Measuring Welfare with Income Effects using Cross-Sectional Data

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August 2022
Preliminary and Incomplete

Abstract
We show how to recover the money-metric utility function, which converts income at one point in time into equivalent income at another point in time, using repeated cross-sectional household level data. Our procedure allows unrestricted preferences, but requires that households’ preferences be the same in both the cross-section and the time-series. Our idea is to trace out Hicksian (or compensated) demand curves through time by matching households with the same utility at different points in time. Given Hicksian demand curves, we can construct ideal price indices and money-metric utility for every matched income level. We apply our method to household consumption survey data from the United Kingdom from 1974 to 2017 and find relatively small, but nonzero, biases relative to an aggregate chained index. This problem was recently studied by Jaravel and Lashkari (2022). Although we answer the same question under similar assumptions, our procedure is mathematically different to theirs and can give different estimates in practice.
1 Introduction

In this paper, we show how using repeated cross-sectional household level data, and the assumption of common and unchanging preferences, one can recover the money-metric utility function that converts income at one point in time into equivalent income at other points in time (i.e. the income that makes households indifferent between their budget sets at both points in time). This problem was recently investigated in an important paper by Jaravel and Lashkari (2022). Although we answer the same question under more or less the same assumptions, our procedure is mathematically different and can yield different estimates in practice.

Our procedure, which builds on Baqaee and Burstein (2021), has an intuitive interpretation using Hicksian, or compensated, demand curves. To recover a money-metric utility function, one must integrate Hicksian demand curves with respect to changes in prices. More prosaically, this means that to construct a welfare-relevant inflation index, we must weigh changes in prices at each point in time using Hicksian demand for each good. In practice, one does not observe the Hicksian demand curves. Therefore, standard price indices, like the chain-weighted inflation index, weigh changes in prices by observed — that is, Marshallian — demand curves instead. When preferences are homothetic, this is innocuous since for homothetic preferences, income effects affect demand for all goods in the same way.

Our idea is the following. Suppose we observe repeated cross-sections of households with identical preferences. To construct the money-metric utility value in terms of some base $t_0$ prices for a household with income $I$ at time $t$, we must know the compensated demand of this household when prices were different. This can be done at each point in time $s \neq t$ by finding another household with a different income level $I' \neq I$ at time $s$ whose utility level matches that of the household with income $I$ at $t$. If we can find such a household at every point in time $t_0 \leq s < t$, then we can trace the compensated Hicksian demand curve through time, and by using these demand curves, rather than observed demand, we can calculate the money-metric utility function.

Like Jaravel and Lashkari (2022), our procedure is non-parametric in the sense that it works for any utility function. Similar to Jaravel and Lashkari (2022), the most important assumption we make is that all households in the sample have the same preferences and that these preferences are not changing through time.

We apply our method to construct a money-metric utility function using household consumption survey data from the United Kingdom from 1974 to 2017. We find that aggregate chain-weighted measures of inflation (following procedures of official statistics)
understate the true inflation rate for all households below the 75th percentile of income in 2017. In other words, for any income level in 2017 under the 75th percentile, the 1974 equivalent income is less than real income implied by an aggregate chain-weighted inflation index. The size of this gap is greatest for the poorest households, at around 7 percentage points, and declines to less than 1/2 percentage point for households close to the 75th percentile. We are unable to compute ideal inflation rates for incomes above the 75th percentile in 2017. The reason is that for the richest households in 2017, there did not exist equally well-off consumers in the past whose demand can be used in place of the compensated demand curves of the richest consumers today.

2 Money-Metrics and the Cost-of-Living

Consider a preference relation $\succeq$ defined over consumption bundles $c$ in $\mathbb{R}^N$. Suppose that we represent these preferences using a utility function $U(c)$ that maps consumption bundles to utility values. Given this utility function, we can define the indirect utility function

$$v(p, I) = \max_c\{U(c) : p \cdot c \leq I\},$$

mapping a vector of prices $p$ and income $I$ to utility values. We can also define the expenditure function

$$e(p, u) = \min_c\{p \cdot c : U(c) \leq u\}.$$

The expenditure and indirect utility functions are useful because they can be used to construct money-metrics and cost-of-living indices.

