Appendix A  Proofs

Proof of Lemma 1. By definition,

\[
EV^m = \log \frac{e(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1})}{e(p_{t_0}, v(p_{t_0}, I_{t_0}; x_{t_1}); x_{t_1})} \\
= \log \frac{e(p_{t_0}, v(p_{t_0}, I_{t_1}; x_{t_1}); x_{t_1}) e(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1})}{e(p_{t_0}, v(p_{t_0}, I_{t_0}; x_{t_1}); x_{t_1}) e(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1})} \\
= \log \frac{I_{t_0}}{I_{t_0}} \\
\]

To finish, rewrite

\[
\log \frac{e(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1})}{e(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1})} = - \int_{t_0}^{t_1} \frac{\partial}{\partial \log p} e(p, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}) \frac{d}{dt} p \, dt,
\]

and use the Shephard’s lemma to express the price elasticity of the expenditure function in terms of budget shares. If the path of prices between \( t_0 \) and \( t_1 \) is not differentiable, then construct a new a modified path of prices that is differentiable, and apply the integral to this modified path. Since the integral is path independent, it only depends on \( p_{t_0} \) and \( p_{t_1} \). Therefore any path that connects \( p_{t_0} \) and \( p_{t_1} \) gives the same integral.

Proof of Proposition 1. If the path of prices is continuously differentiable, we can combine Lemma 1 with the definition of real consumption.

Proof of Corollary 1. From Proposition 1, given any change in prices, income, and preferences, \( EV^m - \Delta \log Y = 0 \) if, and only if, \( b = b^{ev} \), which is the case if, and only if, budget shares do not depend on the level of utility and \( x \) – that is preferences are homothetic and stable.

Proof of Lemma 2. Differentiate real consumption

\[
\Delta \log Y = \int_{t_0}^{t_1} \frac{d}{dt} \log I(t) - \sum_{i \in N} b_i(p(t), u(t); x(t)) \frac{d}{dt} p_i \, dt
\]

twice with respect to \( t_1 \) and evaluate the derivative at \( t_1 = t_0 \). This yields the desired expression.
Proof of Proposition 2. By Lemma 1:

\[ EV = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} \frac{\partial \log e(p, v(p_{i_{1}}, x_{i_{1}}), x_{i_{1}})}{\partial \log p_{i}} d \log p \frac{d \log p}{d \log t} \]

Differentiate \( EV \) twice with respect to \( t_1 \) and evaluate the derivative at \( t_1 = t_0 \)

\[
\frac{d EV}{dt_1} = \frac{d \log I}{dt} - \sum_{i \in N} \frac{\partial \log e(p, v(p_{i_{1}}, x_{i_{1}}), x_{i_{1}})}{\partial \log p_{i}} \frac{d \log p_{i}}{d \log t} - \\
- \int_{t_0}^{t_1} \sum_{i \in N} d \log v \frac{\partial^2 \log e(p, v(p_{i_{1}}, x_{i_{1}}), x_{i_{1}})}{\partial \log u \partial \log p_{i}} d \log p_{i} - \\
- \int_{t_0}^{t_1} \sum_{i \in N} d \log x \frac{\partial^2 \log e(p, v(p_{i_{1}}, x_{i_{1}}), x_{i_{1}})}{\partial \log x \partial \log p_{i}} d \log p_{i}
\]

\[
\frac{d^2 EV}{dt_1^2} = - \sum_{i \in N} b_i \frac{d^2 \log p}{d \log t^2} - \sum_{i \in N} \sum_{j \in N} \frac{\partial^2 \log e(p, v(p_{i_{1}}, x_{i_{1}}), x_{i_{1}})}{\partial \log p_{i} \partial \log p_{j}} d \log p_{i} d \log p_{j} - \\
2 \sum_{i \in N} d \log v \frac{\partial^2 \log e(p, v(p_{i_{1}}, x_{i_{1}}), x_{i_{1}})}{\partial \log p \partial \log u} d \log p_{i} - 2 \sum_{i \in N} d \log x \frac{\partial^2 \log e(p, v(p_{i_{1}}, x_{i_{1}}), x_{i_{1}})}{\partial \log p \partial \log x} d \log p_{i} - \\
- \sum_{i \in N} \sum_{j \in N} b_i \frac{d \log p_{i} d \log p_{j}}{d \log t^2} - \sum_{i \in N} b_i \frac{d^2 \log p}{d \log t^2}
\]

By Lemma 2, the first two terms are equal to the second-order expansion of \( \Delta \log Y \), and the remaining terms are the bias.

Proof of Proposition 3. By Lemma 2, we have

\[ \Delta \log Y \approx \Delta \log I - \sum_{i} b_i \Delta \log p_i - \frac{1}{2} \sum_{i} \Delta b_i \Delta \log p_i. \]

Substitute (8) in place of \( \Delta b \) to get the desired expression. For the bias, note that Proposi-
tion 1 implies that
\[
EV - \Delta \log Y \approx -\frac{1}{2} \sum_i \left[ \Delta b_i - \sum_j \frac{\partial b^H_i}{\partial \log p_j} \Delta \log p_j \right] \Delta \log p_i
\]
where \( b^H \) is the Hicksian budget share (holding fixed utility and demand shifters). Using (8) in place of \( \Delta b \) above and the fact that \( \frac{\partial b^H}{\partial \log p_i} = (1 - \theta_0)b_i(1 - b_i) \) for \( i = j \) and \( \frac{\partial b^H}{\partial \log p_j} = \theta_0 b_i b_j \) for \( i \neq j \), yields the following
\[
\Delta \log EV - \Delta \log Y \approx -\frac{1}{2} \sum_{i \in N} \left[ (\varepsilon_i - 1)b_i \left( d \log I - \sum_{j \in N} b_j \Delta \log p_j \right) + b_i \Delta \log x_i \right] \Delta \log p_i,
\]
which can be rearranged to give the desired expression. \( \Box \)

Proof of Lemma 3. Setting nominal GDP to be the numeraire, we can write
\[
\Delta \log Y = -\int_{t_0}^{t_1} b' d \log p
\]
\[
= -\int_{t_0}^{t_1} b' \left[ -\Psi d \log A - \Psi^F d \log L + \Psi^F d \log \Lambda \right]
\]
\[
= \int_{t_0}^{t_1} b' \Psi d \log A + \int_{t_0}^{t_1} b' \Psi^F [d \log \Lambda - d \log L]
\]
\[
= \int_{t_0}^{t_1} \lambda' d \log A + \int_{t_0}^{t_1} \Lambda' d \log L - \int_{t_0}^{t_1} \Lambda d \log \Lambda
\]
\[
= \int_{t_0}^{t_1} \lambda' d \log A + \int_{t_0}^{t_1} \Lambda' d \log L
\]
where the second line uses Proposition 7, and we use the fact that Using \( \lambda' = b' \Psi \), \( \Lambda' = b' \Psi^F \), and \( b' \Psi^F d \log \Lambda = \Lambda' d \log \Lambda = 0 \) because the factor shares always sum to one: \( \sum_{f \in F} \Lambda_f = 1 \). \( \Box \)

Proof of Proposition 4. Recall that the macro equivalent variation at final preferences is defined by \( EV^M = \phi \), where
\[
V \left(A_{t_0}, e^\phi L_{t_0}; x_{t_1}\right) = V \left(A_{t_1}, L_{t_1}; x_{t_1}\right)
\]
Denote by \( p(A, L, x) \) goods prices under technologies \( A \), factor quantities \( L \), and preferences \( x \). Without loss of generality, we fix income at \( I \). We have \( p_{t_1} \equiv p \left(A_{t_1}, L_{t_1}, x_{t_1}\right) \) and
\[
\nu_{t_1} \equiv \nu \left(p_{t_1}, I, x_{t_1}\right) = V \left(A_{t_1}, L_{t_1}; x_{t_1}\right).
\]
Define a hypothetical economy with fictional households that have stable homothetic preferences defined by the expenditure function \( e^{ev}(p,u) = e(p,v_i; x_{t_1}) \frac{u}{v_{t_1}} \). Budget shares of this fictional consumer are \( b_i^{ev}(p) \equiv \frac{\partial e^{ev}(p,u)}{\partial p_i} = \frac{\partial e(p,v_i;x_{t_1})}{\partial p_i} \). Given any technology vector, in this hypothetical economy we denote the Leontief inverse matrix by \( \Psi^{ev} \) and sales shares by \( \lambda^{ev} \). Given technologies \( A_t \) and factor quantities \( L_t \), we denote prices in this hypothetical economy by \( p_t^{ev} \). Changes in prices in this hypothetical economy satisfy

\[
d \log p^{ev} = -\Psi^{ev} d \log A + \Psi^{ev} F d \log \Lambda^{ev}, \tag{24}
\]

where \( \Psi^{ev} \) is the fictitious Leontief inverse. Note that \( p (A_{t_1}, L_{t_1}, x_{t_1}) = p^{ev}(A_{t_1}, L_{t_1}) \) and \( p(A_{t_0}, e^p L_{t_0}, x_{t_1}) = p^{ev}(A_{t_0}, e^p L_{t_0}) \), where we used the fact that \( V(A_{t_0}, e^p L_{t_0}; x_{t_1}) = v_{t_1} \).

