Online Appendix

Measuring Welfare by Matching Households across Time

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A.1 Step-by-Step Intuition for Iterative Procedure

To give more intuition, it helps to explicitly spell out the first few steps of the iterative procedure. For expositional simplicity, we abstract from the numerical refinements discussed in Footnote 15.

Start with the boundary condition $u(I, t_0) = I$ since t_0-equivalent income at t_0 is just initial income. For period t_1, compute

$$\log u(I, t_1) \approx \log I - \boldsymbol{B}(I, t_0) \cdot \Delta \log \boldsymbol{p}_{t_0},$$

where we use the fact that $I_0^* = u^{-1}(u(I, t_0), t_0) = I$. For values of I outside of $[\underline{I}_{t_0}, \bar{I}_{t_0}]$, we cannot compute $u(I, t_1)$.[1] We also exclude $u(I, t_1)$ if there does not exist $I_0^* \in [\underline{I}_{t_0}, \bar{I}_{t_0}]$ such that $u(I_0^*, t_0) = u(I, t_1)$. This is to ensure there exists a suitable match (compensated household) to (I, t_1) in t_0.

Next, calculate

$$\log u(I, t_2) \approx \log I - \boldsymbol{B}(I_1^*, t_1) \cdot \Delta \log \boldsymbol{p}_{t_1} - \boldsymbol{B}(I_0^*, t_0) \cdot \Delta \log \boldsymbol{p}_{t_0},$$

where $I_1^* = u^{-1}(u(I, t_1), t_1) = I$ and $I_0^* = u^{-1}(u(I, t_1), t_0)$. If necessary to form a candidate I_0^*, we extend $u(I, t_0)$ as a function of I using a loglinear approximation. To ensure there is no extrapolation of the data, if I_1^* is not in $[\underline{I}_{t_1}, \bar{I}_{t_1}]$ or I_0^* is not in $[\underline{I}_{t_0}, \bar{I}_{t_0}]$, then we do not calculate $u(I, t_2)$. We also exclude $u(I, t_2)$ if there does not exist $I_m^* \in [\underline{I}_{t_m}, \bar{I}_{t_m}]$ such that $u(I_m^*, t_m) = u(I, t_2)$ for $m = 0$ and $m = 1$. This ensures that there exists a suitable match (compensated household) to (I, t_2) in both t_0 and t_1. Note that, in contrast to $u(I, t_1)$, it is possible to evaluate $u(I, t_2)$ for some I outside of $[\underline{I}_{t_0}, \bar{I}_{t_0}]$ since households are matched on utility rather than nominal income.

Continue this iterative process until t_M.

A.2 Proofs

Proof of Lemma 1. By definition,

$$\begin{aligned} \log e(\boldsymbol{p}, v(\bar{\boldsymbol{p}}, \bar{I})) &= \log e(\bar{\boldsymbol{p}}, v(\bar{\boldsymbol{p}}, \bar{I})) + \log e(\boldsymbol{p}, v(\bar{\boldsymbol{p}}, \bar{I})) - \log e(\bar{\boldsymbol{p}}, v(\bar{\boldsymbol{p}}, \bar{I})) \\ &= \log \bar{I} + \log e(\boldsymbol{p}, v(\bar{\boldsymbol{p}}, \bar{I})) - \log e(\bar{\boldsymbol{p}}, v(\bar{\boldsymbol{p}}, \bar{I})). \end{aligned}$$

[1]This requirement is not very binding if the support of the income distribution is wide or moves slowly from period to period (the latter is automatic if the data is smooth and the interval between each period is short).

Rewrite

$$\log e(\boldsymbol{p}, v(\bar{\boldsymbol{p}}, \bar{I})) - \log e(\bar{\boldsymbol{p}}, v(\bar{\boldsymbol{p}}, \bar{I})) = \int_{t_0}^{t_1} \sum_{i \in N} \frac{\partial \log e(\xi_t, v(\bar{\boldsymbol{p}}, \bar{I}))}{\partial \log \xi_{it}} \frac{\partial \log \xi_{it}}{dt} dt,$$

where $\{\boldsymbol{\xi}_t : t \in [t_0, t_1]\}$ is a smooth path connecting $\bar{\boldsymbol{p}}$ and $\boldsymbol{p}$ as a function of a scalar t. Finally, use Shephard's lemma to express the price elasticity of the expenditure function in terms of budget shares, and obtain (2). To obtain (1), switch $\boldsymbol{p}$ and $\bar{\boldsymbol{p}}$ as well as I and $\bar{I}$. ■

Proof of Proposition 1. This follows immediately from the definition of $u^{-1}(\cdot, s)$ which maps incomes at t_0 to equivalent income at time s. Hence, for some amount of t_0 income, say $u(I, t)$, the equivalent income at time s is $u^{-1}(u(I, t), s)$. The uncompensated budget share $B(u^{-1}(u(I, t), s), s)$ is just $b(u(I, t), s)$. ■

Proof of Proposition 2. Suppose that preferences $\succeq_x$ vary by some observable characteristic x. For example, x could be marital status. In this case, we can split our sample by x and apply Proposition 1 to each subsample separately resulting in $u(I, t|x)$ — money metrics for different levels of expenditures I, at different points in time t, for different values of the characteristic x. ■

To prove Proposition 3 and Proposition 4, we make use of the following lemma.

Lemma A.1. *Define $\tilde{u}(I, t|\kappa)$ to be the solution to the integral equation* (10). *Then*

$$\frac{\partial \log u(I, t)}{\partial \kappa} = \frac{-\int_{t_0}^{t} Cov(\epsilon(u(I, t), s), \frac{d \log p}{ds}) + \int_{t_0}^{t} \frac{\partial u(I^*(I,t,s),s)}{\partial \kappa} Cov_b(\frac{\partial \log b(u(I,t),s)}{\partial \log u(I,t)}, \frac{d \log p}{ds})}{\left[1 + \int_{t_0}^{t} Cov_b(\frac{\partial \log b(u(I,t),s)}{\partial \log u(I,t)}, \frac{d \log p}{ds})\right]},$$

where Cov_b is a covariance using b in place of the probability weights.

