Online appendix for "Innovation, Firm Dynamics, and International Trade"

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This appendix is composed of two parts. In the first part, we provide details for two analytic results from our model that are discussed in Section 4 but not proved in the paper: the model with no productivity dynamics with Pareto distributed productivities, and the model with assymetric countries. In the second part of this appendix, we provide additional details on the quantitative model and its numerical solution . We also provide an overview of the Matlab codes that are used to compute the model, which are available in our website.

1. Analytical results

In this section we present proofs for results that are discussed in Section 4 of the paper.

1.1. Section 4.2: No productivity dynamics, Pareto distribution and $\beta < 1$

We now show that, if we allow for any $\beta < 1$ and assume that G is such that $\exp(z)$ is distributed Pareto, we get that $\zeta = \Pi_d \left[Z_d + Z_x \left(1 + D^{1-\rho}\right)\right]/\Upsilon$ is unchanged with D. Thus, Lemma 1 applies, and hence, L_r is unchanged with D. Therefore, as in the version of the model with no productivity dynamics and $\beta \to 1$, the ratio of the indirect effect to the direct effect of changes in trade costs on aggregate productivity is given by (4.4) in the paper.

In particular, we assume that the cumulative distribution function of $\exp(z)$ is

$$\bar{G}(z) = 1 - \left(\frac{\exp(z_0)}{\exp(z)}\right)^{\sigma}, \text{ for } \exp(z) > \exp(\bar{z}_0)$$

where $\sigma > 1$. Under this distribution, we have that

$$Z_d = \int_{\exp(\bar{z})}^{\exp(\bar{z}_x)} \frac{\sigma}{\delta} \frac{\exp(\bar{z}_0)^{\sigma}}{\exp(z)^{\sigma}} \, \mathrm{d}\exp(z) = \frac{\sigma \exp(\bar{z}_0)^{\sigma}}{\delta(\sigma - 1)} \left[\exp(\bar{z})^{1-\sigma} - \exp(\bar{z}_x)^{1-\sigma}\right] \text{ and}$$

$$Z_x = \int_{\exp(\bar{z}_x)}^{\infty} \frac{\sigma}{\delta} \frac{\exp(\bar{z}_0)^{\sigma}}{\exp(z)^{\sigma}} \mathrm{d}\exp(z) = \frac{\sigma \exp(\bar{z}_0)^{\sigma}}{\delta(\sigma - 1)} \exp(\bar{z}_x)^{1-\sigma}.$$

Using the cutoff definitions, we get that

$$Z_d = \frac{\sigma \exp(\bar{z}_0)^{\sigma}}{\delta(\sigma - 1)} (\Pi_d)^{\sigma - 1} \left[n_f^{1-\sigma} - \left(\frac{n_x}{D^{1-\rho}}\right)^{1-\sigma}\right], \text{ and}$$

[1] We thank Javier Cravino for superb research assistance in putting together this material.

$$Z_x = \frac{\sigma \exp(\bar{z}_0)^{\sigma}}{\delta(\sigma-1)} (\Pi_d)^{\sigma-1} \left(\frac{n_x}{D^{1-\rho}}\right)^{1-\sigma}.$$

Therefore, we have

$$\Pi_d \left[Z_d + Z_x \left(1 + D^{1-\rho}\right)\right] = \frac{\sigma}{\delta(\sigma-1)} \left[\exp(\bar{z}_0)\, \Pi_d\right]^{\sigma} \left[n_f^{1-\sigma} + n_x^{1-\sigma} \left(D^{1-\rho}\right)^{\sigma}\right]. \tag{1.1}$$

Using the cutoff definitions, we can express Υ as

$$\Upsilon = n_e + \frac{1}{\delta} \left[\exp(\bar{z}_0)\, \Pi_d\right]^{\sigma} \left[n_f^{1-\sigma} + n_x^{1-\sigma} \left(D^{1-\rho}\right)^{\sigma}\right]. \tag{1.2}$$

Combining (1.1) and (1.2), we obtain

$$\Pi_d \left[Z_d + Z_x \left(1 + D^{1-\rho}\right)\right] = \frac{\sigma}{\sigma-1} (\Upsilon - n_e)$$

Combined with (A7) in the paper, this implies that both $\Pi_d \left[Z_d + Z_x \left(1 + D^{1-\rho}\right)\right]$ and Υ are independent of D. Therefore, Lemma 1 applies. Therefore, the ratio of the indirect effect to the direct effect of changes in trade costs on aggregate productivity is given by (4.4) in the paper.

Note that in this case, if $\lambda = 1$, M_e is invariant to changes in D (see expression A2 in the paper together with the fact that L_r and Υ are unchanged with D). Hence, it is possible to prove this result without the use of the free-entry condition, but instead fixing the number of firms in each country. One does require the free-entry condition to prove our result when $\lambda < 1$.

1.2. Section 4.5: Asymmetric countries

In section we extend the analytic results to the version of the model with asymmetric countries. We focus on the special cases that we can solve analytically in Section 4 (all firms export, subset of firms export/ no productivity dynamics, elastic process innovation/exogenous selection). We continue to use the assumption that the real interest rate approaches zero, $\beta \to 1$, to ensure that the allocation of labor remains constant. Note that this assumption that $\beta \to 1$ is not necessary in all of our special cases (e.g. it is not necessary when all firms export or with no productivity dynamics and Pareto distributed productivities), but we use it here to unify the presentation.

We first show that, if we assume trade balance between countries, to a first-order-approximation, the ratio of the indirect effect to the direct effect of a change in marginal trade costs on aggregate productivity is the same as with symmetric countries, and given by

$$\frac{\text{indirect effect}}{\text{direct effect}} = \frac{1-\lambda}{\rho+\lambda-2}$$

in all special cases of the model. With asymmetric countries, a new effect arises in our comparative statics across steady-states after a change in trade costs that is not present in the symmetric case. We term this effect the "terms of trade effect". As we will see below, to obtain the result above in the asymmetric case, we must be careful to include this terms of trade effect as part of what we call the "direct effect" of a change in variable trade costs on aggregate productivity.

The magnitude of the terms of trade effect in response to given changes in trade costs can potentially differ across our model specifications. Hence, in general the change in aggregate productivity, output and consumption in each country will vary across model specifications.

We next show, however, that with trade balance, to a first-order-approximation the growth of world output and consumption (defined as an expenditure weighted average of the growth of output and consumption of individual countries) is equal across model specifications. Hence, even though changes in exit, export, and process innovation decisions can lead to different responses of output and consumption in individual countries, once the change in product innovation and terms of trade are taken into account, changes in these decisions do not affect the global growth in output and consumption. That is, changes in these decisions can lead to a redistribution of output and consumption across countries, but not to changes in world output and consumption.

Finally, we consider a version of our model that does not assume trade balance, but instead assumes risk sharing between countries. The equilibrium allocations coincide with those of the planning problem. We show that, to a first-order approximation, the growth of world consumption is also equal across our alternative model specifications. Moreover, in this case our measure of aggregate consumption growth is equal to the change in welfare of a global planner. Hence, to a first-order-approximation, changes in exit, export, and process innovation decisions of firms in response to changes in trade costs do not affect global welfare, once changes in product innovation and terms of trade are taken into account.

1.2.1. The structure of the model in steady-state

We allow our two countries to differ in the size of the labor force (L), trade costs (D and n_x), fixed costs (n_f), entry costs (n_e), and the parameters of the innovation cost function ($c\left(q\right)$ and Δ_z). We will assume that the elasticity of substitution parameter ρ and the share of labor in the research production function λ are the same in both countries. We will first assume trade balance period by period and then assume risk sharing, and we only focus on steady-state equilibria.

As we did in our paper, we find it useful to break the equations characterizing the steady-state into four groups:

(1) those characterizing the Bellman Equation of the firm and free entry used to pin down "profitability" in each country relative to the price of the research good (Π_d/W_r and Π_d^*/W_r^*). It is in differentiating these equations that we invoke Envelope Theorems to argue that the derivatives of these equations do not depend on the specification of the firms' decision problems as long as we are differentiating at a point at which these decisions are chosen optimally.

(2) those static equations relating profitability relative to the price of the research good Π_d/W_r and Π_r^*/W_r^* to domestic real wages and output.

(3) those equations that determine the allocation of labor in steady-state. Here, we use our result that the allocation of labor is fixed in steady-state if $\beta \to 1$.

(4) those equations that relate real wages and output to the components of productivity and the allocation of labor. This is the set of equations in which all of the new action arises in the case of asymmetric countries.

Bellman Equation and Free Entry The only impact that asymmetric countries has on our Bellman Equations and Free Entry conditions is that a term that we call the "terms of trade" effect comes into the definition of variable profits within the period. This term, denoted by u, reflects the impact of differences in market size and price levels across countries. Thus, in the asymmetric case, we can write steady-state variable profits in units of the local research good in the two countries as follows

$$\frac{\Pi(s)}{W_r} = \frac{\Pi_d}{W_r} \exp(z) + \max\left(\frac{\Pi_d}{W_r} u D^{1-\rho} \exp(z) - n_x, 0\right)$$

$$\frac{\Pi^*(s)}{W_r^*} = \frac{\Pi_d^*}{W_r^*} \exp(z) + \max\left(\frac{\Pi_d^*}{W_r^*} \frac{1}{u} D^{*1-\rho} \exp(z) - n_x^*, 0\right).$$

Note that we will relate u to market size and price levels in the second set of equations below. With symmetry, we have $u = 1$. In the asymmetric case, we must solve for it endogenously. Also note that in the paper, we chose the home research good as the numeraire, so $W_r = 1$ by definition and $W_r^* = 1$ by symmetry. Here we can set $W_r = 1$ as a numeraire, but we must solve for W_r^* in equilibrium. To keep the notation easy to read, we will not impose $W_r = 1$.