We will use the property that, with constant returns to scale, homothetic preferences, and constant income \( I \),

\[
p^{ev}(A, aL) = \frac{1}{a} p^{ev}(A, L)
\]

for every \( a > 0 \). Using the previous results,

\[
V(A_{t_0}, e^p L_{t_0}; x_{t_1}) = v(p(A_{t_0}, e^p L_{t_0}, x_{t_1}), I; x_{t_1}) \\
= v(p^{ev}(A_{t_0}, e^p L_{t_0}), I; x_{t_1}) \\
= v(e^{-p} p^{ev}(A_{t_0}, L_{t_0}), I; x_{t_1}) \\
= v(p^{ev}(A_{t_0}, L_{t_0}), e^p I; x_{t_1}),
\]

where the last equality used the fact that the value function is homogeneous of degree 0 in prices and income. We thus have

\[
v(p^{ev}(A_{t_0}, L_{t_0}), e^p I; x_{t_1}) = v(p^{ev}(A_{t_1}, L_{t_1}), I; x_{t_1}),
\]

which can be re-expressed using the expenditure function as

\[
EV^M = \log \frac{e(p^{ev}(A_{t_1}, L_{t_1}), v_{t_1}; x_{t_1})}{e(p^{ev}(A_{t_0}, L_{t_0}), v_{t_1}; x_{t_1})}.
\]

This observation is a key step in the proof. Macro welfare changes can be re-expressed as micro welfare changes given changes in equilibrium prices in a fictional economy with preferences represented by \( e^{ev}(p,u) \). As in the proof of Lemma 1, rewrite \( EV^M \) as

\[
EV^M = -\int_{t_0}^{t_1} \sum_{i \in N + F} \frac{\partial \log e(p, v_{t_1})}{\partial \log p_i} d \log p_i^{ev} = -\int_{t_0}^{t_1} \sum_{i \in N + F} b_i^{ev} d \log p_i^{ev}.
\]
Following the same steps as in the proof of Lemma 3 (for the hypothetical economy), we obtain

\[ EV^M = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i^{ev} d \log A_i + \int_{t_0}^{t_1} \sum_{f \in F} \lambda_i^{ev} d \log L_f. \]

\[ \square \]

In general, macro and micro welfare changes are not the same when preferences are unstable and nonhomothetic. However, when the PPF is linear, the following proposition shows that they coincide.

**Proposition 11 (Macro vs. Micro Welfare).** Macro and micro welfare changes are equal \((EV^m = EV^M)\) if preferences are stable and homothetic, or if factor income shares are constant (as in a one factor economy).

**Proof of Proposition 11.** By the proof of Proposition 4, \(EV^m = EV^M\) if and only if \(p^{ev}(A_t, L_t) = p(A_t, L_t, x_t)\). This condition is immediate if preferences are homothetic and stable. Consider now the case in which preferences are non-homothetic and/or unstable but factor income shares, \(\Lambda\), are constant. Then by Proposition 7, changes in prices in response to changes in \(A, L,\) and \(x\) are given by the following differential equation:

\[ d \log p = -\Psi d \log A - \Psi^F d \log L. \]

Furthermore, note that changes in \(\Psi\) are determined by changes in \(\Omega\) since \(\Psi = (I - \Omega)^{-1}\). Since every \(i \in N\) has constant returns to scale, changes in \(\Omega_{ij}\) depend only on changes in relative prices for every \(i \in N\). This means that changes in \(\Omega\) only depend on changes in relative prices, therefore changes in \(\Psi\) depend only on changes in relative prices. Since \(x\) and utility \(v\) do not appear in any of these expressions, this means that prices and incomes \(p(A, L, x)\) and \(I(A, L, x)\), relative to the numeraire, do not depend on \(x\) and \(v\). Thus, \(p^{ev}(A_t, L_t) = p(A_t, L_t, x_t)\).

\[ \square \]

**Proof of Lemma 4.** Differentiate real GDP,

\[ \Delta \log Y = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i(A(t); x(t)) \frac{d \log A_i}{dt} dt, \]

twice with respect to \(t_1\) and evaluate the derivative at \(t_1 = t_0\). This yields the desired expression.

\[ \square \]
Proof of Proposition 5. Following similar steps as in the proof of Proposition 3,

\[ EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[ \Delta \lambda_i - \sum_{j \in N} \frac{\partial \lambda_i^{cv}}{\partial \log A_j} \Delta \log A_j \right] \Delta \log A_i. \]

The term in square brackets is the change in sales shares due to changes in utility and demand shifters. This expression can be written as

\[ EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[ \Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A_i \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \] (25)

Proof of Proposition 6. Normalize nominal GDP to one. Applying Proposition 7 to a one-factor model yields

\[ d \log p = -\Psi d \log A, \]

so that relative prices do not respond to changes in demand or income.

To solve for \( \Delta \log Y \), use Lemma 4 in combination with the expression for \( d \log p \) and \( d \lambda \) in Proposition 7 in the case of one factor. To solve for \( EV^M \), by Proposition 11, \( EV^M = EV^m \). Solve for \( EV^m - \Delta \log Y \) by plugging the expression for \( d \log p \) into Proposition 2 and noting that \( b' = \Omega^{(0)} \).

Proof of Proposition 7. We normalize nominal GDP to be the numeraire. Then Shephard’s lemma implies that, for each \( i \in N \)

\[ d \log p_i = -d \log A_i + \sum_j \Omega_{ij} d \log p_j. \]

Furthermore, for \( i \in F \)

\[ d \log p_i = -d \log A_i + d \log \Lambda_i. \]

Combining these yields the desired expression for changes in prices

\[ d \log p = -\Psi d \log A + \Psi^F d \log \Lambda. \]
To get changes in sales shares, note that

\[ \lambda = b' \Psi \]
\[ d\lambda = d(b' \Psi) \]
\[ = b' \Psi d\Omega + db' \Psi \]

\[ \Omega_{ij} d \log \Omega_{ij} = (1 - \theta_i) \Omega_{ij} (d \log p_j - \sum_k \Omega_{jk} d \log p_k) \]
\[ d\Omega_{ij} = (1 - \theta_i) \text{Cov}_{\Omega(i)} (d \log p, I(j)) \]
\[ \sum_j d\Omega_{ij} \Psi_{jk} = (1 - \theta_i) \text{Cov}_{\Omega(i)} (d \log p, I(j)) \Psi_{jk} \]
\[ = (1 - \theta_i) \sum_j \text{Cov}_{\Omega(i)} (d \log p, \Psi_{jk} I(j)) \]
\[ [d\Omega \Psi]_{ik} = (1 - \theta_i) \text{Cov}_{\Omega(i)} (d \log p, \Psi(k)) \]

Meanwhile

\[ d \log b_i = (1 - \theta_0) \left( d \log p_i - \sum_i b_i d \log p_i \right) + (\epsilon_i - 1) d \log Y + d \log x_i \]
\[ = (1 - \theta_0) \text{Cov}_{\Omega(0)} (d \log p, I(i)) + \text{Cov}_{\Omega(0)} (\epsilon, I(i)) d \log Y + \text{Cov}_{\Omega(0)} (d \log x, I(i)) \]
\[ \sum_i db_i \Psi_{ik} = \text{Cov}_{\Omega(0)} \left( (1 - \theta_0) d \log p + \varepsilon d \log Y + d \log x, \Psi(k) \right) \]