Proof of Lemma A.1. Define the integral equation

$$\log u(I, t|\kappa) = \log I - \int_{t_0}^{t} \sum_i B_i(I^*(I, t, s|\kappa), s) + \kappa \epsilon_i(I^*(I, t, s|\kappa), s) \frac{d \log p_i}{ds} ds$$

where

$$u(I^*(I, t, s|\kappa), s|\kappa) = u(I, t|\kappa).$$

Now differentiate this with respect to κ:

$$\frac{1}{u(I, t|\kappa)} \frac{\partial u(I, t|\kappa)}{\partial \kappa} = -\int_{t_0}^{t} \sum_i \left[\frac{\partial B_i}{\partial I^*} \frac{\partial I^*}{\partial \kappa} + \epsilon_i(I^*(I, t, s|\kappa), s) + \kappa \frac{\partial \epsilon_i}{\partial I} \frac{\partial I^*}{\partial \kappa}\right] \frac{d \log p_i}{ds} ds$$

where

$$\frac{\partial I^*(I,t,s|\kappa)}{\partial\kappa} = \frac{\frac{\partial u(I,t|\kappa)}{\partial\kappa} - \frac{\partial u(I^*(I,t,s|\kappa),s|\kappa)}{\partial\kappa}}{\frac{\partial u(I^*(I,t,s|\kappa),s|\kappa)}{\partial I}}.$$

At $\kappa = 0$, this is

$$\frac{\partial I^*(I,t,s|\kappa)}{\partial\kappa} = \frac{\frac{\partial u(I,t)}{\partial\kappa} - \frac{\partial u(I^*(I,t,s),s)}{\partial\kappa}}{\frac{\partial u(I^*(I,t,s),s)}{\partial I}}$$

At $\kappa = 0$, we have

$$\begin{aligned}\frac{1}{u(I,t)}\frac{\partial u(I,t)}{\partial\kappa} &= -\int_{t_0}^{t}\sum_i\left[\frac{\partial B_i}{\partial I^*}\frac{\partial I^*}{\partial\kappa}\right]\frac{d\log p_i}{ds}ds - \int_{t_0}^{t}\sum_i \epsilon_i(I^*(I,t,s),s)\frac{d\log p_i}{ds}ds \\ &= -\int_{t_0}^{t}\sum_i\left[\frac{\partial B_i(I^*(I,t,s),s)}{\partial I^*(I,t,s)}\frac{\frac{\partial u(I,t)}{\partial\kappa} - \frac{\partial u(I^*(I,t,s),s)}{\partial\kappa}}{\frac{\partial u(I^*(I,t,s),s)}{\partial I}}\right]\frac{d\log p_i}{ds}ds \\ &\quad - \int_{t_0}^{t}\sum_i \epsilon_i(I^*(I,t,s),s)\frac{d\log p_i}{ds}ds.\end{aligned}$$

Simplifying further gives

$$\begin{aligned}\frac{\partial\log u(I,t)}{\partial\kappa} &= -\frac{\partial u(I,t)}{\partial\kappa}\int_{t_0}^{t}\sum_i\left[\frac{\partial B_i}{\partial I^*}\frac{1}{\frac{\partial u(I^*(I,t,s),s)}{\partial I}}\right]\frac{d\log p_i}{ds}ds \\ &\quad + \int_{t_0}^{t}\sum_i\left[\frac{\partial B_i}{\partial I^*}\frac{\frac{\partial u(I^*(I,t,s),s)}{\partial\kappa}}{\frac{\partial u(I^*(I,t,s),s)}{\partial I}}\right]\frac{d\log p_i}{ds}ds - \int_{t_0}^{t}\sum_i \epsilon_i(I^*(I,t,s),s)\frac{d\log p_i}{ds}ds \\ \frac{\partial\log u(I,t)}{\partial\kappa} &= \frac{\int_{t_0}^{t}\sum_i\left[\frac{\partial B_i}{\partial I^*}\frac{\frac{\partial u(I^*(I,t,s),s)}{\partial\kappa}}{\frac{\partial u(I^*(I,t,s),s)}{\partial I}}\right]\frac{d\log p_i}{ds}ds - \int_{t_0}^{t}\sum_i \epsilon_i(I^*(I,t,s),s)\frac{d\log p_i}{ds}ds}{\left[1 + u(I,t)\int_{t_0}^{t}\sum_i\left[\frac{\partial B_i}{\partial I^*}\frac{1}{\frac{\partial u(I^*(I,t,s),s)}{\partial I}}\right]\frac{d\log p_i}{ds}ds\right]}.\end{aligned}$$

We know that

$$B_i(I^*(I,t,s),s) = b_i(u(I,t),s)$$

Hence

$$\frac{\partial B_i(I^*(I,t,s),s)}{\partial I^*}\frac{\partial I^*}{\partial u(I,t)} = \frac{\partial b_i(u(I,t),s)}{\partial u(I,t)}$$

Therefore, we can write

$$\frac{\partial\log u(I,t)}{\partial\kappa} = \frac{\int_{t_0}^{t}\sum_i\left[\frac{\partial B_i(I^*(I,t,s),s)}{\partial(I^*(I,t,s))}\left[\frac{\partial u(I^*(I,t,s),s)}{\partial I}\right]^{-1}\frac{\partial u(I^*(I,t,s),s)}{\partial\kappa}\right]\frac{d\log p_i}{ds}ds - \int_{t_0}^{t}\sum_i \epsilon_i(I^*(I,t,s),s)\frac{d\log p_i}{ds}ds}{\left[1 + \int_{t_0}^{t}\sum_i\left[\frac{\partial b_i(u(I,t),s)}{\partial\log u(I,t)}\right]\frac{d\log p_i}{ds}ds\right]}$$

$$= \frac{\int_{t_0}^{t} \sum_i \left[\frac{\partial B_i(I^*(I,t,s),s)}{\partial(I^*(I,t,s))}\left[\frac{\partial I^*(I,t,s)}{\partial u}\right]\frac{\partial u(I^*(I,t,s),s)}{\partial \kappa}\right]\frac{d\log p_i}{ds}ds - \int_{t_0}^{t}\sum_i \epsilon_i(I^*(I,t,s),s)\frac{d\log p_i}{ds}ds}{\left[1+\int_{t_0}^{t}\sum_i\left[\frac{\partial b_i(u(I,t),s)}{\partial \log u(I,t)}\right]\frac{d\log p_i}{ds}ds\right]}$$
$$= \frac{\int_{t_0}^{t} \sum_i \left[\frac{\partial b_i(u(I,t),s)}{\partial u(I,t)}\frac{\partial u(I^*(I,t,s),s)}{\partial \kappa}\right]\frac{d\log p_i}{ds}ds - \int_{t_0}^{t}\sum_i \epsilon_i(I^*(I,t,s),s)\frac{d\log p_i}{ds}ds}{\left[1+\int_{t_0}^{t}\sum_i\left[\frac{\partial b_i(u(I,t),s)}{\partial \log u(I,t)}\right]\frac{d\log p_i}{ds}ds\right]}.$$

The adding up constraint requires that $\sum_i \epsilon_i(I^*(I,t,s|\kappa),s) = \sum_i \partial b_i/\partial u = 0$. Hence, we can rewrite some of the inner products above as covariances as in the statement of Lemma A.1 ■

