Welfare and Output with Income Effects and Taste Shocks

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Abstract

We characterize how welfare responds to changes in budget set and technology when preferences are non-homothetic or subject to shocks, in both partial and general equilibrium. We generalize Hulten's theorem, the basis for constructing aggregate quantity indices, to this context. We show that calculating changes in welfare in response to a shock only requires knowledge of expenditure shares and elasticities of substitution and (given these elasticities) not of income elasticities and taste shocks. We also characterize the gap between changes in welfare and changes in chained indices. We apply our results to long-run and short-run phenomena. In the long-run, we show that structural transformation, if caused by income effects or changes in tastes, is roughly twice as important for welfare than what is implied by standard measures of Baumol's cost disease. In the short-run, we show that when firms' demand shocks are correlated with their supply shocks, industry-level price and output indices are biased, and this bias does not disappear in the aggregate. Finally, using the Covid-19 crisis we illustrate the differences between partial and general equilibrium notions of welfare, and we show that real consumption and real GDP are unreliable metrics for measuring welfare or production.

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1 Introduction

In this paper, we study how a change in the economic environment affects welfare. For example, how does the welfare of an individual consumer change when the budget constraint changes, or how does the welfare of a nation change when its technologies change? Under some strong assumptions, chain-weighted real consumption, as measured by statistical agencies, correctly answers both of these questions at the same time.[1] Two important assumptions required for this serendipity are homotheticity and stability of preferences, which are highly convenient but counterfactual assumptions. Homotheticity requires that the income elasticity of demand must equal one for every good. Stability requires that consumers only change spending in response to changes in incomes and relative prices. In this paper, we relax these assumptions and characterize welfare changes, chained-weighted consumption, and the gap between the two in terms of sufficient statistics.[2]

Our baseline welfare measure is the equivalent variation at fixed final preferences, which answers the question: *"holding fixed preferences, how much must the consumers' initial endowment change to make them indifferent between their choice sets at t_0 and t_1?"* where t can refer to, for example, time or space. Although we focus on equivalent variation, our results can easily be modified to also characterize compensating variation.

We first study this problem in partial equilibrium, where choice sets are defined in terms of budget sets (prices and income are exogenous). Our welfare measure answers a microeconomic question, comparing two budget sets for an infinitesimal agent who cannot alter market-level prices through her choices. We then study this problem in general equilibrium, where choice sets are defined in terms of production possibility frontiers determined by technologies and factor endowments (prices and income are endogenous).[3] In this case, our welfare measure answers a macroeconomic question comparing two technologies for a collection of agents whose collective decisions alter market-level prices. When preferences are homothetic and stable, macroeconomic changes in welfare are equal to microeconomic changes in welfare. However, we show that these two

[1]To aggregate data on prices and quantities over multiple goods, chain-weighted indices use good-specific weights that are updated every period. As compared to fixed-weight indices, chain-weighted indices account for substitution by consumers. The continuous time analog to a chain-weighted index is called a *Divisia* index.

[2]As we discuss in detail in Section 2, preference instability is driven by any factor that changes preference rankings over bundles of goods at fixed prices and income, e.g. age, health, advertising, fads. In the literature, preference instability and non-homotheticities are typically studied independently. We analyze them jointly in this paper because both generate the same type of biases in measures of real consumption. Our results are relevant when either of these forces is active.

[3]For the macro problem, we consider neoclassical economies with representative agents.

measures are not equal when household preferences are non-homothetic or unstable. Intuitively, some points on a budget constraint, which may be feasible for an individual agent, are not feasible for society as a whole due to curvature in the production possibility frontier.

We provide exact and approximate characterizations of the change in micro and macro welfare, and provide a generalization of chain-weighted (or more precisely Divisia) indices that can be used to measure both. Our macro welfare index extends Hulten (1978) to environments with non-homotheticities and taste shocks. We compare welfare and real consumption, and clarify the conditions under which real consumption captures welfare-relevant quantities when preferences are non-homothetic and unstable. In contrast to a standard Divisia index, which weights changes in prices or technologies using actual expenditure shares, the welfare-relevant index uses counterfactual "welfare-relevant" expenditure shares calculated under the final indifference curve. We show that this means that real consumption *undercounts* expenditure-switching caused by either income effects or taste shocks. We show that this undercounting is larger if changes in expenditures caused by income effects or taste shocks are correlated with changes in prices in partial equilibrium or changes in technologies in general equilibrium. If taste shocks are orthogonal to price changes or technology shocks, then real consumption will correctly measure changes in welfare.

To understand the intuition for this result, suppose healthcare services grew relative to manufacturing during the post-war era due to aging or income effects. In this case, measured real consumption does not correctly account for changes in expenditure shares. Intuitively, if expenditure shares change due to changes in demand, then when we compare the past to the present, we must use demand curves that are relevant for choices today, not the ones that were relevant in the past. If richer or older households prefer to spend their income on healthcare, and households in 2021 are richer and older than they were in 1950, then when we compare the economy's productive capacity in 1950 to 2021, we must account for the fact that the richer and older households of today demand a different set of goods than the younger and poorer households of 1950. The gap between real consumption and welfare is large if spending on healthcare increases due to aging or income effects and the relative price of healthcare changes relative to that of other sectors.

Our results for welfare and the gap between welfare and real consumption are expressed in terms of measurable sufficient statistics. In both partial and general equilibrium, we show that computing the change in welfare does not require direct knowledge of the taste shocks or income elasticities. Instead, what we must know are the expenditure shares and the elasticities of substitution at the final allocation. For the micro problem, these are the household expenditure shares and the elasticities of substitution in consumption. For the macro problem, these are the input-output table and the elasticities of substitution in both production and consumption. These results can be used both for ex-post accounting and ex-ante counterfactuals.

For very simple economies with one factor, constant returns to scale, and no intermediates, the difference between welfare and real consumption is approximately half the covariance of supply and demand shocks. This formula can be generalized to more complex economies. We show how the details of the production structure, like input-output linkages, complementarities in production, and decreasing returns to scale, interact with non-homotheticities and preference shocks to magnify the gap between welfare and real consumption. We show that the discrepancies between welfare and real consumption that we emphasize do not get "aggregated" away. In fact, the more we disaggregate, the more important these discrepancies are likely to become. In this sense, our results are related to the literature studying the macroeconomic implications of production networks and disaggregation (e.g. Gabaix, 2011; Acemoglu et al., 2012; Baqaee and Farhi, 2019b).

We illustrate the relevance of our results for understanding short-run and long-run phenomena by means of three applications. In our first application, we analyze the importance of non-homotheticity or preference instability for measures of long-run productivity growth. Since Baumol (1967), an enduring stylized fact about economic growth has been the observation that industries with slow productivity growth tend to become larger as a share of the economy over time. This phenomenon, known as Baumol's cost disease, implies that aggregate growth is increasingly determined by productivity growth in slow-growth industries since, over time, the industrial mix of the economy shifts to favor these industries. To be specific, from 1947 to 2014, aggregate TFP in the US grew by 60%. If the US economy had kept its original 1947 industrial structure, then aggregate TFP would have grown by 78% instead. We show that if structural transformation is caused solely by income effects and demand instability, then welfare-relevant TFP grew by only 47%. This is because measured aggregate TFP does not fully account for substitution caused by changes in demand, and hence the increase in the welfare-relevant measure of aggregate TFP is much lower than what is measured. We also find a similar pattern in consumption data.

In our second application, we argue that the gap between real consumption and welfare changes is likely to be present even at high frequencies of the data. Whereas industry-level shares are relatively stable over short-horizons, firm or product-level sales shares are not. We consider a firm-level specification of our model and show that when firms' demand shocks are correlated with their supply shocks, there is a gap between welfare-relevant and measured changes in industry-level output and prices. These biases do not disappear as we aggregate up to the level of real GDP even if firms and industries are infinitesimal. If we start with aggregate industry-level (rather than disaggregated industry, or firm-level) data, we are ruling out the existence of these biases by construction. In work-in-progress, we quantify these gaps using product-level data on non-durable consumer goods and find that standard price indices, like the Sato-Vartia index and chained-weighted price index, understate welfare-relevant inflation rates.

In our final application, we demonstrate the difference between macroeconomic and microeconomic notions of welfare using the Covid-19 recession. During this recession, household expenditures switched to favor certain sectors and those same sectors displayed higher price increases. We show that this implies that microeconomic welfare, taking changes in prices as given, fell by more than macroeconomic welfare, taking into account the fact that that changes in prices are themselves caused by demand shocks.

Furthermore, in episodes in which household spending patterns are driven by taste shocks, as in Covid-19, changes in real consumption generically depend on irrelevant details like the order in which supply and demand shocks hit the economy. In these circumstances, the change in real consumption between two time periods is not a function of only prices and quantities in those two periods. The same logic applies to real GDP, which means that real GDP or TFP are unreliable metrics for measuring changes in productive capacity.

Of course, there are other reasons, besides instability and non-homotheticity, why chained indices fail to accurately measure welfare. Many of the well-known reasons can be thought of as being due to missing prices and quantities. For example, it is well-known that real consumption fails to properly account for the creation and destruction of goods if we cannot measure the quantity of goods continuously as their price falls from or goes to their choke price (Hicks, 1940; Feenstra, 1994; Hausman, 1996; Aghion et al., 2019); real consumption also fails to properly account for changes in the quality of goods (see Syverson, 2017); finally, real consumption fails to properly account for changes in non-market components of welfare, like changes in leisure and mortality (see Jones and Klenow, 2016), or changes in the user cost of durables. In all of these cases, the problem is that some of the relevant prices or quantities in the consumption bundle are missing or mismeasured, and correcting the index involves imputing a value for these missing prices or quantities. Non-homotheticities and taste shocks are different in the sense that they are not caused by mismeasurement of market prices. For this reason, we abstract from these important mismeasurement issues and assume that prices and quantities have been correctly measured. If prices and quantities are mismeasured or missing, then our results would apply to the quality-adjusted, corrected, version of prices instead of observed prices. That is, the corrections we derive are different to the ones that are equivalent to adjustments in prices.[4]

Relatedly, taste shocks and mismeasured prices (i.e. unobserved quality change) are sometimes viewed as alternative means to the same end. This is because they can both be used to justify why demand curves shift over time, even holding prices and incomes fixed. However, while they have similar implications for changes in observed prices and quantities, they have very different implications for welfare. When there are unobserved changes in quality, the gap between welfare and real consumption is caused by a difference between measured and welfare-relevant *prices*. We show that in the case of non-homotheticities and taste shocks, the gap between welfare and real consumption is caused by a difference between measured and welfare-relevant *expenditure shares*.

Other related literature. This paper contributes to the literatures on growth and productivity accounting, multi-sectoral and disaggregated macroeconomics, as well as the literature on structural transformation. We discuss the way our paper complements and relates to these literatures in turn.

A key assumption in growth accounting is the existence of a stable and homothetic final aggregator. As shown by, for example Hulten (1973) among others, chain-linked indices are meaningful if, and only if, a homothetic and stable final aggregator exists. Therefore, this assumption is ubiquitous in growth accounting, and also appears in almost all papers that study aggregate outcomes using disaggregated input-output models.[5] We provide a generalization of growth accounting to environments where preferences are neither homothetic nor stable.

Measuring changes in welfare using a money metric when there are income effects is standard in microeconomic theory. Our results extend these insights to a general equilibrium context, and we relate them to welfare-relevant growth accounting. We provide a generalization of Domar (1961) and Hulten (1978) that measures changes in welfare in general equilibrium allowing for preferences to be unstable or non-homothetic. Using

[4]Our approach to calculate ex-post welfare changes requires well-measured price changes and elasticities of substitution in the final period. For ex-post welfare measurement, when information on prices is missing or mismeasured, if preferences are non-homothetic an alternative approach is to infer changes in welfare by relying on changes in prices, expenditures, price elasticities, and Engel curve slopes for only a subset of goods, given assumptions on separability and stability in preferences (see e.g. Hamilton, 2001 and, more recently, Atkin et al., 2020). In addition to ex-post measurement with non-homotheticities, in this paper we study the implications of instability of preferences (that generate shifts in expenditures correlated with price changes) and we also consider counterfactuals.

[5]See, for example, the review paper by Carvalho and Tahbaz-Salehi (2018) and the references therein.

this, we can construct exact and approximate characterizations of how welfare responds to shocks in general equilibrium, a question which is of central importance in the literature on disaggregated and production network models.[6]

A recent and related paper is Redding and Weinstein (2020), who show that variations in sales are difficult to explain via shifts in supply curves alone, and shifts in demand curves (i.e. taste shocks) are an important source of variation in the data. Their approach to evaluate welfare changes in the presence of taste shocks contrasts with ours because, unlike us, they treat changes in tastes as being equivalent to changes in price. Operationally, this makes the taste shocks behave like quality shocks. They estimate changes in taste/quality necessary to explain variations in product-level data. However, this only determines changes in the *relative* size of demand shocks across goods, and it does not pin down changes in the overall level of these shocks. Redding and Weinstein (2020) pin down the overall level of the shocks by assuming that they are mean zero (see Martin, 2020 for a discussion of this assumption). Our approach is different in that we do not compare utils before and after the taste shocks. Instead we compute changes in equivalent variation keeping preferences constant for the variation, as advocated by Fisher and Shell (1968) and Samuelson and Swamy (1974). This approach does not require any assumptions about the overall level of the taste shocks in terms of utils. Moreover, as mentioned above, in practical terms the adjustments we derive require the use of counterfactual expenditure shares and not counterfactual *taste-adjusted* prices. We compare the two approaches in more detail in Appendix D.[7]

Our paper is also related to the literature on structural transformation and Baumol's cost disease. As explained by Buera and Kaboski (2009) and Herrendorf et al. (2013), this literature advances two microfoundations for structural transformation. The first expla-

[6]The biases we identify, and the failure of Hulten's theorem, are not caused by inefficiencies (e.g. markups, wedges, taxes). Baqaee and Farhi (2019a) analyze how growth accounting must be adjusted in inefficient economies. Whereas incorporating inefficiencies in production does not affect our micro welfare results, how they interact with demand instability and non-homotheticity in general equilibrium is beyond the scope of this paper.

[7]Given CES preferences, Martin (2020) estimates using scanner level data large differences in annual price changes between price indices based on fixed initial tastes and final tastes. Other papers studying the relationship between conventional index numbers and welfare in the presence of preference instability include Balk (1989) who discusses various ways one can define changes in the cost of living, Feenstra and Reinsdorf (2007) who show that the Sato-Vartia index is equal to the CES price index evaluated at some intermediate level of taste shifters, and Caves et al. (1982) who show that when preferences are homothetic, translog, but unstable, Tornqvist-based indices correspond to a geometric average of welfare changes under initial and final preferences. We characterize welfare (in partial and general equilibrium) at either initial or final preferences and using either EV or CV. In contrast to Tornqvist and Sato-Vartia, Divisia indices cannot generically be interpreted as corresponding to any mixture of well-defined preferences. This is because, as we discuss in Section 5, Divisia (or chained) indices are path-dependent, so they can violate basic properties like assigning a higher value to a strictly larger budget set.

nation is all about relative prices differences: if demand curves are not unit-price-elastic, then changes in relative prices change expenditure shares (e.g. Ngai and Pissarides, 2007; Acemoglu and Guerrieri, 2008; Buera et al., 2015). The second explanation emphasizes shifts in demand curves caused by income effects— households spend more of their income on some goods as they become richer (e.g. Kongsamut et al., 2001; Boppart, 2014; Comin et al., 2015; Alder et al., 2019), or taste shocks— households spend more of their income on some goods as they become older (Cravino et al., 2019). Our results suggest that structural transformation driven by relative price changes has different welfare implications than structural transformation driven by non-homotheticity or taste shocks.[8]

The structure of the paper is as follows. In Section 2, we set up the microeconomic problem and provide exact and approximate characterizations of the difference between welfare and measured real consumption. In Section 3, we set up the macroeconomic general equilibrium model and provide exact and approximate characterizations of the difference between welfare and measured real output changes. Whereas in section 3 we present our macro results in terms of endogenous sufficient statistics, in Section 4 we solve for these endogenous sufficient statistics in terms of microeconomic primitives and consider some simple but instructive analytical examples. Our applications are in Section 5. We discuss some extensions in Section 6 and conclude in Section 7. Proofs are in the appendix.

2 Microeconomic Changes in Welfare and Consumption

In this section, we consider changes in budget constraints in partial equilibrium. We ask how consumers value these changes, and compare these measures of welfare with measures of real consumption. We provide exact and approximate results. This section helps build intuition for Section 3, where we model the equilibrium determination of prices.

2.1 Definition of Welfare and Real Consumption

In this subsection we define welfare and real consumption. Measuring changes in welfare using equivalent variation is standard when preferences are stable. However, measuring welfare changes in the presence of unstable preferences is less common and therefore we discuss this issue in some detail.

[8]For welfare analysis with non-homothetic preferences in other contexts such as cross-country real income comparisons and gains from trade, see Feenstra et al. (2009) and Fajgelbaum and Khandelwal (2016).

Consider a set of preference relations, $\{\succeq_x\}$, over bundles of goods. These preferences are indexed by x, which represents anything that affects preference rankings over bundles of goods. For example, x could be calendar time, age, exposure to advertising, or state of nature. For every x, we represent the preference relation $\succeq_x$ by a utility function $u(c;x)$, where $c \in \mathbb{R}^N$ and N is the number of goods in the consumption bundle. Since the consumer makes no choices over x, we do not need to specify how $u(c;x)$ varies with x. Moreover, preferences over x, if they exist, are not revealed by choices.[9]

There are two properties of preferences that are analytically convenient benchmarks throughout the rest of the analysis.

Definition 1 (Homotheticity)**.** Preferences over goods c are *homothetic* if, for every positive scalar $a > 0$ and every feasible c and x, we can write

$$u(ac;x) = au(c;x).$$

Definition 2 (Stability)**.** Preferences over goods c are *stable* if there exists a time-invariant function $\Phi(\cdot)$ such that the utility function can be written as $u(c;x) = U(\Phi(c);x)$ for every feasible c and x.

If preferences are stable, x can change over time (e.g. households get higher or lower utils from all goods) but, since x is separable from c, these changes do not impact preferences over bundles of goods c. If preferences are not stable, we say that they are unstable.

Given preferences encapsulated in u, the indirect utility function of the consumer, for any value of x, is

$$v(p, I; x) = \max_c \{u(c;x) : p \cdot c = I\}.$$

where p is a price vector over goods and I is expenditures (which we interchangeably refer to as income). The vector p includes all relevant prices in the preference relation. If the preference relation is intertemporal, then p includes the path of current and future prices.[10]

Consider shifts in the budget set as prices and income change from p_{t_0} and I_{t_0} to p_{t_1} and I_{t_1}. Here, t_0 and t_1 simply index the vector of prices and income being compared. Motivated by our applications, we refer to this index as time, but it could equally refer to space. This change in the budget set is accompanied by changes in x from x_{t_0} to x_{t_1}.

[9]In Section 6, we discuss situations in which x is endogenously chosen and valued by the consumer, such as leisure, but its price and quantity are not being measured. We also discuss situations in which x is endogenously chosen by firms, such as advertising.

[10]We discuss how to apply our results in dynamic economies in Section 4.3.

Since utility is only defined up to monotone transformations, changes in utility do not have meaningful units. When prices are exogenous, we measure changes in utility using corresponding changes in income. Our baseline measure of microeconomic welfare is defined as follows.

Definition 3 (Micro Welfare)**.** The change in welfare measured using the *micro equivalent variation* with *final preferences* is $EV^m(p_{t_0}, I_{t_0}, p_{t_1}, I_{t_1}; x_{t_1}) = \phi$ where ϕ solves

$$v(p_{t_1}, I_{t_1}; x_{t_1}) = v(p_{t_0}, e^{\phi} I_{t_0}; x_{t_1}). \tag{1}$$

In words, EV^m is the change in income (in logs), under initial prices p_{t_0}, that a consumer with preferences $\succeq_{x_{t_1}}$ would need to be indifferent between the budget set defined by initial prices $(p_{t_0}, e^{\phi} I_{t_0})$ and the new budget set defined by new prices and income (p_{t_1}, I_{t_1}).[11] The new budget set is preferred to the initial one, if and only if, EV^m is positive. The superscript m represents the fact that this is the *micro* equivalent variation, since we take prices as given.

Discussion of our welfare criterion. Following Fisher and Shell (1968), the welfare criterion in Definition 3 measures the change in welfare by presenting the consumer with a hypothetical choice holding fixed their preferences. To be concrete, suppose that x represents the age of the consumer. Clearly, we cannot meaningfully compare the amount of utils an individual derives from watching cartoons as a child to the amount of utils that individual derives from drinking coffee as an adult. Since consumers never make choices about how old they are, their preferences across consumption goods consumed at different ages are not revealed by their choices. In the words of Heraclitus: "No man ever steps in the same river twice, for it's not the same river and he's not the same man." However, if we fix the consumer's age x, we can meaningfully compare the consumer's choices about budget sets they faced at different points in their life or that they may face in the future.

This approach, of holding x constant, is different to the one taken when x represents some form of quality change. Intuitively, quality adjustments are more applicable to situations where the consumer can conceivably make choices between the good at differing

[11] In principle, we could also measure changes in welfare using compensating (instead of equivalent) variation, or by using initial (rather than final) preferences. Combining EV with final preferences (CV with initial preferences) is natural since this requires preserving the shape of the indifference curve at the final (initial) allocation. In the body of the paper, we focus on EV using final preferences since equivalent variation is more commonly used and final preferences are more relevant than initial preferences, but we characterize these other welfare measures in Appendix B. See also Remark 2.

levels of quality. For example, if a box of chocolates undergoes quality change so that each box now contains twice as many chocolates, the consumer can conceivably make choices between the old and new boxes that reveal how much they value the quality change. Taste changes, on the other hand, do not involve meaningful choices from the consumer's perspective. Our approach of holding x constant allows us to study welfare in situations where, either for practical or philosophical reasons, it is not possible to model preferences over x itself.

Real Consumption. Having defined changes in welfare, we now define changes in real consumption. The change in real consumption corresponds to what national income accountants and statistical agencies do when given data on the evolution of prices p and consumption bundles c.

Definition 4 (Real consumption)**.** For some path of prices that unfold as a function of time t, the change in *real consumption* from t_0 to t_1 is defined to be

$$\Delta \log Y = \int_{t_0}^{t_1} \sum_{i \in N} b_{it} \frac{d \log c_i}{dt} dt = \int_{t_0}^{t_1} \sum_{i \in N} b_i d \log c_i, \tag{2}$$

where b_{it} is the budget share of good i given prices, income, and preferences at time t.[12]

The last equation on the right-hand side simplifies notation by suppressing dependence on t in the integral. We use this convention throughout for writing integrals. Equation (2) is called a *Divisia* quantity index. In practice, since perfect data is not available in continuous time, statistical agencies approximate this integral via a (Riemann) sum using chained indices. We abstract from the imperfections of these approximations in this paper.[13] Moreover, we assume that that the data on prices and quantities is *perfect* — completely accurate, comprehensive, adjusted for any necessary quality changes, and available in continuous time. This is because the important and well-studied biases associated with imperfections in the data, like the lack of quality adjustment, missing prices, or infrequent measurement, are different to the biases we study.

Using the budget constraint, we can express changes in real consumption in terms of

[12]For any variable z, we denote by dz its change over infinitesimal time intervals, so that $\Delta z = \int_{t_0}^{t_1} dz$.

[13]In discrete time, one can approximate this Riemann integral in different ways. For example, we can approximate the integral in levels (arithmetic-weighting) or in logs (geometric-weighting). Furthermore, we can use left-Rieman sums (Chained Laspeyres), right-Riemann sums (Chained Paasche), or mid-point Riemann sums (Chained Tornqvist or Fisher). In continuous time, all of these procedures are equivalent and yield the same answer. In practice, however, Boppart et al. (2021) show that these different weighing procedures have large quantitative implications for the value of the indices.

changes in income deflated by price changes,

$$\Delta \log Y = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} b_i d \log p_i. \tag{3}$$

In other words, changes in real consumption are equal to changes in income minus changes in the consumption price deflator. Notice that changes in real consumption (or the consumption price deflator) potentially depend on the entire path of prices and quantities between t_0 and t_1 and not just the initial and final values. This is unlike welfare changes, EV^m, which depend only on initial and final prices and incomes and not on their entire path.