Hence,

\[ d\lambda' = \lambda' d\Omega \Psi + db' \Psi \]

can be written as

\[ d\lambda_k = \sum_i \lambda_i (1 - \theta_i) \text{Cov}_{\Omega(i)} (d \log p, \Psi(k)) + \text{Cov}_{\Omega(0)} (\epsilon, \Psi(k)) d \log Y + \text{Cov}_{\Omega(0)} (d \log x, \Psi(k)). \]

\[ \square \]

Proof of Proposition 8. Consider intertemporal preferences

\[ V(A, L, K_0) = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s). \]
Comparing economies \( t \) and \( t' \), macro EV solves the following equation:

\[
V(A, \phi L, \phi K_0) = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s(A, \phi L, \phi K_0)) = \sum_{s=t'}^{\infty} \beta^{s-t'} u(C_s(A', L', K'_0)) = V(A', L', K'_0).
\]

Since the economy \( t' \) is in steady-state, we are looking for

\[
\sum_{s=t}^{\infty} \beta^{s-t} u(C_s(A, \phi L, \phi K_0)) = \frac{1}{1-\beta} u(C(A, L', K'_0)).
\]

Furthermore, since \((A, \phi L, \phi K_0)\) is also a steady-state (by Lemma 5 below), we are searching for

\[
u(C(A, \phi L, \phi K_0)) = u(C(A', L', K'_0))
\]

or

\[C(A, \phi L, \phi K_0) = C(A', L', K'_0)\].

Let \( v(p, I) \) be the static indirect utility function. Then we know that we are searching for

\[v(p(A, \phi L, \phi K_0), m) = v(p(A', L', K'_0), \phi m) = v(p(A', L', K'_0), m'),\]

where the first equality uses the fact within period relative goods prices do not depend on within period preferences (since the static PPF is linear). Hence,

\[
\phi = \frac{e(p(A, L, K_0), v_{t_1})}{e(p(A, L, K_0), v_{t_0})} = \frac{e(p(A, L, K_0), v_{t_1})}{e(p(A', L', K'_0), v_{t_1})} = \frac{e(p(A', L', K'_0), v_{t_1})}{e(p(A, L, K_0), v_{t_1})} = \exp EV^m.
\]

Hence, we can use micro \( EV^m \) to calculate the change in macro welfare.

\[\phi = \frac{e(p(A, L, K_0), v_{t_1})}{e(p(A, L, K_0), v_{t_0})} = \frac{e(p(A, L, K_0), v_{t_1})}{e(p(A', L', K'_0), v_{t_1})} = \frac{e(p(A', L', K'_0), v_{t_1})}{e(p(A, L, K_0), v_{t_1})} = \exp EV^m.\]

\[\text{Lemma 5.} \quad \text{The steady-state choice of capital (and investment) is the same for any homothetic and stable within-period preferences.}\]

\[\text{Proof.} \quad \text{Suppose intertemporal welfare is given by}
\]

\[
U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s),
\]

where \( C_s \) is some homothetic aggregator of within-period consumption goods. Since all goods are produced with constant-returns to scale and every good uses the same homothetic.
etic bundle of capital and labor, we can write the consumption aggregator as depending on

\[ C_s = G(L_{cs}, K_{cs}) \]

for some function constant-returns-to-scale function \( G \). Similarly, investment goods are created according to some constant returns to scale function

\[ I_s = H(L_{Is}, K_{Is}) \]

and the capital accumulation equation is

\[ K_{s+1} = (1 - \delta)(K_s + I_s). \]

The Lagrangean is

\[ \mathcal{L} = \sum_{s=t}^{\infty} \beta^{s-t} \left[ u(C_s) + \mu_s (G(L_{cs}, K_{cs}) - C_s) + \kappa_s (K_{s+1} - (1 - \delta)(K_s + H(L_{Is}, K_{Is}))) + \rho_s (L_s - L_{cs} - L_{Is}) + \psi_t (K_s - K_{cs} - K_{Is}) \right] \]

The first order conditions are

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial C_s} : u'(C_s) &= \mu_s \\
\frac{\partial \mathcal{L}}{\partial K_{s+1}} : \kappa_s - \beta \kappa_{s+1} (1 - \delta) + \beta \psi_{s+1} &= 0 \\
\frac{\partial \mathcal{L}}{\partial K_{Is}} : -\kappa_s (1 - \delta) \frac{\partial H_s}{\partial K_{Is}} &= \psi_s = \mu_s \frac{\partial G}{\partial K_{cs}} \\
\frac{\partial \mathcal{L}}{\partial K_{cs}} : \mu_s \frac{\partial G}{\partial K_{cs}} &= \psi_s \\
\frac{\partial \mathcal{L}}{\partial L_{cs}} : \mu_s \frac{\partial G}{\partial L_{cs}} &= \rho_s \\
\frac{\partial \mathcal{L}}{\partial L_{Is}} : -\kappa_s (1 - \delta) \frac{\partial H}{\partial L_{Is}} &= \rho_s.
\end{align*}
\]

Hence

\[ -\kappa_s (1 - \delta) = \mu_s \frac{\partial G/\partial K_{cs}}{\partial H_s/\partial K_{Is}} \]
\[ \kappa_s = \beta \kappa_{s+1}(1 - \delta) - \beta \psi_{s+1} \]

\[ u'(C_s) = \beta(1 - \delta) u'(C_{s+1}) \frac{\partial G/\partial K_{cs+1}}{\partial G/\partial K_{cs}} \partial H_s / \partial K_{Is} \left[ (\partial H_s / \partial K_{Is+1})^{-1} + 1 \right]. \]

In steady state we have

\[ 1 = \beta(1 - \delta) \left[ 1 + \partial H_s / \partial K_{Is} \right]. \]

Hence, the capital stock and investment in steady-state are pinned down by the following 5 equations in 5 unknowns \((K_C, K_I, K, L_C, L_I)\):

\[ 1 = \beta(1 - \delta) \left[ 1 + \partial H / \partial K_I \right], \]

\[ K_C / L_C = K_I / L_I, \]

\[ K = K_C + K_I, \]

\[ L = L_I + L_C, \]

\[ \delta K = (1 - \delta) H(L_I, K_I). \]

Since \(G\) does not appear in any of these equations, the steady-state investment and capital stock do not depend on the shape of the within-period utility function \(G\).

Proof of Proposition 9. Start by setting nominal GDP to be the numeraire. To model the industry-structure, for each industry \(I\), add two new CES aggregators. One buys the good for the household and one buys the good for firms. Let the price of the household aggregator be given by \(p^h_I\) and the price of the non-household aggregator be \(p^f_I\) and let \(p_I\) be the price of the original industry. Let firm \(i\)'s share of industry \(I\) from household expenditures be \(b_{iI}\). Let the expenditure share of other firms on firm \(i\) be \(s_{iI}\). We have

\[ \sum_{i \in I} b_{iI} = 1 \]

\[ \sum_{i \in I} s_{iI} = 1. \]

Let \(\lambda^c_I\) and \(\lambda^f_I\) be sales of industry \(I\) to households and firms. Then we have

\[ d\lambda_I = d\lambda^c_I + d\lambda^f_I. \]
The sales of an individual firm $i$ in industry $I$ is given by

$$
\lambda_i = b_i \lambda_i^c + s_i \lambda_i^f
$$

$$
d\lambda_i = db_i \lambda_i^c + b_i d \lambda_i^c + ds_i \lambda_i^f + s_i d \lambda_i^f
$$

$$
\partial \lambda_i \partial \log x = b_i \partial \lambda_i^c \partial \log x + \partial \lambda_i^f \partial \log x
$$

$\partial \lambda_i \partial \log x = Cov_{b_i}(d \log x, (1 - \sigma) d \log A, I(i))$

$\partial \lambda_i \partial \log x = Cov_{b_i}((1 - \sigma) d \log A, I(i))$.