Proof of Proposition 3. Assume that for all I and s, we have

$$Cov(\epsilon(I,s), \frac{d\log p}{ds}) = 0.$$

Assume that for all $s < t$,we have

$$\frac{\partial \log u(I,s)}{\partial \kappa} = 0$$

Then, using Lemma A.1, we know that

$$\frac{\partial \log u(I,t)}{\partial \kappa} = \frac{\int_{t_0}^{t}\sum_i \frac{\partial u(I^*(I,t,s),s)}{\partial \kappa}\left[\frac{\partial b_i(u(I,t),s)}{\partial u(I,t)}\right]\frac{d\log p_i}{ds}ds}{\left[1+\int_{t_0}^{t} Cov_b(\frac{\partial \log b(u(I,t),s)}{\partial \log u(I,t)}, \frac{d\log p}{ds})ds\right]}.$$

This is equal to zero if $\frac{\partial u(I^*(I,t,s),s)}{\partial \kappa}$ is equal to zero for every $s \leq t$. We also know that

$$\frac{\partial \log u(I,t_0)}{\partial \kappa} = 0.$$

Hence

$$\frac{\partial \log u(I,t)}{\partial \kappa} = 0$$

by transfinite induction. ■

Proof of Proposition 4. If, for every s and I, we have

$$Cov_b(\frac{\partial \log B(I,s)}{\partial \log I}, \frac{d\log p}{ds}) = 0,$$

then we know that, for every s, we have

$$Cov_b(\frac{\partial \log b(u(I,t),s)}{\partial \log u(I,t)}, \frac{d \log p}{ds}) = 0.$$

Substituting this into Lemma A.1 yields

$$\frac{\partial \log u(I,t)}{\partial \kappa} = -\int_{t_0}^{t} Cov(\epsilon(u(I,t),s), \frac{d \log p}{ds})ds.$$

■

Proof of Proposition 5. By Euler's theorem of homogeneous functions, we know that

$$\frac{\partial \log e}{\partial \log e^X} + \frac{\partial \log e}{\partial \log e^Y} = 1.$$

Differentiating this identity with respect to e^X and e^Y yields the following equations

$$\frac{\partial^2 \log e}{(\partial \log e^X)^2} = -\frac{\partial^2 \log e}{\partial \log e^X \partial \log e^Y} = \frac{\partial^2 \log e}{(\partial \log e^Y)^2}.$$

Next, we know that

$$b_X = \sum_{i \in X} b_i = \sum_{i \in X} \frac{\partial \log e}{\partial \log e^X} \frac{\partial \log e^X}{\partial \log p_i} = \frac{\partial \log e}{\partial \log e^X} \sum_{i \in X} \frac{\partial \log e^X}{\partial \log p_i} = \frac{\partial \log e}{\partial \log e^X}.$$

Hence, fixing utility, the total derivative of b_X with respect to prices is

$$\begin{aligned}
b_X d \log b_X &= \frac{\partial^2 \log e}{(\partial \log e^X)^2} \sum_{i \in X} \frac{\partial \log e^X}{\partial \log p_i} d \log p_i + \frac{\partial^2 \log e}{\partial \log e^Y \partial \log e^X} \sum_{i \in Y} \frac{\partial \log e^Y}{\partial \log p_i} d \log p_i \\
&= \frac{\partial^2 \log e}{(\partial \log e^X)^2} \left[\sum_{i \in X} \frac{\partial \log e^X}{\partial \log p_i} d \log p_i - \sum_{i \in Y} \frac{\partial \log e^Y}{\partial \log p_i} d \log p_i \right] \\
&= \frac{\partial^2 \log e}{(\partial \log e^X)^2} \left[\sum_{i \in X} b_{Xi} d \log p_i - \sum_{i \in Y} b_{Yi} d \log p_i \right]
\end{aligned}$$

Using the fact that

$$\sigma(\boldsymbol{p}, u) = 1 - \frac{1}{(1 - b_X) b_X} \frac{\partial^2 \log e}{(\partial \log e^X)^2},$$

we can rewrite this as

$$d\log b_X = (1-b_X)(1-\sigma)\left[\sum_{i\in X} b_{Xi} d\log p_i - \sum_{i\in Y} b_{Yi} d\log p_i\right],$$

where we suppress the fact that σ is a function of prices and utility. For the set of values where $\sigma \neq 1$, rearrange this to get

$$-\frac{d\log b_X}{1-\sigma} + (1-b_X)\sum_{i\in X} b_{Xi} d\log p_i + b_X \sum_{i\in X} b_{Xi} d\log p_i = \sum_{i\in X} b_i d\log p_i + \sum_{i\in Y} b_i d\log p_i,$$

or

$$-\frac{d\log b_X}{1-\sigma} + \sum_{i\in X} b_{Xi} d\log p_i = \sum_{i\in X} b_i d\log p_i + \sum_{i\in Y} b_i d\log p_i.$$

Plug this back into Proposition 1 to get the desired result. Since the set of values where $\sigma = 1$ is measure zero, we can ignore those points in the integral. It is important to note that $d\log b_X$ in the expression above is the compensated change in the budget share of X.

■

Proof of Proposition 6. Consider a perturbation to p_k for $k \in X$ holding fixed utils:

$$\begin{aligned}
\frac{\partial \log b_X}{\partial \log p_k} &= \frac{1}{b_X}\frac{\partial}{\partial \log p_k}\left[\sum_{i\in X}\frac{\partial \log e}{\partial \log e^X}\frac{\partial \log e^X}{\partial \log p_i}\right] \\
&= \frac{1}{b_X}\frac{\partial}{\partial \log p_k}\left[\sum_{i\in X}\frac{\partial \log e}{\partial \log e^X} b_{Xi}\right] \\
&= \frac{1}{b_X}\left[\sum_{i\in X}\frac{\partial}{\partial \log p_k}\frac{\partial \log e}{\partial \log e^X} b_{Xi} + \sum_{i\in X}\frac{\partial \log e}{\partial \log e^X}\frac{\partial b_{Xi}}{\partial \log p_k}\right] \\
&= \frac{1}{b_X}\left[\sum_{i\in X}\frac{\partial^2 \log e}{(\partial \log e^X)^2} b_{Xk} b_{Xi} + \sum_{i\in X}\frac{\partial \log e}{\partial \log e^X}\frac{\partial b_{Xi}}{\partial \log p_k}\right] \\
&= \frac{1}{b_X}\left[\sum_{i\in X}\frac{\partial^2 \log e}{(\partial \log e^X)^2} b_{Xk} b_{Xi} + \frac{\partial \log e}{\partial \log e^X}\frac{\partial \sum_{i\in X} b_{Xi}}{\partial \log p_k}\right] \\
&= \frac{1}{b_X}\frac{\partial^2 \log e}{(\partial \log e^X)^2} b_{Xk},
\end{aligned}$$

where the last line uses the fact that $\frac{\partial \sum_{i\in X} b_{Xi}}{\partial \log p_k} = 0$. Using the following relationship

$$\frac{\partial^2 \log e}{(\partial \log e^X)^2} = b_X \frac{\partial \log b_X}{\partial \log e^X} = b_X(1-b_X)(1-\sigma(\boldsymbol{p}, u)),$$

the compensated change in expenditures on X in response to a change in the price of $k \in X$ is given by

$$\frac{\partial \log b_X}{\partial \log p_k} = (1 - b_X)(1 - \sigma(\boldsymbol{p}, u))b_{Xk}.$$