2.2 Relating Welfare and Consumption

We consider how real consumption and welfare change in response to changes in the budget set and the indifference curves of the consumer. We first consider globally exact results and then local approximations. The results are stated in terms of changes in prices and income, which we endogenize in Sections 3 and 4.

Global results. When preferences are unstable or non-homothetic, the following lemma shows that changes in welfare can be expressed as changes in income deflated by a shock-dependent price index. Changes in this price index are equal to budget-share weighted price changes, as in expression (3) for real consumption. However, whereas the price deflator for real consumption is based on observed budget shares (given prices, income, and preferences over time), the price deflator for welfare is based on hypothetical budget shares (at fixed utility level and fixed preferences).

To state this, define the expenditure function for any value of x by

$$e(p, u; x) = \min_c \{p \cdot c : u(c; x) = u\}.$$

The budget share of good i (given prices, preferences, and a level of utility) is

$$b_i(p, u; x) \equiv \frac{p_i c_i(p, u; x)}{e\,(p, u; x)} = \frac{\partial \log e(p, u; x)}{\partial \log p_i}, \tag{4}$$

where the second equality, Shephard's lemma, establishes a connection between budget shares and elasticities of the expenditure function. Note that when preferences are homothetic, then the expenditure function can be written as $e(p, u; x) = e\,(p; x)\, u$ and, hence, budget shares do not depend on u.

The following lemma characterizes changes in microeconomic welfare.

Lemma 1 (Micro Welfare). *Given any change in prices, income, and preferences, micro welfare changes are given by*

$$EV^m = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} b_i^{ev} d \log p_i, \tag{5}$$

where $b_i^{ev}(p) \equiv b_i(p, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1})$ *denotes budget shares at prices* p*, but fixing final preferences* x_{t_1} *and final utility* $v(p_{t_1}, I_{t_1}; x_{t_1})$.

Lemma 1 follows from the observation that EV^m can be re-expressed, using the expenditure function, as

$$EV^m = \log \frac{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_0}, v(p_{t_0}, I_{t_0}; x_{t_1}); x_{t_1}\right)} = \Delta \log I - \log \frac{e\left(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)},$$

and recognizing that the second term can be written as the integral in (5).[14]

Compared to real consumption (3), which weights price changes by observed budget shares, EV^m weights price changes by hypothetical budget shares evaluated at current prices but for fixed final preferences, x_{t_1}, and final utility, $v(p_{t_1}, I_{t_1}; x_{t_1})$. Welfare depends on budget shares at x_{t_1} since only these preferences matter for EV^m. Welfare depends on budget shares evaluated at final utility, $v(p_{t_1}, I_{t_1}; x_{t_1})$, since EV^m adjusts the level of income in t_0 to make consumers as well off as they are in t_1. If welfare increases from t_0 to t_1, consumers must be given more income in t_0 to make them indifferent between t_0 and t_1. As we give consumers more income in t_0, the shape of their indifference curve changes until it mirrors the one in t_1. This means that the shape of the indifference curve relevant for the comparison is the one at t_1.[15]

[14] By definition, EV^m only depends on initial and final prices and income, given t_1 preferences. By the gradient theorem, the integral in (5) is path independent and can be computed under any continuously differentiable path of prices that go from p_{t_0} to p_{t_1}. When comparing EV^m and real consumption, we consider the integral under the realized path of prices over time, which as described in the text is assumed to be available in continuous time.

[15] When preferences are stable, real consumption, defined by (2), is a multi-good version of the change in consumer surplus, which is the area under the Marshallian demand curve. Similarly, by equation (5), welfare is the area under the Hicksian demand curve. Hence, in a partial equilibrium context with stable preferences, the gap between real consumption and welfare is the same as the gap between consumer surplus and welfare, studied by Hausman (1981) and McKenzie and Pearce (1982) amongst others. This equivalence does not hold when preferences are unstable (since Marshallian consumer surplus is not the same as chained real consumption) or in general equilibrium (since micro and macro welfare are not the same, as we discuss in Section 4).

We can reinterpret the hypothetical budget shares $b^{ev}(p)$ as corresponding to those of a fictional consumer with homothetic and stable preferences with expenditure function $e^{ev}(p,u) = e(p, v_{t_1}; x_{t_1}) \frac{u}{v_{t_1}}$, where $v_{t_1} = v(p_{t_1}, I_{t_1}; x_{t_1})$. This implies that we can calculate changes in welfare given changes in prices based on budget shares $b^{ev}(p)$, *without* needing to know income elasticities or the nature of demand shocks. This is because the fictional consumer has homothetic and stable preferences, which means that all income elasticities are equal to one and there are no demand shocks. To compute $b^{ev}(p)$, we need to know the terminal budget shares and the terminal elasticities of substitution, as discussed in the following remark.

Remark 1 (Non-homothetic CES preferences)**.** To illustrate how Lemma 1 can be used we consider a non-homothetic CES example as in Comin et al. (2015) or Fally (2020).[16] The following pins down changes in budget shares over time:

$$d\log b_i = (1-\theta_0)\left(d\log p_i - \mathbb{E}_b[d\log p]\right) + (\varepsilon_i - 1)\left(d\log I - \mathbb{E}_b[d\log p]\right) + d\log x_i, \quad (6)$$

where $\mathbb{E}_b(\cdot)$ is the weighted average using budget shares as probability weights. The elasticity ε_i is the income elasticity of good i and θ_0 is the (constant utility) elasticity of substitution across goods. The term $d\log x_i$ is a demand shifter (i.e. a taste shock), a residual that captures changes in expenditure shares not attributable to changes in income or prices. Note that when ε_i is equal to 1 for every i, final demand is homothetic, and when x_i is constant for all i, final demand is stable.[17]

For ex-post welfare questions, we can construct the unobserved $b^{ev}(p)$ between t_0 and t_1 by iterating on

$$d\log b_i^{ev} = (1-\theta_0)\left(d\log p_i - \mathbb{E}_{b^{ev}}[d\log p]\right), \quad (7)$$

starting at t_1 with initial value $b_{t_1}^{ev} = b_{t_1}$ and going back to t_0.[18] These are changes in budget shares which are only due to substitution effects, and hence omit the last two terms in equation (6). Given b^{ev}, we can apply Lemma 1. For non-homothetic CES, the

[16] The result that only elasticities of substitution are necessary to calculate EV^m is true for arbitrary non-CES functional forms, but since the intuition for the more general case is similar to the CES case, we leave the more general non-parametric results in Appendix F. We use non-homothetic CES in our worked-out examples since it provides a clean separation between substitution elasticities (necessary for computing welfare) and other parameters of the utility function, in contrast to other commonly used non-homothetic demand systems such as PIGL and AIDS. Furthermore, substitution elasticities are symmetric for CES, which also helps keep the examples intuitive.

[17] Since b_i are expenditure shares that always add up to one, it must necessarily be the case that $\mathbb{E}_b[d\log x] = 0$ and $\mathbb{E}_b[\varepsilon] = 1$. See Appendix C for a derivation.

[18] The budget shares in $\mathbb{E}_{b^{ev}}(\cdot)$ must be updated as we iterate between t_1 and t_0.

integral in Lemma 4 has a closed form solution[19]

$$EV^m = \Delta \log I + \log \left(\sum_i b_{it_1} \left(\frac{p_{it_0}}{p_{it_1}} \right)^{1-\theta_0} \right)^{\frac{1}{1-\theta_0}}.$$

This shows that the income elasticities and taste shocks are not directly required.[20]

For ex-ante counterfactuals, where b_{t_1} is not known, we could first iterate on equation (6) from t_0 to t_1 to obtain b_{t_1}. This first step requires full knowledge of demand shocks and income elasticities over time (see Appendix C for more details). Once in possession of b_{t_1}, we repeat the procedure above and apply (7) to get the path of welfare-relevant budget shares b^{ev}.

Remark 2 (Compensating Variation under Initial Preferences)**.** Our baseline measure of welfare changes is equivalent variation under final preferences. An alternative would be to use compensating variation under initial preferences. Every result in the paper can be translated into compensating variation under initial preferences simply by reversing the flow of time. In particular, whereas Lemma 1 preserves the shape of the indifference curve at the final allocation, the compensating variation counterpart to Lemma 1 preserves the shape of the indifference curve at the initial allocation. Hence, calculating compensating variation requires knowledge of initial budget shares and elasticities of substitution, whereas equivalent variation requires knowledge of final budget shares and elasticities of substitution. This means that EV^m is more convenient for ex-post comparisons and CV^m (at initial preferences) is more convenient for ex-ante comparisons or counterfactuals. This is because in these cases, we use "today's" budget shares and elasticities of substitution to undertake the welfare comparisons (without needing knowledge of taste shocks or income elasticities). See Appendix B for more details.[21]

We now contrast changes in real consumption and welfare.

[19] In Appendix C.2, we show that when preferences are non-homothetic CES, changes in the utility index are not the same as changes in equivalent (or compensating) variation. Hence, the non-homothetic CES utility index is not a money-metric for welfare.

[20] In practice, estimating the elasticity of substitution θ_0 may require knowing the income elasticities (via Slutsky's equation). However, if the expenditure share of each good is sufficiently small, then θ_0 can be estimated without knowledge of income elasticities. Auer et al. (2021) estimate the relevant elasticities and apply Lemma 1 to measure the heterogeneous welfare effects of changes in foreign prices in the presence of demand non-homotheticities.

[21] In Appendix B we show that, up to a second-order approximation (but not globally), changes in real consumption equal a simple average of equivalent variation under final preferences and compensating variation under initial preferences.

Proposition 1 (Consumption vs. Welfare)**.** *Given any continuously differentiable change in prices, income, and preferences, the difference between welfare changes and real consumption is*

$$EV^m - \Delta \log Y = \int_{t_0}^{t_1} \sum_{i \in N} \left(b_{it} - b_{it}^{ev}\right) \frac{d \log p_{it}}{dt} dt = (t_1 - t_0)\, \mathbb{E}_t Cov\left(b - b^{ev}, d \log p\right),$$

where the covariance is calculated across goods at a point in time, and the expectation is calculated across time between t_0 and t_1.

An immediate consequence of Proposition 1 is the well-known result that real consumption is equal to changes in equivalent variation if, and only if, preferences are homothetic and stable. This is because when preferences are stable and homothetic, budget shares do not depend on x or changes in utility u over time. Hence, whenever preferences are homothetic and stable, $b_{it}^{ev} = b_{it}$ for every path of shocks and every t.

When preferences are non-homothetic or unstable, observed budget shares not only reflect price changes but also non-price changes (that is, changes in x and changes in u). This generates discrepancies between observed and hypothetical budget shares, and hence between real consumption and welfare. To gain more intuition for this, we characterize changes in real consumption and welfare using a second-order approximation around initial choices.

Local results. Consider local approximations of the objects of interest as the time period goes to zero, $t_1 - t_0 = \Delta t \to 0$.[22] Throughout the rest of the paper, a second-order approximation means that the remainder term is of order Δt^3. We focus on second-order approximations to capture the interaction between price changes and expenditure-switching, which is the source of the gaps between real consumption and welfare changes.

We begin by stating the results in terms of Hicksian budget shares, and then we re-express them in terms of Marshallian (observable) budget shares.

Proposition 2 (Approximate Micro using Hicksian Demand)**.** *To a second-order approximation, the change in real consumption is*

$$\Delta \log Y \approx \Delta \log I - b' \Delta \log p - \sum_{i \in N} \left[\frac{1}{2} \Delta \log p' \frac{\partial b_i}{\partial \log p} + \frac{1}{2} \Delta \log x' \frac{\partial b_i}{\partial \log x} + \frac{1}{2} \Delta \log v \frac{\partial b_i}{\partial \log u} \right] \Delta \log p,$$

[22] For our local approximations, we assume that the exogenous parameters (prices, income, and taste shifters) are smooth functions of t and that the expenditure function is a smooth function of the exogenous parameters.

and the change in welfare is

$$EV^m \approx \Delta \log I - b' \Delta \log p - \sum_{i \in N} \left[\frac{1}{2} \Delta \log p' \frac{\partial b_i}{\partial \log p} + \Delta \log x' \frac{\partial b_i}{\partial \log x} + \Delta \log v \frac{\partial b_i}{\partial \log u} \right] \Delta \log p.$$

Comparing the expression for real consumption $\Delta \log Y$ and welfare EV^m shows that to a first order, they are the same. Discrepancies between the two arise starting at the second-order and involve how expenditure-switching is treated. Real consumption accounts for changes in budget shares in the same way regardless of their cause. The first term in the square brackets reflects changes in budget shares due to changes in relative prices (substitution effects) and the next two terms correspond to changes in budget share due to non-price factors (taste shocks and income effects).[23]

The second line shows that welfare treats changes in budget shares due to substitution effects differently to changes in budget shares due to taste shocks or income effects. To understand the gap between welfare and real consumption changes, consider first the case of homothetic but unstable preferences. Whereas changes in real consumption only take into consideration changes in budget shares in response to changes in utility parameters as the shock unfolds over time, changes in welfare must account for these changes from the start. Therefore, changes in budget shares due to non-price factors are multiplied by $1/2$ in real consumption, but they are multiplied by 1 in welfare. In other words, real consumption does not sufficiently account for substitution caused by preference instability. For example, the additional reduction in welfare (at new preferences) from a price increase in a good i with increasing demand ($d \log x \frac{\partial b_i}{\partial \log x} d \log p_i > 0$) is not fully reflected in real consumption, implying $EV^m < \Delta \log Y$.[24]

Similar reasoning applies in the case of stable but non-homothetic preferences, since changes in budget shares due to non-homotheticities should be incorporated in welfare immediately but are reflected in real consumption only gradually. For example, a reduction in the price of a good for which income effects are relatively weak ($d \log v \frac{\partial b_i}{\partial \log v} d \log p_i >$ 0) implies a smaller increase in welfare than in real consumption ($EV^m < \Delta \log Y$).

Proposition 2 is expressed in terms of Hicksian elasticities. We now re-express these results in terms of Marshallian elasticities using the non-homothetic CES aggregator in-

[23] The terms $\Delta \log x$ and $\Delta \log u$ need only be first-order approximations since they are multiplied by $\Delta \log p$ (and we only need to keep terms that are of order Δt^2). However, for the first term $-b\Delta \log p$, the primitive shock in prices must be approximated up to the second order, that is, $\Delta \log p \approx (\partial \log p / \partial t)\Delta t + 1/2(\partial \log p^2 / dt^2)\Delta t^2$.

[24] A non-zero correlation between prices and demand shifters may emerge endogenously if firms have non-constant returns to scale or if firms invest in advertisement in response to productivity shocks. We consider the first possibility in Example 4 in Section 4 and discuss the second in Section 6.

troduced in Remark 1.

Proposition 3 (Approximate Micro using Marshallian Demand)**.** *Consider some perturbation in demand* $\Delta \log x$*, prices* $\Delta \log p$*, and income* $\Delta \log I$*. Then, to a second-order approximation, the change in real consumption is*

$$\Delta \log Y \approx \Delta \log I - \mathbb{E}_b\left[\Delta \log p\right] - \frac{1}{2}(1-\theta_0) Var_b\left(\Delta \log p\right) \qquad (8)$$
$$- \frac{1}{2} Cov_b\left(\Delta \log x, \Delta \log p\right) - \frac{1}{2}\left(\Delta \log I - \mathbb{E}_b\left[\Delta \log p\right]\right) Cov_b\left(\varepsilon, \Delta \log p\right),$$

and the change in welfare is

$$EV^m \approx \Delta \log Y - \frac{1}{2} Cov_b\left(\Delta \log x, \Delta \log p\right) - \frac{1}{2}\left(\Delta \log I - \mathbb{E}_b\left[\Delta \log p\right]\right) Cov_b\left(\varepsilon, \Delta \log p\right), \qquad (9)$$

where $Cov_b(\cdot)$ *is the covariance using the initial budget shares as the probability weights.*

We begin by considering the change in real consumption in (8). To a first order, the change in real consumption is just the change in income deflated by prices: $\Delta \log I - \mathbb{E}_b[\Delta \log p]$. The remaining terms capture nonlinearities associated with expenditure-switching. Since these are second-order, they are multiplied by $1/2$. We discuss these terms one-by-one. If goods are substitutes, $\theta_0 > 1$, then variance in relative prices boosts expenditure shares of cheaper goods and this increases measured real consumption. The second line captures the changes due to changes in demand. Intuitively, if the composition of demand shifts in favor of goods that happen to become relatively cheap, either due to non-homotheticity $Cov_b\left(\varepsilon, \Delta \log p\right)\left(\Delta \log I - \mathbb{E}_b\left[\Delta \log p\right]\right) < 0$ or demand shocks $Cov_b(\Delta \log x, \Delta \log p) < 0$, then real consumption increases.

Now consider changes in welfare in (9). As expected, the first-order terms are identical. The remaining terms capture the nonlinear response of welfare to price shocks. Note that if preferences are stable and homothetic, then welfare changes coincide with changes in real consumption. However, if preferences are unstable or non-homothetic, real consumption strays from welfare whenever price changes covary with non-price changes in demand. This happens because real consumption "undercounts" expenditure-switching due to the changes in demand.

In Appendix D we extend Proposition 3 to incorporate unobserved changes in quality. We show that the biases causes by non-homotheticities and taste shocks are very different to the ones caused by quality changes. We also explicitly compare the biases that we study to the ones discussed in Redding and Weinstein (2020).

3 Macroeconomic Changes in Welfare and Consumption

In the previous section we showed how changes in budget sets affect welfare when preferences are unstable and non-homothetic. For these problems, the frontier of the consumer's choice set is linear, since prices are assumed to be exogenous. At the level of a whole society however, choice sets need not be linear. The production possibility set associated with an economy may have a nonlinear frontier. In this case, relative prices respond endogenously to choices made by consumers. In this section, we extend our analysis to allow for nonlinear production possibility frontiers (PPFs). The analysis in this section collapses to the one in Section 2 when the PPF of the economy is the same as the budget constraint (as happens in very simple general equilibrium models).

We first update our definitions of welfare, now at the macroeconomic level, and we introduce some basic structure and notation. We then present expressions for real GDP and welfare at the macroeconomic level, first globally and then locally in terms of endogenous sufficient statistics. In the next section, Section 4, we solve for these endogenous objects in terms of observable primitives.

3.1 Definition of Welfare and Real GDP

Consider a perfectly competitive neoclassical closed economy with a representative agent. Each good $i \in N$ has a production function

$$y_i = A_i G_i \left(\{m_{ij}\}_{j \in N}, \{l_{if}\}_{f \in F} \right),$$

where G_i is a neoclassical production function, m_{ij} are intermediate inputs used by i and produced by j, and l_{if} denotes primary factor inputs used by i for each factor $f \in F$. The exogenous scalar A_i is a Hicks-neutral productivity shifter. Without loss of generality, we assume that G_i has constant returns to scale since decreasing returns to scale can be captured by adding producer-specific factors. Furthermore A_i is Hicks-neutral without loss of generality. This is because we can capture non-neutral (biased) productivity shocks to input j for producer i by introducing a fictitious producer that buys from j and sells to i with a linear technology. A Hicks-neutral shock to this fictitious producer is equivalent to a non-neutral technology shock to i.

Let A be the $N \times 1$ vector of technology shifters and L be the $F \times 1$ vector of primary (exogenously given) factor endowments. The production possibility set (and its associated frontier) is the set of feasible consumption bundles that can be attained given A and L. Given our assumption that production functions have constant returns to scale, the

PPF is linear if there is only one factor of production.

For each A, L, and x, we denote equilibrium prices and aggregate income by $p(A, L, x)$ and $I(A, L, x)$. These equilibrium prices and incomes are unique up to the choice of a numeraire.

Define the *macro indirect utility* function as the maximum amount of utility the economy can deliver

$$V(A, L; x) = \max_{c} \{u(c; x) : c \text{ is feasible}\}.$$

Whereas the micro indirect utility takes prices as given and lets consumers pick any point in their budget set (even if such a point is not feasible at the economy-wide level), the macro indirect utility function takes the PPF as the primitive and lets society pick feasible points in the production possibility set. The first welfare theorem implies that the competitive equilibrium decentralizes the planning problem above with prices determined in equilibrium.[25]

We generalize our macroeconomic measure of welfare in the following way.

Definition 5 (Macro Welfare)**.** The change in welfare measured using the *macro equivalent variation* with *final preferences* is $EV^M(A_{t_0}, L_{t_0}, A_{t_1}, L_{t_1}; x_{t_1}) = \phi$ where ϕ solves

$$V(A_{t_1}, L_{t_1}; x_{t_1}) = V(A_{t_0}, e^{\phi} L_{t_0}; x_{t_1}).$$

In words, EV^M is the proportional change in initial factor endowments required to make a planner with preferences $\succeq_{x_{t_1}}$ indifferent between the PPF defined by $(A_{t_0}, e^{\phi} L_{t_0})$ and the new PPF, defined by (A_{t_1}, L_{t_1}). Intuitively, EV^M expresses utility changes in terms of factor endowments. This is convenient in general equilibrium since it can be stated without reference to (endogenous) prices. In this sense, EV^M is similar to consumption-equivalents commonly used to measure welfare in macroeconomics.[26]

Discussion of our macroeconomic welfare criterion. To see the difference between macro and micro notions of welfare, consider a situation where preferences change between t_0 and t_1 but technologies and factor endowments do not. Since the PPF is unchanged, the change in macro welfare is zero. However, if the PPF is nonlinear, the relative price of goods would change between t_0 and t_1. Typically, the price rises for those

[25] When the decentralized equilibrium is inefficient or preferences are non-aggregable, we can still rely on the micro welfare change defined in Section 2, which requires neither assumption. We discuss non-aggregable preferences in Section 6.

[26] When preferences are stable and homothetic, EV^M is the same as consumption equivalents, but we do not define welfare changes in terms of consumption equivalents because when preferences are non-homothetic or unstable, households' desired consumption bundle is not stable.

goods that became more desirable. In this case, microeconomic welfare, at final preferences, would fall even though the PPF is unchanged. Hence, microeconomic changes in welfare are a poor guide for measuring technological change for a society.

Similarly, suppose we are interested in measuring technological progress in an economy with growth and aging. Households are richer and older in t_1 than in t_0, so they prefer to spend more of their income on healthcare. Microeconomic welfare is measured by the endowment a single consumer, living in t_1, would have to be given to make her willing to go back to the economy with t_0 prices. However, since in t_0 households were poor and young, healthcare services are relatively cheap in t_0. This makes t_0 prices seem very attractive to the consumer in t_1. But this is not because technologies in t_0 are any better. If the older and wealthy consumers were transported to the initial economy, the fact that they demand more healthcare would raise healthcare prices, and this would mean that they would not be able to consume as much healthcare services.

The issue is that using the initial budget set to represent the initial PPF is deceptive, since the initial budget set reflects both the PPF and demand in t_0. Our macroeconomic notion of welfare accounts for the endogenous changes in prices by comparing the initial and final PPFs rather than the initial and final budget sets. To compare initial and final PPFs, we scale factor endowments instead of nominal income, since a proportional shift in factor quantities results in a proportional shift in the PPF and is interpretable without reference to base prices.

When relative prices do not respond to consumers' choices (i.e. the PPF is linear), then macro and micro welfare are always the same. Similarly, if preferences are homothetic and stable, then macro and micro welfare are the same. Proposition 11 in Appendix A formalizes this result. For a quantitative illustration of the difference between micro and macro welfare see the Covid-19 case study in Section 5.