Using the same logic of the paper under our special cases in Section 4, differentiating the free-entry condition in the home country and using the envelope conditions of process innovation, exit and exporting decisions, yields

$$\Delta \frac{\Pi_d}{W_r} \left[Z_d + \left(1 + uD^{1-\rho}\right) Z_x \right] + \frac{\Pi_d}{W_r} Z_x \Delta \left(1 + uD^{1-\rho}\right) = 0$$

so

$$\begin{aligned} \Delta \log \frac{\Pi_d}{W_r} &= -\frac{Z_x}{Z_d + \left(1 + uD^{1-\rho}\right) Z_x} \Delta \left(1 + uD^{1-\rho}\right) \\ &= -s_x \Delta \log u + (\rho - 1)\, s_x \Delta \log D, \end{aligned} \tag{1.3}$$

where s_x and s_x^* denote the shares of exports in the output of intermediate good firms, given by

$$s_x = \frac{uZ_x D^{1-\rho}}{(Z_d + Z_x) + uZ_x D^{1-\rho}} \text{ and } s_x^* = \frac{Z_x^* D^{*1-\rho}}{u\left(Z_d^* + Z_x^*\right) + Z_x^* D^{*1-\rho}}. \tag{1.4}$$

It is straightforward that the envelope condition can be applied here even with asymmetric countries.

In the case with endogenous process innovation, we are using the fact that the real interest rate is zero so that the hybrid share of exports $\tilde{s}_x$ equals the actual share of exports s_x. Note that expression (1.3) coincides with the one in the paper, with the addition of the change in relative country sizes, $s_x \Delta \log u$, which affects profits from sales in the foreign country. Analogously, in the foreign country, differentiating the free-entry condition and using the envelope conditions for process innovation, exit and exporting decisions yields

$$\Delta \log \frac{\Pi_d^*}{W_r^*} = s_x^* \Delta \log u + s_x^* (\rho - 1)\, \Delta \log D^*. \tag{1.5}$$

Static relations: profitability, output, and wages The following equations obtained from static profit maximization and the definitions of W_r and W_r^*, do not change with the assumption that countries are asymmetric:

$$\begin{aligned} \frac{\Pi_d}{W_r} &= \frac{\lambda^\lambda (1-\lambda)^{1-\lambda}}{\rho^\rho (\rho-1)^{1-\rho}} (W/P)^{1-\rho-\lambda}\, Y \,, \\ \frac{\Pi_d^*}{W_r^*} &= \frac{\lambda^\lambda (1-\lambda^*)^{1-\lambda}}{\rho^\rho (\rho-1)^{1-\rho}} (W^*/P^*)^{1-\rho-\lambda}\, Y^*. \end{aligned} \tag{1.6}$$

Clearly, these equations are already log-linear and hence any derivatives associated with them do not depend on other aspects of the specification of the model.

The variable u, which summarizes the size and price level differences across countries that are relevant for firms' profits, is defined by

$$u = \frac{Y^* (P^*)^\rho}{Y (P)^\rho}. \tag{1.7}$$

We will refer to changes in relative sizes and prices across countries, as summarized by u, as the "terms of trade effect". This equation is already log-linear as well.

Aggregate allocation of labor Here we follow exactly the same steps as in the symmetric model. From the CES aggregator, payments to production employment is a fixed ratio of variable profits:

$$W (L - L_r) = (\rho - 1) \Pi_d \left[M_e (Z_d + Z_x) + D^{1-\rho} M_e Z_x u \right].$$

Cobb-Douglas production of the research good implies a constant cost share on research labor

$$W L_r = \lambda W_r \Upsilon M_e,$$

where Υ denotes the use of research good per entering firm as defined in equation (3.7) of the paper. Dividing these two equations yields:

$$\frac{L - L_r}{L_r} = \frac{\rho - 1}{\lambda} \frac{\Pi_d \left[M_e (Z_d + Z_x) + D^{1-\rho} M_e Z_x u \right]}{W_r \Upsilon M_e}$$

With $\beta \to 1$, the free entry condition implies $\frac{\Pi_d}{W_r} \frac{\left[(Z_d + Z_x) + D^{1-\rho} M_e Z_x u \right]}{\Upsilon} \beta \to 1$, so

$$\frac{L - L_r}{L_r} = \frac{\rho - 1}{\lambda}. \tag{1.8}$$

The same relation holds in the foreign country.

Hence, the aggregate allocation of labor is unchanged across steady states when the real interest rate approaches zero. Note that in the special versions of our model in which all firms export or with no productivity dynamics and Pareto distributed productivities, one does need to assume $\beta = 1$ for the aggregate allocation of labor to be unchanged across steady-state.

Aggregate output, wages and productivity under trade balance The final set of equations relate aggregate output and wages to productivities. Here, with asymmetric countries, the algebra is messier and new terms related to the change in u enter. The equations differ slightly between the cases of trade balance and risk sharing.

Labor market equilibrium in the home country requires

$$\left(\frac{\rho-1}{\rho}\right)^{\rho} W^{-\rho}\left(P^{\rho} Y M_e\left(Z_d+Z_x\right)+P^{*\rho} Y^{*} M_e Z_x D^{1-\rho}\right)=L-L_r,$$

which using the definition of u yields

$$\left(\frac{\rho-1}{\rho}\right)^{\rho}\left(\frac{W}{P}\right)^{-\rho} Y M_e\left(Z_d+Z_x+u Z_x D^{1-\rho}\right)=L-L_r. \tag{1.9}$$

The price level in the home country is

$$P=\frac{\rho}{\rho-1}\left[M_e\left(Z_d+Z_x\right) W^{1-\rho}+M_e^{*} Z_x^{*}\left(W^{*}\right)^{1-\rho} D^{*1-\rho}\right]^{1/(1-\rho)},$$

or

$$\frac{W}{P}=\frac{\rho-1}{\rho}\left[M_e\left(Z_d+Z_x\right)+M_e^{*} Z_x^{*}\left(\frac{W^{*}}{W}\right)^{1-\rho} D^{*1-\rho}\right]^{1/(\rho-1)}. \tag{1.10}$$

Trade balance requires that exports equal imports, or

$$W^{1-\rho} D^{1-\rho} M_e Z_x P^{*\rho} Y^{*}=W^{*1-\rho} D^{*1-\rho} M_e^{*} Z_x^{*} P^{\rho} Y. \tag{1.11}$$

Substituting (1.11) into (1.10) yields

$$\frac{W}{P}=\frac{\rho-1}{\rho}\left[M_e\left(Z_d+Z_x\right)+D^{1-\rho} M_e Z_x u\right]^{1/(\rho-1)}. \tag{1.12}$$

Combining (1.9) and (1.12) yields

$$Y=\left(M_e\left(Z_d+Z_x\right)+D^{1-\rho} u M_e Z_x\right)^{\frac{1}{\rho-1}}\left(L-L_r\right) \tag{1.13}$$

Note that (1.12) and (1.13) coincide with that in our symmetric model, except that now aggregate productivity of exporting firms is adjusted with the term u. We define aggregate productivity in the home country as:

$$Z=\left(M_e\left(Z_d+Z_x\right)+D^{1-\rho} u M_e Z_x\right)^{\frac{1}{\rho-1}}. \tag{1.14}$$

The analogous expressions for the real wage and aggregate output in the foreign country are

$$\frac{W^{*}}{P^{*}}=\frac{\rho-1}{\rho}\left[M_e^{*}\left(Z_d^{*}+Z_x^{*}\right)+D^{*1-\rho} M_e^{*} Z_x^{*} \frac{1}{u}\right]^{1/(\rho-1)} \tag{1.15}$$

$$Y^{*}=\left(M_e^{*}\left(Z_d^{*}+Z_x^{*}\right)+D^{*1-\rho} \frac{1}{u} M_e^{*} Z_x^{*}\right)^{\frac{1}{\rho-1}}\left(L-L_r^{*}\right). \tag{1.16}$$

Equilibrium consumption in each country is

$$C=Y-\frac{1-\lambda}{\lambda} \frac{W}{P}, \text{ and } C^{*}=Y^{*}-\frac{1-\lambda}{\lambda} \frac{W^{*}}{P^{*}}. \tag{1.17}$$

1.2.2. Effects of changes in trade costs under trade balance

We first calculate, under trade balance, the ratio of the indirect effect to the direct effect of a change in trade costs on aggregate productivity, to a first-order approximation. Next, we calculate the impact of such a change in trade costs on growth of world output and consumption.

Aggregate productivity Here we study the effects of a small change in marginal trade costs on aggregate productivity across steady-states. Following the logic in our paper, to a first-order approximation, the change in aggregate productivity of the home country in response to a change in marginal trade costs is

$$\Delta \log Z = \underbrace{-s_x \Delta \log D + \frac{1}{\rho - 1} s_x \Delta \log u}_{\textbf{Direct Effect}} \tag{1.18}$$

$$\underbrace{+\frac{1}{\rho-1}\left[s_x \frac{1 + uD^{1-\rho}}{uD^{1-\rho}} \Delta \log Z_x + \left(1 - s_x \frac{1 + uD^{1-\rho}}{uD^{1-\rho}}\right) \Delta \log Z_d + \Delta \log M_e \right]}_{\textbf{Indirect Effect}}.$$

The direct effect corresponds to the change in aggregate productivity from the change in trade costs and the subsequent change in relative country sizes and price levels as summarized by u (i.e. we can refer to this as the terms of trade effect), with firms' exit, export, process, and product innovation decisions held fixed. With symmetric countries, u remains constant. The *indirect effect* arises from changes in firm decisions, which are themselves responding to the trade cost and the change in u.

We first evaluate the size of the indirect effect relative to the direct effect. Log-differentiating (1.6) and using the fact that with zero interest rates the aggregate allocation of labor remains unchanged across steady-states, we obtain:

$$\begin{aligned} \Delta \log \frac{\Pi_d}{W_r} &= \frac{2 - \rho - \lambda}{\rho - 1} \Delta \log \left[M_e \left(Z_d + Z_x \right) + D^{1-\rho} u M_e Z_x \right] \\ &= (2 - \rho - \lambda) \left[\text{Direct} + \text{Indirect} \right] \end{aligned} \tag{1.19}$$

Combining (1.3) and (1.19) as in our paper, we obtain:

$$\frac{\text{Indirect Effect}}{\text{Direct Effect}} = \frac{1 - \lambda}{\rho + \lambda - 2}$$

The same expression holds in the foreign country. So, the ratio of the indirect to the direct effect is the same as with symmetric countries, with the caveat that direct effect now includes the change in relative country sizes and price levels as summarized by u (i.e. the terms of trade effect).