**Definition 1** (Money-Metric). For some reference vector of prices $\bar{p}$, the money-metric function is

$$m(p, I; \bar{p}) = e(\bar{p}, v(p, I)).$$

The money-metric $m(\cdot; \bar{p})$ is specific cardinalization of the indirect utility function in the sense that a budget set $(p, I) \succeq (p', I')$ if, and only if, $m(p, I; \bar{p}) \geq m(p', I'; \bar{p})$. That is, $m(p, I; \bar{p})$ expresses the value of the budget set $(p, I)$ in terms of $\bar{p}$ prices.

**Definition 2** (Cost-of-living). For some reference budget constraint $(\bar{p}, \bar{I})$, the cost-of-living function is

$$r(p; \bar{p}, \bar{I}) = e(p, v(\bar{p}, \bar{I})).$$

Note that $r(p; \bar{p}, \bar{I})$ converts the value of budget constraint $(\bar{p}, \bar{I})$ into equivalent income under prices $p$. 
To summarize, the money-metric is useful for converting different budget constraints into a common price system so they can be compared. On the other hand, the cost-of-living index is useful for converting a given budget constraint into equivalent income under different price systems. This is necessary for providing consumers with cost-of-living adjustments.

Next, denote the Hicksian budget share for good $i$ to be $b_i(p, u)$ where $p$ is a vector of prices and $u$ is a utility level. Similarly, denote the Marshallian budget share for good $i$ to be $B_i(p, I)$, where $p$ is a vector of prices and $I$ is an income level.

The following proposition, building on Lemma 1 from Baqae and Burstein (2021), provides a characterization of both the cost-of-living index and the money-metric.

**Proposition 1** (Money-Metric and Cost-of-Living). For any smooth path of prices and income that unfold as a function of time $t$, the money-metric of a budget set $(p_{t_1}, I_{t_1})$ in terms of $t_0$ prices is given by

$$\log m(p_{t_1}, I_{t_1}; p_{t_0}) = \log e(p_{t_0}, u_{t_1}) = \log I_{t_1} - \int_{t_0}^{t_1} \sum_{i \in N} b_i(p_{t}, u_{t_1}) \frac{d \log p_{ii}}{dt} dt,$$

where $u_{t_1} = v(p_{t_1}, I_{t_1})$.

For any smooth path of prices and income that unfold as a function of time $t$, the cost-of-living of a budget set $(p_{t_0}, I_{t_0})$ in terms of $t_1$ prices is given by

$$\log r(p_{t_1}; p_{t_0}, I_{t_0}) = \log e(p_{t_1}, u_{t_0}) = \log e(p_{t_0}, u_{t_0}) + \int_{t_0}^{t_1} \sum_{i \in N} b_i(p_{t}, u_{t_0}) \frac{d \log p_{ii}}{dt} dt,$$

where $u_{t_0} = v(p_{t_0}, I_{t_0})$.

Suppose that we observe repeated cross-sections of consumers with common and stable preferences $\succeq$ who face a common vector of prices $p_t$ at each time $t \in [t_0, T]$ but have different incomes. We observe the vector of prices $p_t$ at each point in time $t$ and, for consumers with income level $I \in [I_t, \bar{I}_t]$ at time $t$ we observe expenditure shares $B(I, t)$ across all goods.

For any indirect utility function and associate Hicksian budget share, the following always holds:

$$b_i(p, v(p, I)) = B_i(p, I).$$

Since $m(p_t, I; p_{t_0})$ is itself an indirect utility function, we can write

$$b_i(p, m(p, I; p_{t_0})) = B_i(p, I).$$
Using this identity, Proposition 1 can be rewritten as the following integral representation.

**Proposition 2.** For \( t \in [t_0, T] \), the money-metric \( m(p_t, I; p_{t_0}) = \log u(I, t) \), where \( \log u(I, t) \) solves the following integral equation

\[
\log u(I, t) = \log I - \int_{t_0}^{t} \sum_i B_i(u^{-1}(u(I, t), s), s) \frac{d \log p_{is}}{ds} ds
\]

with boundary condition \( u(I, t_0) = I \). Here, \( u^{-1}(\cdot, s) \) is the inverse of \( u \) with respect to its first argument given its second argument equal to \( s \).