3.2 Relating Welfare and Real GDP

We now characterize changes in real GDP and welfare, first globally and then locally. The results in this subsection are the general equilibrium counterparts to those in Section 2. They are "reduced-form" in the sense that they are not expressed in terms of primitives. In Section 4, we explicitly solve for these sufficient statistics in terms of observable primitives.

As in Section 2, to study this problem we index the path of technologies, factor inputs, and preferences $A(t)$, $L(t)$ and $x(t)$, by time t. The definition of ΔY is the same as before: $\Delta Y = \int_{t_0}^{t_1} \sum_{i \in N} p_i dc_i$. In the general equilibrium model and (its applications), we refer to

ΔY as real GDP.

Define the sales shares relative to GDP of each good or factor i to be

$$\lambda_i = \frac{p_i y_i}{I} \mathbf{1}(i \in N) + \frac{w_i L_i}{I} \mathbf{1}(i \in F),$$

where w_i and L_i are the price and quantity of factor i. The sales share λ_i is often referred to as a *Domar* weight. Note that referring λ_i as a "share" is an abuse of language since $\sum_{i \in N} \lambda_i > 1$ whenever there are intermediate inputs. On the other hand $\sum_{i \in F} \lambda_i = 1$.

Global Results. The following results show that changes in real GDP and welfare can both be represented as sales-weighted averages of technology changes. Real GDP uses actual sales shares over time, while welfare uses sales shares in an artificial economy in which budget shares only respond to price changes.

Lemma 2 (Real GDP). *Given a change in technologies, factor quantities, and preferences, the change in real GDP is*

$$\Delta \log Y = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i d \log A_i + \int_{t_0}^{t_1} \sum_{i \in F} \lambda_i d \log L_i, \tag{10}$$

where λ are sales shares which are functions of A, L, and x and can change as a function of time inside the integral.

In (10), the first N summands are equal to measured TFP, and the last F summands are the growth in real GDP caused by changes in factor inputs. Lemma 2 shows that changes in real GDP are equal to sales-weighted changes in technology and factor inputs. This is a slight generalization of Hulten (1978) to environments with unstable and non-homothetic final demand.

Next, we show that a Hulten-style result also exists for changes in welfare. Define $\lambda^{ev}(A, L)$ to be sales shares in a fictional economy with the PPF (A, L) but where consumers have stable homothetic preferences represented by the expenditure function $e^{ev}(p, u) = e(p, v_{t_1}, x_{t_1}) \frac{u}{v_{t_1}}$ where $v_{t_1} = v(p_{t_1}, I_{t_1}; x_{t_1})$, similar to Section 2. We call λ^{ev} the *welfare-relevant* sales share.

Proposition 4 (Macro Welfare). *Changes in macro welfare are*

$$EV^M = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i^{ev} d \log A_i + \int_{t_0}^{t_1} \sum_{i \in F} \lambda_i^{ev} d \log L_i. \tag{11}$$

According to Proposition 4, growth accounting for welfare should be based on hypothetical sales shares evaluated at current technology but for fixed final preferences and final utility. This should be contrasted with real GDP in (10), which uses sales shares evaluated at current technology and current preferences. As with real GDP, the first N summands of (11) are changes in *welfare-relevant TFP* and the last F summands are changes in welfare due to changes in factor inputs. We discuss some salient implications of this proposition below.

The first implication is that for welfare questions, the only information we need about preferences are expenditure shares and elasticities of substitution at the final allocation, since the fictional consumer in Proposition 4 has stable preferences with income elasticities all equal to one.[27]

Second, Proposition 4 implies that if the path of technologies and factor quantities is continuously differentiable, then real GDP is equal to the change in welfare if, and only if, preferences are homothetic and stable (in which case $\lambda(A, L, x) = \lambda^{ev}(A, L)$ for every A, L, and x).

Third, as stated in the following corollary, movements on the surface of a PPF driven by changes in preferences have no effect on macroeconomic welfare or real GDP.

Corollary 1 (Demand Shocks Only). *In response to changes in preferences, x, that keep the PPF unchanged, $A(t) = A(t_0)$ and $L(t) = L(t_0)$ for $t \in [t_0, t_1]$,*

$$\Delta \log Y = EV^M = 0.$$

However, micro welfare changes, EV^m, may be nonzero.

Since the production possibility set is not changing, macro welfare (defined for fixed preferences) does not change. Quantities and prices do, however, change between t_0 and t_1 in response to changes in preferences over these goods. Micro welfare changes are typically non-zero when prices change, as shown in Section 2. These results are not contradictory: the micro welfare metric assumes that consumers can choose any bundle in their budget set at given prices (hence welfare changes as prices change). On the other hand, the macro welfare metric takes into account the fact that such choices may not be feasible for society as a whole. Finally, movements along the surface of a PPF have no effect on real GDP because demand-driven changes in output raise some quantities and reduce others, and these effects exactly cancel out.

[27] Following the observation made in Remark 2, for compensating variation at initial preferences, we need to know elasticities of substitution at the initial allocation instead of the final one.

While real GDP and macroeconomic welfare changes are the same so long as we stay on the surface of a given PPF, the two are not equal when the PPF shifts. This is because real GDP is based on a path of sales shares λ that take into consideration technology shocks as well as changes in preferences and non-homotheticities in final demand. However, changes in welfare are based on a path of sales shares λ^{ev} that only take into consideration technology shocks. Therefore, if productivity rises for goods for which sales shares fall due to non-technological factors, then $EV^M < \Delta \log Y$.

To get more intuition for Proposition 4, in the following section, we use a second-order approximation to characterize changes in real GDP and welfare.

Local Results. We characterize, up to a second order approximation (as $t_1 - t_0 = \Delta t \to 0$), the response of real GDP and welfare to technology and preference shocks, now taking into account the endogenous evolution of sales shares. To make the formulas more compact and without loss of generality, when we write local approximations we abstract from shocks to factor endowments.[28]

Proposition 5 (Approximate Macro Welfare and Real GDP)**.** *Up to to a second order approximation, the change in real GDP is*

$$\Delta \log Y \approx \lambda' \Delta \log A + \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A' \frac{\partial \lambda_i}{\partial \log A} \right] \Delta \log A_i, \tag{12}$$

and the change in welfare is

$$EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{13}$$

Equation (12) resembles the one in Proposition 2, but it is based on sales shares and technology shocks rather than budget shares and price changes. The first term in (12) corresponds to the Hulten-Domar formula. The terms in square brackets reflect nonlinearities due to changes in sales shares. Intuitively, if sales shares decrease for those goods with higher productivity growth, then real GDP growth slows down due to substitution effects. This type of effect, known as Baumol's cost disease, is an important driver of the slow-down in aggregate productivity growth.

[28]Shocks to factor endowments are a special case of TFP shocks. To represent a factor endowment shock as a TFP shock, we add fictitious producers that buy the factor endowments on behalf of the other producers and shock their productivity.

The intuition underlying the gap between macro welfare and real GDP in Proposition 5 is similar to that in Proposition 2 for our micro results (the signs are flipped because a positive productivity shock reduces prices). Specifically, real GDP takes into consideration changes in sales shares *along* the equilibrium path. These changes in sales shares could be induced by technology shocks but they could also be due to changes in preferences and non-homotheticities. However, welfare measures treat changes in shares due to technology shocks differently than changes in shares due to demand shocks or non-homotheticities. In both cases, real GDP "undercorrects" for changes in shares caused by non-homotheticities or changes in preferences. In particular, welfare is lower than real GDP if technology growth is lower in goods where sales shares rise due to preference changes or non-homotheticities.

Note that there can be a gap between real GDP and welfare even if all productivity shocks are the same. Specifically, suppose that productivity growth is common across all goods $\Delta \log A_i = \Delta \log A$ and denote the gross output to GDP ratio by $\lambda^{sum} = \sum_{i \in N} \lambda_i \geq 1$. Then Proposition 5 implies that the gap between real GDP and welfare is

$$EV^M - \Delta \log Y \approx \frac{1}{2} \Delta \log A \left[\Delta \lambda^{sum} - \frac{\partial \lambda^{sum}}{\partial \log A} \Delta \log A \right], \tag{14}$$

where the term in square brackets is the change in the gross output to GDP ratio due to demand-side forces only. In particular, if demand shifts towards sectors with higher value-added as a share of sales, then $EV^M < \Delta \log Y$ when technology shocks are positive. Intuitively, this happens because welfare is less reliant on intermediates than real GDP, and hence real GDP is more sensitive to productivity shocks. Of course, in the absence of intermediate inputs, this effect disappears because λ^{sum} will always equal one.

4 Structural Macro Results and Analytic Examples

The results in Section 3 are reduced-form in the sense that they take changes in observed and welfare-relevant sales shares as given. In this section, we solve for changes in these endogenous objects in terms of observable sufficient statistics. For clarity, we restrict attention to nested-CES economies. The general case is in Appendix F, and the intuition is very similar. After providing a characterization to solve for changes in prices and shares in general equilibrium, we go over some analytical examples to provide more intuition. We also discuss how our results can be applied in dynamics economies.

Nested-CES economies. Household preferences are represented by a non-homothetic CES aggregator, which imply that budget shares vary according to (6). Recall that θ_0 is the elasticity of substitution across consumption goods and ε is the vector of income-elasticities. Production also uses nested-CES aggregators. Nested-CES economies can be written in many different equivalent ways, since they may have arbitrary patterns of nests. We adopt the following representation. We assume that each good $i \in N$ is produced with the production function

$$y_i = A_i G_i \left(\left\{ m_{ij} \right\}_{j \in N}, \left\{ l_{if} \right\}_{f \in F} \right) = A_i \left(\sum_{j \in N} \omega_{ij} m_{ij}^{\frac{\theta_i - 1}{\theta_i}} + \sum_{f \in F} \omega_{if} l_{if}^{\frac{\theta_i - 1}{\theta_i}} \right)^{\frac{\theta_i}{\theta_i - 1}},$$

where the parameters ω_{ij} and ω_{if} are constants. Any nested-CES production network can be represented in this way if we treat each CES aggregator as a separate producer (see Baqaee and Farhi, 2019b).

Input-output matrix. We stack the expenditure shares of the representative household, all producers, and all factors into the $(1 + N + F) \times (1 + N + F)$ input-output matrix Ω. The first row corresponds to the household. To highlight the special role played by the representative agent, we index the household by 0, which means that the first row of Ω is equal to the household's budget shares introduced above ($\Omega_0 = b'$, with $b_i = 0$ for $i \notin N$).[29] The next N rows correspond to the expenditure shares of each producer on every other producer and factor. The last F rows correspond to the expenditure shares of the primary factors (which are all zeros, since primary factors do not require any inputs).

Leontief inverse matrix. The Leontief inverse matrix is the $(1 + N + F) \times (1 + N + F)$ matrix defined as

$$\Psi \equiv (I - \Omega)^{-1} = I + \Omega + \Omega^2 + \ldots,$$

where I is the identity matrix. The Leontief inverse matrix $\Psi \geq I$ records the *direct and indirect* exposures through the supply chains in the production network. We partition Ψ

[29] We expand the vector of demand-shifters $\Delta \log x$ and income elasticities ε to be $(1 + N + F) \times 1$, where $\Delta \log x_i = \varepsilon_i = 0$ if $i \notin N$.

in the following way:

$$\Psi = \left[\begin{array}{c|ccc|ccc} 1 & \lambda_1 & \cdots & \lambda_N & \Lambda_1 & \cdots & \Lambda_F \\ \hline 0 & \Psi_{11} & \cdots & \Psi_{1N} & \Psi_{1N+1} & \cdots & \Psi_{1N+F} \\ 0 & & \ddots & & & & \\ 0 & \Psi_{N1} & & \Psi_{NN} & \Psi_{NN+1} & \cdots & \Psi_{NN+F} \\ \hline 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right].$$

The first row and column correspond to final demand (good 0). The first row is equal to the vector of sales shares for goods and factors λ'. To highlight the special role played by factors, we interchangeably denote their sales share by the $F \times 1$ vector Λ. The next N rows and columns correspond to goods, and the last F rows and columns correspond to the factors. Define the $(1 + N + F) \times F$ matrix Ψ^F as the submatrix consisting of the right F columns of Ψ, representing the network-adjusted factor intensities of each good. The sum of network-adjusted factor intensities for every good i is equal to one, $\sum_{f \in F} \Psi_{if} = 1$ because the factor content of every good is equal to one. In our results below we will use the identities $\lambda' = b'\Psi$ and $\Lambda' = b'\Psi^F$.

4.1 General characterization for nested-CES economies

Changes in real GDP and welfare can be computed by weighing technology shocks by observed and welfare-relevant sales shares (λ and λ^{ev}, respectively) and cumulating the results (see Lemma 2 and Proposition 5). The following proposition pins down λ and λ^{ev} along the transition path. This proposition can then be used in combination with Lemma 2 and Proposition 5 to calculate (globally) changes in real GDP and welfare.

To simplify notation, we again assume away shocks to factor endowments and consider some path of taste and technology shocks $\{x(t), A(t)\}_{t \in [t_0, t_1]}$.

Proposition 6 (Characterization for nested-CES economies)**.** *Changes in observed variables are pinned down by the following coupled system of differential equations:*

$$d \log p_i = -\sum_j \Psi_{ij} d \log A_j + \sum_{f \in F} \Psi_{if} d \log \lambda_f,$$

$$d\Psi_{il} = \sum_j \Psi_{ij}(\theta_j - 1) Cov_{\Omega^{(j)}}\left(-d \log p, \Psi_{(l)}\right) + \mathbf{1}_{\{i=0\}} Cov_{\Omega^{(0)}}\left(d \log x + \varepsilon d \log Y, \Psi_{(l)}\right),$$

where changes in observed sales shares are given by $d\lambda_i = d\Psi_{0i}$ *for every* i *and changes in real GDP are given by* $d\log Y = \sum_i \lambda_i d\log A_i$.

On the other hand, changes in welfare-relevant variables are pinned down by the following system of differential equations

$$d\log p_i^{ev} = -\sum_j \Psi_{ij}^{ev} d\log A_j + \sum_{f\in F} \Psi_{if}^{ev} d\log \lambda_f^{ev},$$

$$d\Psi_{il}^{ev} = \sum_j \Psi_{ij}^{ev}(\theta_j - 1)Cov_{\Omega^{ev,(j)}}\left(-d\log p^{ev}, \Psi_{(l)}^{ev}\right), \tag{15}$$

where changes in welfare-relevant sales shares are given by $d\lambda_i^{ev} = d\Psi_{0i}^{ev}$ *for every* i *and changes in welfare are given by* (4).

For all of these expressions, the summations are evaluated over all goods and factors, so that i *and* $j \in \{0\} + N + F$, $Cov_{\Omega^{(j)}}(\cdot)$ *is the covariance using the* j*th row of* Ω *as the probability weights, and* $\Psi_{(i)}$ *is the* i*th column of the Leontief inverse.*

These differential equations can be solved by repeated iteration. This allows us to compute the path of λ^{ev} and λ. Once in possession of these paths, the change in real GDP and welfare are straightfoward to calculate by cumulating the λ and λ^{ev}-weighted sum of technology shocks. For this iterative procedure, the boundary conditions are that at the end point, prices satisfy $p(t_1) = p^{ev}(t_1) = \mathbf{1}$ and the Leontief inverse matches $\Psi^{ev}(t_1) = \Psi(t_1)$.

For ex-post welfare questions, where the Leontief inverse Ψ is observed at t_1, we can calculate Ψ^{ev} between t_0 and t_1 by starting at t_1 and going backwards to t_0. This process does not require knowledge of either the income elasticities ε nor the taste shocks $\Delta\log x$ since they do not appear in either the equation for $d\log p^{ev}$ nor the equation for $d\Psi^{ev}$.

Each term in Proposition 6 has a clear interpretation. We start by discussing the equation determining prices $d\log p$. This equation captures the fact that the price of each good $d\log p_i$ is determined by its (direct and indirect) exposure to the price of inputs j and factors f. The equation for $d\log\Psi_{il}$, in turn, shows that changes in the Leontief inverse are determined by substitutions by j, if j is an intermediary between i and l, as well as income and substitution effects if i is the household ($i = 0$). Finally, the welfare-relevant versions of these equations, $d\log p^{ev}$ and $d\Psi^{ev}$ are identical except that they do not account for expenditure-switching due to income effects or taste shocks.

Remark 3 (Micro Welfare)**.** Proposition 6 can also be used to compute changes in microeconomic welfare EV^m. To this, we compute the path of prices using Proposition 6 and then plug these price changes into equation (7) to get the welfare-relevant budget shares

b^{ev}. These can then be used in conjunction with Proposition 2 to get EV^m. Note that, unlike macroeconomic welfare EV^M, this calculation requires knowledge of both income elasticities and taste shocks, unless there is a single factor of production as discussed bellow.

Remark 4 (Compensating Variation)**.** Proposition 6 can also be used to compute changes in compensating variation at initial preferences (instead of equivalent variation at final preferences). To do this, we would still solve the differential equations for $d\log p^{ev}$ and $d\Psi^{ev}$, however, we would use different boundary conditions. The boundary conditions for compensating variation at initial preferences would match the data at t_0 instead of t_1, setting $p^{ev}(t_0) = \mathbf{1}$ and $\Psi^{ev} = \Psi(t_0)$. We would then solve the differential equations forward from t_0 to t_1 and use the resulting welfare-relevant shares to weight the technology shocks. This procedure effectively makes use of the fact that compensating variation at initial preferences going from t_0 to t_1 is equal to equivalent variation at final preferences if we go from t_1 to t_0. As with EV^M, conditional on the boundary conditions, we do not need to know the income elasticities or the taste shocks.

To build more intuition, we focus on the economies with only a single factor of production. In this case, the differential equations for $d\log p$ and $d\log p^{ev}$ are decoupled from the ones for $d\Psi$ and $d\Psi^{ev}$. This follows from the fact that the economy's single primary factor must have a sales share of unity. In other words, the following set of equations always hold: $\lambda_f = \lambda_f^{ev} = \Psi_{0f} = \Psi_{0f}^{ev} = 1$. This allows for a simple closed-form characterization of both welfare and real GDP up to a second-order approximation.

Proposition 7 (Approximate Macro Welfare vs GDP: Single Factor)**.** *Consider some perturbation in technology, $\Delta\log A$, and final demand, $\Delta\log x$. When the economy has one factor of production, the change in real GDP is*

$$\Delta\log Y \approx \sum_{i\in N}\lambda_i\Delta\log A_i + \frac{1}{2}\sum_j \lambda_j(\theta_j-1)Var_{\Omega^{(j)}}\left(\sum_{i\in N}\Psi_{(i)}\Delta\log A_i\right) + \frac{1}{2}Cov_{\Omega^{(0)}}\left(\Delta\log x + (\sum_{i\in N}\lambda_i\Delta\log A_i)\varepsilon, \sum_{i\in N}\Psi_{(i)}\Delta\log A_i\right), \tag{16}$$

The difference between welfare and GDP is

$$EV^M - \Delta\log Y \approx \frac{1}{2}Cov_{\Omega^{(0)}}\left(\Delta\log x + (\sum_{i\in N}\lambda_i\Delta\log A_i)\varepsilon, \sum_{i\in N}\Psi_{(i)}\Delta\log A_i\right). \tag{17}$$

Proposition 7 is a general equilibrium counterpart to Proposition 3. We discuss (16) and (17) in turn, starting with (16). The first term in Equation (16) is the Hulten-Domar term. The other terms are second-order terms resulting from the fact that sales shares change in response to shocks. The first one of these terms captures nonlinearities due to the fact that sales shares can respond to changes in relative prices caused by technology shocks (these effects were emphasized by Baqaee and Farhi, 2019b). The terms on the second line of (16), which are the ones we focus on in this paper, capture changes in sales shares due to changes in preferences or non-homotheticities.

Equation (17) shows that while real GDP correctly accounts for substitution due to supply shocks, it needs to be corrected for expenditure-switching due to changes in final demand caused by taste shocks or non-homotheticities. Whereas in partial equilibrium, the gap between welfare and real GDP is proportional to the covariance of supply and demand shocks (see Proposition 3), equation (17) shows that in general equilibrium, the relevant statistic is the covariance of demand shocks with a network-adjusted notion of supply shocks not supply shocks per se. Furthermore, Proposition 7 shows that the elasticities of substitution are irrelevant for the gap between welfare and real GDP in one-factor models. This is because, in response to demand-driven forces, relative prices do not change as the equilibrium moves along a linear PPF. Therefore, demand shocks do not trigger expenditure switching due to the endogenous response of relative prices. When we relax the linearity of the PPF, we see that the elasticities of substitution in production do, in general, affect the gap between welfare and GDP.

4.2 Analytical Examples

We now work through some simple examples to illustrate the forces that drive a gap between λ and λ^{ev} and, by extension, real GDP and welfare.

Example 1 (Correlated Supply and Demand Shocks)**.** We start with the simplest possible example, a one sector model without any intermediates. In this case, sales shares are just budget shares $\lambda_i = b_i = \Omega_{0i}$ and $\Psi_{(i)}$ is the ith column of the $1 + N + F$ identity matrix $I_{(i)}$. Therefore, Proposition 7 implies

$$EV^M - \Delta \log Y \approx \frac{1}{2} \left(Cov_b \left(\Delta \log x, \Delta \log A \right) + Cov_b \left(\varepsilon, \Delta \log A \right) \mathbb{E}_b[\Delta \log A] \right).$$

Hence, welfare changes are greater than the change in real GDP if productivity and demand shocks are positively correlated. This could happen either because preferences exogenously change to favor high productivity goods, $Cov_b \left(\Delta \log x, \Delta \log A \right) > 0$, or preferences endogenously change to favor high productivity growth goods due to non-homotheticities, $Cov_b(\varepsilon, \Delta \log A) \Delta \log Y > 0$. When shifts in demand are orthogonal to shifts in supply, to a second-order approximation, real GDP measures welfare correctly.

Example 2 (Input-Output Connections)**.** For models with linear PPFs, input-output connections affect the gap between real GDP and welfare in two ways: (1) the impact of technology shocks is bigger when there are input-output linkages because $\Psi_{(i)} \geq I_{(i)}$ and $\lambda_i \geq b_i$; (2) the production network "mixes" the shocks, and this may reduce the correlation of supply and demand shocks by making the technology shocks more uniform. However, since it is the covariance (not the correlation) of the shocks that matters, this means the effects are, at least theoretically ambiguous.

To see these two forces, consider the three economies depicted in Figure 1. Each of these economies has a roundabout structure. Panel 1a depicts a situation where each producer uses only its own output as an input, Panel 1b a situation where all producers use the same basket of goods (denoted by M) as an intermediate input, and Panel 1c a situation where each producer uses the output of the other producer as an input. We compute the correction to GDP necessary to arrive at welfare for each of these cases using Proposition 7. For clarity, we focus on demand shocks caused by instability rather than non-homotheticity, though it should be clear that this does not affect any of the intuitions.

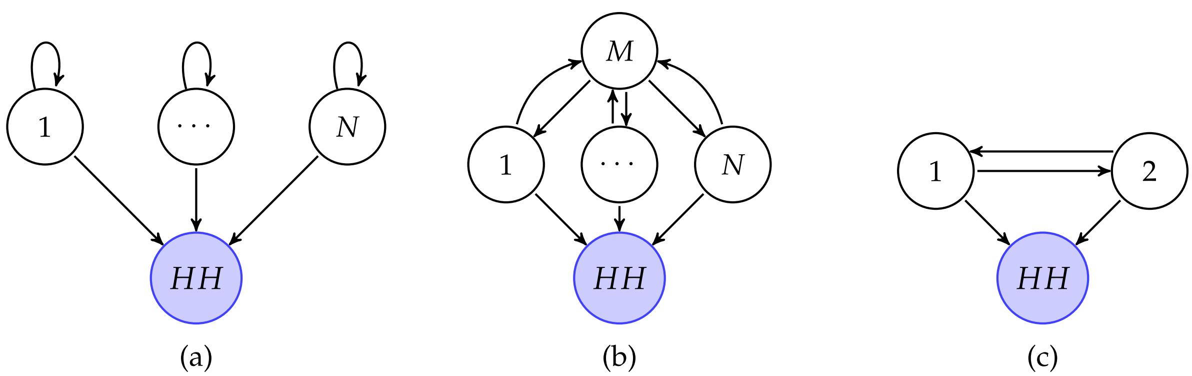

Figure 1: Three different kinds of round-about economy. The arrows represent the flow of goods. The only factor is labor which is not depicted in the diagram.

