. For every i, good $1i$ is used for consumption by household $h = 1$

and as intermediate input by all producers. For any good i specific factor f, $\sum l_{hif}$ must not exceed L_f in the primitive economy. In the expanded economy, $\lambda_i^{ev} = \sum_h \lambda_{hi}^{ev}$.

□

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