In Proposition 2, \( u(I, t) \) converts the value of the budget constraint defined by prices \( p_t \) and income \( I \) into income under \( p_{t_0} \). That is, \( u(I, t) = e(p_{t_0}, v(p_t, I)) \). This is the money-metric for \((p_t, I)\) in terms of \( p_{t_0} \). Note that \( t \) plays the same role as \( t_1 \) in Proposition 1. Note that once in possession of the money-metric, we can also recover the cost-of-living function. To see this, define \( I_{t_0} \equiv m(p_t, I; p_{t_0}) \). Invert this relationship for \( I = m^{-1}(p_t, I_{t_0}; p_{t_0}) = r(p_t, p_{t_0}, I_{t_0}) \).

Proposition 2 follows immediately from Proposition 1 if we recognize that in the integral equation above, \( B_i(u^{-1}(\cdot, s), s) : \mathbb{R}_+ \rightarrow [0, 1] \) maps utility values into budget shares at time \( s \). That is, it is the Hicksian budget share.

If we can solve the integral equation in Proposition 2, then we can recover the money-metric and cost-of-living functions without direct knowledge of the elasticities of substitution or income elasticities. This is because we can compute the Hicksian budget shares \( b(u_t, p_s) \) of a household with utility \( u_t \) at time \( t \) under prices \( p_s \) at time \( s \) by using the budget shares \( B(u^{-1}(u_t, s), s) \) of a household with the same level of utility at time \( s \). Following Proposition 1, we use \( B(u^{-1}(u_t, s), s) \) to weigh prices changes at time \( s \) for a household with utility \( u_t \). Proposition 2 can be used to measure changes in microeconomic welfare if we have repeated cross-sectional data and common stable preferences. However, we cannot use this procedure to answer counterfactual or macroeconomic welfare questions.

Below, we provide a simple numerical procedure for solving this integral equation.

**Numerical Procedure.** Denote the Hicksian budget shares at time \( s \) by \( b_i(\cdot, s) = B_i(u^{-1}(\cdot, s), s) \). We can solve the integral equation numerically using a simple iterative procedure like the one below. For any \( t \in [t_0, T] \), discretize time between \([t_0, t]\) into a grid of points \([t_0, \ldots, t_M]\) where \( t_n < t_{n+1} \), with \( t_M = T \). Then, use the following iterative procedure for
each \( n \in \{1, \ldots, M\} \):

\[
\log u(I, t_n) = \log I - \sum_{s=0}^{n-1} b(u(I, t_{n-1}), t_s) \cdot \Delta \log p_{t_s},
\]

\[
b(v, t_s) = B(u^{-1}(v, t_s), t_s).
\]

By the boundary condition, we know that \( u(I, t_0) = I \) and hence \( b(u, t_0) = B(I, t_0) \). For those values of \( u \) that can be inverted, this procedure recovers welfare as the grid size converges to zero.\(^1\)

To give more intuition, it helps to explicitly spell out the first few steps of this iterative procedure. We start with the boundary condition \( u(I, t_0) = I \) since \( t_0 \)-equivalent income in \( t_0 \) is just initial income. At time \( t_1 \), we use

\[
u(I, t_1) = \log I - b(u(I, t_0), t_0) \cdot \Delta \log p_{t_1}
\]

where the boundary condition implies that \( b(u(I, t_0), t_0) = B(I, t_0) \). Given \( u(I, t_1) \), we can now construct

\[
b(u, t_1) = B(u^{-1}(u, t_1), t_1).
\]

That is, to each budget share \( B(I, t_1) \) in \( t_1 \), we assign a utility value based on \( u(I, t_1) \). Hence, we now have Hicksian budget shares \( b(u, t_0) \) and \( b(u, t_1) \). Next, we construct

\[
u(I, t_2) = \log I - b(u(I, t_1), t_1) \cdot \Delta \log p_{t_2} - b(u(I, t_1), t_0) \cdot \Delta \log p_{t_1},
\]

and given \( u(I, t_2) \), we construct Hicksian budget shares in \( t_2 \):

\[
b(u, t_2) = B(u^{-1}(u, t_2), t_2).
\]

That is, for each budget share \( B(I, t_2) \) in \( t_2 \), we assign a utility value based on \( u(I, t_2) \). We continue this process ad infinitum.