The following identity links the uncompensated and compensated budget share of X goods:

$$B_X(p, e(\boldsymbol{p}, u)) = b_X(\boldsymbol{p}, u).$$

Differentiating both sides of this identity with respect to the price of some good $k \in X$ yields

$$\begin{aligned}
\frac{\partial \log B_X}{\partial \log p_k} &= \frac{\partial \log b_X}{\partial \log p_k} - \frac{\partial \log B_X}{\partial \log I}\frac{\partial \log e}{\partial \log p_k}, \\
&= \frac{\partial \log b_X}{\partial \log p_k} - \sum_{i \in X} b_{Xi}\frac{\partial \log b_i}{d \log I} b_K, \\
&= (1 - b_X)(1 - \sigma)b_{Xk} - b_X b_{Xk} \sum_{i \in X} b_{Xi}(\eta_i - 1),
\end{aligned}$$

where we use the fact that $\partial \log e / \partial \log p_k = b_k$. Summing over all $k \in X$, we get

$$\sum_{k \in X} \frac{\partial \log B_X}{\partial \log p_k} d \log p_k = \left[(1 - b_X)(1 - \sigma) - b_X \sum_{i \in X} b_{Xi}(\eta_i - 1)\right]\left(\sum_{k \in X} b_{Xk} d \log p_k\right).$$

Meanwhile, we also have $\sum_{k \in X} \frac{\partial \log B_X}{\partial \log p_k} d \log p_k = \epsilon_X d \log p_X$, where $d \log p_X = \sum_{k \in X} b_{Xk} d \log p_k$ and $\epsilon_X = (1 - b_X)(1 - \sigma(\boldsymbol{p}, u)) - b_X \sum_{i \in X}(\eta_i - 1)b_{Xi}$. Rearranging this for $\sigma(\boldsymbol{p}, u)$ yields the desired result

$$\sigma(\boldsymbol{p}, u) = 1 - \frac{\epsilon_X + b_X \sum_{i \in X}(\eta_i - 1)b_{Xi}}{1 - b_X}.$$

■

A.3 Existence and Uniqueness

Proposition A.1 (Uniqueness and Convergence)**.** *Consider the integral equation*

$$u(I, t) = \log I - \int_{t_0}^{t} \sum_i b_i(s, u(I, t)) \frac{d \log p_i}{ds} ds.$$

Suppose that b_i and $\partial b_i / \partial u$ are smooth functions in all of their arguments and that $\boldsymbol{p}$ is absolutely continuous in time. Then the integral equation has a unique solution in some closed interval

$[t_0, t_0 + h]$ *where* $h > 0$. *Furthermore, the iterations defined by*

$$u_{n+1}(I,t) = \log I - \int_{t_0}^{t} \sum_i b_i(s, u_n(I,t)) \frac{d \log p_i}{ds} ds$$

produces a sequence that converges uniformly to this solution on $[t_0, t_0 + h]$.

Before showing the proof, we note that local uniqueness implies global uniqueness. Suppose there exist two solutions to the integral equation $u(I,t)$ and $v(I,t)$. Pick the largest s such that $u(I,s) = v(I,s)$. Such an s must exist since $u(I,t_0) = v(I,t_0) = I$. We then apply Proposition A.1 starting at s, and conclude that $u(I, s+h) = v(I, s+h)$ for some $h > 0$. By transfinite induction, $u(I,t) = v(I,t)$ for all t and for every I.

Proof. To prove uniqueness, we use the contraction mapping theorem. We begin by showing that there exists a sufficiently small compact set, around the boundary condition, over which the integral equation is a continuous self-map. We then show that this self-map is a contraction mapping if the compact set is sufficiently small. This shows local uniqueness inside that set. Using the argument above, we can extend this to global uniqueness.

Part (i): To begin, adopt the infinity norm, and define the operator:

$$T(v(I,t)) = \log I - \int_{t_0}^{t} \sum_i b_i(s, v(I,t)) \frac{d \log p_i}{ds} ds.$$

Choose h_1 and α_1 such that

$$R_1 = \{(t,y) : |t - t_0| \leq h_1, |y - I| \leq \alpha_1\}.$$

It follows that b_i, $\partial b_i / \partial u$, and p_i all attain their supremum on R_1. It follows that there exist $M > 0$ and $L > 0$ such that

$$\forall (t,y) \in R_1, \sum_i |b_i \frac{d \log p_i}{ds}| \leq M \text{ and } \left| \frac{\partial b_i}{\partial u} \frac{d \log p_i}{ds} \right| \leq L.$$

Let g be a continuous function on R_1 satisfying $g(t,I) \leq \alpha_1$ for all $(t,I) \in R_1$. Then

$$\begin{aligned} \left| T(g(I,t)) - \log I \right| &= \left| \int_{t_0}^{t} \sum_i b_i(s, g(I,t)) \frac{d \log p_i}{ds} ds \right| \\ &\leq \int_{t_0}^{t} \sum_i \left| b_i(s, g(I,t)) \frac{d \log p_i}{ds} ds \right| \end{aligned}$$

$$\leq M|t - t_0|.$$

Choose h such that $0 < h < \min\{h_1, \frac{\alpha_1}{M}, \frac{1}{L}\}$. Hence

$$\left|T(g(I,t)) - \log I\right| \leq \alpha_1.$$

Hence, for the set

$$S = \{g \in C([t_0, t_0 + h]) : \|g - \log I\| \leq \alpha_1\},$$

the operator T is a self-map of continuous functions satisfying $g(t, I) \leq \alpha_1$ over R_1.