The gap between macro welfare and real GDP, $EV^M - \Delta \log Y$, is approximately given by

$$
\frac{1}{2} d \log x \frac{\partial \lambda_i}{\partial \log x} d \log A = \frac{1}{2} \sum_{i \in N} \left[ \sum_{j \in N} d \log x \frac{\partial \lambda_i}{\partial \log x} \right] d \log A_i
$$

where the sums can be re-written as

$$
\sum_{i \in N} \left[ \sum_{j \in N} d \log x \frac{\partial \lambda_i}{\partial \log x} \right] d \log A_i = \sum_{i \in N} \left[ d \log x \frac{\partial b_i}{\partial \log x} \lambda_i^c d \log A_i + b_i d \log x \frac{\partial \lambda_i^c}{\partial \log x} d \log A_i
$$

$$
+ d \log x \frac{\partial s_i}{\partial \log x} \lambda_i^f d \log A_i + s_i d \log x \frac{\partial \lambda_i^f}{\partial \log x} d \log A_i \right].
$$

The individual terms of this expression are:

$$
\sum_{i \in N} \left[ d \log x \frac{\partial b_i}{\partial \log x} \lambda_i^c d \log A_i \right] = \sum_{i \in N} Cov_{b_i}(d \log x, I(i), \lambda_i^c d \log A_i
$$

$$
= Cov_{b_i}(d \log x, \sum_{i \in N} I(i) d \log A_i \lambda_i^c
$$

$$
= Cov_{b_i}(d \log x, I(i), d \log A_i \lambda_i^c);
$$

$$
\sum_{i \in N} \left[ b_i d \log x \frac{\partial \lambda_i^c}{\partial \log x} d \log A_i \right] = E_{b_i}(d \log A_i) d \log x \frac{\partial \lambda_i^c}{\partial \log x};
$$

$$
\sum_{i \in N} d \log x \frac{\partial s_i}{\partial \log x} \lambda_i^f d \log A_i = 0;
$$

and

$$
\sum_{i} s_i d \log x \frac{\partial \lambda_i^f}{\partial \log x} d \log A_i = E_{s_i}(d \log A_i) d \log x \frac{\partial \lambda_i^f}{\partial \log x}.
$$

Of the four terms, two depend on changes on industry-level sales shares (the sum of which is denoted by $\Theta$ in the proposition), one of them is zero, and the remaining one
(the first term) is the within-industry covariance of supply and demand shocks in the proposition.

Proof of Proposition 10. Consider a household with preferences given by

\[ C = \left( \int_0^{x^*} c(x) \frac{e^{-1}}{\sigma} dx \right)^{\frac{\sigma}{\sigma - 1}}. \]

Note that budget shares are

\[ \lambda^{ev}(x, t, t_1) = \frac{p(x, t)^{1-\sigma}}{\left( \int_0^{x^*(t_1)} p(x, t)^{1-\sigma} dx \right)} \]

\[ \lambda(x, t) = \frac{p(x, t)^{1-\sigma}}{\left( \int_0^{x^*(t)} p(x)^{1-\sigma} dx \right)} \]

Hence

\[ EV^m = \frac{I}{\left( \int_0^{x^*(t_1)} p(x)^{1-\sigma} dx \right)^{1-\sigma}}. \]

Next

\[ \Delta \log EV^m = \Delta \log I - \int_{t_0}^{t_1} \int_0^{x^*(t_1)} \lambda^{ev}(x, t, t_1) \frac{d \log p(x, t)}{dt} dx dt. \]

Without loss of generality, let's normalize changes in nominal income to zero. Let \( \partial_i \lambda^{ev} \) refer to the partial derivative of \( \lambda^{ev} \) with respect to its \( i \)th argument. Differentiating and evaluating at the initial point, we get

\[ \frac{d \log EV^m}{dt_1} = - \int_{t_0}^{t_1} \int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x, t, t_1) \frac{d \log p(x, t)}{dt} dx dt \]

\[ - \int_{t_0}^{t_1} \lambda^{ev}(x^*(t_1), t, t_1) \frac{d \log p(x^*(t_1), t)}{dt_1} dx^* dt_1 - \lambda^{ev}(x^*(t_1), t, t_1) \frac{d \log p(x^*(t_1), t_1)}{dt_1} dx^* dt_1 \]

\[ \frac{d^2 \log EV^m}{dt_1^2} = - \int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x, t_1, t_1) \frac{d \log p(x, t_1)}{dt_1} dx - \lambda^{ev}(x^*(t_1), t_1, t_1) \frac{d \log p(x^*(t_1), t_1)}{dt_1} dx^* \]

\[ - \lambda^{ev}(x^*(t_1), t_1, t_1) \frac{d x^* d \log p(x^*, t_1)}{dt_1 dt_1} - \int_0^{x^*(t_1)} \frac{d \lambda^{ev}(x, t_1, t_1)}{dt_1} \frac{d \log p(x, t_1)}{dt_1} dx \]

\[ - \int_{t_0}^{t_1} \lambda^{ev}(x, t_1, t_1) \frac{d^2 \log p(x, t_1)}{dt_1^2} dx. \]
Evaluating at the initial point this simplifies to

\[
\begin{align*}
\frac{d \log EV^m}{dt_1} &= - \int_0^{x^*} \lambda(x) \frac{d \log p(x, t)}{dt} dx \\
\frac{d^2 \log EV^m}{dt_1^2} &= - \int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x, t_1, t_1) \frac{d \log p(x, t_1)}{dt_1} dx - \lambda^{ev}(x^*(t_1), t_1, t_1) \frac{d \log p(x^*(t_1), t_1)}{dt_1} \frac{dx^*}{dt_1} \\
&- \lambda^{ev}(x^*(t_1), t_1, t_1) \frac{dx^*}{dt_1} \frac{d \log p(x^*(t_1))}{dt_1} dx - \int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x, t_1, t_1) \frac{d \log p(x, t_1)}{dt_1} dx \\
&- \int_0^{x^*(t_1)} \lambda^{ev}(x, t_1, t_1) \frac{d^2 \log p(x, t_1)}{dt_1^2} dx
\end{align*}
\]

We note that

\[
\lambda^{ev}(x, t, t_1) = \frac{p(x, t)^{1-\sigma}}{\left( \int_0^{x^*(t)} p(x, t)^{1-\sigma} dx \right)}
\]

\[
\frac{\partial \log \lambda^{ev}(x, t, t_1)}{\partial t} = (1 - \sigma) \left( \frac{d \log p(x, t)}{dt} \right) - \int_0^{x^*(t_1)} \lambda(x, t) \frac{d \log p(x, t)}{dt} dx.
\]

\[
\frac{\partial_3 \log \lambda^{ev}(x, t, t_1)}{\partial t_1} = \left( -\lambda(x^*, t) \frac{dx^*}{dt_1} \right).
\]

Meanwhile, real consumption changes are given by

\[
\begin{align*}
\log Y &= - \int_{t_0}^{t_1} \int_0^{x^*(t)} \lambda(x, t) \frac{d \log p}{dt} dx dt \\
\frac{d \log Y}{dt_1} &= - \int_0^{x^*(t_1)} \lambda(x, t_1) \frac{d \log p}{dt_1} dx \\
\frac{d^2 \log Y}{dt_1^2} &= -\lambda(x^*(t_1), t_1) \frac{dx^*}{dt_1} \frac{d \log p}{dt_1} dx - \int_0^{x^*(t_1)} \frac{d \lambda(x, t_1)}{dt_1} \frac{d \log p}{dt_1} dx - \int_0^{x^*(t_1)} \lambda(x, t_1) \frac{d^2 \log p}{dt_1^2} dx
\end{align*}
\]

where

\[
\frac{d \log \lambda(x, t_1)}{dt_1} = (1 - \sigma) \left( \frac{d \log p(x, t)}{dt} \right) - \int_0^{x^*(t_1)} \lambda(x, t) \frac{d \log p(x, t)}{dt} dx - \lambda(x^*, t) \frac{dx^*}{dt}.
\]
\[
\begin{align*}
\frac{d \log \text{EV}^m}{dt_1} &= \frac{d \log Y}{dt_1} \\
\frac{d^2 \log \text{EV}^m}{dt_1^2} &= \frac{d^2 \log Y}{dt_1^2} - \int_0^{x^*(t_1)} \partial_3 \lambda_{\text{ev}}(x, t_1, t_1) \frac{d \log p(x, t_1)}{dt_1} dx - \lambda_{\text{ev}}(x^*(t_1), t_1, t_1) \frac{d \log p(x^*(t_1), t_1)}{dt_1} dx^* \\
&= \frac{d^2 \log Y}{dt_1^2} + \lambda(x^*) \frac{dx^*}{dt_1} \left[ \int_0^{x^*(t_1)} \lambda(x) \frac{d \log p(x, t_1)}{dt_1} dx - \frac{d \log p(x^*(t_1), t_1)}{dt} \right] \\
&= \frac{d^2 \log Y}{dt_1^2} + \lambda(x^*) \frac{dx^*}{dt_1} \left[ \mathbb{E}_{\lambda} \left[ \frac{d \log p}{dt} \right] - \frac{d \log p(x^*)}{dt} \right].
\end{align*}
\]

\[\square\]