Note that when $\lambda = 1$, the indirect effect is equal to zero. Therefore, changes in product innovation $\Delta \log M_e$ exactly offset changes in exit, export, and process innovation decisions as summarized by $\Delta \log Z_d$ and $\Delta \log Z_x$.

The magnitude of the change in aggregate productivity and output in each country now depends on the terms of trade effect, as summarized by $\Delta \log u$, which is determined in general equilibrium. The magnitude of the change in the terms of trade can potentially vary across model specifications. To see this, note that the trade balance condition, (1.11), depends on changes in exporters' aggregate productivity $\Delta \log Z_x$ and $\Delta \log Z_x^*$ and not on changes in domestic producers' aggregate productivity $\Delta \log Z_d$ and $\Delta \log Z_d^*$, and these can vary across model specifications. One can pick parameters across some of the model specifications to ensure that the change in $\Delta \log Z_x$ and $\Delta \log Z_x^*$ are the same (i.e.: for example elastic process innovation - inelastic export participation, or inelastic process innovation - elastic export participation).

We now show that even though the change in output and consumption of each individual country can vary across model specifications due to the terms of trade effect, the growth in world output and consumption (and the welfare of a global planner under complete markets) does not.

World output and consumption We define a measure of the growth of world consumption as an expenditure-weighted average of the growth of consumption in each country,

$$\Delta \log C^W = \frac{1}{1+k} \Delta \log C + \frac{k}{1+k} \Delta \log C^* \tag{1.20}$$

where k is the ratio of foreign consumption expenditures to home consumption expenditures:

$$k = \frac{P^* C^*}{PC}.$$

Using (1.8), (1.12), (1.13), (1.15), and (1.16), and (1.17), we have that the ratio of consumption expenditures is equal to the ratio of final output expenditures (this is straightforward with $\lambda = 1$, but otherwise it relies on the fact that real wages are proportional to final output):

$$k=\frac{P^{*}C^{*}}{PC}=\frac{P^{*}Y^{*}}{PY}$$

These expressions also imply that $\Delta\log Y=\Delta\log W/P$ in each country, so using (1.17), we have

$$\Delta\log C=\Delta\log Y,\ \Delta\log C^{*}=\Delta\log Y^{*},\text{ and }\Delta\log Y^{W}=\Delta\log C^{W}$$

We obtain $\Delta\log Y$ and $\Delta\log Y^{*}$ using (1.3), (1.5), (1.19) and the analogous of expression (1.19) in the foreign country:

$$-s_x\Delta\log u+(\rho-1)\,s_x\Delta\log D=\Delta\log\frac{\Pi_d}{W_r}=(2-\rho-\lambda)\,\Delta\log Y \tag{1.21}$$

$$s_x^{*}\Delta\log u+(\rho-1)\,s_x^{*}\Delta\log D^{*}=\Delta\log\frac{\Pi_d^{*}}{W_r^{*}}=(2-\rho-\lambda)\,\Delta\log Y^{*} \tag{1.22}$$

Using the definitions s_x =Exports$/PY$ and s_x^{*} =Exports$^{*}/P^{*}Y^{*}$, and the trade balance condition (Exports=Exports*) we have $k=s_x/s_x^{*}$. Hence, we can calculate $\Delta\log C^{W}$ by multiplying (1.22) by s_x/s_x^{*} and adding it to (1.21), noting that the terms with $\Delta\log u$ cancel-out:

$$\Delta\log C^{W}=\frac{\Delta\log Y+\frac{s_x}{s_x^{*}}\Delta\log Y^{*}}{1+\frac{s_x}{s_x^{*}}}=-\frac{\rho-1}{\rho+\lambda-2}\frac{s_x s_x^{*}}{s_x+s_x^{*}}\left[\Delta\log D+\log D^{*}\right]$$

Note that with symmetric countries, $\Delta\log C^{W}=-(\rho-1)/(\rho+\lambda-2)\,\Delta\log D$, as in our paper.

Therefore, conditional on choosing the model parameters to match export shares s_x and s_x^{*} in each country, the steady-state world growth in output and consumption in response to changes in trade costs are, to a first-order approximation, equal across our alternative model specifications. Changes in exit, export, and process innovation decisions that have differential impact on the terms of trade affect the distribution of world output across countries, but not the growth in world output and consumption.

1.2.3. Effects of changes in trade costs under risk sharing

The previous discussion assumed trade balance between countries (i.e. financial autarky) We now substitute this assumption with risk sharing, which emerges under complete financial markets between countries. We show that, to a first-order approximation, the steady-state world growth in consumption is still the same across our model specifications as long as we choose parameters to target the same trade shares and relative expenditures across countries.

Moreover, the growth in world consumption in this case is proportional to the change in the welfare of a global planner.

We assume preferences over consumption in each country of the form $U(C) = \frac{1}{1-\sigma} C^{1-\sigma}$, with $\sigma \geq 0$. The risk sharing condition that emerges under complete markets is

$$\frac{C^{-\sigma}}{P} = \chi \frac{C^{*-\sigma}}{P^{*}} \tag{1.23}$$

where χ is a constant that is determined by the initial level of wealth in each country. Equation (1.23) replaces the trade balance condition (1.11). The ratio of foreign consumption expenditures to domestic consumption expenditures is given by

$$k = \chi \frac{C^{*1-\sigma}}{C^{1-\sigma}}$$

Define the welfare function of a world social planner by $U^W = \frac{1}{1-\sigma} C^{1-\sigma} + \chi \frac{1}{1-\sigma} C^{*1-\sigma}$. Note that the weight on foreign utility depends on the initial level of wealth in each country. The log-percentage change in welfare is proportional to the expenditure-weighted average of consumption growth in each country:

$$\begin{aligned}
\Delta \log U^W &= (1-\sigma) \left\{ \frac{C^{1-\sigma}}{C^{1-\sigma} + \chi C^{*1-\sigma}} \Delta \log C + \frac{\chi C^{*1-\sigma}}{C^{1-\sigma} + \chi C^{*1-\sigma}} \Delta \log C^{*} \right\} \\
&= (1-\sigma) \left\{ \frac{1}{1+k} \Delta \log C + \frac{k}{1+k} \Delta \log C^{*} \right\} = (1-\sigma) \Delta \log C^W
\end{aligned}$$

We now show that, conditional on targeting export shares in each country s_x and s_x^* and relative expenditures k, the steady-state world growth of consumption, $\Delta \log C^W$, is independent of the response of exit, export, and process innovation decisions, and hence is equal across our alternative model specifications.

In deriving our results, we assume that firms receive a per-unit production subsidy $\tau \geq 1$, so that revenues of a firm selling y units at a price p are equal to $\tau p y$. With this per-unit subsidy, the domestic price set by a home firm with productivity index $\exp(z)$ is $p = \frac{1}{\tau} \frac{\rho}{\rho-1} \frac{W}{\exp(z)^{1/(\rho-1)}}$.

If $\lambda = 1$, the level of τ does not affect the steady-state growth in world consumption (and hence, we can assume $\tau = 1$ as in our baseline model). If $\lambda < 1$, we need to set $\tau = \tau^* = \rho/(\rho-1)$ for our result to hold. This is related to the fact that, with $\lambda < 1$, markups distort the equilibrium level of entry. With $\tau = \tau^*$, markups are eliminated and the efficient level of entry is restored. Our central equivalence result holds for any value of τ if $\lambda = 1$, and $\tau = \tau^*$ if $\lambda < 1$.

Loglinearized system of equations With risk sharing, we can make use of many of the previous steps under balanced trade. In particular, the change in the constant on variable profits is still given by (1.3) and (1.5), the constant in variable profits is related to output and the real wage by (1.6), the labor market clearing condition is given by (1.9), the price level is determined by (1.10) with the addition of a multiplicative constant that reflects the production subsidy τ, and the level of consumption is determined by (1.17). When real interest rates are zero, the aggregate allocation is constant and given by (1.8). Note, however, that we cannot make use of equations (1.12) and (1.15) because in deriving these we made use of the trade balance equation (1.11), which is now replaced by (1.23).

With this in mind, the log-linear equations that determine the first-order aggregate effects of changes in trade costs are: (1.3), (1.5), the loglinear versions of (1.6)

$$\Delta \log \frac{\Pi_d}{W_r} = (1-\rho-\lambda)\,\Delta \log (W/P) + \Delta \log Y \tag{1.24}$$

$$\Delta \log \frac{\Pi_d^*}{W_r^*} = (1-\rho-\lambda)\,\Delta \log (W^*/P^*) + \Delta \log Y^* \ , \tag{1.25}$$

the loglinear version of (1.9) and the analogous expression in the foreign country

$$\begin{aligned}\rho\Delta \log (W/P) \;&=\; \Delta \log Y + s_x \Delta \log u + s_x(1-\rho)\Delta \log D + \Delta \log M_e + \\ &\quad + \left(1 - s_x \frac{1+uD^{1-\rho}}{uD^{1-\rho}}\right) \Delta \log Z_d + s_x \frac{1+uD^{1-\rho}}{uD^{1-\rho}} \Delta \log Z_x \ ,\end{aligned} \tag{1.26}$$

$$\begin{aligned}\rho\Delta \log (W^*/P^*) \;&=\; \Delta \log Y^* - s_x^* \Delta \log u + s_x^*(1-\rho)\Delta \log D^* + \Delta \log M_e^* + \\ &\quad + \left(1 - s_x^* \frac{1+u^{-1}D^{*1-\rho}}{u^{-1}D^{*1-\rho}}\right) \Delta \log Z_d^* + s_x^* \frac{1+u^{-1}D^{*1-\rho}}{u^{-1}D^{*1-\rho}} \Delta \log Z_x^*\end{aligned} \tag{1.27}$$

the loglinear version of (1.10) and the analogous expression in the foreign country

$$\begin{aligned}\Delta \log (W/P) \;&=\; \frac{1}{\rho-1}(1-s_m)\left(\Delta \log M_e + \frac{Z_d}{Z_d+Z_x}\Delta \log Z_d + \frac{Z_x}{Z_d+Z_d}\Delta \log Z_x\right) \\ &\quad + \frac{1}{\rho-1} s_m \left(\Delta \log M_e^* + \Delta \log Z_x^*\right) - s_m \Delta \log (W^*/W) - s_m \Delta \log D^*\end{aligned} \tag{1.28}$$