### 3 Empirical Results

In this section, we apply our algorithm to long-run household cross-section data. Our goal is to compare changes in welfare, as measured by the money-metric, with changes

\(^1\)Invertibility at \((u, s)\) means that we observe an income level \( I \) such that \( u(I, s) = u \). When applying the algorithm to the data in the next section, the value of the budget share in period \( s \) corresponding to \( u(I, t_{n-1}) \) is obtained by linear interpolation of the grid of \( u \) at the time \( s \).
in real consumption as measured by a chain-weighted inflation index. For this purpose, we use the *Family Expenditure Survey, and Living Costs and Food Survey Derived Variables* for the UK, Oldfield et al. (2020). This data is a repeated cross-sectional data of UK household expenditures from 1974 to 2017, which has a longer history than the US Consumer Expenditure Survey available from 1984.

We used retail price index (RPI) categories of spending, which provide the most extended consecutive series (1974-2017). For the definition of goods categories, we selected 14 high-level item categories that could be used for the entire period and merged them with the corresponding RPI.

### 3.1 Mapping Household Data to the Algorithm

Our procedure requires the income $I_t$ and the budget shares $B(I, t)$ at time $t$ across all goods. Since budget shares in the data are noisy, we fit a smooth curve for each good at each time point $t$ and use these curves as $B(I, t)$ (ignoring the noise). That is, we assume that the theoretical value of the budget share for each item is present with measurement noise, and we estimate the true $B_i(I, t)$ function for some good $i$ by fitting the following curve for each $t$ using OLS

$$\log B_{ih} = \alpha_{it} + \beta_{it} \log I_{ht} + \gamma_{it} (\log I_{ht})^2 + \varepsilon_{ih},$$

where $i$ is the good, $h$ is the household, and $t$ is the time period.

Finally, computing $u(I, t)$ requires that for each $s < t$, $b(u(I, t), s)$ be observable. That is, for each income level $I$ at time $t$, we must be able to find consumers at $s < t$ whose utility values were the same as that delivered by $I$ at time $t$. This naturally means that we can only compute $u(I, t)$ if $u(I, t)$ is less than the upper bound and more than the lower bound of utility levels at all past times $s(< t)$. Thus, at each time $t$, we remove households that do not satisfy the condition. Although this may appear to be restrictive, it turns out that only the upper bound matters in practice since incomes have increased over time as the economy has grown. In the end, our dataset covers households from the bottom 0.5th percentile households in the 2017 sample to about the 75th percentile. We applied our procedure sequentially from 1974 to 2017 for the UK cross-sectional data constructed in the manner described above to compute $m(p_t; I; p_{t_0}) = u(I, t)$. 
3.2 Results

The blue line in Figure 1 plots the expenditure function $e(p_{1974}, v(p_{2017}, I))$ for different values of income. This expresses different incomes in 2017 in terms of 1974 pounds. For comparison, the red line shows the equivalent if all households faced the same effective inflation rate, as given by the Tornqvist chain-weighted aggregate inflation rate. Since the red line is above the blue line, this means that real income based on chain-weighted aggregate inflation is higher than equivalent income using the money metric for households in the sample, and the size of the bias is largest for the poorest households. That is, the poorest households are not as well off as would be implied by relying on the official statistics. This gap shrinks for richer households at the top of our sample (the 75th percentile of income).

![Figure 1: The money-metric $m(p_{2017}, I_{2017}; p_{1974})$ and real income using aggregate chain-weighted inflation, from 1974 to 2017.](image)

Figure 2 instead displays log difference between real income growth, as measured by the chain-weighted aggregate inflation, and the growth in income as measured by the money-metric at 1974 base prices. As expected, the difference is positive, meaning that the official statistics are biased upwards. The size of the bias is around 7% for the poorest households and declines as households get richer. Intuitively, over time, as households get
Figure 2: The log difference between the inflation implied by the cost-of-living function and aggregate chain-weighted inflation between 1974 to 2017 for different percentiles of the income distribution in 2017.

richer, their expenditures shift due to income effects. For a given relatively poor household in 2017, consumers with the same utility level who lived in the 1970s spent relatively more on sectors with higher inflation rates than consumers as a whole. Therefore, the inflation rate for these consumers is higher than the aggregate inflation rate.

4 Conclusion

In this paper, we provide a simple and intuitive procedure for constructing money-metric representations of utility using repeated cross-sectional data. Our insight is that one can trace out Hicksian, or compensated, demand curves through time by matching households whose incomes are equivalent in utility terms. Although our approach is non-parametric, it relies on the assumption that preferences are the same in both the cross-section and the time-series dimensions. Relaxing this assumption is an interesting avenue for future work.
References