**Appendix B  Extension to Other Welfare Measures**

Our baseline measure of welfare changes is equivalent variation under final preferences. Alternatively, we could measure changes in welfare using compensating (instead of equivalent) variation, or by using initial (rather than final) preferences. We focus on equivalent variation with final preferences since it uses indifference curves in the final allocation to make welfare comparisons (that is, preferences “today” for growth-accounting purposes). In this appendix, we show that our methods generalize to the other welfare measures. If preferences are homothetic, then the expenditure function can be written as \( e(p, u; x) = e(p; x) u \), so equivalent and compensating variation are equal. If preferences are stable, then the expenditure function can be written as \( e(p, u; x) = e(p, u) \), so equivalent variation under initial and final preferences are equal (and the same is the case for compensating variation).

Recall that when preferences are homothetic, then the expenditure function can be written as \( e(p, u; x) = e(p; x) u \). Hence, in this case, for any fixed \( x \), compensating variation is equal to equivalent variation.

**B.1 Micro welfare changes**

We consider four alternative measures of micro welfare changes. For each measure, we present expression for global welfare changes and the approximate gap with real consumption.
The compensating variation with initial preferences is $CV^m(p_{t_0}, I_{t_0}, p_{t_1}, I_{t_1}; x_{t_0}) = \phi$, where $\phi$ solves

$$v(p_{t_1}, e^{-\phi} I_{t_1}; x_{t_0}) = v(p_{t_0}, I_{t_0}; x_{t_0}).$$

(26)

The analog to (7) in Lemma 1 is

$$CV^m = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} b^c v_i d \log p_i,$$

(27)

where $b^c_i(p) \equiv b_i(p, v(p_{t_0}, I_{t_0}; x_{t_0}); x_{t_0})$.

Whereas $EV^m$ weights price changes by hypothetical budget shares evaluated at current prices for fixed final preferences and final utility, $CV^m$ uses budget shares evaluated at current prices for fixed initial preferences and initial utility. An alternative way of calculating $CV^m$ is to reverse the flow of time (the final period corresponds to the initial period), calculate the baseline EV measure under this alternative timeline, and then set $CV^m = -EV^m$.

We now briefly describe how to calculate $b^c$ to apply (33). For ex-ante counterfactuals, where $b(t_0)$ is known, we can construct $b^c(p)$ between $t_0$ and $t_1$ by iterating on (9) starting at $t_0$ and going forward to $t_1$. For ex-post counterfactuals, $b(t_0)$ can be obtained from past data, so we can construct $b^c(p)$ by iterating on (9) starting at $t_0$ and going forward to $t_1$.

To a second-order approximation

$$\Delta \log CV^m \approx \Delta \log I - b' \Delta \log p - \frac{1}{2} \sum_{i \in N} \left[ \Delta \log p \frac{\partial b_i}{\partial \log p} \right] \Delta \log p$$

(28)

$$\approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[ \Delta \log x \frac{\partial b_i}{\partial \log x} + \Delta \log v \frac{\partial b_i}{\partial \log u} \right] \Delta \log p.$$  

(29)

Recall that changes in budget shares due to non-price factors are multiplied by $1/2$ in real consumption. However, they are multiplied by 0 in $CV^m$, since $CV^m$ is based on budget shares at initial preferences and initial utility.

Combining (10) and (28), we see that up to a second order approximation,

$$0.5 (EV^m + CV^m) \approx \Delta \log Y.$$  

That is, locally (but not globally) changes in real consumption equal a simple average of equivalent variation under final preferences and compensating variation under initial preferences.

Alternatively, we can measure the change in welfare using the micro equivalent variation
with initial preferences, \( EV^m(p_{t0}, I_{t0}, p_{t1}, I_{t1}; x_{t0}) = \phi \) where \( \phi \) solves
\[
v(p_{t1}, I_{t1}; x_{t0}) = v(p_{t0}, e^{\phi} I_{t0}; x_{t0}).
\] (30)

Globally, changes in welfare are
\[
EV^m = \Delta \log I - \int_{t0}^{t1} \sum_{i \in N} b_{i}^{cv} d \log p_i,
\] (31)

where \( b_{i}^{cv} (p) \equiv b_{i} (p, v(p_{t1}, I_{t1}; x_{t0}); x_{t0}) \). The gap between changes in welfare and real consumption is, up to a first order approximation,
\[
\Delta \log EV^m - \Delta \log Y \approx \frac{1}{2} \sum_{i \in N} \left[ -\Delta \log x' \frac{\partial b_{i}}{\partial \log x} + \Delta \log v \frac{\partial b_{i}}{\partial \log u} \right] \Delta \log p.
\]

Finally, the change in welfare measured using the micro compensating variation with final preferences is \( CV^m(p_{t0}, I_{t0}, p_{t1}, I_{t1}; x_{t1}) = \phi \) where \( \phi \) solves
\[
v(p_{t1}, e^{-\phi} I_{t1}; x_{t1}) = v(p_{t0}, I_{t0}; x_{t1}).
\] (32)

Globally, changes in welfare are given by
\[
CV^m = \Delta \log I - \int_{t0}^{t1} \sum_{i \in N} b_{i}^{cv} d \log p_i,
\] (33)

where \( b_{i}^{cv} (p) \equiv b_{i} (p, v(p_{t0}, I_{t0}; x_{t0}); x_{t0}) \). The gap between changes in welfare and real consumption is, up to a first order approximation,
\[
\Delta \log CV^m - \Delta \log Y \approx \frac{1}{2} \sum_{i \in N} \left[ \Delta \log x' \frac{\partial b_{i}}{\partial \log x} + \Delta \log v \frac{\partial b_{i}}{\partial \log u} \right] \Delta \log p.
\]

Note for EV with initial preferences or CV with final preferences, we must be able to separate demand instability from income effects. For this reason, to compute welfare changes, the elasticities of substitution are not sufficient — we must also know income elasticities or the demand shocks.

Finally, we note that real consumption can be interpreted as representing an alternative measure of welfare, defined as the sum of instantaneous welfare changes using current preferences at each point in time. In particular, real consumption can be written
\[
\Delta \log Y = \int_{t_0}^{t_1} \frac{\partial EV^m(p(t), I(t), p(t), I(t); x(t))}{\partial p_{t_i}} dp(t) + \frac{\partial EV^m(p(t), I(t), p(t), I(t); x(t))}{\partial I_{t_i}} dI(t),
\]

where, at every \( t \in [t_0, t_1] \), the integrand is the instantaneous welfare change in response to changes in prices and income, \( dp \) and \( dI \), measured using equivalent variation with preferences \( x(t) \). In contrast to our welfare measures, this measure does not represent the welfare change over a single preference ordering, and is path dependent (it does not only depend only on initial and final income and prices).

### B.2 Macro welfare changes

For each alternative micro welfare measure there is a corresponding macro welfare measure. For example, the macro compensating variation with initial preferences is

\[
CV^M(A_{t_0}, L_{t_0}, A_{t_1}, L_{t_1}; x_{t_0}) = \phi,
\]

where \( \phi \) solves

\[
V(A_{t_0}, L_{t_0}; x_{t_0}) = V(A_{t_1}, e^{-\phi} L_{t_1}; x_{t_0}).
\]

In words, \( CV^M \) is the proportional change in final factor endowments necessary to make a planner with preferences \( \succeq_{x_{t_0}} \) indifferent between the initial PPF \( (A_{t_0}, L_{t_0}) \) and PPF defined by \( (A_{t_1}, e^{-\phi} L_{t_1}) \).