For Panel 1a, we get

$$EV^M - \Delta \log Y \approx \frac{1}{2} Cov_b(\Delta \log x_i, \Omega_{iL}^{-1} \Delta \log A_i),$$

where the covariance is computed across goods $i \in N$ and Ω_{iL} is the labor share for i. Hence, as intermediate inputs become more important, the necessary adjustment becomes larger. This is because, for a given vector of preference shocks, the movement in sales shares is now larger due to the roundabout nature of production.[30]

On the other hand, for Panel 1b, we get[31]

$$EV^M - \Delta \log Y \approx \frac{1}{2}\left(Cov_b(\Delta \log x_i, \Delta \log A_i) - Cov_b(\Delta \log x_i, \Omega_{iL})\frac{\sum_{i\in N} \Delta \log A_i}{\sum_{i\in N} \Omega_{iL}}\right).$$

Hence, in this case, if the labor share Ω_{iL} is the same for all $i \in N$, then the intermediate input share is irrelevant. Intuitively, in this case, all producers buy the same share of materials, so a shock to the composition of household demand does not alter the sales of any producer through the supply chain, and hence only the first-round non-network component of the shocks matters.[32]

Finally, consider Panel 1c. For clarity, focus on the case where only producer 1 gets a productivity shock ($\Delta \log A_2 = 0$). In this case, the difference between real GDP and welfare is

$$EV^M - \Delta \log Y \approx \frac{1}{2}\frac{1}{1-\Omega_{12}\Omega_{21}} Cov_b\left(\Delta \log x, \begin{bmatrix} 1 \\ \Omega_{21} \end{bmatrix}\right) \Delta \log A_1.$$

As the intermediate input share Ω_{21} approaches one, the adjustment goes to zero (since the covariance term goes to zero). Intuitively, as Ω_{21} goes to one, the increase in demand for the first producer from a change in preferences is exactly offset by a reduction in demand from the second producer who buys inputs from the first producer. In this limiting case, changes in consumer preferences have no effect on the overall sales share of the first producer.

These three examples serve to illustrate that the effect of input-output networks on the adjustment are theoretically ambiguous but potent.

We now work through some simple examples with multiple factors of production to illustrate how nonlinear PPFs affect the previous results.

Example 3 (Decreasing Returns to Scale)**.** Consider the one-sector model without intermediate inputs in Example 1 but now suppose that production functions are non-

[30] As discussed after Equation (14), if all productivity shocks are the same, there may still be an adjustment due to heterogeneity in labor shares. In particular, if demand shocks are higher for sectors with higher labor shares, then $EV^M < \Delta \log Y$ when technology shocks are positive.

[31] For this example, we assume that there are no productivity shocks to the intermediate bundle $\Delta \log A_M = 0$ and we assume that $\Omega_{iM} = 1/N$ for each $i \in N$.

[32] As in Footnote 30, if the labor share is heterogeneous across producers, there is an additional adjustment which depends on the covariance between demand shocks and labor shares. If the demand shocks reallocate expenditures towards sectors with high labor shares, then welfare becomes less sensitive to productivity shocks than real GDP.

constant-returns-to-scale. Specifically, the production for good i is

$$y_i = A_i L_i^{\gamma},$$

where L_i is labor and γ need not equal 1. Furthermore, suppose that preferences are homothetic ($\varepsilon_i = 1$ for every i), but potentially unstable ($\Delta \log x \neq 0$). To apply our theorems to this economy, where producers have non-constant-returns production functions, we introduce a set of producer-specific factors in inelastic supply, and suppose that each producer has a Cobb-Douglas production function that combines a common factor with elasticity γ and a producer-specific factor with elasticity $1 - \gamma$. This means that our economy has $1 + N$ factors.

We apply Proposition 5 to compute the difference between welfare and real GDP. To do this, we first use Proposition 6 to compute changes in sales shares due to demand shocks:

$$\frac{\partial \lambda_i}{\partial \log x} \cdot d \log x = Cov_{\Omega^{(0)}} \left(d \log x, \Psi_{(i)} \right) + (\theta_0 - 1) Cov_{\Omega^{(0)}} \left(-\frac{d \log p}{d \log x} d \log x, \Psi_{(i)} \right).$$

The factor content of every good $i \in N$ is given by $\Psi^F_{if} = \gamma$ when f is the common factor and $\Psi^F_{if} = 1 - \gamma$ when f is the producer-specific factor. Since the factor shares of each producer are constant, the log change in the producer-specific factor share is the same as the log change in the sales share of that producer. Therefore, we can replace $d \log p_i / d \log x$ in the covariance with$(\gamma - 1) \left(\partial \log \lambda_i / \partial \log x \right) d \log x$ (the other components of price changes are common to all producers and drop out of the covariance). Plugging this into the expression above and solving yields a closed-form expression for $\partial \lambda_i / \partial \log x$. This allows us to apply Proposition 5 to get the difference between welfare and real GDP up to a second order approximation:

$$EV^M - \Delta \log Y \approx \frac{1}{2} \frac{Cov_{\Omega^{(0)}} (\Delta \log x, \Delta \log A)}{1 + (\theta_0 - 1)(1 - \gamma)}.$$

Note that the denominator disappears when we have constant-returns to scale ($\gamma = 1$) *or* the elasticity of substitution across goods is one ($\theta_0 = 1$). Outside of these cases, complementarities ($\theta_0 < 1$) amplify the impact of preference shocks under decreasing returns to scale ($\gamma < 1$). Intuitively, if preferences shift in favor of some good, the price of that good rises due to decreasing returns to scale. The fact that the price of the good increases raises the sales share of that good due to complementarities, which creates a feedback loop, raising prices of the good further, and causing additional substitution. In other words, in the decreasing returns to scale model with complementarities, sales shares respond more strongly to demand shocks. Given that sales shares respond more strongly to demand shocks, the necessary adjustment to correct real GDP is larger.

Example 4 (Macro vs. Micro Welfare Change)**.** Finally, we demonstrate the difference between macro and micro welfare changes using the previous example. The economy in the previous example has multiple factors and unstable preferences. Therefore, macro and micro notions of welfare are different since the PPF is no longer linear.

To illustrate this difference, suppose that only preference shocks are active (there are no supply shocks $\Delta \log A = 0$ and $\Delta \log L = 0$). By Corollary 1, real GDP changes are zero. Since the PPF is being held constant, macro-welfare changes are also zero. Micro-welfare changes, on the other hand, are not equal to zero. Specifically, by Proposition 2, micro welfare improves $EV^m > 0$ if preference shocks negatively covary with price changes. By equation (44), changes in prices are

$$d \log p_i = \sum_{f \in F} \Psi^F_{if} \frac{\partial \log \Lambda_f}{\partial \log x} d \log x.$$

Using the derivations above, for each $i \in N$, we obtain

$$d \log p_i = \frac{(1-\gamma)}{(1 + (\theta_0 - 1)(1-\gamma))} \frac{1}{\lambda_i} Cov_{\Omega^{(0)}} \left(d \log x, I_{(i)} \right),$$

where $I_{(i)}$ is the ith column of the identity matrix. If there are decreasing returns, $\gamma < 1$, then a positive demand shock for i raises the price of i. The change in the price is amplified if goods are complements and mitigated if goods are substitutes. We can now apply Proposition 3 to obtain the change in micro welfare, up to a second order,

$$EV^m \approx -\frac{1}{2} \frac{(1-\gamma)}{(1 + (\theta_0 - 1)(1-\gamma))} Var_{\Omega^{(0)}} (\Delta \log x) \neq 0 = EV^M.$$

With decreasing returns to scale ($\gamma < 1$), micro welfare decreases since the demand shock increases the prices of goods the consumer now values more. From a micro perspective, where the agent takes the budget sets as given, the agent is worse off.

On the other hand, when the economy has increasing returns to scale ($\gamma > 1$), micro welfare increases in response to demand shocks. Intuitively, in this case, increased demand for a good lowers the price of that good, which makes the consumer better off.[33] Of

[33]If the economy has increasing returns to scale, then the decentralized equilibrium is potentially inefficient. However, the propositions regarding micro welfare changes, which take changes in prices as given, do not require that the decentralized equilibrium be efficient.

course, from a societal perspective, welfare has not changed, since the production possibility set of the economy has not changed.

4.3 Dynamic Economies

As mentioned earlier, at an abstract level, all of our results can be applied to dynamic economies by using the Arrow-Debreu formalism. In particular, we can index goods by period of time and state of nature and apply our results to these economies (see e.g. Basu et al., 2012). In a dynamic economy the utility function is intertemporal and capital accumulation is an intertemporal intermediate good. Proposition 4 implies that, in such a model, macro welfare can be computed using the final (intertemporal) indifference curve of the representative agent.

In this subsection, we instead show how Proposition 4 can be used to make steady-state to steady-state welfare comparisons in models with unstable and non-homothetic preferences without requiring us to solve a dynamic model. For simplicity, we restrict our discussion to non-homothetic CES preferences, though this logic can be extended.

To this end, consider a perfectly competitive dynamic economy indexed by the initial period t with a representative agent whose intertemporal preferences are given by

$$\mathcal{U}_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s), \qquad \sum_i \omega_{i0} x_{it} \left(\frac{c_{is}}{C_s^{\xi_i}} \right)^{\frac{\theta_0 - 1}{\theta_0}} = 1,$$

where C_s is a non-homothetic (and potentially unstable) CES aggregator.The economy has the same set of goods every period, and every good i in period s is produced according to constant returns production technology

$$y_{is} = A_{is} G_i \left(\left\{ m_{ijs} \right\}_{j \in N}, H(l_{is}, k_{is}) \right),$$

where A_{is} is a productivity shifter, l_{is} are capital and labor inputs, and H is constant returns to scale.

Labor L_s in each period is inelastically supplied, and capital is accumulated according to a capital accumulation technology

$$K_{s+1} = (1 - \delta) \left(K_s + I_s \right),$$

where I_s is aggregate investment. Investment goods are produced according to a constant returns technology

$$I_s = A_{Is} I \left(\left\{ m_{Ijs} \right\}_{j \in N}, H(l_{Is}, k_{Is}) \right).$$

The intertemporal PPF of economy t is defined by an initial capital stock inherited from the past, a path of future labor endowments, and a path of vectors of productivities: $(K_t, \{L_s\}_{s=t}^{\infty}, \{A_s\}_{s=t}^{\infty})$. This economy has infinitely many factors: the path of labor endowments as well as the initial capital stock $(K_t, \{L_s\}_{s=t}^{\infty})$. The welfare change between t_0 and t_1 is the proportional change in factor endowments of the t_0 economy required to make the household indifferent between that and the t_1 economy. We say that economy t is in *steady-state* if the vector of productivities A_s, labor endowments L_s, per-period utility $u(C_s)$, and capital stocks K_s are constant over time.

The following proposition shows that computing the welfare change between t_0 and t_1 is straightforward if the economy is in steady-state in both t_0 and t_1.

Proposition 8 (Dynamic Welfare Change)**.** *Consider two dynamic economies, denoted t_0 and t_1, that are in steady-state. The change in macro welfare is given by*

$$EV^M = \log\left(\frac{\sum_i p_{it_1} c_{it_1}}{\sum_i p_{it_0} c_{it_0}}\right) + \log\left(\sum_i b_{it_1} \left(\frac{p_{it_0}}{p_{it_1}}\right)^{1-\theta_0}\right)^{\frac{1}{1-\theta_0}}.$$

In words, the change in nominal consumption deflated by the ideal price index associated with the t_1 indifference curve is equal to the macroeconomic change in welfare. If preferences are stable and homothetic in both t_0 and t_1, then the change in welfare is equivalent to the change in real consumption. In the next section, we use Proposition 8 to compute welfare changes for the US under alternative assumptions about substitution and income elasticities.

5 Applications

In this section, we consider three applications of our results. The first application is to the problem of long-run growth and the difference between welfare-relevant and measured aggregate productivity growth as well as the difference between welfare and measured real consumption, in the presence of income effects and demand instability. The second application shows that correlated firm-level supply and demand shocks drive a wedge between measured real GDP and welfare even in the short-run. Our last application considers how demand instability can make measured real GDP an unreliable metric for changes in production, and we illustrate this point for the Covid-19 crisis.

5.1 Long-Run Growth and Structural Transformation

Baumol (1967) showed that, as economies grow, sectors with lower relative productivity growth rates expand (in terms of sales and value-added) relative to sectors with faster productivity growth. This means that over time, aggregate productivity growth is increasingly determined by those sectors whose productivity growth is slowest. This phenomenon is oftentimes called Baumol's cost disease.

Following Nordhaus et al. (2008), aggregate productivity growth can be decomposed into two terms:

$$\Delta \log TFP = \sum_{t=t_0}^{t_1} \sum_{i \in N} \lambda_{i,t_0} \Delta \log A_{i,t} + \underbrace{\sum_{t=t_0}^{t_1} \sum_{i \in N} \left(\lambda_{i,t} - \lambda_{i,t_0} \right) \Delta \log A_{i,t}}_{\text{Baumol Adjustment}},$$

where $\lambda_{i,t}$ is the sales shares of industry i in period t and $\Delta \log A_{i,t}$ is the growth in gross-output productivity over that time period.[34] The first term captures changes in aggregate TFP if industry-structure had remained fixed, and the second term is the adjustment attributed to the fact that sales shares change over time. The second-term captures the importance of Baumol's cost disease.

Proposition 4 implies that, for the purposes of welfare, changes in sales shares due to income effects or demand instability must be treated differently to changes in sales shares due to substitution effects. In particular, the welfare-relevant measure of the change in TFP is

$$\Delta \log TFP^{w} = \sum_{t=t_0}^{t_1} \sum_{i \in N} \lambda_{i,t_0} \Delta \log A_{i,t} + \underbrace{\sum_{t=t_0}^{t_1} \sum_{i \in N} \left(\lambda_{i,t} - \lambda_{i,t_0} \right) \Delta \log A_{i,t}}_{\text{Baumol Adjustment}} + \underbrace{\sum_{t=t_0}^{t_1} \sum_{i \in N} \left(\lambda_{i,t}^{ev} - \lambda_{i,t} \right) \Delta \log A_{i,t}}_{\text{Welfare Adjustment}},$$

where λ^{ev} is the hypothetical sales-shares of each industry holding fixed final preferences and income-level — that is, sales shares after they have been purged from changes due to factors other than changes in relative prices.[35]

[34] Technically, this is an approximation, since we define aggregate TFP in continuous time but the data is measured in discrete time (at annual frequency). However, this approximation error, resulting from the fact that the Riemann sum is not exactly equal to the integral is likely to be negligible in practice. At our level of disaggregation, long run TFP growth is very similar if we weight sectors using sales shares at time t or time t and $t+1$ averages.

[35] We abstract from investment and apply our formulas statically. This means that we assume a reduced form representation whereby preference relations are defined over all final goods in a given period (including government spending, net exports, and investment) and calculate welfare changes between two time periods taking technologies and factor quantities as given. When calculating welfare using consumption

Two polar extremes. Computing these terms requires an explicit structural model of the economy.[36] However, there are two polar cases in which the TFP adjustment term can be calculated without specifying the detailed model. On the one hand, demand is stable and homothetic, and changes in sales shares are due only to relative price changes (substitution effects). On the other hand, there are no substitution effects (as in a Cobb-Douglas economy) and changes in sales shares are only due to income effects or demand instability. If structural transformation is driven by a combination of substitution effects and non-homotheticities or demand instability, then the change in welfare TFP will be somewhere in between these two cases. The following corollary of Proposition 4 summarizes the change in welfare-TFP in these two polar cases.

Corollary 2. *If changes in sales shares are due only due only to substitution effects, then*

$$\Delta \log TFP^{w} = \Delta \log TFP = \sum_{t=t_0}^{t_1} \sum_{i \in N} \lambda_{i,t} \Delta \log A_{i,t}.$$

If changes in sales shares are due only to non-homotheticity or instability of demand, then

$$\Delta \log TFP^{w} = \Delta \log TFP + \sum_{t=t_0}^{t_1} \sum_{i \in N} \left(\lambda_{i,t_1} - \lambda_{i,t}\right) \Delta \log A_{i,t} = \sum_{t=t_0}^{t_1} \sum_{i \in N} \lambda_{i,t_1} \Delta \log A_{i,t}.$$

In the first case, since preferences are homothetic and stable, welfare-TFP is equal to TFP in the data. In the second case, since there are no substitution effects in production or demand, sales shares do not respond to productivity changes. In order to hold utility and preferences fixed at their final value, we must compute welfare-TFP using terminal sales shares.

To quantify Corollary 2, we use US-KLEMS data on sales shares and TFP growth for 61 private-sector industries between 1947 and 2014. We calculate changes in industry-level gross-output TFP following the methodology of Jorgenson et al. (2005) and Carvalho and Gabaix (2013).[37]

Figure 2 plots EV^{M} comparing 2014 to previous years under alternative assumptions

data below, we apply the steady-state results of a dynamic economy implied by Proposition 8.

[36]For this exercise, we abstract from investment decisions and apply our formulas statically. This means that we assume a reduced-form representation whereby preference relations are defined over all final goods in a given period (including government spending, net exports, and investment) and calculate welfare changes between two time periods taking preferences, technologies, and factor quantities as given. When calculating welfare using consumption data below, we apply the steady-state results of a dynamic economy implied by Proposition 8.

[37]For each industry, the change in TFP is itself a chain-weighted index calculated as output growth minus share-weighted input growth. Inputs are industry-level measures of materials, labor, and capital services.

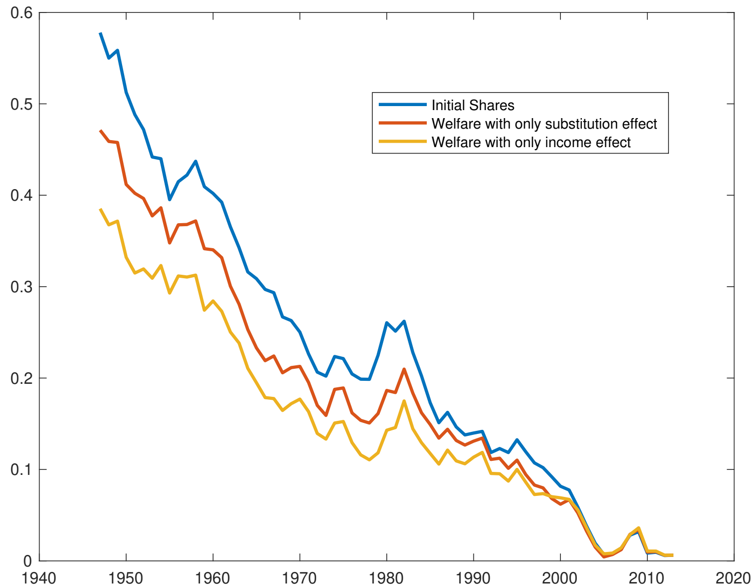

Figure 2: Growth in welfare-relevant TFP (in logs) from 1947 to 2014 using US-KLEMS. The blue line uses initial shares (in each year t between 1947 and 2014) to calculate TFP changes. The red and yellow line measure the increase in welfare-relevant TFP between t and 2014 under alternative assumptions about income and substitution elasticities. The red line assumes that sales shares change only due to substitution effects (welfare-relevant TFP is equal to measured chained-aggregate TFP). The yellow line assumes that sales shares change only due to income effects (or demand instability).

about substitution and income elasticities. For comparisons that are relatively close to 2014, the change in welfare is not very sensitive to our assumptions about elasticities. This is because at high frequency, the shocks are small and the sales shares are reasonably stable. However, the assumptions about substitution and income elasticities do start to play a role as we roll the comparison back farther in time. Comparing 1947 to 2014, the constant-initial-sales-share term grows by around 58 log points (or 78%), whereas the chain-linked change in aggregate TFP grew by around 47 log points (or 60%). Hence, Baumol's cost-disease caused aggregate TFP to fall by -10 log points, reducing aggregate productivity growth by around 23 percent (from 78% to 60%).

If we assume that structural transformation is due solely to non-homotheticities or demand instability, then by Corollary 2 the growth in welfare-relevant TFP from 1947-2014 has been 37 log points (or 46%) instead of the measured 47 log points (or 60%) — that is, to say, a 23 percent additional reduction in the growth rate.[38]

[38]The gap between constant-initial-sales shares TFP and (chained-linked) aggregate TFP growth and the gap between aggregate TFP and welfare-relevant TFP growth are driven by two forces. First, reallocation

Intuitively, welfare-based productivity increases less than TFP because, relative to 1947, preferences in 2014 favor low productivity growth sectors such as services (due to either income effects or demand instability). This means that, at 1947 prices, households require less income growth to be indifferent between their budget constraint in 1947 and the one in 2014. This is because sectors with high income elasticities or that consumers prefer in 2014, like services, were cheaper compared to manufacturing in 1947 than in 2014.[39]

To sum up, structural transformation between caused by income effects or demand instability reduced welfare, EV^M, by roughly twice as much as structural transformation caused by substitution effects. To understand why the necessary adjustment is roughly twice as big using the second-order approximation in Proposition 4, see Appendix E. In this appendix, we also provide some quantitative illustrations away from the two polar extremes we discussed above. In Appendix E, we compute welfare changes for different values of elasticities of substitution in consumption and production using Proposition 6. Recall that this does not require taking a stance on the income elasticities. We show that welfare-relevant TFP is closer to measured TFP if the elasticity of substitution across disaggregated industries (in consumption or production) is lower than one.

The Baumol effect in real consumption. So far in this application we have examined aggregate productivity. We now show that similar conclusions apply if we measure welfare changes using data on consumer prices and budget shares across goods in the US between 1947 and 2019.

Specifically, we measure the change in microeconomic welfare using Lemma 1 under alternative assumptions about income and substitution elasticities. We apply this formula statically and calculate welfare changes between two time periods taking as given changes in prices and nominal expenditures. Under the assumptions of Proposition 8, these static numbers also represent the change in macroeconomic welfare in a dynamic

of sales towards sectors with lower relative productivity growth rates (the standard Baumol's cost-disease mechanism). Second, reallocation in sales towards sectors with lower intermediate input use (see equation 14). In our quantitative results, the second force accounts for roughly 18% of the gap between constant-initial-sales shares TFP and aggregate TFP growth, and 35% of the gap between aggregate TFP growth and welfare-relevant TFP growth.

[39]This intuition is flipped for compensating variation. As households become poorer in 2014 to be made equally well-off as under their budget constraint in 1947, they favor goods which are relatively cheap in 2014 such as manufacturing, so their income must be reduced by more. More generally, if structural transformation is purely due to income effects or preference instability, then welfare-based productivity growth using CV at initial preferences is given by initial sale-share weighted productivity growth, $\sum_{t=t_0}^{t_1} \sum_{i \in N} \lambda_{i,t_0} \Delta \log A_{i,t}$ (which corresponds to the Initial Shares line in Figure 2), so in this case the Baumol adjustment is not welfare-relevant.

model between two steady-states.

If changes in budget shares are driven by substitution effects only, then welfare changes are equal to growth in real consumption per capita. If changes in budget shares are driven by income effects and demand instability only, then welfare changes between any year and 2019 are given by changes in nominal expenditures deflated by a price index using 2019 budget shares.

Figure 3 shows that for comparisons that are close to 2019, the change in welfare is not very sensitive to the assumptions on demand instability and income effects versus substitution effects because, at high frequency, the shocks are small and the sales shares at our level of aggregation (66 goods and services) are stable. On the other hand, for longer time periods, welfare growth is smaller if changes in budget shares took place due to income effects (or demand instability) rather than substitution effects. That is, comparing 1947 and 2019, the change in welfare per capita is 145 log points if preferences are homothetic and stable, but it is only 126 log points if changes in budget shares were entirely due to demand shocks and income effects. As before, structural transformation in consumption caused by demand shocks and income effects is roughly twice as important for welfare as structural transformation caused by substitution effects.