Part (ii): Now we show that T is a contraction mapping.

$$\begin{aligned}
|T(v(I,t)) - T(u(I,t))| &= |\int_{t_0}^{t} \sum_i [b_i(s, v(I,t)) - b_i(s, u(I,t))] \frac{d \log p_i}{ds} ds| \\
&\leq \int_{t_0}^{t} \sum_i \left| [b_i(s, v(I,t)) - b_i(s, u(I,t))] \frac{d \log p_i}{ds} ds \right|.
\end{aligned}$$

By the mean value theorem, there exists $\tilde{u}(I,t) \in [v(I,t), u(I,t)]$ such that

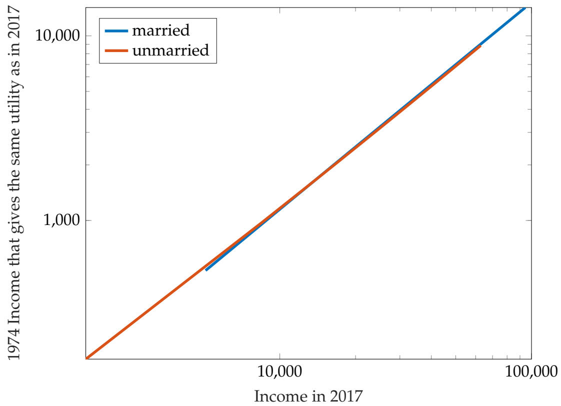
$$\begin{aligned}
|T(v(I,t)) - T(u(I,t))| &\leq \int_{t_0}^{t} \sum_i \left| \frac{\partial b_i(s, \tilde{u}(I,t))}{\partial u} \left(u(I,t) - v(I,t)\right) \frac{d \log p_i}{ds} ds \right| \\
&\leq \int_{t_0}^{t} \sum_i L \left|(u(I,t) - v(I,t))\right| ds \\
&\leq \sum_i L \left|(u(I,t) - v(I,t))\right| |t - t_0| \\
&= \kappa \left|(u(I,t) - v(I,t))\right|
\end{aligned}$$

where $\kappa = \sum_i L|t - t_0|$. This holds if we choose $h < 1/LN$, so we have $\sum_i L|t - t_0| < hNL < 1$. Hence, T is a contraction mapping and we can apply the contraction mapping theorem. ■

A.4 Additional Figures

Figure A.1: Money metric $e(\boldsymbol{p}_{1974}, v(\boldsymbol{p}_{2017}, I_{2017}))$ by household characteristic (annualized pounds, log scale) for the UK data in Section 4.

(a) Married and unmarried



(b) Above and below median age

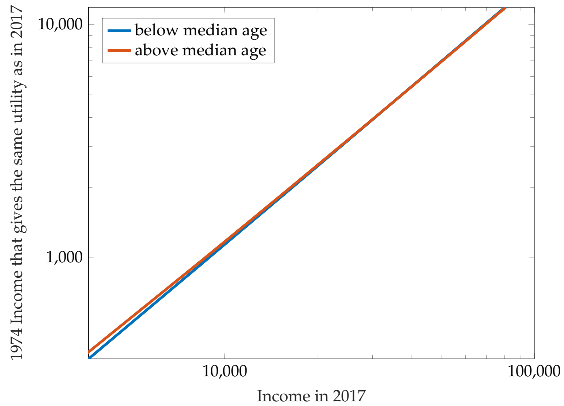
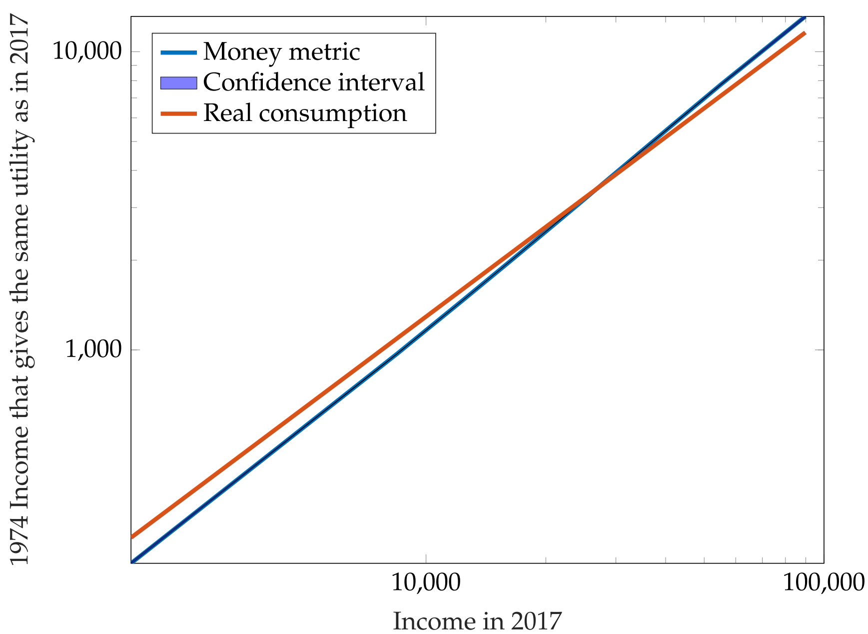


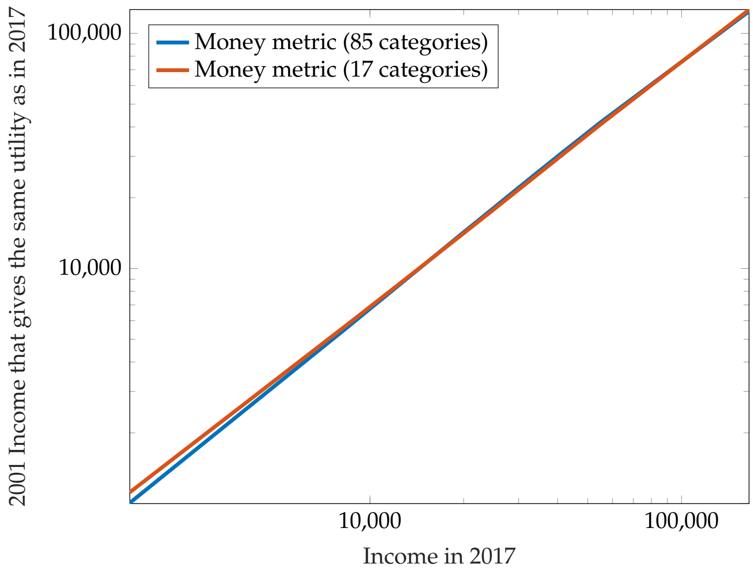
Figure A.2: Money metric $e(\boldsymbol{p}_{1974}, v(\boldsymbol{p}_{2017}, I))$ and real consumption as a function of I in 2017 using LOWESS



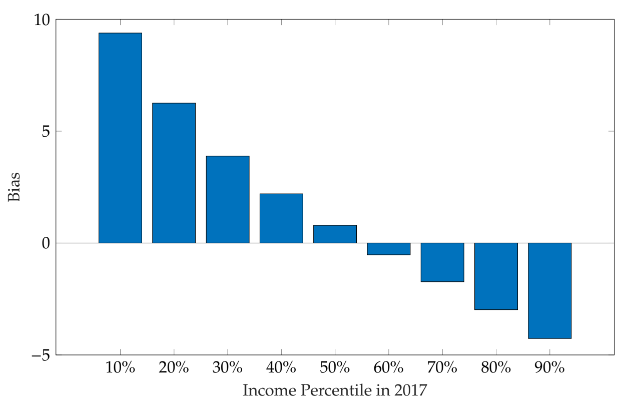
Notes: This figure is calculated using the recursive solution method rather than the iterative one. The 95% confidence intervals are bootstrapped using 500 draws with replacement.