$$\begin{aligned}\Delta \log (W^*/P^*) \;&=\; \frac{1}{\rho-1}(1-s_m^*)\left(\Delta \log M_e^* + \frac{Z_d^*}{Z_d^*+Z_x^*}\Delta \log Z_d^* + \frac{Z_x^*}{Z_d^*+Z_x^*}\Delta \log Z_x^*\right) \\ &\quad + \frac{1}{\rho-1} s_m^* \left(\Delta \log M_e + \Delta \log Z_x\right) + s_m^* \Delta \log (W^*/W) - s_m^* \Delta \log D \ ,\end{aligned} \tag{1.29}$$

the loglinear versions of (1.17)

$$\Delta \log Y = \frac{C}{Y}\Delta \log C + \left(1 - \frac{C}{Y}\right) \Delta \log \log (W/P) \tag{1.30}$$

$$\Delta \log Y^* = \frac{C^*}{Y^*}\Delta \log C + \left(1 - \frac{C^*}{Y^*}\right)\Delta \log \left(W^*/P^*\right) \ , \tag{1.31}$$

and the loglinear versions of (1.7) and (1.23)

$$\Delta \log \left(W^*/W\right) = \sigma \left[\Delta \log C - \Delta \log C^*\right] + \Delta \log \left(W^*/P^*\right) - \Delta \log \left(W/P\right) \tag{1.32}$$

$$\Delta \log u = \Delta \log Y^* - \Delta \log Y - \sigma\rho \left[\Delta \log C^* - \Delta \log C\right] \tag{1.33}$$

Here, we define s_m and s_m^* to be the share of imports in final good expenditures:

$$s_m = \frac{\text{Imports}}{PY} = \frac{M_e^* Z_x^* \left(\frac{W^*}{W}\right)^{1-\rho} D^{*1-\rho}}{M_e \left(Z_d + Z_x\right) + M_e^* Z_x^* \left(\frac{W^*}{W}\right)^{1-\rho} D^{*1-\rho}}$$

$$s_m^* = \frac{\text{Imports}^*}{P^* Y^*} = \frac{M_e Z_x \left(\frac{W}{W^*}\right)^{1-\rho} D^{1-\rho}}{M_e^* \left(Z_d^* + Z_x^*\right) + M_e Z_x \left(\frac{W}{W^*}\right)^{1-\rho} D^{1-\rho}}$$

Under trade balance, $s_m = s_x$ and $s_m^* = s_x^*$.

Moreover, $s_n = s_x \frac{1+uD^{1-\rho}}{uD^{1-\rho}}$ in (1.26) denotes the employment share of exporters in the home country, and $s_n^* = s_x \frac{1+u^{-1}D^{1-\rho}}{u^{-1}D^{1-\rho}}$ in (1.27) denotes the employment share of exporters in the foreign country. The ratio Z_d/Z_x in (1.28) is equal to $\frac{1-s_n}{s_n - s_x}$, and the ratio Z_d^*/Z_x^* in (1.29) is equal to $\frac{1-s_n^*}{s_n^* - s_x^*}$.

The 12 loglinear equations (1.3), (1.5), (1.24), (1.25), (1.26), (1.27), (1.28), (1.29), (1.30), (1.31), (1.32), and (1.33) can be used to solve for the twelve variables $\Delta \log \frac{\Pi_d}{W_r}$, $\Delta \log \frac{\Pi_d^*}{W_r^*}$, $\Delta \log C, \Delta \log C^*, \Delta \log Y, \Delta \log Y^*, \Delta \log \left(W/P\right), \Delta \log \left(W^*/P^*\right), \Delta \log M_e, \Delta \log M_e^*, \Delta \log u$, $\Delta \log S$, given changes in trade cost $\{\Delta \log D, \Delta \log D^*\}$, changes in aggregate productivity indices $\{\Delta \log Z_d, \Delta \log Z_x, \Delta \log Z_d^*, \Delta \log Z_x^*\}$, the model parameters $\{\rho, \lambda, \sigma\}$, export shares s_x, s_x^*, import shares s_m, s_m^*, ratio of consumption expenditures k, and employment share of exporters s_n, s_n^*. Given $\Delta \log C$ and $\Delta \log C^*$, we can obtain $\Delta \log C^W$ using (1.20).

At the end of this document (Additional appendix 1) we show that import shares s_m, s_m^*, and the ratio of consumption to output, C/Y, C^*/Y^* are related to export shares s_x, s_x^* and the ratio of consumption expenditures across countries k as follows:

$$s_m = \frac{s_x^* \left(k - \frac{k+1}{b} s_x\right)}{1 - s_x + s_x^* \left(k - \frac{k+1}{b}\right)} \ , \ s_m^* = \frac{s_x^* \left(k^{-1} - \frac{k^{-1}+1}{b} s_x^*\right)}{1 - s_x^* + s_x \left(k^{-1} - \frac{k^{-1}+1}{b}\right)}$$

$$\frac{C}{Y} = \frac{b + \frac{s_x^*}{1 - \frac{k+1}{b} s_x^*}\left(k - \frac{k+1}{b} s_x\right) - s_x}{1 + \frac{s_x^*}{1 - \frac{k+1}{b} s_x^*}\left(k - \frac{k+1}{b} s_x\right) - s_x} \ ,$$

$$\frac{C^*}{Y^*} = \frac{b + \frac{s_x}{1 - \frac{k^{-1}+1}{b} s_x}\left(k^{-1} - \frac{k^{-1}+1}{b} s_x^*\right) - s_x^*}{1 + \frac{s_x}{1 - \frac{k^{-1}+1}{b} s_x}\left(k^{-1} - \frac{k^{-1}+1}{b} s_x^*\right) - s_x^*}$$

where $b = 1 - \frac{1-\lambda}{\rho}\tau$. Hence, conditional on matching export shares and a ratio of consumption expenditures across countries, import shares and the ratio of consumption to output are also uniquely pinned down.

World consumption Assume that the production subsidy is given by $\tau = \rho/(\rho-1)$ if $\lambda < 1$, and takes any value $\tau > 0$ if $\lambda = 1$. To a first-order approximation, the growth in world consumption, $\Delta \log C^W$, in response to changes in trade costs is given by

$$\begin{aligned}
\Delta \log C^W \quad = \quad & \frac{(\rho+\lambda-1)(\rho-1)}{(1+k)(2-\rho-\lambda)\left[(1-\lambda)(s_m+s_m^*)+\lambda+\rho-2\right]} \times \\
& \{(\rho-1)(s_m \Delta \log D^* + k s_m^* \Delta \log D) + \\
& (1-\lambda)\left[(s_m-(1-s_m)k)s_m^* \Delta \log D + (k s_m^* - (1-s_m^*)) s_m \Delta \log D^*\right] + \\
& (\rho+\lambda-2)(s_m - k s_m^*)\frac{(\rho-1)}{\rho+\lambda-1}(s_x \Delta \log D - s_x^* \Delta \log D^*)\} + \\
& + \frac{(\rho-1)}{1+k}(s_x \Delta \log D + k s_x^* \Delta \log D^*)
\end{aligned} \tag{1.34}$$

Note that with symmetric countries, $\Delta \log C^W = -(\rho-1)/(\rho+\lambda-2)\Delta \log D$, as in our paper.

Hence, $\Delta \log C^W$ is only a function of the model parameters ρ, λ, export shares s_x, s_x^*, and the ratio of consumption expenditures across countries k (recall that s_m and s_m^* are pinned down by s_x, s_x^*, and k). Hence, changes in exit, export and process innovation decisions, which impact $\{\Delta \log Z_d, \Delta \log Z_x, \Delta \log Z_d^*, \Delta \log Z_x^*\}$, have no effects on $\Delta \log C^W$ and welfare. The derivation of (1.34) is presented at the end of this document (Additional appendix 2)

2. Quantitative model

This section is divided in two parts. In the first part, we first present results that we use when numerically solving our model. In the second part, we describe the algorithms used to compute the steady-state and the transition dynamics of our model.

2.1. Preliminaries

Discrete time approximation Recall that Δ_z denotes the step size in logs and q denotes the probability of taking a step up. We choose these two parameters as functions of the length of a time period as follows. Let Δt denote the time interval for one step. Let α denote the expected change in z per unit time and σ^2 the variance of z per unit time. We choose Δ_z and q so that $z_{t+\Delta t} - z_t$ is distributed normal with mean $\alpha \Delta t$ and variance $\sigma^2 \Delta t$. To do that, we set

$$\Delta_z = \sigma \sqrt{\Delta t}$$

and

$$q = \frac{1}{2}\left[1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}\right]$$

This is because

$$E z_{t+\Delta t} - z_t = q\Delta_z - (1-q)\Delta_z = (2q-1)\,\Delta_z = \alpha \Delta t$$

and

$$E\left(z_{t+\Delta t} - z_t\right)^2 = q\Delta_z^2 + (1-q)\Delta_z^2 = \Delta_z^2$$

so

$$Var\left(z_{t+\Delta t} - z_t\right) = \Delta_z^2\left(1 - (2q-1)^2\right) = 4q(1-q)\Delta_z^2 = \sigma^2 \Delta t - \alpha^2 \left(\Delta t\right)^2$$

Note that this formula works exactly if $\alpha = 0$ (no drift) and it is approximate if the time interval is small (so $\left(\Delta t\right)^2$ is really small). Thus, the real parameters that we choose are α, the mean growth of size, and σ^2, the variance of growth of size.

In our Matlab codes, we define the parameter $per = 1/\left(\Delta t\right)$ as the number of periods in a year. The higher is per, the shorter is each time period.

Choosing elasticity parameter b Recall that in our formulation, the cost of process innovation is given by $\exp(bq)$. We want to calibrate b so that the relevant elasticity (defined below) is unchanged as we vary the length of the time period, Δ.