5.2 Aggregation Bias with Firm-Level Shocks

In the previous application, we considered a long-run industry-level application. Since industry-level sales shares are relatively stable over short-horizons, given industry-level data, biases in real GDP and consumption are likely to be modest at high frequency. However, this does not mean that these biases are necessarily absent from short-run data.

Whereas industry sales shares are stable at high frequency, firm or product-level sales shares are highly volatile even over the very short-run. If firms' or products' supply and demand shocks are correlated, then measured industry-level output is biased relative to what is relevant for welfare. In what follows we show that the biases in industry-level data are not diversified away as we aggregate, even if all firms are infinitesimal in their industry and all industries are infinitesimal in the aggregate economy.

We first formalize this logic and then provide an empirical example using product-level data from the Nielsen Consumer Panel data. We introduce a specification of our model with an explicit industrial structure. For simplicity, we abstract from non-homotheticities.

Definition 6 (Industrial Structure)**.** An economy has an *industry structure* if the following conditions hold:

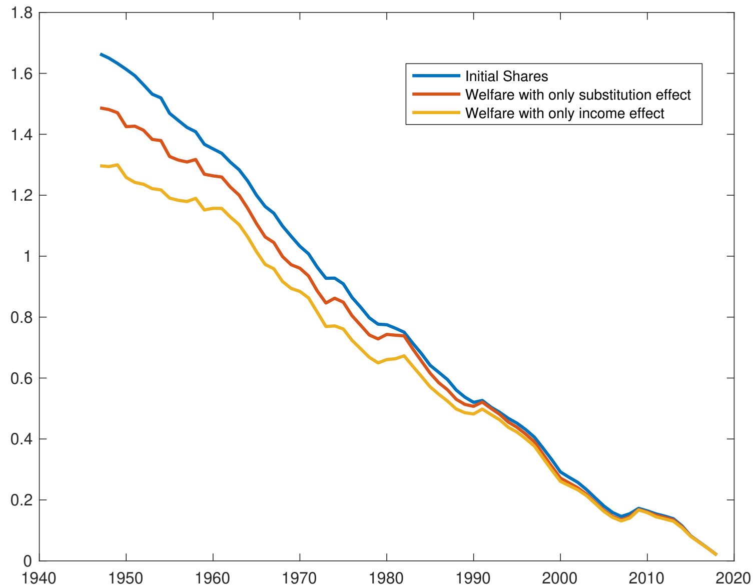

Figure 3: Change in welfare per capita from 1947 to 2019 using Personal Consumption Expenditure (PCE) prices and expenditures for 66 goods and services from the BEA. The blue line uses initial shares (in each year t between 1947 and 2019) to calculate the deflator. The red and yellow line measure the increase in welfare between t and 2019 under alternative assumptions about income and substitution elasticities. The red line assumes that budget shares change only due to substitution effects (welfare is equal to measured chained-real consumption). The yellow line assumes that budget shares change only due to income effects (or demand instability).

i. Each firm i belongs to one, and only one, industry I. Firms in the same industry share the same constant-returns-to-scale production function up to a firm-specific Hicks-neutral productivity shifter A_i.

ii. The representative household has homothetic preferences over industry-level goods, where the Ith industry-level consumption aggregator is

$$c_I = \left(\sum_{i \in I} \bar{b}_{iI} x_i c_i^{\frac{\zeta_I - 1}{\zeta_I}} \right)^{\frac{\zeta_I}{\zeta_I - 1}},$$

where c_i are consumption goods purchased by the household from firm i in industry I and x_i are firm-level demand shocks.

iii. Inputs purchased by any firm j from firms i in industry I are aggregated according to

$$m_{jI} = \left(\sum_{i \in I} \bar{s}_{iI} m_{ji}^{\frac{\sigma_I - 1}{\sigma_I}} \right)^{\frac{\sigma_I}{\sigma_I - 1}},$$

where m_{ji} are inputs purchased by firm j from firm i, and $\bar{s}_{iI}$ is a constant.

Input-output and production network models that are disciplined by industry-level data typically have an industry structure of the form defined above. For such economies, the following proposition characterizes the bias in real GDP relative to welfare.

Proposition 9 (Aggregation Bias). *For models with an industry structure, in response to firm-level supply shocks $\Delta \log A$ and demand shocks $\Delta \log x$, we have*

$$\Delta \log EV^M \approx \Delta \log Y + \frac{1}{2} \sum_I b_I Cov_{b_{(I)}} (\Delta \log x, \Delta \log A) + \Theta,$$

where b_I is industry I's share of final demand and $b_{(I)}$ is a vector whose ith element is b_i / b_I if i belongs to industry I and zero otherwise. The scalar Θ is the gap between real GDP and welfare in a version of the model with only industry-level shocks defined in the proof of the proposition.

In words, Proposition 9 implies that if firms' productivity and demand shocks are correlated with each other (but not necessarily across firms), then there is a gap between real GDP and welfare that does not appear in an industry-level specification of the model. Furthermore, this bias is, to a second-order, additive. That is, the overall bias is the sum of the industry-level bias (that we studied in the previous section) plus the additional bias driven by within-industry covariance of supply and demand shocks. Note that if supply and demand shocks at the firm level are correlated and persistent, then the bias grows over time, as in our product-level data discussed below.

Application to Nielsen Consumer Panel Data. This will be added pending approval from data provider.

5.3 Case Study: the Covid-19 Recession

Our final application concerns the Covid-19 recession, and asks how real GDP, microeconomic welfare, and macroeconomic welfare were affected. The Covid-19 recession is a good case study since sectoral expenditure shares changed dramatically during this time, these changes were not explainable via changes in observed prices alone, and the movements in demand curves were correlated with movements in supply curves. These are exactly the conditions under which micro welfare, macro welfare, and real GDP can diverge from each other.

Cavallo (2020) argues that, during this episode, the fact that price indices were not being chained at high enough frequency led to “biases” in official measures of inflation. However, since final demand was unstable during this period, chaining is not theoretically justified. As we have argued, chaining is only theoretically valid if expenditure-switching is caused by substitution effects, and not if expenditure-switching is caused by shocks to demand. Furthermore, if changes in prices are themselves caused by changes in demand (due to decreasing returns to scale), then microeconomic welfare and macroeconomic welfare changes are different.

In this section, we do not attempt to measure the welfare costs of Covid-19 itself. This is because households do not make choices over whether or not they live in a world with Covid-19. Therefore, their preferences about Covid-19 itself are not revealed by their choices. Instead, we ask a more modest question: how does the household value changes in prices (micro welfare) and changes in production (macro welfare), holding *fixed* the presence of Covid-19.

To study this episode, we use a modified version of the quantitative model introduced in Section 4. Since we are interested in a short-run application, we assume that factor markets are segmented by industry, so that labor and capital in each industry is inelastically supplied. We calibrate share parameters to match the 71 industry input-output table in 2018, and consider a range of elasticities of substitution. Following Baqaee and Farhi (2020), we model the Covid-19 recession as a combination of negative sectoral employment shocks and sectoral taste shifters. We hit the economy with a vector of primitive supply and demand shocks calibrated to match the reductions in employment and personal consumption expenditures by industry.[40] The reductions in sectoral employment are calibrated to match peak-to-trough reductions in hours worked by sector from January, 2020 to May, 2020. The primitive demand shifters are calibrated to match the observed peak-to-trough reductions in personal consumption expenditures by sector from January, 2020 to May, 2020 (conditional on the supply shocks and the elasticities of substitution).

We consider three different calibrations informed by empirical estimates from Atalay (2017) and Boehm et al. (2015): high complementarities, medium complementarities, and no complementarities (Cobb-Douglas). The high complementarity scenario sets the elasticity of substitution across consumption goods to be 0.7, the one across intermedi-

[40]Changes in labor by sector and personal consumption expenditures, used to calibrate supply and demand shocks, are taken from Baqaee and Farhi (2020). For related analysis of Covid-19 induced supply shocks, see e.g. Bonadio et al. (2020) and Barrot et al. (2020). For related analysis of Covid-19 induced demand shocks, see Cakmakli et al. (2020).

ates to be 0.01, across value-added and materials to be 0.3, and the one between labor and capital to be 0.2. The medium complementarities case sets the elasticity of substitution across consumption goods to be 0.95, the one across intermediates to be 0.01, across value-added and materials to be 0.5, and the one between labor and capital to be 0.5. The Cobb-Douglas calibration sets all elasticities of substitution equal to unity.

Table 1 displays welfare changes between January 2020 and May 2020 in the calibrated model. We report separately micro and macro welfare based on pre-Covid (initial) and post-Covid (final) preferences. Recall that micro and macro welfare are not equal in this economy because the PPF is nonlinear. For comparison, we also report the change in real consumption assuming supply and demand shocks arrive simultaneously.

Table 1: The change in micro and macro welfare with pre-Covid and post-Covid preferences given the supply and demand shocks between February 2020 to May 2020. Chained real consumption is computed assuming supply and demand shocks arrive simultaneously.

Elasticities	High compl.	Medium compl.	Cobb-Douglas
Micro pre-Covid preferences	-11.7%	-9.1%	-8.7%
Micro post-Covid preferences	-13.2%	-12.3%	-10.9%
Macro pre-Covid preferences	-16.2%	-12.5%	-10.8%
Macro post-Covid preferences	-10.1%	-9.4%	-9.0%
Chained real consumption	-12.1%	-10.6%	-9.8%

Table 1 shows that the drop in micro welfare is larger under post-Covid preferences than under pre-Covid preferences. This is because, as shown in our analytic example 4, demand shocks reduce welfare in the presence of decreasing returns to scale. Intuitively, demand shocks increase the price of goods that consumers value more over time and this causes micro welfare to drop since whatever households value becomes more expensive relative to the past.

This pattern is exactly reversed for macro welfare. Macro welfare is higher at post-Covid preferences than at pre-Covid preferences. This is because the negative supply shocks were biggest in those sectors where demand also fell more drastically (e.g. transportation and energy). Hence, the reduction in welfare is smaller with post-Covid preferences because those goods that the economy is less capable of producing are less desirable. This illustrates that micro and macro welfare answer different questions, and the answers to these questions can be quantitatively very different. Furthermore, comparing columns of Table 1 shows that the magnitude of these differences depend on the details of the production structure like the extent of complementarities in production. As we raise the elasticities of substitution in production closer to unity (Cobb-Douglas), the differences between macro and micro notions become less dramatic. This is because the PPF is become less curves.

In Table 1, we also compute real consumption assuming supply and demand shocks arrive simultaneously and linearly over time. Interestingly, chained real consumption in Table 1 does not exactly measure any of the different welfare notions. This is because supply and demand shocks are not orthogonal along the path. In fact, if we change the order or path of supply and demand shocks, real consumption changes value (even though the initial and final allocation are not changing). For example, if the supply shocks arrive before the demand shocks, then real consumption equals macro welfare changes at pre-Covid preferences. On the other hand, if demand shocks arrive before the supply shocks, then real consumption equals macro welfare changes at post-Covid preferences.

Hence, if the supply and demand shocks do not disappear in exactly the same way as they arrived, measured real consumption (or GDP) after the recovery can be higher or lower than it was before the crisis, even if the economy returns exactly to its pre-Covid allocation. For example, if in the downturn, demand shocks arrive before supply shocks and, in the recovery, demand shocks disappear before the supply shocks, then real consumption can be as much as 6% higher when comparing pre-shock real consumption to post-recovery real consumption. This is despite the fact that every price and quantity is the same when comparing the pre-shock allocation to the post-recovery allocation. Hence, during episodes where final demand is unstable, real GDP and consumption are unreliable guides for measuring output or welfare, even if we chain in continuous time.[41]

6 Extensions

In this section, we briefly summarize how our theoretical results can be extended in different ways.

Extensive margin. If preference instability or non-homotheticity causes a consumer to begin purchasing a good in t_1 that she did not consume in t_0 (or to stop consuming a

[41] This is related to a problem known as *"chain drift"* bias in national accounting. Chain drift occurs when a chained index registers an overall change between t_0 and t_1 even though all prices and quantities in t_0 and t_1 are identical. This is a specific manifestation of path dependence and, by the gradient theorem for line integrals, it must be driven by either demand instability, income effects, or approximation errors due to discreteness. Chain drift bias can thus appear when movements in prices and quantities are oscillatory, where changes that take place over some periods are reversed in subsequent periods. Welfare changes do not exhibit chain drift since, by definition, they depend only on t_0 and t_1 variables.

good that she was previously consuming), then our global and local formulas apply to that consumer without change.

To make this more explicit, consider a consumer whose preferences are represented by the utility function

$$u(c;x^*) = \left(\int_0^{x^*} c(z)^{\frac{\sigma-1}{\sigma}} dz\right)^{\frac{\sigma}{\sigma-1}}, \tag{18}$$

where goods are indexed by $z \in [0,1]$ but the consumer only values goods $z \in [0, x^*]$. In this situation, x^* is a preference parameter, where goods $z \in (x^*, 1]$ are available at a finite price, but the consumer chooses not to consume them.

Consider how the welfare of the consumer changes accounting for the fact that x^* can change between t_0 and t_1. The following is an application of Lemma 1.

Proposition 10 (New Goods Due to Taste Shocks)**.** *Consider a household with preferences defined by* (18)*. Up to a second-order approximation,*

$$\Delta \log EV^m \approx \Delta \log Y + \frac{1}{2} b(x^*)\Delta x^* \left[\mathbb{E}_b \left[\Delta \log p\right] - \Delta \log p(x^*)\right].$$

In words, the gap between welfare and real GDP depends on product of sales shares at the cut-off $b(x^*)$, the change in the cut-off Δx^*, and the difference between inflation at the cut-off versus average inflation. If new goods are added $\Delta x^* > 0$, and the new goods experienced lower than average inflation, then welfare is higher than what is detected by real GDP. However, this adjustment is second-order (since it involves products of Δ), and to a first-order, real GDP is equal to the true change in the cost of living.

It is interesting to contrast Proposition 10 to the well-known new-goods adjustment due to Feenstra (1994), which, to a first-order approximation, is

$$\Delta \log EV^m = \Delta \log Y + \frac{1}{1-\sigma} \Delta \log \left[\int_{\mathcal{C}} b(z) dz\right], \tag{19}$$

where $\mathcal{C}$ is the set of continuing goods and the integral is their share in expenditures. The difference in these results is due to a difference in interpretation. Under the interpretation in Proposition 10, the change in the extensive margin is caused by a change in tastes — that is, the goods were previously available to the consumer in the initial period but the consumer chose not to consume them (or goods are available in the final period, but the consumer chose to stop consuming them). Therefore, when we calculate welfare changes, we simply need to adjust the price index so that it accounts for the price of goods that the consumer is choosing to consume in the final period. On the other hand, under (19), when we compute the change in welfare, we assume that the consumer is unable to consume the new goods in the past or can no longer consume the disappearing goods in the present. That is, under (19), when goods are not consumed they are valued by the consumer but the implicit price is infinity.

Therefore, if a good is available in t_0 and t_1, but the consumer does not consume the good in period t_0 and does consume the good in t_1 (due to, for example, advertising), an application of (19) is not innocuous. If the change in consumer behavior is due to a change in tastes, as opposed to a change in availability, then no adjustment is necessary to a first-order, and to a second-order, the relevant adjustment is the one in Proposition 10.

Endogenous separable arguments in the utility function (e.g. leisure or home-production). If there are goods in the utility function that are endogenously chosen but not measured, then an all-encompassing welfare measure must impute shadow prices for these goods (see Jones and Klenow, 2016). For example, suppose that leisure is the non-measured argument in the utility function. If these are separable from market goods, so that preferences over c are stable when the quantity of leisure changes, then our baseline results apply to the market-good component of welfare, even if leisure changes.

Endogenous non-separable arguments in the utility function (e.g. advertising). If the parameters of the utility function x are not separable from goods c, then our welfare questions ask how changes in constraints over c affect welfare holding fixed x. That is, we do not attempt to answer how a change in x itself affects welfare, which may or may not be a question that can be answered. A salient example of x can be advertising, which can change households ranking over different consumption bundles, and is obviously non-separable from market goods. In principle, advertising may have value to the consumer — that is, the consumer can have preferences over the amount of advertising they wish to be exposed to. If advertising is informative, the consumer's utility may be increasing in advertising, and equally plausibly, if advertising is manipulative, then utility may be decreasing in advertising. We do not attempt to answer the question of how much the household values advertising, instead, we hold fixed the amount of advertising[42] (or indeed the weather, chemicals in the brain, and whatever else that affects valuations over consumption bundles), and measure how changes in the availability of market goods affects welfare.

[42]In this sense, our approach is related to Dixit and Norman (1978), who study the welfare implications of advertising at either pre- or post-advertising preferences. As argued by Fisher and McGowan (1979), this does not answer the question of what is the value of advertisement taking into account the change in tastes.

Interestingly, unlike random fluctuations in tastes, advertising is a purposeful economic activity, and therefore, models of advertising and consumer acquisition, for example Arkolakis (2016), explicitly predict that changes in tastes induced by advertising will be correlated with changes in physical productivity, whereby more productive firms will expend more resources on advertising. This positive correlation means that we should expect real GDP or real consumption measures to be systematically biased in situations where advertising plays a large role in consumption choices.

Beyond CES. Our results in Section 4 can be generalized beyond CES functional forms relatively easily. In Appendix F, we discuss how Proposition 6 must be adjusted to allow for non-CES production and utility functions.

Heterogeneous agents. Our microeconomic welfare results can be applied to individual households in economies without representative agents. Furthermore, if one commits to a particular social welfare function (SWF), we can extend the results to calculate the change in social welfare and to characterize the gap between social welfare and real GDP. With heterogeneous preferences, there are other reasons why social welfare and real GDP do not coincide in addition to the ones that we have focused on. For example, in Appendix G we consider a utilitarian SWF and show that there is a gap between social welfare and changes in real GDP even if preferences at the individual level are stable and homothetic.

7 Conclusion

In this paper, we characterize welfare and the gap between standard measures of consumption and welfare that appear when preferences are non-homothetic or unstable. We do this in both partial and general equilibrium. We show that the gap between welfare and real consumption can be large over long horizons relevant for long-run growth as well as for short-horizons, if expenditure shares at the firm and product-level change rapidly, and if demand-driven changes in expenditures covary with prices.

Although our motivation and applications have focused on shocks across time, our results can also be applied to compare welfare across locations in space (see e.g. Deaton, 2003, and Argente et al., 2020). Variation in tastes and income effects are likely to be even more significant across space than across time. To measure the difference in micro welfare between two countries, we need to compute how much income consumers of country A must be given in country B so that the resulting budget in country B makes them as well off as they are given their budget in country A. This can be calculated using equation

(5), where the welfare-relevant budget shares (as a function of prices) use country A's demand.

The distinction between macroeconomic and microeconomic notions of welfare are also relevant in a spatial context. Comparing budget constraints in one location to another may be misleading as a way to compare two economies. This is because, even if PPFs in both locations are exactly the same, the relative price of goods households value more in one location will be lower in the other location. Applying our results in a spatial context is an interesting avenue for future work.

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Appendix A Proofs

Proof of Lemma 1. By definition,

$$\begin{aligned}
EV^m &= \log \frac{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_0}, v(p_{t_0}, I_{t_0}; x_{t_1}); x_{t_1}\right)} \\
&= \log \frac{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_0}, v(p_{t_0}, I_{t_0}; x_{t_1}); x_{t_1}\right)} \frac{e\left(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)} \\
&= \log \frac{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{I_{t_0}} \frac{I_{t_1}}{e\left(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}.
\end{aligned}$$

To finish, rewrite

$$\log \frac{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)} = -\int_{t_0}^{t_1} \frac{\partial \log e(p, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1})}{\partial \log p} \frac{d \log p}{dt} dt,$$

and use the Shephard's lemma to express the price elasticity of the expenditure function in terms of budget shares. If the path of prices between t_0 and t_1 is not differentiable, then construct a new a modified path of prices that is differentiable, and apply the integral to this modified path. Since the integral is path independent, it only depends on p_{t_0} and p_{t_1}. Therefore any path that connects p_{t_0} and p_{t_1} gives the same integral. □

Proof of Proposition 1. If the path of prices is continuously differentiable, we can combine Lemma 1 with the definition of real consumption. □

Proof of Proposition 2. For real consumption, differentiate real consumption

$$\Delta \log Y = \int_{t_0}^{t_1} d \log I_t - \sum_{i \in N} b_i(p(t), u(t); x(t)) \frac{d \log p_i}{dt} dt$$

twice with respect to t_1 and evaluate the derivative at $t_1 = t_0$. This yields the desired expression.