Figure A.3: Results using more disaggregated spending categories

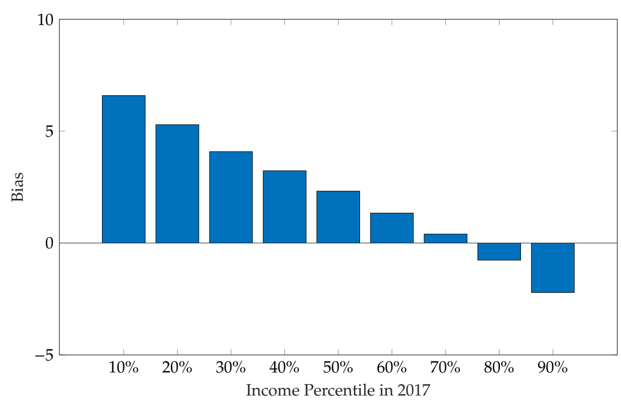
(a) Comparison of $e(\boldsymbol{p}_{2001}, v(\boldsymbol{p}_{2017}, I))$ computed using 17 and 85 spending categories.



(b) Log difference between chain-weighted inflation and true cost-of-living inflation using 85 spending categories.



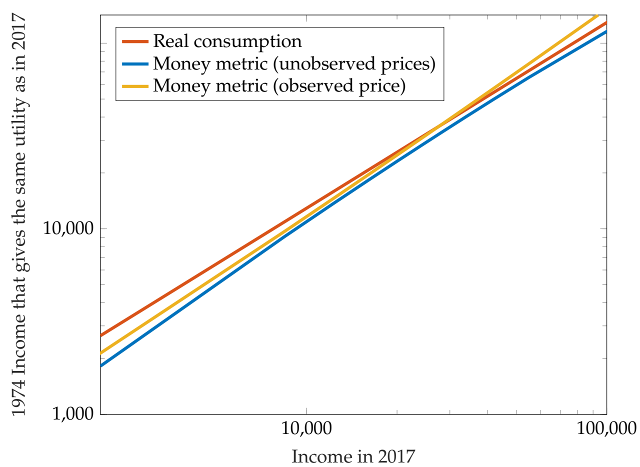
(c) Log difference between chain-weighted inflation and true cost-of-living inflation using 17 spending categories.



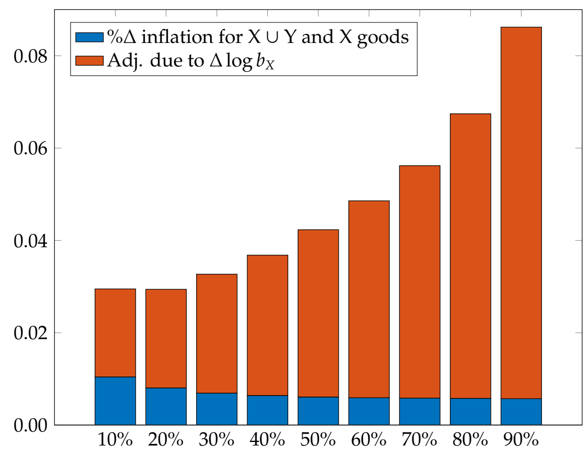
Notes: Figure A.3 uses the restricted sample from 2001 – 2017 using CPI price data.

Figure A.4: Replication of Section 5 using a constant σ and the IV estimates.

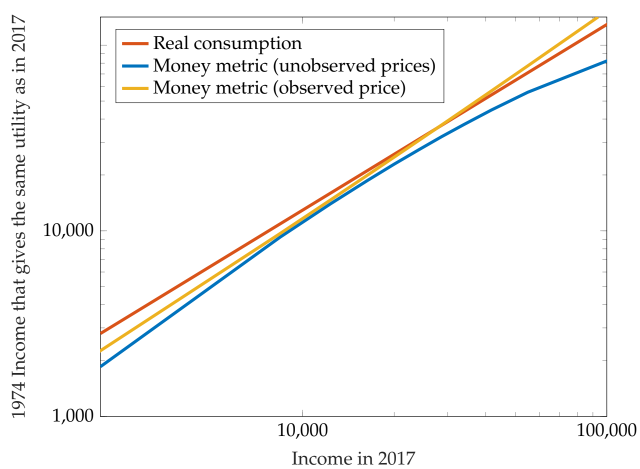
(a) Money metric $e(\boldsymbol{p}_{1974}, v(\boldsymbol{p}_{2017}, I))$ and real consumption as a function of I in 2017 assuming $\sigma = 0.5$.



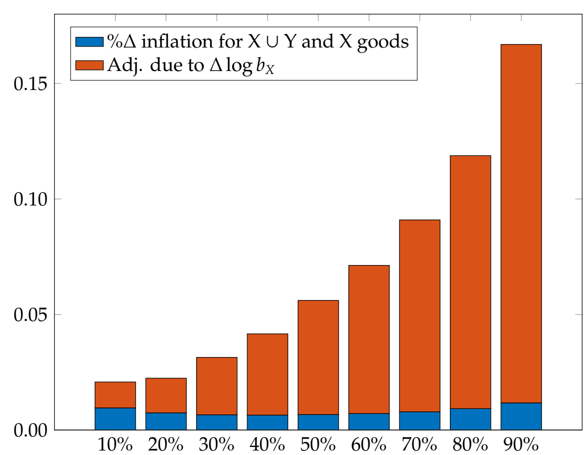
(b) Percent difference in money-metric values with observed and unobserved prices for different percentiles of the I distribution assuming $\sigma = 0.5$



(c) Money metric $e(\boldsymbol{p}_{1974}, v(\boldsymbol{p}_{2017}, I))$ and real consumption as a function of I in 2017 using IV estimates.



(d) Percent difference in money-metric values with observed and unobserved prices for different percentiles of the I distribution using IV estimates.



A.5 Additional Details of the UK Data Used in Section 4

We use two different datasets. One is a household-level expenditure survey and the other is data on prices of different categories of goods. The first data set is *Family Expenditure Survey and Living Costs and Food Survey Derived Variables*, which is a dataset of annual household expenditures with demographic information compiled from various household surveys conducted in the UK. Each sample includes about 5,000-7,000 households. The spending categories in the survey correspond to RPI (Retail Price Index) categories. We have continuous data from 1974 to 2017. Starting in 1995, the data are split into separate files for adults and children, so we merge them into households by adding up their expenditures.