Equation (14) in Proposition 4 applies using \( \lambda^{cv}(A) \), the sales shares in a fictional economy with the PPF \( A, L \) but where consumers have stable homothetic preferences represented by the expenditure function \( e^{cv}(p, u) = e(p, v_{t_0}, x_{t_0}) \frac{u}{v_{t_0}} \) where \( v_{t_0} = v(p_{t_0}, I_{t_0}; x_{t_0}) \).

Growth accounting for welfare is based on hypothetical sales shares evaluated at current technology but for fixed initial preferences and initial utility. The only information on preferences we need to know is elasticities of substitution at the final allocation. As discussed above, \( CV^M \) is equal to \( -EV^M \) if we reverse the flow of time.

The gap between changes in welfare and real GDP is, to a second-order approximation (the analog of equation 16 in Proposition 5) is

\[
CV^M \approx \Delta \log Y - \frac{1}{2} \sum_{i \in N} \left[ \Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{34}
\]

We can also define macro equivalent variation with initial preferences, \( EV^M(A_{t_0}, L_{t_0}, A_{t_1}, L_{t_1}; x_{t_0}) = \phi \), where \( \phi \) solves

\[
V(A_{t_1}, L_{t_1}; x_{t_0}) = V(A_{t_0}, e^\phi L_{t_0}; x_{t_0}).
\]

Growth accounting for welfare is based on hypothetical sales shares evaluated at cur-
rent technology for fixed initial preferences and final utility. In contrast to our previous measures, in order to implement this measure we must know initial demand shifters or income effects. The gap between changes in welfare and real GDP is

\[
EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[ -\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{35}
\]

Finally, define macro compensating variation with final preferences, \(CV^M(A_{t0}, L_{t0}, A_{t1}, L_{t1}; x_{t1}) = \phi\), where \(\phi\) solves

\[
V(A_{t0}, L_{t0}; x_{t1}) = V(A_{t1}, e^{-\phi}L_{t1}; x_{t1}).
\]

Growth accounting for welfare is based on hypothetical sales shares evaluated at current technology for fixed final preferences and initial utility, which requires information on demand shifters or income effects. The gap between changes in welfare and real GDP is

\[
CV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[ \Delta \log x' \frac{\partial \lambda_i}{\partial \log x} - \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{36}
\]

**Appendix C  Non-homothetic CES preferences**

This appendix provides a derivation of the log-linearized expression (8). Changes in Marshallian budget share are given by

\[
d \log b_i^M = d \log p_i - d \log I + \sum_j \epsilon_{ij}^M d \log p_j + \epsilon_i^w d \log I + d \log x_i,
\]

\[
= d \log p_i - d \log I + \sum_j \left( \epsilon_{ij}^H - \epsilon_i^w b_j \right) d \log p_j + \epsilon_i^w d \log I + d \log x_i,
\]

where \(\epsilon^H\) and \(\epsilon^M\) are the Hicksian and Marshallian price elasticities, \(\epsilon^w\) are the income elasticities, and \(d \log x_i\) is a residual that captures changes in shares not attributed to changes in prices or income. The third line is an application of Slutsky’s equation. When preferences are non-homothetic CES, then the Hicksian demand curve can be written as

\[
c_i = \gamma_i \left( \frac{p_i}{\sum_j p_j c_j} \right)^{-\theta_0} u_i,
\]

where \(\gamma_i\) and \(\zeta_i\) are some parameters. The Hicksian price elasticity for \(j \neq i\) is

\[
\frac{\partial \log c_i}{\partial \log p_j} = \epsilon_{ij}^H = \theta_0 \frac{p_j c_j}{I} = \theta_0 b_j.
\]
Using this fact and the identity $e^H_{ij} = -\sum_{j \neq i} e^H_{ij}$, we can rewrite changes in budget shares as

$$d \log b^M_i = \sum_j \left( e^H_{ij} - e^w_i b_j \right) d \log p_j + d \log p_i + (e^w_i - 1) d \log I + d \log x$$

$$= \left( 1 - \sum_{j \neq i} e^H_{ij} \right) d \log p_i + \sum_{j \neq i} e^H_{ij} d \log p_I + e^w_i \left[ d \log I - \sum_j b_j d \log p_j \right] + d \log x_i$$

$$= \left( 1 - \sum_{j \neq i} \theta_0 b_j \right) d \log p_i + \sum_{j \neq i} \theta_0 b_j d \log p_I + e^w_i \left[ d \log I - \sum_j b_j d \log p_j \right] + d \log x_i$$

$$= \left( 1 - \theta_0 (1 - b_i) \right) d \log p_i + \sum_{j \neq i} \theta_0 b_j d \log p_I + e^w_i \left[ d \log I - \sum_j b_j d \log p_j \right] + d \log x_i$$

$$= \left( 1 - \theta_0 \right) \left[ d \log p_i - \sum_j b_j d \log p_j \right] + (e^w_i - 1) \left[ d \log I - \sum_j b_j d \log p_j \right] + d \log x_i.$$

### Appendix D  Additional details on the Baumol application

According to our results in Section 5, structural transformation caused by income effects or demand instability reduced welfare by roughly twice as much as structural transformation caused by substitution effects. To understand why the necessary adjustment is roughly twice as big, consider the second-order approximation in Proposition 4:

$$\Delta \log TFP^{welfare} \approx \Delta \log TFP + \frac{1}{2} \left[ \sum_{i \in N} \frac{\partial \lambda_i}{\partial \log x} \Delta \log x + \frac{\partial \lambda_i}{\partial \log v} \Delta \log v \right] \Delta \log A_i,$$  \(37\)

where

$$\Delta \log TFP \approx \sum_{i \in N} \lambda_{i,0} \Delta \log A_i + \frac{1}{2} \sum_{i \in N} \Delta \lambda_i \Delta \log A_i.$$

If changes in sales shares are due entirely to demand-driven factors, then the term in square brackets in \(37\) is equal to $\sum_{i \in N} \Delta \lambda_i \Delta \log A_i$, so

$$\Delta \log TFP^{welfare} \approx \sum_{i \in N} \lambda_{i,0} \Delta \log A_i + \sum_{i \in N} \Delta \lambda_i \Delta \log A_i.$$
In other words, the adjustment to the initial sales shares must be roughly twice as large as the adjustment to the initial sales shares caused by substitution effects.\footnote{These second-order approximations are more accurate if changes in sales shares are well-approximated by linear time trends, and the surprising accuracy of the second-order approximation is a result of this fact.}

In practice, both substitution effects and non-homotheticities are likely to play an important role in explaining structural transformation. To dig deeper into the size of the welfare adjustment outside our two polar cases, we use a simplified version of the model introduced in Section 4 calibrated to the US economy, accounting for input-output linkages and complementarities, and use the model to quantify the size of the welfare-adjustment as a function of the elasticities of substitution.

Remarkably, Proposition 4 implies that to compute the welfare-relevant change in TFP, we must only supply the information necessary to compute $\lambda^{CV}$. That is, since we know sales shares in the terminal period 2014, we do not need to model the non-homotheticities or demand-shocks themselves, and the exercise requires no information on the functional form of non-homotheticities or the slope of Engel curves or magnitude of income elasticities conditional on knowing the elasticities of substitution.

We map the model to the data as follows. We assume that the constant-utility final demand aggregator has a nested-CES form. There is an elasticity $\theta_0$ across the three groups of industries: primary, manufacturing, and service industries. The inner nest has elasticity of substitution $\theta_1$ across industries within primary (2 industries), manufacturing (24 industries), and services (35 industries).\footnote{In order to map this nested structure to our baseline model, good 0 is a composite of good 1-3, where good 1 is a composite of primary industries, good 2 is a composite of manufacturing industries, and good 3 is a composite of service industries. Goods 4-65 are the disaggregated industries. Finally, good 66 is the single factor of production.} Production functions are also assumed to have nested-CES forms: there is an elasticity of substitution $\theta_2$ between the bundle of intermediates and value-added, and an elasticity of substitution $\theta_3$ across different types of intermediate inputs. For simplicity, we assume there is only one primary factor of production (a composite of capital and labor). We solve the non-linear model by repeated application of Proposition 7 in the fictional economy with stable and homothetic preferences.