In order to do so, suppose that the cost of process innovation is expressed in terms of the expected growth rate of z per unit of time, α: $\exp\left(\tilde{b}\alpha\right)$. In this case, the optimal choice of α for large firms solves the problem:

$$\begin{aligned}
&\max_{\alpha,q} -h\exp\left(\tilde{b}\alpha\right) + \beta\left(1-\delta\right)\bar{V}\left[q\exp\left(\Delta_z\right) + (1-q)\exp\left(\Delta_z\right)\right] \\
&\text{s.t. } q = \frac{1}{2}\left[1 + \frac{\alpha}{\Delta_z}\Delta t\right]
\end{aligned}$$

where we used the fact that $s = gs\sqrt{\Delta t}$, and that for these large firms, $V\left(z\right) = \bar{V}\exp\left(z\right)$ because fixed costs for these firms represent a trivial portion of revenues and hence can be ignored in the value functions. The FOC w.r.t. q is:

$$\frac{2\Delta_z}{\Delta t}\tilde{b}h\exp\left(\tilde{b}\alpha\right)=\beta\left(1-\delta\right)\bar{V}\left[\exp\left(\Delta_z\right)-\exp\left(-\Delta_z\right)\right]$$

where we used $\frac{\partial\alpha}{\partial q}=\frac{2\Delta_z}{\Delta t}$. Hence,

$$\frac{\partial\alpha}{\partial\log\bar{V}}=\frac{1}{\tilde{b}}$$

In our current formulation, the FOC with respect to q is

$$hb\exp\left(bq\right)=\beta\left(1-\delta\right)\bar{V}\left[\exp\left(\Delta_z\right)-\exp\left(-\Delta_z\right)\right]$$

Substituting the definition of q :

$$hb\exp\left(\frac{b}{2}\left[1+\frac{\alpha}{\Delta_z}\Delta t\right]\right)=\beta\left(1-\delta\right)\bar{V}\left[\exp\left(\Delta_z\right)-\exp\left(-\Delta_z\right)\right]$$

Therefore,

$$\frac{\partial\alpha}{\partial\log\bar{V}}=\frac{1}{b}\frac{2\Delta_z}{\Delta t}$$

So, we will choose b so that

$$b=\frac{2\Delta_z}{\Delta t}\tilde{b}$$

Inelastic process innovation In the paper, we refer to the case with inelastic process innovation as that with $\tilde{b}=1200$ (high curvature of the process innovation cost function). We also considered an alternative specification in which q is assumed to be exogenously given and equal for all firms. That is, we impose $q\left(z\right)=q^{calibrated}$ for all z, and we set the costs of process innovation equal to zero for all firms. In the codes we call this case by setting $\tilde{b}=1111$. Both alternative assumptions give almost identical results (the key is that with $\tilde{b}=1200$, $q\left(z\right)$ is almost invariant with z).

Timing of product innovation In our quantitative model, as we reduce the period length (i.e. smaller Δ), we keep the entry period of new firms at one year and we set the annual entry cost at $\frac{n_e}{\Delta t}$ so that n_e denotes the entry cost per unit of time.

Initial guess of value functions In "Fixed.m" we use $V\left(z\right)=\bar{V}\exp\left(z\right)$ as the initial guess when solving for the fixed point of $V\left(z\right)$ in the initial steady state.[2] Here we are using

[2] For the new steady state, we use as our initial guess V_{SS0}, the value function in the initial steady state.

the fact that, for large firms, fixed costs become insignificant as a share of total value, so value functions are proportional to $\exp(z)$. We solve for $\bar{V}$ as follows.

We have

$$\bar{V} = \Pi_d \left(1 + D^{1-\rho}\right) - h \exp(bq) + \bar{V} \beta (1-\delta) \left[q \exp(\Delta_z) + (1-q) \exp(-\Delta_z)\right]$$

and the foc for q is

$$hb \exp(bq) = \bar{V} \beta (1-\delta) \left[\exp(\Delta_z) - \exp(-\Delta_z)\right]$$

Replacing the FOC into $\bar{V}$, we have:

$$\bar{V} = \Pi_d \left(1 + D^{1-\rho}\right) - \frac{\bar{V}}{b} \beta (1-\delta) \left[\exp(\Delta_z) - \exp(-\Delta_z)\right] + V \beta (1-\delta) \left[q \exp(\Delta_z) + (1-q) \exp(-\Delta_z)\right]$$

or

$$\bar{V} = \frac{\Pi_d \left(1 + D^{1-\rho}\right)}{1 + \beta (1-\delta) \left[\exp(\Delta_z) - \exp(-\Delta_z)\right] \left(\frac{1}{b} - q\right) - \beta (1-\delta) \exp(-\Delta_z)}$$

So, given Π_d, we set $V(z) = \bar{V} \exp(z)$ as the initial guess.

Calibrating h We calibrate h to target a slope of the employment-based distribution of -0.2 for firms ranging between $1,000$ and $5,000$ employees in the version of the model with inelastic process innovation (high b). The model then implies a value of process innovation q for large firms. As we lower b, we adjust the model parameters to keep the value of q for large firms constant and thus keep the dynamics of large firms unchanged.

When we implement this procedure, we guess for a slope coefficient for large firms, then obtain the value of q consistent with this slope coefficient, and calibrate h so that large firms choose this value of q. We follow this indirect procedure because it depends less on our choice of the length of a time period. To see this, here we derive an expression of the slope coefficient for sufficiently large firms. We define the slope coefficient as the slope of the line $\log\left[1 - \int_z^\infty dM(z) \exp(z)\right] / z$, where we used the fact that employment is proportional to $\exp(z)$, so the log of employment is proportional to z.

Denote by m_n de mass of firms with productivity index $z = z_0 + n\Delta_z$, normalized by the mass of entering firms (i.e. $\partial \tilde{M}(z)$). The slope coefficient is

$$\begin{aligned}
& \lim_{k\to\infty} \frac{\left(\log(\sum_{n=k+1}^{\infty} \exp(n\Delta_z)\, m_n) - \log(\sum_{n=k}^{\infty} \exp(n\Delta_z)\, m_n)\right)}{\Delta_z} \\
= & \lim_{k\to\infty} \frac{\log\left(\sum_{n=k+1}^{\infty} \frac{\exp(n\Delta_z)}{\exp(k\Delta_z)} \frac{m_n}{m_k}\right) - \log\left(\sum_{n=k}^{\infty} \frac{\exp(n\Delta_z)}{\exp(k\Delta_z)} \frac{m_n}{m_k}\right)}{\Delta_z} \\
= & \lim_{k\to\infty} \frac{\log\left(\sum_{n=k+1}^{\infty} \frac{\exp(n\Delta_z)}{\exp(k\Delta_z)} \frac{m_n}{m_k}\right) - \log\left(1+\sum_{n=k+1}^{\infty} \frac{\exp(n\Delta_z)}{\exp(k\Delta_z)} \frac{m_n}{m_k}\right)}{\Delta_z}
\end{aligned}$$

We need to compute $\lim_{k\to\infty} \sum_{n=k+1}^{\infty} \frac{\exp(n\Delta_z)}{\exp(k\Delta_z)} \frac{m_n}{m_k}$.

We know that sufficiently large firms have a constant q. Then, for large enough n,

$$m_n = (1-\delta)\, q m_{n-1} + (1-\delta)(1-q)\, m_{n+1}$$

and

$$\frac{m_n}{m_{n-1}} = (1-\delta)\, q + (1-\delta)(1-q) \frac{m_{n+1}}{m_{n-1}}$$

where $\frac{m_n}{m_{n-1}}$ is independent of n for large firms. Define $y = \frac{m_n}{m_{n-1}}$, so

$$y = (1-\delta)\, q + (1-\delta)(1-q)\, y^2$$

The slope coefficient is

$$\begin{aligned}
& \frac{\log\left(\sum_{n=1}^{\infty} (\exp(\Delta_z)\, y)^n\right) - \log\left(1+\sum_{n=1}^{\infty} (\exp(\Delta_z)\, y)^n\right)}{\Delta_z} \\
= & \frac{\log \frac{\exp(\Delta_z) y}{1-\exp(\Delta_z) y} - \log \frac{1}{1-\exp(\Delta_z) y}}{\Delta_z} \\
= & \frac{\log(\exp(\Delta_z)\, y)}{\Delta_z} \\
= & 1 + \frac{\log(y)}{\Delta_z}
\end{aligned}$$

Suppose we want to hit a slope coefficient of $x < 0$. Then, we need

$$1 + \frac{\log(y)}{\Delta_z} = x$$

so

$$y = \exp((x-1)\, \Delta_z)$$

What value of q do we need to hit a given y? From

$$y = (1-\delta)\, q + (1-\delta)(1-q)\, y^2$$

we obtain

$$q = \frac{\frac{y}{1-\delta} - y^2}{1-y^2}$$

2.2. Algorithms to compute model

The codes "Master.m", "Steady.m" and "Fixed.m" solve the steady-state and transition dynamics of our model. Master.m is the main code to choose parameter values, and to call the other codes. The output is stored in the file "Storeresults" in the active directory. We include an excel file, "results.xls" that displays the output for most of the parameterizations in the paper.

We included comments in the code to explain various steps. However, some steps might still be not 100% clean in the codes, so please do not hesitate to let us know if you have any questions.

Steady-state As described in section 3.4 of the paper, we solve for the steady-state in two steps. First, we find the level of Π_d that is consistent with free entry, which in turn gives exit, export and process innovation decisions. We then use these policy functions to solve for the aggregate variables. These steps are taken in the functions "Fixed.m" and "Steady.m".

"Fixed.m" finds the steady-state value function V associated with any given Π_d using value function iteration, and calculates the difference between the entry cost and the discounted value of entry (denoted by F). The free entry condition requires $F = 0$. In the initial steady state, "Fixed.m" also calibrates the parameter h to match a value of q for large firms, as described above.

"Steady.m" solves for the steady-state. It first uses "fsolve" to find the value of Π_d such that $F = 0$ in "Fixed.m". This is the only steady state value of Π_d that is consistent with the free-entry condition. Associated with Π_d are the firms' policy functions, which are used to find the stationary distribution of firms. Given these policy functions and the stationary distribution of firms, Steady.m solves for the remaining aggregate variables.