Our algorithm does not require a representative sampling of the entire distribution of households, and can recover the money metric for a subsample of observed households, even if that subsample does not sample incomes at the same frequency as the population. The expenditure survey samples from the entire income distribution except for top earners and some pensioners. In order to correct for possible nonresponse bias, household weights are provided since 1997.[2] We use these weights to calculate the chained aggregate price index, which we use to calculate real consumption as in the official statistics. However, our approach for the money metric does not use household weights.

For the prices, we use the underlying data for the consumer price index (CPI) and the retail price index (RPI). To construct the consumption deflator in the national accounts, the Office of National Statistics switched from the Retail Price Index (RPI) to the Consumer Price Index (CPI).[3] By comparing the RPI and CPI with the consumption deflator provided by the Office of National Statistics, we identify the switching point as 1998 and do the same for our price data.

Because the CPI and RPI consider different baskets of goods and services, we merged various sub-categories to obtain a consistent set of categories over time. For example, "alcohol" in the RPI includes some items served outdoors, which is included in "restaurants" in the CPI. In this case, we merged "Catering and Alcohol" in the RPI and matched it with "Restaurant and Alcohol" in the CPI. We end up with 17 categories that are available for the entire period for both RPI and CPI. Table A.2 summarizes how we integrated the CPI

[2] Prior to 1997, benefit unit weights are provided instead of household weights. Since a benefit unit is a single person or a couple with any dependent children, there can be more than one benefit unit weight in a household. For example, if a couple with their children and the father's parents live together, then two benefit unit weights are recorded. In this case, we use the simple average as the household weight.

[3] `https://webarchive.nationalarchives.gov.uk/ukgwa/20151014001957mp_/http://www.ons.gov.uk/ons/guide-method/user-guidance/prices/cpi-and-rpi/mini-triennial-review-of-the-consumer-prices-index-and-retail-prices-index.pdf`.

and RPI baskets.

Figure A.5: Comparison of aggregate annual inflation reported by the UK Office of National Statistics and aggregate inflation calculated in our dataset following the same methodology.

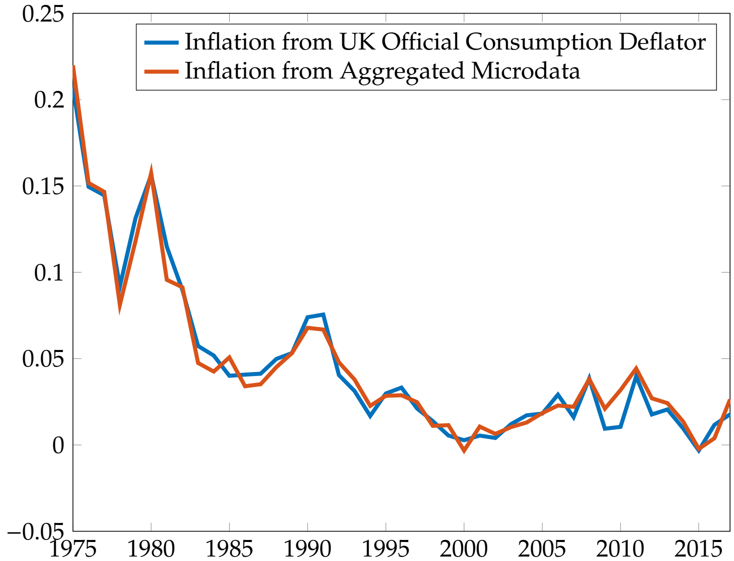


Table A.1: Comparison of ONS and our microdata.

	Decile								Difference
	2	3	4	5	6	7	8	9	D9-D2
ONS	2.8%	2.7%	2.6%	2.5%	2.4%	2.4%	2.3%	2.3%	-0.5%
Microdata	2.6%	2.6%	2.5%	2.4%	2.4%	2.3%	2.2%	2.1%	-0.5%

Notes: We report average annual inflation 2005-2017, in percentages. The ONS data is from Table 9 of "Data tables for the CPI consistent inflation rate estimates for UK household groups" Release date: 15th February 2023. We do not compare the 1st and 10th decile since those deciles are sensitive to how the tails of the distribution are treated. The last column is the difference between the ninth and second deciles.

Figure A.5 shows that our aggregated microdata closely matches the official consumption price deflator series for the UK. Table A.1 compares average chain-weighted inflation by expenditure decile reported by the ONS to similar statistics calculated using our microdata. We do not compare the 1st and 10th decile since those deciles are sensitive to how the tails of the distribution are treated. Once again, our microdata matches the official rates reasonably closely.

Integrated Categories	RPI	CPI
Bread & Cereals	Bread, Cereals and Biscuits	Bread & cereals
Meat & Fish	Meat, Fish, Beef, Lamb and Pork	Meat & fish
	Poultry and Other meat	-
Milk & Eggs	Butter, Cheese and Eggs	Milk, cheese & eggs
	Fresh milk and Milk products	-
Oils & fats	Oils & fats	Oils & fats
Fruit	Fruit	Fruit
Vegetable	Potatoes and Other vegetables	Vegetables including potatoes & other tubers
Other food	Sweets & Chocolates	Food Products
	Other Foods	Sugar, jam, honey, syrups, chocolate & confectionery
Non-Alcoholic Beverages	Tea and Soft drinks	Non-Alcoholic Beverages
	Coffee & other hot drinks	
Tobacco	Cigarettes & tobacco	Tobacco
Catering	Catering	Catering services
	Alcoholic drink	Alcoholic beverage
Household & Fuel	Housing except mortgage interest	Housing, water and fuels
	Fuel & light	
	(-)Dwelling insurance & ground rent	
Clothing	Clothing & footwear	Clothing & footwear
Household Goods	Household goods	Furniture and household equipment & routine repair of house
	domestic services	
Postage & Telecom	Postage	Communication
	Telephones & Telemessages	
Personal Goods	Personal goods & services	Health
	Fees & subscriptions	Miscellaneous goods and service
	Dwelling insurance & ground rent	-
Transport	Motoring expenditure	Transport
	Fares & other travel costs	-
Leisure Goods & Service	Leisure goods	Recreation & culture
	Leisure services	Education
	-	Accommodation service

Table A.2: RPI and CPI Correspondence Table

A.6 Testing for Separability Between X and Y goods

In this appendix, we sketch-out one way to test separability between X and Y goods, expanding on Footnote 30. After running our method, we bin households by money metric values. Then, for each money metric bin h, we run regressions of the form

$$\Delta \log b_{hit} - \Delta \log b_{hjt} = \beta_k \Delta \log p_{kt} + \text{controls} + \text{error}_{ht},$$

where $i, j \in Y$ and $k \in X$, and t is time. If this regression can be estimated without omitted variable bias, then we expect that the estimates for β should be equal to zero for every k. Intuitively, the relative compensated budget shares of i and j should not respond to changes in the price of k. The same should hold if we swap the role of Y and X, although the latter is not testable if prices in Y are missing.