By Lemma 1:

$$EV = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} \frac{\partial \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p} \frac{d \log p}{d \log t} d \log t$$

Differentiate EV twice with respect to t_1 and evaluate the derivative at $t_1 = t_0$

$$\begin{aligned}
\frac{dEV}{dt_1} &= \frac{d\log I}{dt} - \sum_{i\in N} \frac{\partial \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p_i} \frac{d\log p_i}{d\log t} - \\
&- \int_{t_0}^{t_1} \sum_{i\in N} d\log v \frac{\partial^2 \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log u \partial \log p_i} d\log p_i \\
&- \int_{t_0}^{t_1} \sum_{i\in N} d\log x \frac{\partial^2 \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log x \partial \log p_i} d\log p_i \\
\frac{d^2 EV}{dt_1^2} &= -\sum_{i\in N} b_i \frac{d^2 \log p}{d\log t^2} - \sum_{i\in N}\sum_{j\in N} \frac{\partial^2 \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p_i \partial \log p_j} d\log p_i d\log p_j \\
&- 2\sum_{i\in N} d\log v \frac{\partial^2 \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p_i \partial \log u} d\log p_i - 2\sum_{i\in N} d\log x' \frac{\partial \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p_i \partial \log x} d\log p_i \\
&= -\sum_{i\in N}\sum_{j\in N} \frac{\partial b_i}{\partial \log p_i} d\log p_i d\log p_j - \sum_{i\in N} b_i \frac{d^2 \log p}{d\log t^2} \\
&- 2\sum_{i\in N} d\log v \frac{\partial b_i}{\partial \log u} d\log p_i - 2\sum_{i\in N} d\log x' \frac{\partial b_i}{\partial \log x} d\log p_i \\
&= -\sum_{i\in N} \left[\sum_{j\in N} \frac{\partial b_i}{\partial \log p_j} d\log p_j + d\log v \frac{\partial b_i}{\partial \log u} + d\log x' \frac{\partial b_i}{\partial \log x} \right] d\log p_i - \sum_{i\in N} b_i \frac{d^2 \log p}{d\log t^2} \\
&- \sum_{i\in N} d\log v \frac{\partial b_i}{\partial \log u} d\log p_i - \sum_{i\in N} d\log x' \frac{\partial b_i}{\partial \log x} d\log p_i \\
&= -\sum_{i\in N} db_i d\log p_i - \sum_{i\in N} b_i \frac{d^2 \log p}{d\log t^2} - \sum_{i\in N} d\log v \frac{\partial b_i}{\partial \log u} d\log p_i - \sum_{i\in N} d\log x' \frac{\partial b_i}{\partial \log x} d\log p_i
\end{aligned}$$

The first two terms are equal to the second-order expansion of $\Delta \log Y$, and the remaining terms are the bias. □

Proof of Proposition 3. By Proposition 2, we have

$$\Delta \log Y \approx \Delta \log I - \sum_i b_i \Delta \log p_i - \frac{1}{2}\sum_i \Delta b_i \Delta \log p_i.$$

Substitute (6) in place of Δb to get the desired expression. For the bias, note that Proposition 1 implies that

$$EV - \Delta \log Y \approx -\frac{1}{2}\sum_i \left[\Delta b_i - \sum_j \frac{\partial b_i^H}{\partial \log p_j} \Delta \log p_j \right] \Delta \log p_i$$

where b^H is the Hicksian budget share (holding fixed utility and demand shifters). Using

(6) in place of Δb above and the fact that $\frac{\partial b_i^H}{\partial \log p_i} = (1-\theta_0)b_i(1-b_i)$ for $i = j$ and $\frac{\partial b_i^H}{\partial \log p_j} = \theta_0 b_i b_j$ for $i \neq j$, yields the following

$$\Delta \log EV - \Delta \log Y \approx -\frac{1}{2}\sum_{i\in N}\left[(\varepsilon_i - 1)b_i\left(d\log I - \sum_{j\in N} b_j \Delta \log p_j\right) + b_i \Delta \log x_i\right]\Delta \log p_i,$$

which can be rearranged to give the desired expression. □

Proof of Lemma 2. Setting nominal GDP to be the numeraire, we can write

$$\begin{aligned}
\Delta \log Y &= -\int_{t_0}^{t_1} b' d\log p \\
&= -\int_{t_0}^{t_1} b'\left[-\Psi d\log A - \Psi^F d\log L + \Psi^F d\log \Lambda\right] \\
&= \int_{t_0}^{t_1} b'\Psi d\log A - \int_{t_0}^{t_1} b'\Psi^F [d\log \Lambda - d\log L] \\
&= \int_{t_0}^{t_1} \lambda' d\log A + \int_{t_0}^{t_1} \Lambda' d\log L - \int_{t_0}^{t_1} \Lambda d\log \Lambda \\
&= \int_{t_0}^{t_1} \lambda' d\log A + \int_{t_0}^{t_1} \Lambda' d\log L
\end{aligned}$$

where the second line uses Proposition 6, and we use the fact that Using $\lambda' = b'\Psi$, $\Lambda' = b'\Psi^F$, and $b'\Psi^F d\log \Lambda = \Lambda' d\log \Lambda = 0$ because the factor shares always sum to one: $\sum_{f\in F}\Lambda_f = 1$. □

Proof of Proposition 4. Recall that the macro equivalent variation at final preferences is defined by $EV^M = \phi$, where

$$V\left(A_{t_0}, e^{\phi} L_{t_0}; x_{t_1}\right) = V\left(A_{t_1}, L_{t_1}; x_{t_1}\right)$$

Denote by $p\left(A, L, x\right)$ goods prices under technologies A, factor quantities L, and preferences x. Without loss of generality, we fix income at I. We have $p_{t_1} \equiv p\left(A_{t_1}, L_{t_1}, x_{t_1}\right)$ and

$$v_{t_1} \equiv v\left(p_{t_1}, I; x_{t_1}\right) = V\left(A_{t_1}, L_{t_1}; x_{t_1}\right).$$

Define a hypothetical economy with fictional households that have stable homothetic preferences defined by the expenditure function $e^{ev}\left(p, u\right) = e\left(p, v_{t_1}; x_{t_1}\right)\frac{u}{v_{t_1}}$. Budget shares of this fictional consumer are $b_i^{ev}\left(p\right) \equiv \frac{\partial e^{ev}(p,u)}{\partial p_i} = \frac{\partial e\left(p, v_{t_1}; x_{t_1}\right)}{\partial p_i}$. Given any technology vector, in this hypothetical economy we denote the Leontief inverse matrix by Ψ^{ev} and sales shares by λ^{ev}. Given technologies A_t and factor quantities L_t, we denote prices in this hypothetical economy by p_t^{ev}. Changes in prices in this hypothetical economy satisfy

$$d \log p^{ev} = -\Psi^{ev} d \log A + \Psi^{evF} d \log \Lambda^{ev}, \tag{20}$$

where Ψ^{ev} is the fictitious Leontief inverse. Note that $p\left(A_{t_1}, L_{t_1}, x_{t_1}\right) = p^{ev}\left(A_{t_1}, L_{t_1}\right)$ and $p\left(A_{t_0}, e^{\phi} L_{t_0}, x_{t_1}\right) = p^{ev}\left(A_{t_0}, e^{\phi} L_{t_0}\right)$, where we used the fact that $V\left(A_{t_0}, e^{\phi} L_{t_0}; x_{t_1}\right) = v_{t_1}$. We will use the property that, with constant returns to scale, homothetic preferences, and constant income I,

$$p^{ev}(A, aL) = \frac{1}{a} p^{ev}(A, L)$$

for every $a > 0$. Using the previous results,

$$\begin{aligned} V\left(A_{t_0}, e^{\phi} L_{t_0}; x_{t_1}\right) &= v\left(p\left(A_{t_0}, e^{\phi} L_{t_0}, x_{t_1}\right), I; x_{t_1}\right) \\ &= v\left(p^{ev}\left(A_{t_0}, e^{\phi} L_{t_0}\right), I; x_{t_1}\right) \\ &= v\left(e^{-\phi} p^{ev}\left(A_{t_0}, L_{t_0}\right), I; x_{t_1}\right) \\ &= v\left(p^{ev}\left(A_{t_0}, L_{t_0}\right), e^{\phi} I; x_{t_1}\right), \end{aligned}$$

where the last equality used the fact that the value function is homogeneous of degree 0 in prices and income. We thus have

$$v\left(p^{ev}\left(A_{t_0}, L_{t_0}\right), e^{\phi} I; x_{t_1}\right) = v\left(p^{ev}\left(A_{t_1}, L_{t_1}\right), I; x_{t_1}\right),$$

which can be re-expressed using the expenditure function as

$$EV^M = \log \frac{e\left(p^{ev}\left(A_{t_1}, L_{t_1}\right), v_{t_1}; x_{t_1}\right)}{e\left(p^{ev}\left(A_{t_0}, L_{t_0}\right), v_{t_1}; x_{t_1}\right)}.$$

This observation is a key step in the proof. Macro welfare changes can be re-expressed as micro welfare changes given changes in equilibrium prices in a fictional economy with preferences represented by $e^{ev}(p, u)$. As in the proof of Lemma 1, rewrite EV^M as

$$EV^M = -\int_{t_0}^{t_1} \sum_{i \in N+F} \frac{\partial \log e(p, v_{t_1})}{\partial \log p_i} d \log p_i^{ev} = -\int_{t_0}^{t_1} \sum_{i \in N+F} b_i^{ev} d \log p_i^{ev}.$$

Following the same steps as in the proof of Lemma 2 (for the hypothetical economy), we obtain

$$EV^M = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i^{ev} d \log A_i + \int_{t_0}^{t_1} \sum_{f \in F} \lambda_i^{ev} d \log L_f.$$

□

In general, macro and micro welfare changes are not the same when preferences are unstable and nonhomothetic. However, when the PPF is linear, the following proposition shows that they coincide.

Proposition 11 (Macro vs. Micro Welfare)**.** *Macro and micro welfare changes are equal ($EV^m = EV^M$) if preferences are stable and homothetic, or if factor income shares are constant (as in a one factor economy).*

Proof of Proposition 11. By the proof of Proposition 4, $EV^m = EV^M$ if and only if $p^{ev}(A_t, L_t) = p(A_t, L_t, x_t)$. This condition is immediate if preferences are homothetic and stable. Consider now the case in which preferences are non-homothetic and/or unstable but factor income shares, Λ, are constant. Then by Proposition 6, changes in prices in response to changes in A, L, and x are given by the following differential equation:

$$d \log p = -\Psi d \log A - \Psi^F d \log L.$$

Furthermore, note that changes in Ψ are determined by changes in Ω since $\Psi = (I - \Omega)^{-1}$. Since every $i \in N$ has constant returns to scale, changes in Ω_{ij} depend only on changes in relative prices for every $i \in N$. This means that changes in Ω only depend on changes in relative prices, therefore changes in Ψ depend only on changes in relative prices. Since x and utility v do not appear in any of these expressions, this means that prices and incomes $p(A, L, x)$ and $I(A, L, x)$, relative to the numeraire, do not depend on x and v. Thus, $p^{ev}(A_t, L_t) = p(A_t, L_t, x_t)$.

□

Proof of Proposition 5. Differentiate real GDP,

$$\Delta \log Y = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i(A(t); x(t)) \frac{d \log A_i}{dt} dt,$$

twice with respect to t_1 and evaluate the derivative at $t_1 = t_0$. This yields the desired expression. Following similar steps as in the proof of Proposition 3,

$$EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[\Delta \lambda_i - \sum_{j \in N} \frac{\partial \lambda_i^{ev}}{\partial \log A_j} \Delta \log A_j \right] \Delta \log A_i.$$

The term in square brackets is the change in sales shares due to changes in utility and demand shifters. This expression can be written as

$$EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{21}$$

□

Proof of Proposition 7. Normalize nominal GDP to one. Applying Proposition 6 to a one-factor model yields

$$d \log p = -\Psi d \log A,$$

so that relative prices do not respond to changes in demand or income.

To solve for $\Delta \log Y$, use Proposition 5 in combination with the expression for $d \log p$ and $d\lambda$ in Proposition 6 in the case of one factor. To solve for EV^M, by Proposition 11, $EV^M = EV^m$. Solve for $EV^m - \Delta \log Y$ by plugging the expression for $d \log p$ into Proposition 2 and noting that $b' = \Omega^{(0)}$. □

Proof of Proposition 6. We normalize nominal GDP to be the numeraire. Then Shephard's lemma implies that, for each $i \in N$

$$d \log p_i = -d \log A_i + \sum_j \Omega_{ij} d \log p_j.$$

Furthermore, for $i \in F$

$$d \log p_i = -d \log A_i + d \log \Lambda_i.$$

Combining these yields the desired expression for changes in prices

$$d \log p = -\Psi d \log A + \Psi^F d \log \Lambda.$$

To get changes in sales shares, note that

$$\begin{aligned} \lambda &= b'\Psi \\ d\lambda &= d(b'\Psi) \\ &= b'\Psi d\Omega\Psi + db'\Psi \end{aligned}$$

$$\begin{aligned}
\Omega_{ij} d\log\Omega_{ij} &= (1-\theta_i)\Omega_{ij}(d\log p_j - \sum_k \Omega_{ik} d\log p_k) \\
d\Omega_{ij} &= (1-\theta_i) Cov_{\Omega^{(i)}}(d\log p, I_{(j)}) \\
\sum_j d\Omega_{ij}\Psi_{jk} &= (1-\theta_i) Cov_{\Omega^{(i)}}(d\log p, I_{(j)})\Psi_{jk} \\
&= (1-\theta_i)\sum_j Cov_{\Omega^{(i)}}(d\log p, \Psi_{jk} I_{(j)}) \\
[d\Omega\Psi]_{ik} &= (1-\theta_i) Cov_{\Omega^{(i)}}(d\log p, \Psi_{(k)})
\end{aligned}$$

Meanwhile

$$\begin{aligned}
d\log b_i &= (1-\theta_0)\left(d\log p_i - \sum_i b_i d\log p_i\right) + (\varepsilon_i - 1) d\log Y + d\log x_i \\
&= (1-\theta_0) Cov_{\Omega^{(0)}}\left(d\log p, I_{(i)}\right) + Cov_{\Omega^{(0)}}\left(\varepsilon, I_{(i)}\right) d\log Y + Cov_{\Omega^{(0)}}\left(d\log x, I_{(i)}\right) \\
\sum_i db_i \Psi_{ik} &= Cov_{\Omega^{(0)}}\left((1-\theta_0) d\log p + \varepsilon d\log Y + d\log x, \Psi_{(k)}\right)
\end{aligned}$$

Hence,

$$d\lambda' = \lambda' d\Omega\Psi + db'\Psi$$

can be written as

$$d\lambda_k = \sum_i \lambda_i (1-\theta_i) Cov_{\Omega^{(i)}}(d\log p, \Psi_{(k)}) + Cov_{\Omega^{(0)}}\left(\varepsilon, \Psi_{(k)}\right) d\log Y + Cov_{\Omega^{(0)}}\left(d\log x, \Psi_{(k)}\right).$$

□

Proof of Proposition 8. Consider intertemporal preferences

$$V(\boldsymbol{A}, \boldsymbol{L}, K_0) = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s).$$

Comparing economies t and t', macro EV solves the following equation:

$$V(\boldsymbol{A}, \phi\boldsymbol{L}, \phi K_0) = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s\left(\boldsymbol{A}, \phi\boldsymbol{L}, \phi K_0\right)) = \sum_{s=t'}^{\infty} \beta^{s-t'} u(C_s\left(\boldsymbol{A}', \boldsymbol{L}', K_0'\right)) = V(\boldsymbol{A}', \boldsymbol{L}', K_0').$$

Since the economy t' is in steady-state, we are looking for

$$\sum_{s=t}^{\infty} \beta^{s-t} u\left(C_s\left(\boldsymbol{A}, \phi \boldsymbol{L}, \phi K_0\right)\right) = \frac{1}{1-\beta} u\left(C\left(\boldsymbol{A}', \boldsymbol{L}', K_0'\right)\right).$$

Furthermore, since $(\boldsymbol{A}, \phi \boldsymbol{L}, \phi K_0)$ is also a steady-state (by Lemma 3 below), we are searching for

$$u\left(C\left(\boldsymbol{A}, \phi \boldsymbol{L}, \phi K_0\right)\right) = u\left(C\left(\boldsymbol{A}', \boldsymbol{L}', K_0'\right)\right)$$

or

$$C\left(\boldsymbol{A}, \phi \boldsymbol{L}, \phi K_0\right) = C\left(\boldsymbol{A}', \boldsymbol{L}', K_0'\right).$$

Let $v(\boldsymbol{p}, I)$ be the static indirect utility function. Then we know that we are searching for

$$v(p(\boldsymbol{A}, \phi \boldsymbol{L}, \phi K_0), m) = v(p(\boldsymbol{A}, \boldsymbol{L}, K_0), \phi m) = v(p(\boldsymbol{A}', \boldsymbol{L}', K_0'), m'),$$

where the first equality uses the fact within period relative goods prices do not depend on within period preferences (since the static PPF is linear). Hence,

$$\begin{aligned}
\phi &= \frac{e(p(\boldsymbol{A}, \boldsymbol{L}, K_0), v_{t_1})}{e(p(\boldsymbol{A}, \boldsymbol{L}, K_0), v_{t_0})} = \frac{e(p(\boldsymbol{A}, \boldsymbol{L}, K_0), v_{t_1})}{e(p(\boldsymbol{A}, \boldsymbol{L}, K_0), v_{t_0})} \frac{e(p(\boldsymbol{A}', \boldsymbol{L}', K_0'), v_{t_1})}{e(p(\boldsymbol{A}', \boldsymbol{L}', K_0'), v_{t_1})} \\
&= \frac{e(p(\boldsymbol{A}', \boldsymbol{L}', K_0'), v_{t_1})}{e(p(\boldsymbol{A}, \boldsymbol{L}, K_0), v_{t_0})} \frac{e(p(\boldsymbol{A}, \boldsymbol{L}, K_0), v_{t_1})}{e(p(\boldsymbol{A}', \boldsymbol{L}', K_0'), v_{t_1})} \\
&= \exp EV^m.
\end{aligned}$$

Hence, we can use micro EV^m to calculate the change in macro welfare. □

Lemma 3. *The steady-state choice of capital (and investment) is the same for any homothetic and stable within-period preferences.*

Proof. Suppose intertemporal welfare is given by

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s),$$

where C_s is some homothetic aggregator of within-period consumption goods. Since all goods are produced with constant-returns to scale and every good uses the same homothetic bundle of capital and labor, we can write the consumption aggregator as depending on

$$C_s = G(L_{cs}, K_{cs})$$

for some function constant-returns-to-scale function G. Similarly, investment goods are created according to some constant returns to scale function

$$I_s = H(L_{Is}, K_{ls}),$$

and the capital accumulation equation is

$$K_{s+1} = (1-\delta)(K_s + I_s).$$

The Lagrangean is

$$\mathcal{L} = \sum_{s=t}^{\infty} \beta^{s-t} \left[u(C_s) + \mu_s (G(L_{cs}, K_{cs}) - C_s) + \kappa_s (K_{s+1} - (1-\delta)(K_s + H(L_{Is}, K_{Is}))) \right.$$
$$\left. + \rho_s (L_s - L_{cs} - L_{Is}) + \psi_t (K_s - K_{cs} - K_{Is}) \right]$$

The first order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_s} &: u'(C_s) = \mu_s \\
\frac{\partial \mathcal{L}}{\partial K_{s+1}} &: \kappa_s - \beta \kappa_{s+1}(1-\delta) + \beta \psi_{s+1} = 0 \\
\frac{\partial \mathcal{L}}{\partial K_{Is}} &: -\kappa_s (1-\delta) \frac{\partial H_s}{\partial K_{Is}} = \psi_s = \mu_s \frac{\partial G}{\partial K_{cs}} \\
\frac{\partial \mathcal{L}}{\partial K_{cs}} &: \mu_s \frac{\partial G}{\partial K_{cs}} = \psi_s \\
\frac{\partial \mathcal{L}}{\partial L_{cs}} &: \mu_s \frac{\partial G}{\partial L_{cs}} = \rho_s \\
\frac{\partial \mathcal{L}}{\partial L_{Is}} &: -\kappa_s (1-\delta) \frac{\partial H}{\partial L_{Is}} = \rho_s.
\end{aligned}$$

Hence

$$-\kappa_s (1-\delta) = \mu_s \frac{\partial G / \partial K_{cs}}{\partial H_s / \partial K_{Is}}$$

$$\begin{aligned}
\kappa_s &= \beta \kappa_{s+1}(1-\delta) - \beta \psi_{s+1} \\
u'(C_s) &= \beta(1-\delta) u'(C_{s+1}) \frac{\partial G / \partial K_{cs+1}}{\partial G / \partial K_{cs}} \partial H_s / \partial K_{Is} \left[(\partial H_s / \partial K_{Is+1})^{-1} + 1 \right].
\end{aligned}$$

In steady state we have

$$1 = \beta(1-\delta) \left[1 + \partial H_s / \partial K_{Is} \right].$$

Hence, the capital stock and investment in steady-state are pinned down by the following

5 equations in 5 unkowns (K_C, K_I, K, L_C, L_I) :

$$\begin{aligned}
1 &= \beta(1-\delta)\left[1+\partial H/\partial K_I\right], \\
\frac{K_C}{L_C} &= \frac{K_I}{L_I}, \\
K &= K_C + K_I, \\
L &= L_I + L_C, \\
\delta K &= (1-\delta)H(L_I, K_I).
\end{aligned}$$

Since G does not appear in any of these equations, the steady-state investment and capital stock do not depend on the shape of the within-period utility function G. □

Proof of Proposition 9. Start by setting nominal GDP to be the numeraire. To model the industry-structure, for each industry I, add two new CES aggregators. One buys the good for the household and one buys the good for firms. Let the price of the household aggregator be given by p_I^c and the price of the non-household aggregator be p_I^f and let p_I be the price of the original industry. Let firm $i's$ share of industry I from household expenditures be b_{iI}. Let the expenditure share of other firms on firm i be s_{iI}. We have

$$\begin{aligned}
\sum_{i\in I} b_{iI} &= 1 \\
\sum_{i\in I} s_{iI} &= 1.
\end{aligned}$$

Let λ_I^c and λ_I^f be sales of industry I to households and firms. Then we have

$$d\lambda_I = d\lambda_I^c + d\lambda_I^f.$$

The sales of an individual firm i in industry I is given by

$$\lambda_i = b_{iI}\lambda_I^c + s_{iI}\lambda_I^f, \tag{22}$$

$$d\lambda_i = db_{iI}\lambda_I^c + b_{iI}d\lambda_I^c + ds_{iI}\lambda_I^f + s_{iI}d\lambda_I^f, \tag{23}$$

$$\begin{aligned}
db_{iI} &= Cov_{b_I}(d\log x + (1-\sigma_I)d\log A, I_{(i)}), \\
ds_{iI} &= Cov_{b_I}((1-\sigma_I)d\log A, I_{(i)}),
\end{aligned}$$

where $I_{(i)}$ is a vector of all zeros except for its ith element which is equal to one.

The gap between macro welfare and real GDP, $EV^M - \Delta\log Y$, is approximately given by

$$\frac{1}{2}d\log x\frac{\partial\lambda}{\partial\log x}d\log A=\frac{1}{2}\sum_{i\in N}\left[\sum_{j\in N}d\log x_j\frac{\partial\lambda_i}{\partial\log x_j}\right]d\log A_i.$$

Using (23), the sums can be re-written as

$$\sum_{i\in N}\left[\sum_{j\in N}d\log x_j\frac{\partial\lambda_i}{\partial\log x_j}\right]d\log A_i=\sum_{i\in N}\left[d\log x\frac{\partial b_{iI}}{\partial\log x}\lambda_I^c d\log A_i+b_{iI}d\log x\frac{\partial\lambda_I^c}{\partial\log x}d\log A_i\right.$$
$$\left.+d\log x\frac{\partial s_{iI}}{\partial\log x}\lambda_I^f d\log A_i+s_{iI}d\log x\frac{\partial\lambda_I^f}{\partial\log x}d\log A_i\right],$$

where now the subscript I indicates the industry that the firm i belongs to.