Transition dynamics After finding the initial and final steady states, we use the following algorithm to compute the transition across steady-states:

- Guess the number of periods it takes to reach the new steady state (denoted by $pertran$). Guess sequences for $\{Y_t^{tranit}, Z_t^{tranit}, C_t^{tranit}, W_t^{tranit}\}_{t=1}^{pertran}$, where $\{Y_t, Z_t, C_t, W_t\}$ denote output, aggregate productivity , consumption and the wage, respectively.

- Using the value function from the final period (new steady state) and the guess $\{Y_t^{tranit}, Z_t^{tranit}, C_t^{tranit}, W_t^{tranit}\}_{t=1}^{pertran}$, iterate backwards to find the policy functions every period. Use the free-entry condition to find the value of the wage $\{W_t^{tran}\}_{t=1}^{pertran}$ consistent with free-entry in each period.

- Using the initial guess for $\{Y_t^{tranit}, Z_t^{tranit}, C_t^{tranit}, W_t^{tranit}\}_{t=1}^{pertran}$, the initial distribution of productivities and the policy functions we found in the previous step, iterate forward and solve the distribution of firms over productivities and the aggregate variables. Find updated values for $\{Y_t^{tran}, Z_t^{tran}, C_t^{tran}, W_t^{trainit}\}_{t=1}^{pertran}$.

- Make a new guess using a weighted average of our initial guess and the updated values for the aggregate variables. Make sure to give a lot of weight to the old guess (low speedconv).

- Iterate until convergence.

Remarks Choice of the numeraire: When computing the transition, we used the price of the final good P as the numeraire instead of using the price of the research good W_r. This choice is made so that we can update the wage W directly from the free-entry condition as we iterate backwards in step 2. In particular, when the value of a firm V is written in terms of the final good, the free-entry condition is:

$$W_{r,t-per} n_e per = \frac{1}{\left(1 + r_{t-per}^{annual}\right)} V_t\left(z_0\right),$$

where per is the number of time periods in a year. As we iterate backwards, we can use this equation to find the value of $P_{r,t-per}$ consistent with free entry, and use $W_{r,t} = \frac{1}{\lambda^{\lambda}(1-\lambda)^{1-\lambda}} W_t^{\lambda}$ to find W_t^{tran}.

Finding L_p: To solve for L_p, we use a weighted average of the value implied by our guess on ZZ^{tranit} and our updated value of ZZ^{tran}. This makes the algorithm less sensitive to deviations in the updated value of ZZ, making the convergence more stable.

3. Additional Appendix on Model with Assymetric countries

Appendix 1: Deriving import shares and consumption-output ratios under risk sharing

The value of production wages is

$$WL_p = \frac{\rho - 1}{\rho} \tau Out \tag{3.1}$$

where Out denotes the value of output by intermediate good producers (exclusive of the subsidy τ). To see this, note that for a firm with productivity 1, $p = \frac{\rho}{\rho-1}\frac{1}{\tau}w$, $y = p^{-\rho}YP^{\rho}$, $py = \left(\frac{1}{\tau}\frac{\rho}{\rho-1}W\right)^{1-\rho} YP^{\rho}$, $Wl = W^{1-\rho}\left(\frac{1}{\tau}\frac{\rho}{\rho-1}\right)^{-\rho} YP^{\rho}$, so $Wl = py\frac{\rho-1}{\rho}\tau$. Summing over home and foreign sales and aggregating across producers we obtain (3.1).

Aggregate variable profits (inclusive of the subsidy) are

$$\tau\Pi_d\left[M_e\left(Z_d + Z_x\right) + D^{1-\rho}M_eZ_xu\right] = \tau Out - WL_p$$

so

$$WL_p = (\rho - 1)\,\tau\Pi_d\left[M_e\left(Z_d + Z_x\right) + D^{1-\rho}M_eZ_xu\right]$$

The value of research wages is

$$WL_r = \lambda W_r \Upsilon M_e,$$

Dividing the last two two equations yields

$$\frac{L - L_r}{L_r} = \frac{\rho - 1}{\lambda}\frac{\tau\Pi_d\left[M_e\left(Z_d + Z_x\right) + D^{1-\rho}M_eZ_xu\right]}{W_r\Upsilon M_e}$$

With free-entry and $\beta = 1$, $\frac{\tau\Pi_d\left[M_e(Z_d+Z_x)+D^{1-\rho}M_eZ_xu\right]}{W_r\Upsilon M_e} = 1$, and we obtain (1.8).

The value of final goods used in research activities, PX , is

$$\begin{aligned} PX &= \frac{1-\lambda}{\lambda}WL_r = \frac{1-\lambda}{\lambda}\frac{\lambda}{\rho-1}WL_p = \\ &= \frac{1-\lambda}{\lambda}\frac{\lambda}{\rho-1}\frac{\rho-1}{\rho}\tau Out = \frac{1-\lambda}{\rho}\tau Out \end{aligned}$$

The total value of final goods in the home country is

$$PY = Out + \text{Imports-Exports}$$

The value of consumptions is

$$\begin{aligned} PC &= PY - PX = Out + \text{Imports-Exports-}\frac{1-\lambda}{\rho}\tau Out \\ &= bOut + \text{Imports-Exports} \end{aligned}$$

where $b = 1 - \frac{1-\lambda}{\rho}\tau$.

In the foreign country

$$P^*C^* = bOut^* + \text{Imports}^*\text{-Exports}^* = bOut^* + \text{Exports-Imports}$$

where we used the fact that Exports* =Imports, and Imports* =Exports.

Recall that

$$P^*C^* = kPC$$

so

$$bOut^* + \text{Exports-Imports} = k\left[bOut + \text{Imports-Exports}\right]$$

or

$$Out^* = kOut + \frac{k+1}{b}\left(\text{Imports-Exports}\right)$$

Recall that s_x =Exports/Out and s_x^* =Imports/Out*. So,

$$s_x^* = \frac{\text{Imports}}{Out^*} = \frac{\text{Imports}}{kOut + \frac{k+1}{b}\left(\text{Imports-Exports}\right)} = \frac{\text{Imports}/Out}{k + \frac{k+1}{b}\left(\text{Imports}/Out - s_x\right)}$$

thus

$$\text{Imports}/Out = \frac{s_x^*}{1 - \frac{k+1}{b}s_x^*}\left(k - \frac{k+1}{b}s_x\right)$$

$$\begin{aligned} s_m &= \frac{\text{Imports}}{PY} = \frac{\text{Imports}}{Out + \text{Imports} - \text{Exports}} = \frac{\text{Imports}/Out}{1 + \text{Imports}/Out - s_x} \\ &= \frac{\frac{s_x^*}{1-\frac{k+1}{b}s_x^*}\left(k - \frac{k+1}{b}s_x\right)}{1 + \frac{s_x^*}{1-\frac{k+1}{b}s_x^*}\left(k - \frac{k+1}{b}s_x\right) - s_x} = \frac{s_x^*\left(k - \frac{k+1}{b}s_x\right)}{1 - s_x + s_x^*\left(k - \frac{k+1}{b}\right)} \end{aligned}$$

By symmetry

$$s_m^* = \frac{s_x^*\left(k^{-1} - \frac{k^{-1}+1}{b}s_x^*\right)}{1 - s_x^* + s_x\left(k^{-1} - \frac{k^{-1}+1}{b}\right)}$$

Finally

$$\begin{aligned} \frac{C}{Y} &= \frac{bOut + \text{Imports-Exports}}{Out + \text{Imports-Exports}} = \frac{b + \text{Imports}/Out - s_x}{1 + \text{Imports}/Out - s_x} \\ &= \frac{b + \frac{s_x^*}{1-\frac{k+1}{b}s_x^*}\left(k - \frac{k+1}{b}s_x\right) - s_x}{1 + \frac{s_x^*}{1-\frac{k+1}{b}s_x^*}\left(k - \frac{k+1}{b}s_x\right) - s_x} \end{aligned}$$

and similarly

$$\frac{C^*}{Y^*} = \frac{b + \frac{\text{Imports}^*\text{-Exports}^*}{Out^*}}{1 + \frac{\text{Imports}^*\text{-Exports}^*}{Out^*}} = \frac{b + \frac{s_x}{1-\frac{k^{-1}+1}{b}s_x}\left(k^{-1} - \frac{k^{-1}+1}{b}s_x^*\right) - s_x^*}{1 + \frac{s_x}{1-\frac{k^{-1}+1}{b}s_x}\left(k^{-1} - \frac{k^{-1}+1}{b}s_x^*\right) - s_x^*}$$

Appendix 2: Deriving expression (1.34) under risk sharing

To save on notation, define $W = \Delta \log (W/P)$, $W^* = \Delta \log(W^*/P^*)$, $Y = \Delta \log (Y)$, $Y^* = \Delta \log (Y^*)$, $C = \Delta \log (C)$, $C^* = \Delta \log (C^*)$, $M = \Delta \log (M_e)$, $M^* = \Delta \log (M_e^*)$, $U = \Delta \log U$, $S = \Delta \log (W^*/W)$, $d = \Delta \log D$, $d^* = \Delta \log D^*$, $z_d = \Delta \log Z_d$, $z_x = \Delta \log Z_x$, $z_d^* = \Delta \log (Z_d^*)$, $z_x^* = \Delta \log (Z_x^*)$, $c = C/Y$, $c^* = C^*/Y^*$.