We calibrate the CES share parameters so that the model matches the 2014 input-output tables provided by the BEA. For different values of the elasticities of substitution $(\theta_0, \theta_1, \theta_2, \theta_3)$ we feed changes in industry-level TFP (going backwards, from 2014 to 1947) into the model and compute the resulting change in aggregate TFP. This number represents the welfare-relevant change in aggregate TFP. We report the results in Table 2.

The first column in Table 2 shows the change in welfare-relevant TFP assuming that
Table 2: Percentage change in measured and welfare-relevant TFP in the US from 1947 to 2014.

<table>
<thead>
<tr>
<th>(θ₀, θ₁, θ₂, θ₃)</th>
<th>(1,1,1,1)</th>
<th>(0.5,1,1,1)</th>
<th>(1,0.5,1,1)</th>
<th>(1,1,0.5,1)</th>
<th>(1,1,1,0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Welfare TFP</td>
<td>46%</td>
<td>46%</td>
<td>54%</td>
<td>48%</td>
<td>55%</td>
</tr>
<tr>
<td>Measured TFP</td>
<td>60%</td>
<td>60%</td>
<td>60%</td>
<td>60%</td>
<td>60%</td>
</tr>
</tbody>
</table>

there are no substitution effects (all production and consumption functions are Cobb-Douglas). In this case, all changes in sales shares in the data are driven by non-homotheticities or demand-instability, and hence welfare-relevant TFP has grown more slowly than measured TFP, exactly as discussed in the previous section. The other columns show how the results change given lower elasticities of substitution. As we increase the strength of complementarities (so that substitution effects are active), the implied non-homotheticities required to match changes in sales shares in the data are weaker. This in turn reduces the gap between measured and welfare-relevant productivity growth.

Table 2 also shows that not all elasticities of substitution are equally important. The results are much more sensitive to changes in the elasticity of substitution across more disaggregated categories, like materials, than aggregated categories, like agriculture, manufacturing, and services.

To see why the results in Table 2 are differentially sensitive to changes in different elasticities of substitution, combine Propositions 6 and 11 to obtain the following second-order approximation:

\[
\Delta \log TFP_{\text{welfare}} \approx \sum_i \lambda_i \Delta \log A_i + \frac{1}{2} \sum_{j \in \{0\} + N} (\theta_j - 1) \lambda_j \text{Var}_{\Omega(i)} \left( \sum_{k \in N} \Psi(k) \Delta \log A_i \right). \tag{38}
\]

The second term is half the sum of changes in Domar weights due to substitution effects (i.e. changes in welfare-relevant sales shares) times the change in productivities. Note that changes in these welfare-relevant sales shares are linear in the microeconomic elasticities of substitution. The importance of some elasticity \( \theta \) depends on

\[
\sum_j \lambda_j \text{Var}_{\Omega(i)} \left( \sum_{k \in N} \Psi(k) \Delta \log A_i \right),
\]

where the index \( j \) sums over all CES nests whose elasticity of substitution is equal to \( \theta \) (i.e. all \( j \) such that \( \theta_j = \theta \)). Therefore, elasticities of substitution are relatively more potent if: (1) they control substitution over many nests with high sales shares, or (2) if the
nests corresponding to those elasticities are heterogeneously exposed to the productivity shocks.

We compute the coefficients in (38) for our model’s various elasticities using the IO table at the end of the sample. The coefficient on \((\theta_0 - 1)\), the elasticity of substitution between agriculture, manufacturing, and services in consumption is only 0.01. This explains why the results in Table 2 are not very sensitive to this elasticity. On the other hand, the coefficient on \((\theta_1 - 1)\), the elasticity across disaggregated consumption goods, is much higher at 0.21. The coefficient on \((\theta_2 - 1)\), the elasticity between materials and value-added bundles is 0.07. Finally, the coefficient on \((\theta_3 - 1)\), the elasticity between disaggregated categories of materials is 0.25. This underscores the fact that elasticities of substitution are more important if they control substitution in CES nests which are very heterogeneously exposed to productivity shocks — that is, nests that have more disaggregated inputs.

According to equation (38), setting \(q_1 = q_2 = q_3 = 1\) (which is similar to abstracting from heterogeneity within the three broader sectors and heterogeneity within intermediate inputs), then \(q_0\) is the only parameter that can generate substitution effects in the model. This may help understand why more aggregated models of structural transformation (e.g. Buera et al., 2015 and Alder et al., 2019) require low values of \(q_0\) to account for the extent of sectoral reallocation in the data.

Appendix E  Micro and Macro Welfare in the Covid-19 Application

Table 3 displays welfare changes between January 2020 and May 2020 in the calibrated model of section 5.3. We report separately micro and macro welfare based on initial and final preferences. Recall that micro and macro welfare are not equal in this economy because the PPF is nonlinear (because there are multiple factors). For comparison, we also report the change in real consumption assuming supply and demand shocks arrive simultaneously (as in the last row of Table 3).

The numbers for macro welfare coincide with the changes in real GDP reported in Table 3 under different assumptions on the timing of the demand and supply shocks. Recall that, according to Corollary 4, welfare at initial preferences is equal to real GDP when supply shocks arrive first, and welfare at final preferences is equal to real GDP when demand shocks arrive first and then the supply shocks. Since supply and demand shocks are positively correlated, the decline in welfare is larger under initial preferences.
than under final preferences.

On the other hand Table 3 shows that the drop in micro welfare is larger under final preferences than under initial preferences. This is because, as shown in our analytic example 4, demand shocks reduce welfare in the presence of decreasing returns to scale (since demand shocks increase the price of goods that consumers value more over time).

Focusing on final preferences, which are more relevant, we see that chained real consumption under-measures welfare losses for the microeconomic change in welfare (comparing initial and final budget sets) and it over-measures welfare losses for the macroeconomic change in welfare (comparing initial and final PPFs). This example also illustrates that micro and macro welfare answer different questions, and the answers to these questions can be quantitatively very different.

Table 3: The change in micro and macro welfare with initial and final preferences given the supply and demand shocks between February 2020 to May 2020. Chained real consumption is computed assuming supply and demand shocks arrive simultaneously.

<table>
<thead>
<tr>
<th>Elasticities</th>
<th>High compl.</th>
<th>Medium compl.</th>
<th>Cobb-Douglas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Micro initial preferences</td>
<td>-11.7%</td>
<td>-9.1%</td>
<td>-8.7%</td>
</tr>
<tr>
<td>Micro final preferences</td>
<td>-13.2%</td>
<td>-12.3%</td>
<td>-10.9%</td>
</tr>
<tr>
<td>Macro initial preferences</td>
<td>-16.2%</td>
<td>-12.5%</td>
<td>-10.8%</td>
</tr>
<tr>
<td>Macro final preferences</td>
<td>-10.1%</td>
<td>-9.4%</td>
<td>-9.0%</td>
</tr>
<tr>
<td>Chained real consumption</td>
<td>-12.1%</td>
<td>-10.6%</td>
<td>-9.8%</td>
</tr>
</tbody>
</table>
Appendix F  Non-CES Functional Forms

In this appendix, we generalize Proposition 7 beyond CES functional forms. To do this, for each producer $k$ with cost function $C_k$, we define the Allen-Uzawa elasticity of substitution between inputs $x$ and $y$ as

$$
\theta_k(x, y) = \frac{C_k d^2 C_k / (dp_x dp_y)}{(dC_k / dp_x)(dC_k / dp_y)} = \frac{e_k(x, y)}{\Omega_{ky}},
$$

where $e_k(x, y)$ is the elasticity of the demand by producer $k$ for input $x$ with respect to the price $p_y$ of input $y$, and $\Omega_{ky}$ is the expenditure share in cost of input $y$. For the household $k = 0$, we use the household’s expenditure function in place of the cost function (where the Allen-Uzawa elasticities are disciplined by Hicksian cross-price elasticities and expenditure shares).

Following Baqaee and Farhi (2019c), define the input-output substitution operator for producer $k$ as

$$
\Phi_k(\Psi(i), \Psi(j)) = - \sum_{1 \leq x, y \leq N+1+F} \Omega_{ky} [\delta_{xy} + \Omega_{ky}(\theta_k(x, y) - 1)] \Psi_{xi} \Psi_{yj},
$$

and

$$
= \frac{1}{2} E_{\Omega^{(k)}} ((\theta_k(x, y) - 1)(\Psi_i(x) - \Psi_i(y))(\Psi_j(x) - \Psi_j(y))),
$$

where $\delta_{xy}$ is the Kronecker delta, $\Psi_i(x) = \Psi_{xi}$ and $\Psi_j(x) = \Psi_{xj}$, and the expectation on the second line is over $x$ and $y$. The second line can be obtained from the first using the symmetry of Allen-Uzawa elasticities of substitution and the homogeneity identity.