The individual terms can be written out as

$$\begin{aligned}\sum_{i\in N}\left[d\log x\frac{\partial b_{iI}}{\partial\log x}\lambda_I^c d\log A_i\right]&=\sum_{i\in N}Cov_{b_I}(d\log x,I_{(i)})\lambda_I^c d\log A_i\\&=Cov_{b_I}(d\log x,\sum_{i\in N}I_{(i)}d\log A_i)\lambda_I^c\\&=Cov_{b_I}(d\log x,d\log A)\lambda_I^c;\end{aligned}$$

$$\sum_{i\in N}\left[b_{iI}d\log x\frac{\partial\lambda_I^c}{\partial\log x}d\log A_i\right]=\mathbb{E}_{b_I}\left(d\log A\right)d\log x\frac{\partial\lambda_I^c}{\partial\log x};$$

$$\sum_{i\in N}d\log x\frac{\partial s_{iI}}{\partial\log x}\lambda_I^f d\log A_i=0;$$

and

$$\sum_i s_{iI}d\log x\frac{\partial\lambda_I^f}{\partial\log x}d\log A_i=\mathbb{E}_{s_I}\left(d\log A\right)d\log x\frac{\partial\lambda_I^f}{\partial\log x}.$$

Of the four terms, two depend on changes on industry-level sales shares, one of them is zero, and the remaining one (the first term) is the within-industry covariance of supply and demand shocks that is highlighted in the statement of the proposition. Hence, the remaining terms in the statement of the proposition are

$$\Theta=\sum_I\left[\mathbb{E}_{s_I}\left(d\log A\right)d\log x\frac{\partial\lambda_I^f}{\partial\log x}+\mathbb{E}_{b_I}\left(d\log A\right)d\log x\frac{\partial\lambda_I^c}{\partial\log x}\right].$$

□

Proof of Proposition 10. Consider a household with preferences given by

$$C = \left(\int_0^{x^*} c(x)^{\frac{\sigma-1}{\sigma}} dx \right)^{\frac{\sigma}{\sigma-1}}.$$

Note that budget shares are

$$\lambda^{ev}(x,t,t_1) = \frac{p(x,t)^{1-\sigma}}{\left(\int_0^{x^*(t_1)} p(x,t)^{1-\sigma} dx \right)}$$

$$\lambda(x,t) = \frac{p(x,t)^{1-\sigma}}{\left(\int_0^{x^*(t)} p(x)^{1-\sigma} dx \right)}$$

Hence

$$EV^m = \frac{I}{\left(\int_0^{x^*(t_1)} p(x)^{1-\sigma} dx \right)^{\frac{1}{1-\sigma}}}.$$

Next

$$\Delta \log EV^m = \Delta \log I - \int_{t_0}^{t_1} \int_0^{x^*(t_1)} \lambda^{ev}(x,t,t_1) \frac{d \log p(x,t)}{dt} dx dt.$$

Without loss of generality, let's normalize changes in nominal income to zero. Let $\partial_i \lambda^{ev}$ refer to the partial derivative of λ^{ev} with respect to its ith argument. Differentiating and evaluating at the initial point, we get

$$\begin{aligned}
\frac{d \log EV^m}{dt_1} &= - \int_{t_0}^{t_1} \int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x,t,t_1) \frac{d \log p(x,t)}{dt} dx dt \\
&\quad - \int_{t_0}^{t_1} \lambda^{ev}(x^*(t_1),t,t_1) \frac{d \log p(x^*(t_1),t)}{dt} \frac{dx^*}{dt_1} dt - \int_0^{x^*(t_1)} \lambda^{ev}(x,t_1,t_1) \frac{d \log p(x,t_1)}{dt_1} dx \\
\frac{d^2 \log EV^m}{dt_1^2} &= - \int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x,t_1,t_1) \frac{d \log p(x,t_1)}{dt_1} dx - \lambda^{ev}(x^*(t_1),t_1,t_1) \frac{d \log p(x^*(t_1),t_1)}{dt} \frac{dx^*}{dt_1} \\
&\quad - \lambda^{ev}(x^*(t_1),t_1,t_1) \frac{dx^*}{dt_1} \frac{d \log p(x^*,t_1)}{dt_1} - \int_0^{x^*(t_1)} \frac{d\lambda^{ev}(x,t_1,t_1)}{dt_1} \frac{d \log p(x,t_1)}{dt_1} dx \\
&\quad - \int_0^{x^*(t_1)} \lambda^{ev}(x,t_1,t_1) \frac{d^2 \log p(x,t_1)}{dt_1^2} dx
\end{aligned}$$

Evaluating at the initial point this simplifies to

$$
\begin{aligned}
\frac{d \log EV^m}{dt_1} &= -\int_0^{x^*} \lambda(x) \frac{d \log p(x,t)}{dt} dx \\
\frac{d^2 \log EV^m}{dt_1^2} &= -\int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x, t_1, t_1) \frac{d \log p(x, t_1)}{dt_1} dx - \lambda^{ev}(x^*(t_1), t_1, t_1) \frac{d \log p(x^*(t_1), t_1)}{dt} \frac{dx^*}{dt_1} \\
&\quad - \lambda^{ev}(x^*(t_1), t_1, t_1) \frac{dx^*}{dt_1} \frac{d \log p(x^*, t_1)}{dt_1} - \int_0^{x^*(t_1)} \frac{d\lambda^{ev}(x, t_1, t_1)}{dt_1} \frac{d \log p(x, t_1)}{dt_1} dx \\
&\quad - \int_0^{x^*(t_1)} \lambda^{ev}(x, t_1, t_1) \frac{d^2 \log p(x, t_1)}{dt_1^2} dx
\end{aligned}
$$

We note that

$$
\lambda^{ev}(x, t, t_1) = \frac{p(x,t)^{1-\sigma}}{\left(\int_0^{x^*(t_1)} p(x,t)^{1-\sigma} dx \right)}
$$

$$
\frac{\partial \log \lambda^{ev}(x,t,t_1)}{\partial t} = (1-\sigma) \left(\frac{d \log p(x,t)}{dt} - \int_0^{x^*(t_1)} \lambda(x,t) \frac{d \log p(x,t)}{dt} dx \right).
$$

$$
\partial_3 \log \lambda^{ev}(x,t,t_1) = \frac{\partial \log \lambda^{ev}(x,t,t_1)}{\partial t_1} = \left(-\lambda(x^*, t) \frac{dx^*}{dt_1} \right).
$$

Meanwhile, real consumption changes are given by

$$
\begin{aligned}
\log Y &= -\int_{t_0}^{t_1} \int_0^{x^*(t)} \lambda(x,t) \frac{d \log p}{dt} dx dt \\
\frac{d \log Y}{dt_1} &= -\int_0^{x^*(t_1)} \lambda(x, t_1) \frac{d \log p}{dt_1} dx \\
\frac{d^2 \log Y}{dt_1^2} &= -\lambda(x^*(t_1), t_1) \frac{dx^*}{dt_1} \frac{d \log p}{dt_1} \\
&\quad - \int_0^{x^*(t_1)} \frac{d\lambda(x, t_1)}{dt_1} \frac{d \log p}{dt_1} dx - \int_0^{x^*(t_1)} \lambda(x, t_1) \frac{d^2 \log p}{dt_1^2} dx
\end{aligned}
$$

where

$$
\frac{d \log \lambda(x, t_1)}{dt_1} = (1-\sigma) \left(\frac{d \log p(x,t)}{dt} - \int_0^{x^*(t_1)} \lambda(x,t) \frac{d \log p(x,t)}{dt} dx \right) - \lambda(x^*, t) \frac{dx^*}{dt}.
$$

Hence

$$
\begin{aligned}
\frac{d\log EV^m}{dt_1} &= \frac{d\log Y}{dt_1} \\
\frac{d^2\log EV^m}{dt_1^2} &= \frac{d^2\log Y}{dt_1^2} - \int_0^{x^*(t_1)} \partial_3\lambda^{ev}(x,t_1,t_1)\frac{d\log p(x,t_1)}{dt_1}dx - \lambda^{ev}(x^*(t_1),t_1,t_1)\frac{d\log p(x^*(t_1),t_1)}{dt}\frac{dx^*}{dt_1} \\
&= \frac{d^2\log Y}{dt_1^2} + \lambda(x^*)\frac{dx^*}{dt_1}\int_0^{x^*(t_1)} \lambda(x)\frac{d\log p(x,t_1)}{dt_1}dx - \lambda(x^*)\frac{d\log p(x^*(t_1),t_1)}{dt}\frac{dx^*}{dt_1} \\
&= \frac{d^2\log Y}{dt_1^2} + \lambda(x^*)\frac{dx^*}{dt_1}\left[\int_0^{x^*(t_1)} \lambda(x)\frac{d\log p(x,t_1)}{dt_1}dx - \frac{d\log p(x^*(t_1),t_1)}{dt}\right] \\
&= \frac{d^2\log Y}{dt_1^2} + \lambda(x^*)\frac{dx^*}{dt_1}\left[\mathbb{E}_\lambda\left[\frac{d\log p}{dt}\right] - \frac{d\log p(x^*)}{dt}\right].
\end{aligned}
$$

□

Appendix B Extension to Other Welfare Measures

Our baseline measure of welfare changes is equivalent variation under final preferences. Alternatively, we could measure changes in welfare using compensating (instead of equivalent) variation, or by using initial (rather than final) preferences. We focus on equivalent variation with final preferences since it uses indifference curves in the final allocation to make welfare comparisons (that is, preferences "today" for growth-accounting purposes). In this appendix, we show that our methods generalize to the other welfare measures. If preferences are homothetic, then the expenditure function can be written as $e(p,u;x) = e(p;x)\,u$, so equivalent and compensating variation are equal. If preferences are stable, then the expenditure function can be written as $e(p,u;x) = e(p,u)$, so equivalent variation under initial and final preferences are equal (and the same is the case for compensating variation).

Recall that when preferences are homothetic, then the expenditure function can be written as $e(p,u;x) = e(p;x)\,u$. Hence, in this case, for any fixed x, compensating variation is equal to equivalent variation.

B.1 Micro welfare changes

We consider four alternative measures of micro welfare changes. For each measure, we present expression for global welfare changes and the approximate gap with real consumption.

The *compensating variation* with *initial preferences* is $CV^m(p_{t_0}, I_{t_0}, p_{t_1}, I_{t_1}; x_{t_0}) = \phi$, where ϕ solves

$$v(p_{t_1}, e^{-\phi} I_{t_1}; x_{t_0}) = v(p_{t_0}, I_{t_0}; x_{t_0}). \tag{24}$$

The analog to (5) in Lemma 1 is

$$CV^m = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} b_i^{cv} d \log p_i, \tag{25}$$

where $b_i^{cv}(p) \equiv b_i(p, v(p_{t_0}, I_{t_0}; x_{t_0}); x_{t_0})$.

Whereas EV^m weights price changes by hypothetical budget shares evaluated at current prices for fixed *final preferences and final utility*, CV^m uses budget shares evaluated at current prices for fixed *initial preferences and initial utility*. An alternative way of calculating CV^m is to reverse the flow of time (the final period corresponds to the initial period), calculate the baseline EV measure under this alternative timeline, and then set $CV^m = -EV^m$.

We now briefly describe how to calculate b^{cv} to apply (31). For ex-ante counterfactuals, where $b(t_0)$ is known, we can construct $b^{cv}(p)$ between t_0 and t_1 by iterating on (7) starting at t_0 and going forward to t_1. For ex-post counterfactuals, $b(t_0)$ can be obtained from past data, so we can construct $b^{cv}(p)$ by iterating on (7) starting at t_0 and going forward to t_1.

To a second-order approximation

$$\Delta \log CV^m \approx \Delta \log I - b' \Delta \log p - \frac{1}{2} \sum_{i \in N} \left[\Delta \log p' \frac{\partial b_i}{\partial \log p} \right] \Delta \log p \tag{26}$$

$$\approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial b_i}{\partial \log x} + \Delta \log v \frac{\partial b_i}{\partial \log u} \right] \Delta \log p. \tag{27}$$

Recall that changes in budget shares due to non-price factors are multiplied by $1/2$ in real consumption. However, they are multiplied by 0 in CV^m, since CV^m is based on budget shares at initial preferences and initial utility.

Combining Proposition 2 and (26), we see that up to a second order approximation,

$$0.5\,(EV^m + CV^m) \approx \Delta \log Y.$$

That is, locally (but not globally) changes in real consumption equal a simple average of equivalent variation under final preferences and compensating variation under initial preferences.

Alternatively, we can measure the change in welfare using the *micro equivalent variation* with *initial preferences*, $EV^m(p_{t_0}, I_{t_0}, p_{t_1}, I_{t_1}; x_{t_0}) = \phi$ where ϕ solves

$$v(p_{t_1}, I_{t_1}; x_{t_0}) = v(p_{t_0}, e^{\phi} I_{t_0}; x_{t_0}). \tag{28}$$

Globally, changes in welfare are

$$EV^m = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} b_i^{ev} d \log p_i, \tag{29}$$

where $b_i^{ev}(p) \equiv b_i(p, v(p_{t_1}, I_{t_1}; x_{t_0}); x_{t_0})$. The gap between changes in welfare and real consumption is, up to a first order approximation,

$$\Delta \log EV^m - \Delta \log Y \approx \frac{1}{2} \sum_{i \in N} \left[-\Delta \log x' \frac{\partial b_i}{\partial \log x} + \Delta \log v \frac{\partial b_i}{\partial \log u} \right] \Delta \log p.$$

Finally, the change in welfare measured using the *micro compensating variation* with *final preferences* is $CV^m(p_{t_0}, I_{t_0}, p_{t_1}, I_{t_1}; x_{t_1}) = \phi$ where ϕ solves

$$v(p_{t_1}, e^{-\phi} I_{t_1}; x_{t_1}) = v(p_{t_0}, I_{t_0}; x_{t_1}). \tag{30}$$

Globally, changes in welfare are given by

$$CV^m = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} b_i^{cv} d \log p_i, \tag{31}$$

where $b_i^{cv}(p) \equiv b_i(p, v(p_{t_0}, I_{t_0}; x_{t_0}); x_{t_1})$. The gap between changes in welfare and real consumption is, up to a first order approximation,

$$\Delta \log CV^m - \Delta \log Y \approx \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial b_i}{\partial \log x} - \Delta \log v \frac{\partial b_i}{\partial \log u} \right] \Delta \log p.$$

Note for EV with initial preferences or CV with final preferences, we must be able to separate demand instability from income effects. For this reason, to compute welfare changes, the elasticities of substitution are not sufficient — we must also know income elasticities or the demand shocks.

B.2 Macro welfare changes

For each alternative micro welfare measure there is a corresponding macro welfare measure. For example, the *macro compensating variation* with *initial preferences* is

$$CV^M(A_{t_0}, L_{t_0}, A_{t_1}, L_{t_1}; x_{t_0}) = \phi,$$

where ϕ solves

$$V(A_{t_0}, L_{t_0}; x_{t_0}) = V(A_{t_1}, e^{-\phi} L_{t_1}; x_{t_0}).$$

In words, CV^M is the proportional change in final factor endowments necessary to make a planner with preferences $\succeq_{x_{t_0}}$ indifferent between the initial PPF (A_{t_0}, L_{t_0}) and PPF defined by $(A_{t_1}, e^{-\phi} L_{t_1})$.

Equation (11) in Proposition 4 applies using $\lambda^{cv}(A)$, the sales shares in a fictional economy with the PPF A, L but where consumers have stable homothetic preferences represented by the expenditure function $e^{cv}(p, u) = e(p, v_{t_0}, x_{t_0}) \frac{u}{v_{t_0}}$ where $v_{t_0} = v(p_{t_0}, I_{t_0}; x_{t_0})$. Growth accounting for welfare is based on hypothetical sales shares evaluated at current technology but for fixed initial preferences and initial utility. The only information on preferences we need to know is elasticities of substitution at the final allocation. As discussed above, CV^M is equal to $-EV^M$ if we reverse the flow of time.

The gap between changes in welfare and real GDP is, to a second-order approximation (the analog of that in Proposition 5) is

$$CV^M \approx \Delta \log Y - \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{32}$$

We can also define *macro equivalent variation* with *initial preferences*, $EV^M(A_{t_0}, L_{t_0}, A_{t_1}, L_{t_1}; x_{t_0}) = \phi$, where ϕ solves

$$V(A_{t_1}, L_{t_1}; x_{t_0}) = V(A_{t_0}, e^{\phi} L_{t_0}; x_{t_0}).$$

Growth accounting for welfare is based on hypothetical sales shares evaluated at current technology for fixed initial preferences and final utility. In contrast to our previous measures, in order to implement this measure we must know initial demand shifters or income effects. The gap between changes in welfare and real GDP is

$$EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[-\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{33}$$

Finally, define *macro compensating variation* with *final preferences*, $CV^M(A_{t_0}, L_{t_0}, A_{t_1}, L_{t_1}; x_{t_1}) =$

ϕ, where ϕ solves

$$V(A_{t_0}, L_{t_0}; x_{t_1}) = V(A_{t_1}, e^{-\phi} L_{t_1}; x_{t_1}).$$

Growth accounting for welfare is based on hypothetical sales shares evaluated at current technology for fixed final preferences and initial utility, which requires information on demand shifters or income effects. The gap between changes in welfare and real GDP is

$$CV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} - \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \quad (34)$$

Appendix C Non-homothetic CES preferences

C.1 Derivation of Marshallian budget shares

This appendix provides a derivation of the log-linearized expression (6). Changes in Marshallian budget share are given by

$$\begin{aligned} d \log b_i &= d \log p_i - d \log I + \sum_j \varepsilon_{ij}^M d \log p_j + \varepsilon_i^w d \log I + d \log x_i, \\ &= d \log p_i - d \log I + \sum_j \left(\varepsilon_{ij}^H - \varepsilon_i^w b_j \right) d \log p_j + \varepsilon_i^w d \log I + d \log x_i, \end{aligned}$$

where ε^H and ε^M are the Hicksian and Marshallian price elasticities, $\varepsilon^w$are the income elasticities, and $d \log x_i$ is a residual that captures changes in shares not attributed to changes in prices or income. The third line is an application of Slutsky's equation. When preferences are non-homothetic CES, then the Hicksian demand curve can be written as

$$c_i = \gamma_i \left(\frac{p_i}{\sum_j p_j c_j} \right)^{-\theta_0} u^{\xi_i},$$

where γ_i and ξ_i are some parameters. The Hicksian price elasticity for $j \neq i$ is

$$\frac{\partial \log c_i}{\partial \log p_j} = \varepsilon_{ij}^H = \theta_0 \frac{p_j c_j}{I} = \theta_0 b_j.$$

Using this fact and the identity $\varepsilon_{ii}^H = -\sum_{j\neq i}\varepsilon_{ij}^H$, we can rewrite changes in budget shares as

$$
\begin{aligned}
d\log b_i &= \sum_j \left(\varepsilon_{ij}^H - \varepsilon_i^w b_j\right) d\log p_j + d\log p_i + (\varepsilon_i^w - 1)\, d\log I + d\log x \\
&= \left(1 - \sum_{j\neq i}\varepsilon_{ij}^H\right) d\log\frac{p_i}{I} + \sum_{j\neq i}\varepsilon_{ij}^H d\log\frac{p_j}{I} + \varepsilon_i^w\left[d\log I - \sum_j b_j d\log p_j\right] + d\log x_i \\
&= \left(1 - \sum_{j\neq i}\theta_0 b_j\right) d\log\frac{p_i}{I} + \sum_{j\neq i}\theta_0 b_j d\log\frac{p_j}{I} + \varepsilon_i^w\left[d\log I - \sum_j b_j d\log p_j\right] + d\log x_i \\
&= (1 - \theta_0(1-b_i))\, d\log\frac{p_i}{I} + \sum_{j\neq i}\theta_0 b_j d\log\frac{p_j}{I} + \varepsilon_i^w\left[d\log I - \sum_j b_j d\log p_j\right] + d\log x_i \\
&= (1-\theta_0)\left[d\log p_i - \sum_j b_j d\log p_j\right] + (\varepsilon_i^w - 1)\left[d\log I - \sum_j b_j d\log p_j\right] + d\log x_i.
\end{aligned}
\tag{35}
$$

In the body of the paper we use ε_i in place of ε_i^w. This completes the derivation of (6) where $d\log x$ could be any perturbation to budget shares not explained by income and substitution effects.

For ex-ante counterfactuals, we can also use (35) as a differential equation to solve for budget shares in the future. To do this, we must put some structure on $d\log x$. For example, if $d\log x$ is being driven by changes in taste parameters γ, and θ_0 and ξ_i are constant, then we can show that

$$
d\log x_i = \left[d\log\gamma_{it} - \frac{\xi_i}{\sum_i b_i\xi_i}\sum_i b_i d\log\gamma_i\right]. \tag{36}
$$

Next, we use the identities

$$
\frac{\partial\log e}{\partial\log u} = \frac{\sum_i b_i\xi_i}{1-\theta}
$$

and

$$
\frac{\partial\log b_i}{\partial\log u} = \xi_i - (1-\theta)\frac{\partial\log e}{\partial\log u}
$$

to rewrite (35) as

$$
d\log b_i = (1-\theta_0)\left[d\log p_i - \mathbb{E}_b[d\log p]\right] + (1-\theta_0)\left(\frac{\xi_i}{\mathbb{E}_b[\xi]} - 1\right)\left[d\log I - \mathbb{E}_b[d\log p]\right] + d\log x_i,
$$

where $d\log x$ is given by (36). This is a differential equation that pins down budget shares

b as a function of prices, incomes and primitives γ. Hence, it can be solved numerically for any path of prices, incomes, and γ to arrive at b.

C.2 Comparison of welfare and changes in utility index

In this appendix, we discuss the difference between changes in welfare as measured by the equivalent variation and changes in the utility index under non-homothetic CES preferences. We show that there is no normalization of the parameters such that the equivalent variation is equal to changes in the utility index, up to a second order approximation, unless preferences are homothetic. The same result applies for compensating variation.

The expenditure function can we expressed as

$$e(p,u) = \left(\sum_i \gamma_i p_i^{1-\theta_0} u^{\xi_i}\right)^{\frac{1}{1-\theta_0}}$$

where for brevity we have assumed away taste shocks. The micro equivalent variation is given by

$$EV^m = \log \frac{e(p_0, v(p_1, I_1))}{e(p_0, v(p_0, I_0))},$$

where $v(p, I)$ is the indirect utility function, initial prices and income are p_0 and I_0, and final prices and income are p_1 and I_1.

Consider now the change in the utility index u in the expenditure function introduced above, which is given by

$$U \equiv \log \frac{v(p_1, I_1)}{v(p_0, I_0)}.$$

As this definition makes clear, EV and U are not generically the same. In particular, whereas EV can be defined in terms of a hypothetical choice and is independent of the utility function chosen to represent preferences (how much income would the household need to be given to make them indifferent), U will depend on the cardinal properties of the utility function.

If preferences are homothetic ($\xi_{ii} = \bar{\xi}$ for all i), then $e(p,u) = \left(\sum_i \gamma_i p_i^{1-\theta_0}\right)^{\frac{1}{1-\theta_0}} u^{\frac{\bar{\xi}}{1-\theta_0}}$ and we can write

$$EV^m = \frac{\bar{\xi}}{1-\theta_0} U.$$

So, when preferences are homothetic, in order for $EV^m = U$ we must cardinalize utility in such a way that the expenditure function is homogeneous of degree 1 in u ($d \log e / d \log u = 1$). In other words, although there are infinitely many utility functions that represent these preferences, when preferences are homothetic, there is one representation where $EV^m = U$.

We now consider the non-homothetic case, and we characterize the difference between EV^m and U to a first and second order. We write these results in terms of primitive shocks (that is, changes in income and prices) rather than in terms of changes in endogenous objects like budget shares (i.e. as in equation 45 of Comin et al. (2015)).

Define the budget share of good i by

$$b_i \equiv \frac{p_i c_i}{I} = \frac{\partial \log e}{\partial \log p_i}.$$

Furthermore, let ε_i^w be the income elasticity of demand for good i.

Using Proposition 3, we have that to a first-order EV^m is

$$dEV^m = d\log e - bd\log p = d\log Y,$$

where $d\log Y$ is the first-order change in real consumption as measured by Tornqvist. Hence, to a first order, Tornqvist and EV are the same. The second-order change in EV^m is, by Proposition 3, equal to

$$\begin{aligned} d^2EV^m &= d^2\log e - dbd\log p - (d\log e - bd\log p)\, Cov_b(\varepsilon^w, d\log p) \\ &= d^2\log Y - (d\log e - bd\log p)\, Cov_b(\varepsilon^w, d\log p), \end{aligned}$$

where ε^w is the vector of income elasticities and $d^2\log Y$ is the change in real consumption as measured by a Tornqvist or Divisa index (to a second-order, they are equivalent). On the other hand, the first and second-order changes in the utility index are given by (derivations are available upon request)

$$dU = \frac{1-\theta_0}{\sum_i b_i \bar{\xi}}\,(d\log e - bd\log p)$$

and

$$\begin{aligned} d^2U = \frac{1-\theta_0}{\sum_i b_i \xi_i} \Bigg[& d^2\log e - dbd\log p - (d\log e - bd\log p) \sum_i b_i(\varepsilon_i^w - 1) d\log p_i \\ & - \frac{1}{1-\theta_0} \sum_i b_i \varepsilon_i^w \left((\varepsilon_i^w - 1)\right) (d\log e - bd\log p)^2 \Bigg] \end{aligned}$$

The derivatives EV^m and U are in general different. Whereas EV^m is a function of observables, U depends on a normalization since $\sum_i b_i \xi_i$ is not pinned down by observables (scaling all ξ proportionally does not change any observable, but it does change how U responds).

Now we compare the first and second-order derivatives in turn. The first order difference is

$$dU - dEV^m = \left(\frac{1-\theta_0}{\sum_i b_i \xi_i} - 1 \right) (d \log e - b d \log p) .$$

If we impose a normalization on utility parameters such that, in the initial point,

$$\frac{1-\theta_0}{\sum_i b_i \xi_i} = 1,$$

we have that $dU = dEV^m = d \log Y$. This normalization is effectively ensuring that $\partial \log e / \partial \log u = 1$.