Using this notation, we can re-write the system of log-linearized equations that is used to solve for the endogenous variables as

$$(1-\rho-\lambda) W + Y = -s_x U + (\rho - 1) s_x d \tag{3.2}$$

$$(1-\rho-\lambda) W^* + Y^* = s_x^* U + (\rho - 1) s_x^* d^* \tag{3.3}$$

$$Y + M + s_x U + s_x(1-\rho) d + (1-s_n) z_d + s_n z_x = \rho W \tag{3.4}$$

$$Y^* + M^* - s_x^* U + s_x^*(1-\rho) d^* + (1-s_n^*) z_d^* + s_n^* z_x^* = \rho W^* \tag{3.5}$$

$$W = \frac{1}{\rho-1}\left[(1-s_m) M + s_m M^*\right] + \frac{1}{\rho-1}\left[(1-s_m)\left(\frac{1-s_n}{1-s_x} z_d + \frac{s_n - s_x}{1-s_x} z_x\right) + s_m z_x^*\right] - s_m S - s_m d^* \tag{3.6}$$

$$W^* = \frac{1}{\rho-1}\left[(1-s_m^*) M^* + s_m^* M\right] + \frac{1}{\rho-1}\left[(1-s_m^*)\left(\frac{1-s_n^*}{1-s_x^*} z_d^* + \frac{s_n^* - s_x^*}{1-s_x^*} z_x^*\right) + s_m^* z_x\right] + s_m^* S - s_m^* d \tag{3.7}$$

$$Y = cC + (1-c) W \tag{3.8}$$

$$Y^* = c^* C + (1-c^*) W^* \tag{3.9}$$

$$U = Y^* - Y - \sigma\rho\left[C^* - C\right] \tag{3.10}$$

$$S = \sigma (C - C^*) + W^* - W \tag{3.11}$$

Case $\lambda = 1$

Combining (3.2) and (3.4) we obtain

$$M = -(1-s_n) z_d - s_n z_x$$

and combining (3.3) and (3.5) we obtain

$$M^* = -(1-s_n^*) z_d^* - s_n^* z_x^*$$

So, we can write (3.6) as

$$\begin{aligned} W &= \frac{1}{\rho-1}\left[-(1-s_m)\left((1-s_n) z_d + s_n z_x\right) - s_m\left((1-s_n^*) z_d^* + s_n^* z_x^*\right)\right] + \\ &\quad \frac{1}{\rho-1}\left[(1-s_m)\left(\frac{1-s_n}{1-s_x} z_d + \frac{s_n - s_x}{1-s_x} z_x\right) + s_m z_x^*\right] - s_m S - s_m d^* \end{aligned}$$

and combining terms

$$W = \frac{1 - s_m}{\rho - 1} \frac{s_x}{1 - s_x} (1 - s_n) (z_d - z_x) + \frac{s_m}{\rho - 1} (1 - s_n^*) (z_x^* - z_d^*) - s_m S - s_m d^* \tag{3.12}$$

By symmetry

$$W^* = \frac{1 - s_m^*}{\rho - 1} \frac{s_x^*}{1 - s_x^*} (1 - s_n^*) (z_d^* - z_x^*) + \frac{s_m^*}{\rho - 1} (1 - s_n) (z_x - z_d) + s_m^* S - s_m^* d \tag{3.13}$$

To simplify the notation further, we summarize the changes in the aggregate productivity indices by two variables, a and a^*:

$$a = \frac{1}{\rho - 1} (1 - s_n) (z_d - z_x) \ , \ a^* = \frac{1}{\rho - 1} (1 - s_n^*) (z_d^* - z_x^*) \tag{3.14}$$

Then, (3.12) and (3.13) can be re-expressed as

$$W = (1 - s_m) \frac{s_x}{1 - s_x} a - s_m a^* - s_m S - s_m d^*$$

$$W^* = (1 - s_m^*) \frac{s_x^*}{1 - s_x^*} a^* - s_m^* a + s_m^* S - s_m^* d$$

Substituting the expression for W into (3.2) we obtain

$$\begin{aligned} Y &= \rho W - s_x U + (\rho - 1) s_x d \\ &= \rho \left[(1 - s_m) \frac{s_x}{1 - s_x} a - s_m a^* \right] - s_x U - \rho s_m S + (\rho - 1) s_x d - \rho s_m d^* \end{aligned}$$

and similarly substituting the expression for W^* into (3.4) we obtain

$$Y^* = \rho \left[(1 - s_m^*) \frac{s_x^*}{1 - s_x^*} a^* - s_m^* a \right] + s_x^* U + \rho s_m^* S + (\rho - 1) s_x^* d^* - \rho s_m^* d$$

With $\lambda = 1$, $\Delta \log C^w = \Delta \log Y^w$, so

$$\begin{aligned} (1 + k) \Delta \log C^w &= (1 + k) \Delta \log Y^w = Y + kY^* \\ &= \rho \left[(1 - s_m) \frac{s_x}{1 - s_x} a - s_m a^* + k (1 - s_m^*) \frac{s_x^*}{1 - s_x^*} a^* - k s_m^* a \right] \\ &\quad + (k s_x^* - s_x) U + \rho (k s_m^* - s_m) S + ((\rho - 1) s_x - \rho k s_m^*) d + ((\rho - 1) k s_x^* - \rho s_m) d^* \end{aligned}$$

Using the definitions of s_m and s_m^* , we can show

$$k = \frac{(1 - s_m) s_x}{s_m^* (1 - s_x)} \text{ and } \frac{1}{k} = \frac{(1 - s_m^*) s_x^*}{s_m (1 - s_x^*)}$$

Then,

$$\begin{aligned}
&\left(1-s_m\right)\frac{s_x}{1-s_x}a-s_ma^*+k\left(1-s_m^*\right)\frac{s_x^*}{1-s_x^*}a^*-ks_m^*a\\
=\;&\left(\frac{\left(1-s_m\right)s_x}{s_m^*\left(1-s_x\right)}-k\right)s_m^*a+\left(k\frac{\left(1-s_m^*\right)s_x^*}{s_m\left(1-s_x^*\right)}-1\right)s_ma^*=0
\end{aligned}$$

so a and a^* cancel-out from $\Delta\log C^w$. Therefore,

$$Y+kY^*=\left(ks_x^*-s_x\right)U+\rho\left(ks_m^*-s_m\right)S+\left(\left(\rho-1\right)s_x-\rho ks_m^*\right)d+\left(\left(\rho-1\right)ks_x^*-\rho s_m\right)d^*$$

We now focus on the term $\left(ks_x^*-s_x\right)U+\rho\left(ks_m^*-s_m\right)S$. Using (3.2), (3.3), we obtain:

$$Y-Y^*-\rho\left(W-W^*\right)+\left(s_x+s_x^*\right)U=\left(\rho-1\right)\left(s_xd-s_x^*d^*\right)$$

combined with (3.10)

$$\rho\left(W^*-W\right)=-\left(Y-Y^*\right)\left(1+\left(s_x+s_x^*\right)\left(\rho\sigma-1\right)\right)+\left(\rho-1\right)\left(s_xd-s_x^*d^*\right) \tag{3.15}$$

Therefore, using (3.10), (3.11) and (3.15),

$$\begin{aligned}
&\left(ks_x^*-s_x\right)U+\rho\left(ks_m^*-s_m\right)S\\
=\;&\left(ks_x^*-s_x\right)\left(Y^*-Y\right)\left(1-\sigma\rho\right)+\rho\sigma\left(ks_m^*-s_m\right)\left(Y-Y^*\right)+\rho\left(ks_m^*-s_m\right)\left(W^*-W\right)\\
=\;&\left[\left(ks_x^*-s_x\right)\left(\sigma\rho-1\right)+\rho\sigma\left(ks_m^*-s_m\right)-\left(ks_m^*-s_m\right)-\left(ks_m^*-s_m\right)\left(s_x+s_x^*\right)\left(\rho\sigma-1\right)\right]\left(Y-Y^*\right)+\\
&+\left(ks_m^*-s_m\right)\left(\rho-1\right)\left(s_xd-s_x^*d^*\right)\\
=\;&\left(ks_m^*-s_m\right)\left(\rho-1\right)\left(s_xd-s_x^*d^*\right)
\end{aligned}$$

In the final step, we used

$$\begin{aligned}
&\left(ks_x^*-s_x\right)\left(\sigma\rho-1\right)+\left(\rho\sigma-1\right)\left(ks_m^*-s_m\right)-\left(ks_m^*-s_m\right)\left(s_x+s_x^*\right)\left(\rho\sigma-1\right)\\
=\;&\left(\sigma\rho-1\right)\left[ks_x^*-s_x+\left(ks_m^*-s_m\right)\left(1-s_x-s_x^*\right)\right]\\
=\;&\left(\sigma\rho-1\right)\left[ks_x^*-s_x+\left(k\frac{s_x\left[k^{-1}-\left(k^{-1}+1\right)s_x^*\right]}{1-s_x^*-s_x}-\frac{s_x^*\left[k-\left(k+1\right)s_x\right]}{1-s_x-s_x^*}\right)\left(1-s_x-s_x^*\right)\right]\\
=\;&\left(\sigma\rho-1\right)\left[ks_x^*-s_x+s_x-s_xs_x^*\left(k+1\right)-s_x^*k+s_x^*\left(k+1\right)s_x\right]\\
=\;&\left(\sigma\rho-1\right)\left[ks_x^*-s_x+s_x-s_x^*k\right]=0
\end{aligned}$$

$$s_m=\frac{s_x^*\left[k-\left(k+1\right)s_x\right]}{1-s_x-s_x^*}\;,\;s_m^*=\frac{s_x\left[k^{-1}-\left(k^{-1}+1\right)s_x^*\right]}{1-s_x^*-s_x}$$

Therefore

$$\begin{aligned}
Y+kY^*\;&=\;\left(\left(\rho-1\right)s_x-\rho ks_m^*\right)d+\left(\left(\rho-1\right)ks_x^*-\rho s_m\right)d^*+\left(ks_m^*-s_m\right)\left(\rho-1\right)\left(s_xd-s_x^*d^*\right)\\
&=\;\left[\left(\rho-1\right)s_x\left(1+ks_m^*-s_m\right)-\rho ks_m^*\right]d+\left[\left(\rho-1\right)s_x^*\left(k+s_m-ks_m^*\right)-\rho s_m\right]d^*
\end{aligned}$$

which coincides with expression (1.34) when $\lambda = 1$.