Then, Proposition 7 generalizes as follows:

**Proposition 12.** Consider some perturbation in final demand $d \log x$ and technology $d \log A$. Then changes in prices of goods and factors are

$$
d \log p_i = - \sum_{j \in N} \Psi_{ij} d \log A_j + \sum_{f \in F} \Psi^F_{if} d \log \lambda_f.
$$

Changes in sales shares for goods and factors are

$$
\lambda_i d \log \lambda_i = \sum_{j \in \{0\} + N} \lambda_j \Phi_j \left( -d \log p, \Psi(i) \right)
$$

$$
+ \text{Cov}_{\Omega^{(0)}} \left( d \log x, \Psi(i) \right) + \text{Cov}_{\Omega^{(0)}(\varepsilon, \Psi(i))} \left( \sum_{k \in N} \lambda_k d \log A_k \right).
$$

Since $\Phi_j$ shares many of the same properties as a covariance (it is bilinear and symmet-
ric in its arguments, and is equal to zero whenever one of the arguments is a constant), the intuition for Proposition 12 is very similar to that of Proposition 7. Computing the equilibrium response in Proposition 12 requires solving a linear system exactly as in Proposition 7.

Appendix G  Heterogeneous Agents

We consider a utilitarian SWF that sums the welfare of each agent \( h \), measured in terms of initial prices, relative to initial aggregate income\(^{49}\)

\[
W = \frac{\sum_{h \in I} e_h(p, u_h)}{\sum_{j \in I} e_j(p, u_j)},
\]

where \( I \) is the set of agents. Preferences can vary across households but, for simplicity, we assume that each \( h \)'s preferences are homothetic and stable.

For this case, in this appendix we show that to a second-order approximation,

\[
\Delta \log W = \Delta \log Y + \frac{1}{2} \text{Cov}_\chi (-\mathbb{E}_b[\Delta \log p], \Delta \log I - \mathbb{E}_b[\Delta \log p]),
\]

where \( \chi \) is the initial distribution of expenditures by each agent, \( \mathbb{E}_b[\Delta \log p] \) is the vector of inflation rates for each household, and \( \Delta \log I \) is the vector of nominal income changes. The change in social welfare is greater than the change in real GDP if changes in real income negatively covary with inflation across households. There is no bias if preferences are aggregable, in which case \( \mathbb{E}_b[\Delta \log p] \) is uniform across households, if real income growth is uniform or, more generally, if changes in household-level inflation rates are uncorrelated with changes in real income.

While the specific form of the gap between welfare and real GDP depends on the social welfare function, if individuals’ preferences are not aggregable, then there will always be a gap even if preferences at the individual level are stable and homothetic.

To establish these results, we first write the utilitarian social welfare function as

\[
W = \sum_h \bar{\chi}_h Y_h,
\]

where \( \bar{\chi}_h \) is the ratio of agent \( i \) expenditures in total expenditures in the initial point,\(^{49}\)

Footnote: This social welfare function implements a version of the Kaldor-Hicks compensation principle whereby a change is deemed socially desirable if the winners can hypothetically compensate the losers. In this case, this hypothetical compensation is measured in terms of initial prices.
and \( Y_h \equiv \frac{e_h(\bar{\beta},\mu_h)}{e_j(\bar{\beta},\mu_j)} \) denotes the change in real consumption of agent \( h \) (or \( \exp(EV^m_h) \), since preferences are stable and homothetic). We show that to a second order approximation

\[
\Delta \log W \approx \Delta \log Y - \frac{1}{2} \text{Cov}_\chi(\mathbb{E}_{b_h}[\Delta \log p], \Delta \log Y_h),
\]

where the covariance is applied across individuals using the probabilities implies by the vector \( \chi \) at the initial point.

To see this,

\[
d \log W = \frac{1}{W} \sum_h \bar{x}_h Y_h d \log Y_h,
\]

\[
d^2 \log W = \frac{1}{W} \sum_h \bar{x}_h Y_h (d \log Y_h)^2 + \frac{1}{W} \sum_h \bar{x}_h Y_h d^2 \log Y_h - \frac{1}{W} \sum_h \bar{x}_h Y_h d \log Y_h d \log W,
\]

\[
= \frac{1}{W} \sum_h \bar{x}_h Y_h (d \log Y_h)^2 + \frac{1}{W} \sum_h \bar{x}_h Y_h d^2 \log Y_h - (d \log W)^2,
\]

\[
= \frac{1}{W} \sum_h \bar{x}_h Y_h (d \log Y_h)^2 + \frac{1}{W} \sum_h \bar{x}_h Y_h d^2 \log Y_h - (d \log Y)^2,
\]

\[
= \sum_h \bar{x}_h (d \log Y_h)^2 + \sum_h \bar{x}_h d^2 \log Y_h - (d \log Y)^2,
\]

where we use the fact that \( W = Y_h = 1 \) and \( d \log W = d \log Y \) at the initial point.

Next consider the change in real GDP:

\[
\log Y = \int_{t_0}^{t_1} \sum_i \frac{p_i(t)q_i(t)}{\sum_j p_j(t)q_j(t)} d \log q_i(t),
\]

and

\[
p_i q_i = \sum_h \bar{x}_h b_{hi} GDP,
\]

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where $b_{hi}$ is the budget share of agent $h$ on good $i$ and GDP is nominal GDP. We have

$$d \log q_i = \sum_h q_{hi} d \log q_{hi}$$

$$= \sum_h \frac{\chi_h b_{hi}}{\sum_g \chi_g b_{gi}} d \log q_{hi}$$

$$\log Y = \int_{t_0}^{t_1} \sum_h \chi_h \sum_i b_{hi} d \log q_{hi}$$

$$= \int_{t_0}^{t_1} \sum_h \chi_h d \log Y_h.$$

Differentiating this with respect to $t_1$ gives

$$d \log Y = \chi \cdot d \log Y(h).$$

Differentiating again gives

$$d^2 \log Y = d \chi \cdot d \log Y(h) + \chi \cdot d^2 \log Y(h)$$

$$= d \chi \cdot d \log Y(h) + d^2 \log W - \sum_h \chi_h (d \log Y_h)^2 + d \log Y^2.$$

Using the fact that

$$d \chi_h = \chi_h d \log \chi_h = \chi_h \left[ E_{b_h}[d \log p] + d \log Y_h - \sum_j \chi_j \left[ E_{b_j}[d \log p] + d \log Y_j \right] \right],$$

we have evaluating at $t_1$,

$$d^2 \log Y = \sum_h \chi_h \left[ E_{b_h}[d \log p] + d \log Y_h - \sum_j \chi_j \left[ E_{b_j}[d \log p] + d \log Y_j \right] \right] d \log Y_h$$

$$+ d^2 \log W - \sum_h \chi_h (d \log Y_h)^2 + d \log Y^2$$

$$= \sum_h \chi_h E_{b_h}[d \log p] d \log Y_h + \sum_h \chi_h d \log Y_h d \log Y_h - \sum_h \chi_h d \log Y_h \sum_j \chi_j \left[ E_{b_j}[d \log p] + d \log Y_j \right]$$

$$+ d^2 \log W - \sum_h \chi_h (d \log Y_h)^2 + d \log Y^2$$

$$= \text{Cov}_\chi (E_{b_h}[d \log p], d \log Y(h)) + d^2 \log W.$$
This implies that
\[ d^2 \log W = d^2 \log Y - \text{Cov}_x \left( E_{bh} [d \log p], d \log Y_{(h)} \right). \]

Combining the first order and second order terms, changes in the social welfare function are given to a second order approximation by
\[ d \log W + \frac{1}{2} d^2 \log W = d \log Y + \frac{1}{2} d^2 \log Y - \frac{1}{2} \text{Cov}_x \left( E_{bh} [d \log p], d \log Y_{(h)} \right). \]