Now let's consider the second-order difference and let's impose the same normalization

$$\begin{aligned}
d^2U - d^2EV^m &= -\frac{1}{1-\theta_0} \sum_i b_i \varepsilon_i^w (\varepsilon_i^w - 1) (d \log e - b d \log p)^2 \\
&\quad - (d \log e - b d \log p) \left[\sum_i b_i (\varepsilon_i^w - 1) \sum_i b_i d \log p_i \right] \\
&= -\frac{1}{1-\theta_0} \sum_i b_i \varepsilon_i^w (\varepsilon_i^w - 1) (d \log e - b d \log p)^2 \\
&= -\frac{1}{1-\theta_0} Var_b(\varepsilon_i^w) (d \log e - b d \log p)^2 \\
&\neq 0,
\end{aligned}$$

where we used $\sum_i b_i \varepsilon_i^w = 1$. Hence, unless preferences are homothetic (in which case $\varepsilon_i^w = 1$ for every i), the change in U and EV^m are not the same even under the normalization. This is not to mention that globally, we cannot ensure that the normalization

$$\frac{1-\theta_0}{\sum_i b_i \xi_i} = 1$$

always holds. This means that the gap between EV^m and U, which exists at the initial equilibrium, only gets more severe if, once we commit to a specific normalization of utility, $\frac{1-\theta_0}{\sum_i b_i \xi_i}$ starts to change from 1.

Recall from Appendix B that changes in real consumption are equal to an average of equivalent and compensating variation, up to a second order approximation. Since changes in the utility index are not equal to a Tornquist real consumption index index, it follows that the utility index is not equal to an average of EV and CV.

Appendix D Quality changes

In this appendix, we discuss how our results can be extended to environments with quality changes that are not reflected in market prices. In this appendix, we also contrast the bias we identify with the "taste shock bias" discussed by Redding and Weinstein (2020).

If quality changes are equivalent to changes in prices, then we can write the quality-adjusted price as $p = \tilde{p}/q$, where $\tilde{p}$ is the market price and q is the quality adjustment. In other words, changes in quality-adjusted prices are given by $\Delta \log p = \Delta \log \tilde{p} - \Delta \log q$. Substituting this into our various propositions allows us to isolate the way quality changes affect our results. For example, Proposition 3 becomes the following. For brevity, we assume homothetic preferences below.

Proposition 12 (Approximate Micro with Quality Change)**.** *Consider some perturbation in demand $\Delta \log x$, market prices $\Delta \log \tilde{p}$, quality $\Delta \log q$, and income $\Delta \log I$. Then, to a second-order approximation, the change in real consumption is*

$$\Delta \log Y \approx \Delta \log I - \mathbb{E}_b \left[\Delta \log \tilde{p}\right] - \frac{1}{2}(1-\theta_0) Var_b(d \log \tilde{p}) + \frac{1}{2}(1-\theta_0) Cov_b(d \log q, d \log \tilde{p}) - \frac{1}{2} Cov_b(d \log x, d \log \tilde{p}),$$

and the change in welfare is

$$EV^m \approx \Delta \log I - \mathbb{E}_b \left[\Delta \log \tilde{p} - \Delta \log q\right] - \frac{1}{2}(1-\theta_0) Var_b \left(\Delta \log \tilde{p}\right) - \frac{1}{2}(1-\theta_0) Var_b \left(\Delta \log q\right) + (1-\theta_0) Cov_b \left(\Delta \log \tilde{p}, \Delta \log q\right) - Cov_b \left(\Delta \log x, \Delta \log p\right),$$

where $Cov_b(\cdot)$ is the covariance using the initial budget shares as the probability weights.

Hence, by subtracting these two expressions, we can derive the gap between real consumption and welfare up to a second order approximation as

$$EV^m - \Delta \log Y \approx \underbrace{\mathbb{E}_b\left[\Delta \log q\right]}_{\text{average quality}} + \frac{1}{2}\underbrace{(\theta_0 - 1)Var_b\left(\Delta \log q\right)}_{\text{dispersion in quality}} + \frac{1}{2}\underbrace{(1-\theta_0)Cov_b\left(\Delta \log \tilde{p}, \Delta \log q\right)}_{\text{covariance of price and quality}}$$
$$- \frac{1}{2}\underbrace{Cov_b\left(\Delta \log x, \Delta \log \tilde{p}\right)}_{\text{covariance of taste and price}} + \underbrace{Cov_b\left(\Delta \log x, \Delta \log q\right)}_{\text{covariance of taste and quality}} . \quad (37)$$

The first term on the right-hand side captures how the average increase in quality raises welfare relative to real consumption. The second term captures the fact that dispersion in quality raises welfare if the elasticity of substitution is greater than one (since the consumer substitutes towards goods with relatively higher quality, but quality is not captured by market prices in real consumtption). The third term is an interaction (cross-partial) effect that raises welfare if market prices fall for goods whose quality rose, as long as the elasticity of substitution is greater than one. The fourth term is the bias we have been emphasizing in the paper so far. The final term is the interaction between quality and taste changes — welfare is higher, at final preferences, if tastes increase for goods whose quality also increased.

In our analysis, we assume that prices have already been adjusted for quality, so the only non-zero term is the fourth one. Welfare is higher than real consumption if the covariance between taste shocks and prices is negative. This is independent of the value of the elasticity of substitution. In other words, in the body of the paper, we assume that $\Delta \log q = 0$, which means that (37) simplifies to

$$EV^m - \Delta \log Y \approx -\frac{1}{2}Cov_b\left(\Delta \log x, \Delta \log \tilde{p}\right). \quad (38)$$

An important challenge for future research is to separately identify demand shocks $\Delta \log x$ versus quality changes $\Delta \log q$.

Comparison to Redding and Weinstein (2020). We can use (37) to contrast our results with those of Redding and Weinstein (2020).[43] The "taste shifters" in that paper are equivalent to quality shocks $\Delta \log q \neq 0$, whereas $\Delta \log x = 0$. Hence, in their case, (37) simplifies

$$EV^m - \Delta \log Y \approx \mathbb{E}_b\left[\Delta \log q\right] + \frac{1}{2}(\theta_0 - 1)Var_b\left(\Delta \log q\right) - \frac{1}{2}(\theta_0 - 1)Cov_b\left(\Delta \log \tilde{p}, \Delta \log q\right). \quad (39)$$

[43]We abstract from product creation and destruction, which Redding and Weinstein (2020) also consider.

Comparing (38) to (39) elucidates the differences. First, the average level of $\Delta \log q$ affects welfare but the average level of $\Delta \log x$ does not. Redding and Weinstein (2020) neutralize this effect by assuming it is zero. Second, for shocks to $\Delta \log q$, even when they are mean zero, dispersion in q can raise or lower welfare depending on the elasticity of substitution. Hence, shocks to q on their own can change welfare, even if choice sets have not changed, and the sign of this effect depends on the elasticity of substitution. This is in contrast to shocks to x which do not change welfare on their own. Third, in both (38) and (39), the covariance of taste shifters and market prices matters, however, in (39) the sign of the covariance depends on whether the elasticity of substitution is greater than or less than one, whereas in (38), the sign is always the same.

Appendix E Additional details on the Baumol application

In this appendix, we provide some intuition for why, from a welfare perspective, structural transformation caused by income effects or taste shocks is roughly twice as important as that caused by substitution effects. We also use a structural nested-CES model to explore the change in welfare-relevant TFP outside of the two polar extremes in Section 5.

E.1 Intuition for size of welfare-adjustment

According to our results in Section 5, structural transformation caused by income effects or demand instability reduced welfare by roughly twice as much as structural transformation caused by substitution effects. To understand why the necessary adjustment is roughly twice as big, consider the second-order approximation in Proposition 4:

$$\Delta \log TFP^{\text{welfare}} \approx \Delta \log TFP + \frac{1}{2} \left[\sum_{i \in N} \frac{\partial \lambda_i}{\partial \log x} \Delta \log x + \frac{\partial \lambda_i}{\partial \log v} \Delta \log v \right] \Delta \log A_i, \qquad (40)$$

where

$$\Delta \log TFP \approx \sum_{i \in N} \lambda_{i,t_0} \Delta \log A_i + \frac{1}{2} \sum_{i \in N} \Delta \lambda_i \Delta \log A_i.$$

If changes in sales shares are due entirely to demand-driven factors, then the term in square brackets in (40) is equal to $\sum_{i \in N} \Delta\lambda_i \Delta \log A_i$, so

$$\Delta \log TFP^{\text{welfare}} \approx \sum_{i \in N} \lambda_{i,t_0} \Delta \log A_i + \sum_{i \in N} \Delta\lambda_i \Delta \log A_i.$$

In other words, the adjustment to the initial sales shares must be roughly twice as large as the adjustment to the initial sales shares caused by substitution effects.[44]

E.2 Welfare-TFP outside of polar extremes

In practice, both substitution effects and non-homotheticities are likely to play an important role in explaining structural transformation. To dig deeper into the size of the welfare adjustment outside our two polar cases, we use a simplified version of the model introduced in Section 4 calibrated to the US economy, accounting for input-output linkages and complementarities, and use the model to quantify the size of the welfare-adjustment as a function of the elasticities of substitution. We calculate TFP by industry in the data allowing for cross-industry variation in capital and labor shares. For simplicity, we feed these TFP shocks as primitive shocks into a one-factor model. Recomputing these numbers for a model with multiple factors would be straightforward.

Remarkably, Proposition 4 implies that to compute the welfare-relevant change in TFP, we must only supply the information necessary to compute λ^{ev}. That is, since we know sales shares in the terminal period 2014, we do not need to model the non-homotheticities or demand-shocks themselves, and the exercise requires no information on the functional form of non-homotheticities or the slope of Engel curves or magnitude of income elasticities *conditional* on knowing the elasticities of substitution.

We map the model to the data as follows. We assume that the constant-utility final demand aggregator has a nested-CES form. There is an elasticity θ_0 across the three groups of industries: primary, manufacturing, and service industries. The inner nest has elasticity of substitution θ_1 across industries within primary (2 industries), manufacturing (24 industries), and services (35 industries).[45] Production functions are also assumed to have nested-CES forms: there is an elasticity of substitution θ_2 between the bundle of interme-

[44]These second-order approximations are more accurate if changes in sales shares are well-approximated by linear time trends, and the surprising accuracy of the second-order approximation is a result of this fact.

[45]In order to map this nested structure to our baseline model, good 0 is a composite of good 1-3, where good 1 is a composite of primary industries, good 2 is a composite of manufacturing industries, and good 3 is a composite of service industries. Goods 4-65 are the disaggregated industries. Finally, good 66 is the single factor of production.

diates and value-added, and an elasticity of substitution θ_3 across different types of intermediate inputs. For simplicity, we assume there is only one primary factor of production (a composite of capital and labor). We solve the non-linear model by repeated application of Proposition 6 in the fictional economy with stable and homothetic preferences.

We calibrate the CES share parameters so that the model matches the 2014 input-output tables provided by the BEA. For different values of the elasticities of substitution $(\theta_0, \theta_1, \theta_2, \theta_3)$ we feed changes in industry-level TFP (going backwards, from 2014 to 1947) into the model and compute the resulting change in aggregate TFP. This number represents the welfare-relevant change in aggregate TFP. We report the results in Table 2.

Table 2: Percentage change in measured and welfare-relevant TFP in the US from 1947 to 2014.

$(\theta_0, \theta_1, \theta_2, \theta_3)$	(1,1,1,1)	(0.5,1,1,1)	(1,0.5,1,1)	(1,1,0.5,1)	(1,1,1,0.5)
Welfare TFP	46%	46%	54%	48%	55%
Measured TFP	60%	60%	60%	60%	60%

The first column in Table 2 shows the change in welfare-relevant TFP assuming that there are no substitution effects (all production and consumption functions are Cobb-Douglas). In this case, all changes in sales shares in the data are driven by non-homotheticities or demand-instability, and hence welfare-relevant TFP has grown more slowly than measured TFP, exactly as discussed in the previous section. The other columns show how the results change given lower elasticities of substitution. As we increase the strength of complementarities (so that substitution effects are active), the implied non-homotheticities required to match changes in sales shares in the data are weaker. This in turn reduces the gap between measured and welfare-relevant productivity growth.

Table 2 also shows that not all elasticities of substitution are equally important. The results are much more sensitive to changes in the elasticity of substitution across more disaggregated categories, like materials, than aggregated categories, like agriculture, manufacturing, and services.

To see why the results in Table 2 are differentially sensitive to changes in different elasticities of substitution, combine Propositions 7 and 11 to obtain the following second-order approximation:

$$\Delta \log TFP^{\text{welfare}} \approx \sum_i \lambda_i \Delta \log A_i + \frac{1}{2} \sum_{j \in \{0\}+\mathcal{N}} (\theta_j - 1) \lambda_j Var_{\Omega^{(j)}} \left(\sum_{k \in \mathcal{N}} \Psi_{(k)} \Delta \log A_i \right). \quad (41)$$

The second term is half the sum of changes in Domar weights due to substitution effects (i.e. changes in welfare-relevant sales shares) times the change in productivities. Note that changes in these welfare-relevant sales shares are linear in the microeconomic elasticities of substitution. The importance of some elasticity θ depends on

$$\sum_j \lambda_j Var_{\Omega^{(j)}} \left(\sum_{k \in \mathcal{N}} \Psi_{(k)} \Delta \log A_i \right),$$

where the index j sums over all CES nests whose elasticity of substitution is equal to θ (i.e. all j such that $\theta_j = \theta$). Therefore, elasticities of substitution are relatively more potent if: (1) they control substitution over many nests with high sales shares, or (2) if the nests corresponding to those elasticities are heterogeneously exposed to the productivity shocks.

We compute the coefficients in (41) for our model's various elasticities using the IO table at the end of the sample. The coefficient on $(\theta_0 - 1)$, the elasticity of substitution between agriculture, manufacturing, and services in consumption is only 0.01. This explains why the results in Table 2 are not very sensitive to this elasticity. On the other hand, the coefficient on $(\theta_1 - 1)$, the elasticity across disaggregated consumption goods, is much higher at 0.21. The coefficient on $(\theta_2 - 1)$, the elasticity between materials and value-added bundles is 0.07. Finally, the coefficient on $(\theta_3 - 1)$, the elasticity between disaggregated categories of materials is 0.25. This underscores the fact that elasticities of substitution are more important if they control substitution in CES nests which are very heterogeneously exposed to productivity shocks — that is, nests that have more disaggregated inputs.

According to equation (41), setting $\theta_1 = \theta_2 = \theta_3 = 1$ (which is similar to abstracting from heterogeneity within the three broader sectors and heterogeneity within intermediate inputs), then θ_0 is the only parameter that can generate substitution effects in the model. This may help understand why more aggregated models of structural transformation (e.g. Buera et al., 2015 and Alder et al., 2019) require low values of θ_0 to account for the extent of sectoral reallocation in the data.

Appendix F Non-CES Functional Forms

In this appendix, we generalize Proposition 6 beyond CES functional forms. To do this, for each producer k with cost function $\mathbf{C}_k$, we define the Allen-Uzawa elasticity of substitution between inputs x and y as

$$\theta_k(x,y) = \frac{\mathbf{C}_k d^2 \mathbf{C}_k / (dp_x dp_y)}{(d\mathbf{C}_k/dp_x)(d\mathbf{C}_k/dp_y)} = \frac{\epsilon_k(x,y)}{\Omega_{ky}},$$

where $\epsilon_k(x,y)$ is the elasticity of the demand by producer k for input x with respect to the price p_y of input y, and Ω_{ky} is the expenditure share in cost of input y. For the household $k = 0$, we use the household's expenditure function in place of the cost function (where the Allen-Uzawa elasticities are disciplined by Hicksian cross-price elasticities and expenditure shares).

Following Baqaee and Farhi (2019b), define the *input-output substitution operator* for producer k as

$$\Phi_k(\Psi_{(i)}, \Psi_{(j)}) = - \sum_{1 \le x,y \le N+1+F} \Omega_{kx}[\delta_{xy} + \Omega_{ky}(\theta_k(x,y) - 1)]\Psi_{xi}\Psi_{yj}, \tag{42}$$

$$= \frac{1}{2} E_{\Omega^{(k)}} \left((\theta_k(x,y) - 1)(\Psi_i(x) - \Psi_i(y))(\Psi_j(x) - \Psi_j(y)) \right), \tag{43}$$

where δ_{xy} is the Kronecker delta, $\Psi_i(x) = \Psi_{xi}$ and $\Psi_j(x) = \Psi_{xj}$, and the expectation on the second line is over x and y. The second line can be obtained from the first using the symmetry of Allen-Uzawa elasticities of substitution and the homogeneity identity.

Then, Proposition 6 generalizes as follows:

Proposition 13. *Consider some perturbation in final demand $d\log x$ and technology $d\log A$. Then changes in prices of goods and factors are*

$$d\log p_i = - \sum_{j \in N} \Psi_{ij} d\log A_j + \sum_{f \in F} \Psi^F_{if} d\log \lambda_f. \tag{44}$$

Changes in sales shares for goods and factors are

$$\lambda_i d\log\lambda_i = \sum_{j \in \{0\}+N} \lambda_j \Phi_j \left(-d\log p, \Psi_{(i)} \right) \tag{45}$$

$$+ Cov_{\Omega^{(0)}} \left(d\log x, \Psi_{(i)} \right) + Cov_{\Omega^{(0)}}(\varepsilon, \Psi_{(i)}) \left(\sum_{k \in N} \lambda_k d\log A_k \right).$$

Since Φ_j shares many of the same properties as a covariance (it is bilinear and symmetric in its arguments, and is equal to zero whenever one of the arguments is a constant), the intuition for Proposition 13 is very similar to that of Proposition 6. Computing the equilibrium response in Proposition 13 requires solving a linear system exactly as in Proposition

6.

Appendix G Heterogeneous Agents

We consider a utilitarian SWF that sums the welfare of each agent h, measured in terms of initial prices, relative to initial aggregate income[46]

$$W = \frac{\sum_{h \in I} e_h(\bar{p}, u_h)}{\sum_{j \in I} e_j(\bar{p}, \bar{u}_j)},$$

where I is the set of agents. Preferences can vary across households but, for simplicity, we assume that each h's preferences are homothetic and stable.

For this case, in this appendix we show that to a second-order approximation,

$$\Delta \log W = \Delta \log Y + \frac{1}{2} Cov_{\chi} \left(-\mathbb{E}_b[\Delta \log p], \Delta \log I - \mathbb{E}_b[\Delta \log p] \right),$$

where χ is the initial distribution of expenditures by each agent, $\mathbb{E}_b[\Delta \log p]$ is the vector of inflation rates for each household, and $\Delta \log I$ is the vector of nominal income changes. The change in social welfare is greater than the change in real GDP if changes in real income negatively covary with inflation across households. There is no bias if preferences are aggregable, in which case $\mathbb{E}_b[\Delta \log p]$ is uniform across households, if real income growth is uniform or, more generally, if changes in household-level inflation rates are uncorrelated with changes in real income.

While the specific form of the gap between welfare and real GDP depends on the social welfare function, if individuals' preferences are not aggregable, then there will always be a gap even if preferences at the individual level are stable and homothetic.

To establish these results, we first write the utilitarian social welfare function as

$$W = \sum_h \bar{\chi}_h Y_h,$$

where $\bar{\chi}_h$ is the ratio of agent i expenditures in total expenditures in the initial point, and $Y_h \equiv \frac{e_h(\bar{p}, u_h)}{e_j(\bar{p}, \bar{u}_j)}$ denotes the change in real consumption of agent h (or $\exp(EV_h^m)$, since

[46]This social welfare function implements a version of the Kaldor-Hicks compensation principle whereby a change is deemed socially desirable if the winners can hypothetically compensate the losers. In this case, this hypothetical compensation is measured in terms of initial prices.

preferences are stable and homothetic). We show that to a second order approximation

$$\Delta \log W \approx \Delta \log Y - \frac{1}{2} Cov_{\chi}(\mathbb{E}_{b_h}[\Delta \log p], \Delta \log Y_h),$$

where the covariance is applied across individuals using the probabilities implies by the vector χ at the initial point.

To see this,

$$\begin{aligned} d \log W &= \frac{1}{W} \sum_h \bar{\chi}_h Y_h d \log Y_h, \\ d^2 \log W &= \frac{1}{W} \sum_h \bar{\chi}_h Y_h (d \log Y_h)^2 + \frac{1}{W} \sum_h \bar{\chi}_h Y_h d^2 \log Y_h - \frac{1}{W} \sum_h \bar{\chi}_h Y_h d \log Y_h d \log W, \\ &= \frac{1}{W} \sum_h \bar{\chi}_h Y_h (d \log Y_h)^2 + \frac{1}{W} \sum_h \bar{\chi}_h Y_h d^2 \log Y_h - (d \log W)^2, \\ &= \frac{1}{W} \sum_h \bar{\chi}_h Y_h (d \log Y_h)^2 + \frac{1}{W} \sum_h \bar{\chi}_h Y_h d^2 \log Y_h - (d \log Y)^2, \\ &= \sum_h \bar{\chi}_h (d \log Y_h)^2 + \sum_h \bar{\chi}_h d^2 \log Y_h - (d \log Y)^2, \end{aligned}$$

where we use the fact that $W = Y_h = 1$ and $d \log W = d \log Y$ at the initial point.

Next consider the change in real GDP:

$$\log Y = \int_{t_0}^{t_1} \sum_i \frac{p_i(t) q_i(t)}{\sum_j p_j(t) q_j(t)} d \log q_i(t),$$

and

$$p_i q_i = \sum_h \chi_h b_{hi} GDP,$$

where b_{hi} is the budget share of agent h on good i and GDP is nominal GDP. We have

$$
\begin{aligned}
d\log q_i &= \sum_h \frac{q_{hi}}{q_i} d\log q_{hi} \\
&= \sum_h \frac{\chi_h b_{hi}}{\sum_g \chi_g b_{gi}} d\log q_{hi} \\
\log Y &= \int_{t_0}^{t_1} \sum_h \chi_h \sum_i b_{hi} d\log q_{hi} \\
&= \int_{t_0}^{t_1} \sum_h \chi_h d\log Y_h.
\end{aligned}
$$

Differentiating this with respect to t_1 gives

$$
d\log Y = \chi \cdot d\log Y_{(h)}.
$$

Differentiating again gives

$$
\begin{aligned}
d^2\log Y &= d\chi \cdot d\log Y_{(h)} + \chi \cdot d^2\log Y_{(h)} \\
&= d\chi \cdot d\log Y_{(h)} + d^2\log W - \sum_h \chi_h \left(d\log Y_h\right)^2 + d\log Y^2.
\end{aligned}
$$

Using the fact that

$$
d\chi_h = \chi_h d\log\chi_h = \chi_h \left[\mathbb{E}_{b_h}[d\log p] + d\log Y_h - \sum_j \chi_j \left[\mathbb{E}_{b_j}[d\log p] + d\log Y_j\right]\right],
$$

we have evaluating at t_1,

$$
\begin{aligned}
d^2\log Y = & \sum_h \chi_h \left[\mathbb{E}_{b_h}[d\log p] + d\log Y_h - \sum_j \chi_j \left[\mathbb{E}_{b_j}[d\log p] + d\log Y_j\right]\right] d\log Y_h \\
& + d^2\log W - \sum_h \chi_h \left(d\log Y_h\right)^2 + d\log Y^2 \\
& = \sum_h \chi_h \mathbb{E}_{b_h}[d\log p] d\log Y_h + \sum_h \chi_h d\log Y_h d\log Y_h - \sum_h \chi_h d\log Y_h \sum_j \chi_j \left[\mathbb{E}_{b_j}[d\log p] + d\log Y_j\right] \\
& + d^2\log W - \sum_h \chi_h \left(d\log Y_h\right)^2 + d\log Y^2 \\
& = Cov_\chi(\mathbb{E}_{b_h}[d\log p], d\log Y_{(h)}) + d^2\log W.
\end{aligned}
$$

This implies that

$$d^2 \log W = d^2 \log Y - Cov_\chi(\mathbb{E}_{b_h}[d \log p], d \log Y_{(h)}).$$

Combining the first order and second order terms, changes in the social welfare function are given to a second order approximation by

$$d \log W + \frac{1}{2} d^2 \log W = d \log Y + \frac{1}{2} d^2 \log Y - \frac{1}{2} Cov_\chi(\mathbb{E}_{b_h}[d \log p], d \log Y_{(h)}).$$