Case $\lambda < 1$

From (3.2) and (3.4) we obtain

$$M = (1-\lambda) W - (1 - s_{Lx}) z_d - s_{Lx} z_x \ , \ M^* = (1-\lambda) W^* - (1 - s^*_{Lx}) z^*_d - s^*_{Lx} z^*_x$$

Replacing these into (3.6)

$$W = \frac{(1-\lambda)}{\rho - 1} [(1 - s_m) W + s_m W^*] + (1 - s_m) \frac{s_x}{1 - s_x} a - s_m a^* - s_m S - s_m d^*$$

where a and a^* are defined in (3.14). Symmetrically in the foreign country

$$W^* = \frac{(1-\lambda)}{\rho - 1} [(1 - s^*_m) W^* + s^*_m W] + (1 - s^*_m) \frac{s^*_x}{1 - s^*_x} a^* - s^*_m a + s^*_m S - s^*_m d$$

Solving for W and W^*, we obtain

$$W = \frac{1}{\left(\frac{2-\rho-\lambda}{\rho-1}\right)^2 \left[1 - \frac{(1-\lambda)}{2-\rho-\lambda}(s_m + s^*_m)\right]} \left\{ \begin{array}{c} \left(1 - \frac{(1-\lambda)}{\rho-1}(1 - s^*_m)\right) \left[(1 - s_m) \frac{s_x}{1-s_x} a - s_m a^* - s_m S - s_m d^*\right] \\ + \frac{(1-\lambda)}{\rho-1} s_m \left[(1 - s^*_m) \frac{s^*_x}{1-s^*_x} a^* - s^*_m a + s^*_m S - s^*_m d\right] \end{array} \right\}$$

$$W^* = \frac{1}{\left(\frac{2-\rho-\lambda}{\rho-1}\right)^2 \left[1 - \frac{(1-\lambda)}{2-\rho-\lambda}(s_m + s^*_m)\right]} \left\{ \begin{array}{c} \left(1 - \frac{(1-\lambda)}{\rho-1}(1 - s_m)\right) \left[(1 - s^*_m) \frac{s^*_x}{1-s^*_x} a^* - s^*_m a + s^*_m S - s^*_m d\right] \\ + \frac{(1-\lambda)}{\rho-1} s^*_m \left[(1 - s_m) \frac{s_x}{1-s_x} a - s_m a^* - s_m S - s_m d^*\right] \end{array} \right\}$$

In calculating $W + kW^*$, one can show that the sum of the terms involving a and a^* are equal to zero:

$$\begin{aligned} &\left(1 - \frac{(1-\lambda)}{\rho-1}(1 - s^*_m)\right) \left[(1 - s_m) \frac{s_x}{1 - s_x} a - s_m a^*\right] + \frac{(1-\lambda)}{\rho-1} s_m \left[(1 - s^*_m) \frac{s^*_x}{1 - s^*_x} a^* - s^*_m a\right] \\ &k \left(1 - \frac{(1-\lambda)}{\rho-1}(1 - s_m)\right) \left[(1 - s^*_m) \frac{s^*_x}{1 - s^*_x} a^* - s^*_m a\right] + \frac{k(1-\lambda)}{\rho-1} s^*_m \left[(1 - s_m) \frac{s_x}{1 - s_x} a - s_m a^*\right] \\ = \ &0 \end{aligned}$$

Therefore

$$\begin{aligned} W + kW^* &= \frac{1}{\left(\frac{2-\rho-\lambda}{\rho-1}\right)^2 \left[\frac{(1-\lambda)}{2-\rho-\lambda}(s_m + s^*_m) - 1\right]} \times \\ &\times \left\{ \begin{array}{c} \left(1 - \frac{(1-\lambda)}{\rho-1}(1 - s^*_m)\right) s_m [S + d^*] + \frac{(1-\lambda)}{\rho-1} s_m s^*_m [-S + d] + \\ k \left(1 - \frac{(1-\lambda)}{\rho-1}(1 - s_m)\right) s^*_m [-S + d] + \frac{k(1-\lambda)}{\rho-1} s^*_m s_m [S + d^*] \end{array} \right\} \\ &= \frac{\rho - 1}{(2 - \rho - \lambda)^2 \left[\frac{(1-\lambda)}{2-\rho-\lambda}(s_m + s^*_m) - 1\right]} \times \\ &\times \left[\begin{array}{c} (\rho - 1 - (1-\lambda)(1 - s^*_m)) s_m [S + d^*] + (1-\lambda) s_m s^*_m [-S + d] + \\ k (\rho - 1 - (1-\lambda)(1 - s_m)) s^*_m [-S + d] + (1-\lambda) s^*_m s_m [+S + d^*] \end{array} \right] \end{aligned}$$

We now calculate $(1+k)\,\Delta\log C^w$:

$$\begin{aligned}
C+kC^* &= Y+kY+(1-c)(C-W)+k(1-c^*)(C^*-W^*)\\
&= (\rho+\lambda-1)(W+kW^*)-s_xU+(\rho-1)s_xd+ks_x^*U+\\
&\quad +k(\rho-1)s_x^*d^*+(1-c)(C-W)+k(1-c^*)(C^*-W^*)\\
&= \frac{(\rho+\lambda-1)(\rho-1)}{(2-\rho-\lambda)^2\left[\frac{(1-\lambda)}{2-\rho-\lambda}(s_m+s_m^*)-1\right]}\times\\
&\quad\times\begin{pmatrix} (\rho-1-(1-\lambda)(1-s_m^*))\,s_m[S+d^*]+(1-\lambda)s_ms_m^*[-S+d]+\\ k(\rho-1-(1-\lambda)(1-s_m))\,s_m^*[-S+d]+(1-\lambda)s_m^*s_m[+S+d^*]\end{pmatrix}\\
&\quad -s_xU+(\rho-1)s_xd+ks_x^*U+k(\rho-1)s_x^*d^*+(1-c)(C-W)+k(1-c^*)(C^*-W^*)
\end{aligned}$$

In the expression for $C+kC^*$, the terms with d and d^* are

$$\begin{aligned}
&\frac{(\rho+\lambda-1)(\rho-1)}{(2-\rho-\lambda)\left[(1-\lambda)(s_m+s_m^*)+\lambda+\rho-2\right]}\times\\
&\times\left[\begin{array}{c}(\rho-1-(1-\lambda)(1-s_m^*))\,s_md^*+\\ +[\rho-1-(1-\lambda)(1-s_m)]\,ks_m^*d+(1-\lambda)s_m^*s_m(d+kd^*)\end{array}\right]
\end{aligned}\tag{3.16}$$

We now focus on the remaining terms of $C+kC^*$, which when collected are given by:

$$\begin{aligned}
&\frac{(\rho+\lambda-1)(\rho-1)}{(2-\rho-\lambda)\left[(1-\lambda)(s_m+s_m^*)+\lambda+\rho-2\right]}\times\\
&\times\{(\rho-1)(s_m-ks_m^*)+(1-\lambda)(k(1-s_m)s_m^*-(1-s_m^*)s_m)+(1-\lambda)s_m^*s_m(k-1)\}\,S\\
&-(s_x-ks_x^*)\,U+(1-c)(C-W)+k(1-c^*)(C^*-W^*)
\end{aligned}\tag{3.17}$$

Using (3.2), (3.3) we obtain

$$Y-Y^*=(\rho+\lambda-1)(W-W^*)-U(s_x+s_x^*)+(\rho-1)(s_xd-s_x^*d^*)$$

so

$$W-W^*=\frac{Y-Y^*}{\rho+\lambda-1}+\frac{(s_x+s_x^*)}{\rho+\lambda-1}U+\frac{(1-\rho)}{\rho+\lambda-1}(s_xd-s_x^*d^*)$$

Using (3.8)-(3.10):

$$\begin{aligned}
U &= Y^*-Y-\sigma\rho\,[C^*-C]\\
&= (Y^*-Y)(1-\sigma\rho)+\sigma\rho\,[(1-c)(C-W)-(1-c^*)(C^*-W^*)]
\end{aligned}\tag{3.18}$$

From expression (3.11)

$$\begin{aligned}
S &= \sigma\left(C-C^*\right)+W^*-W \\
&= \sigma\left(Y-Y^*\right)+\sigma\left[\left(1-c\right)\left(C-W\right)-\left(1-c^*\right)\left(C^*-W^*\right)\right]+ \\
&\quad +\frac{Y^*-Y}{\rho+\lambda-1}-\frac{\left(s_x+s_x^*\right)}{\rho+\lambda-1}U+\frac{\left(\rho-1\right)}{\rho+\lambda-1}\left(s_xd-s_x^*d^*\right) \\
&= \left(\frac{1}{\rho+\lambda-1}-\sigma\right)\left(Y^*-Y\right)-\frac{\left(s_x+s_x^*\right)}{\rho+\lambda-1}U+\frac{\left(\rho-1\right)}{\rho+\lambda-1}\left(s_xd-s_x^*d^*\right)+ \\
&\quad +\sigma\left[\left(1-c\right)\left(C-W\right)-\left(1-c^*\right)\left(C^*-W^*\right)\right]
\end{aligned}$$

Substituting (3.18) we obtain

$$\begin{aligned}
S &= \left(\frac{1-\left(s_x+s_x^*\right)\left(1-\sigma\rho\right)}{\rho+\lambda-1}-\sigma\right)\left(Y^*-Y\right)+\frac{\left(\rho-1\right)}{\rho+\lambda-1}\left(s_xd-s_x^*d^*\right)+ \\
&\quad +\left[\sigma-\frac{\left(s_x+s_x^*\right)\sigma\rho}{\rho+\lambda-1}\right]\left[\left(1-c\right)\left(C-W\right)-\left(1-c^*\right)\left(C^*-W^*\right)\right]
\end{aligned} \tag{3.19}$$

Using (3.18), (3.19), one can show that (3.17) is equal to:

$$\begin{aligned}
&\frac{\left(\rho+\lambda-1\right)\left(\rho-1\right)}{\left(2-\rho-\lambda\right)\left[\left(1-\lambda\right)\left(s_m+s_m^*\right)+\lambda+\rho-2\right]}\frac{\left(\rho-1\right)}{\rho+\lambda-1}\times \\
&\left\{\left(\rho-1\right)\left(s_m-ks_m^*\right)+\left(1-\lambda\right)\left(k\left(1-s_m\right)s_m^*-\left(1-s_m^*\right)s_m\right)+\left(1-\lambda\right)s_m^*s_m\left(k-1\right)\right\}\times \\
&\left(s_xd-s_x^*d^*\right)
\end{aligned} \tag{3.20}$$

Summing up (3.16) and (3.20) and dividing by $1+k$ results in $\Delta\log C^w$ given in expression (1.34).