Table A.3 provides an example, estimated using OLS in the UK data, where Y is "Catering" and "Leisure Goods & Service" and X is the 15 remaining product categories (see table A.2). We find that almost all coefficients are insignificant, except for "personal goods" and "other food" when we include the relative price within Y as a control, which is significant at the 10 percent level. We view this as tentative evidence that separability is not strongly violated in this example.

Table A.3: Illustration of test of separability using UK data

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Bread & Cereals	0.038	0.038	0.040	0.044	0.097	0.097	0.122	0.125
	(0.056)	(0.056)	(0.080)	(0.081)	(0.060)	(0.061)	(0.087)	(0.088)
Meat & Fish	−0.013	−0.012	−0.027	−0.023	0.026	0.027	0.017	0.020
	(0.064)	(0.065)	(0.079)	(0.080)	(0.070)	(0.070)	(0.085)	(0.086)
Milk & Eggs	0.021	0.022	0.019	0.021	0.065	0.065	0.077	0.079
	(0.042)	(0.042)	(0.059)	(0.060)	(0.046)	(0.046)	(0.065)	(0.065)
Oilfats	−0.059	−0.058	−0.072	−0.071	−0.025	−0.025	−0.037	−0.036
	(0.052)	(0.052)	(0.058)	(0.058)	(0.055)	(0.055)	(0.061)	(0.062)
Fruit	0.043	0.043	0.042	0.044	0.083	0.083	0.084	0.085
	(0.073)	(0.073)	(0.079)	(0.079)	(0.078)	(0.078)	(0.084)	(0.084)
Vegetables	−0.018	−0.018	−0.022	−0.022	0.019	0.019	0.016	0.015
	(0.048)	(0.048)	(0.052)	(0.052)	(0.055)	(0.055)	(0.060)	(0.060)
Other food	0.092	0.093	0.107	0.108	0.128*	0.129*	0.147*	0.148*
	(0.064)	(0.064)	(0.074)	(0.075)	(0.068)	(0.068)	(0.079)	(0.079)
Non-Alcoholic Beverages	0.026	0.026	0.025	0.025	0.056	0.055	0.058	0.058
	(0.054)	(0.054)	(0.065)	(0.065)	(0.056)	(0.056)	(0.067)	(0.067)
Tobacco	−0.081	−0.081	−0.115	−0.112	−0.014	−0.014	−0.037	−0.033
	(0.069)	(0.070)	(0.086)	(0.086)	(0.076)	(0.076)	(0.094)	(0.094)
Household & Fuel	−0.052	−0.051	−0.087	−0.083	0.035	0.036	0.031	0.035
	(0.052)	(0.052)	(0.070)	(0.071)	(0.063)	(0.064)	(0.091)	(0.091)
Clothing	0.052	0.051	0.075	0.078	0.042	0.041	0.045	0.049
	(0.046)	(0.047)	(0.082)	(0.083)	(0.046)	(0.047)	(0.084)	(0.084)
Household Goods	0.062	0.063	0.075	0.079	0.096	0.097	0.117	0.121
	(0.067)	(0.067)	(0.093)	(0.093)	(0.069)	(0.069)	(0.095)	(0.096)
Postage & telecoms	0.002	0.003	−0.003	−0.000	0.061	0.062	0.070	0.073
	(0.045)	(0.045)	(0.055)	(0.055)	(0.052)	(0.052)	(0.065)	(0.065)
Personal Goods	0.056	0.056	0.088	0.092	0.103*	0.103*	0.175	0.179
	(0.056)	(0.056)	(0.100)	(0.101)	(0.060)	(0.061)	(0.111)	(0.111)
Transport	0.066	0.067	0.079	0.082	0.111	0.112	0.136	0.139
	(0.070)	(0.070)	(0.089)	(0.090)	(0.075)	(0.075)	(0.097)	(0.097)
Quantile FE	N	Y	Y	N	N	Y	Y	N
Decade FE	N	N	Y	N	N	N	Y	N
Quantile × Decade FE	N	N	N	Y	N	N	N	Y
Relative price within Y	N	N	N	N	Y	Y	Y	Y
N	41,427	41,427	41,427	41,427	41,427	41,427	41,427	41,427

Notes: Standard errors are clustered at the household level.

A.7 Comparison with Blundell et al. (2003)

In this appendix, we exposit and apply the welfare bounds in Blundell et al. (2003) to artificial and real data. We start by discussing how we implement their methodology since, due to a typographical error in the algorithm for the lower-bound in the published paper, we do not exactly implement their procedure.

A.7.1 Description of Bounding Algorithm

To bound the cost-of-living, Blundell et al. (2003) provide an algorithm for an upper-bound and a lower-bound. Following the notation in their paper, let $q_t(I)$ be bundle of goods consumed by a household with income I in period t. Blundell et al. (2003) assume that $q_t(I)$ is an injective function (each I maps to a unique bundle of quantities in each period).

Algorithm A (Upper-bound). To recover an upper-bound for $e(p_s, v(p_t, I_t))$, start by defining $q^* = q_t(I_t)$ and let T be the set of periods for which we have data.

(1) Set $i = 0$ and $F^{(i)} = \{q_s^i = q_s(p_s \cdot q^*)\}_{s \in T}$.

(2) Set $F^{(i+1)} = \{q_s^{i+1} = q_s(\min_{q \in F^{(i)}} p_s \cdot q)\}_{s \in T}$.

(3) If $F^{(i+1)} = F^{(i)}$, then set $Q_B(q^*) = F^{(i)}$ and stop. Else set $i = i + 1$ and go to step (2).

We have that $e(p_s, v(p_t, I_t)) \leq \min_q\{p_s \cdot q : q \in Q_B(q^*)\}$. For the income levels I_t for which $F^{(0)}$ is empty for $s \neq t$ (because there are no households at s whose spending at s is as high or as low as $p_s \cdot q^*$), we cannot calculate an upper-bound.

Intuitively, the cost of living in period s associated with q^*, $e(p_s, v(p_t, I_t))$, is weakly less than $p_s \cdot q^*$. Hence, for every s, we must have that $q_s^0 = q_s(p_s \cdot q^*)$ is weakly preferred to q^*. This collection of bundles, $\{q_s^0\}_{s \in T}$, all of which are preferred to q^*, is $F^{(0)}$ defined in step (1). In step (2), we search across all of these bundles to find the cheapest one in each period s. We update each q_s^i to be the bundle that households with that level of income actually picked in each period (which is still better than q^*). We continue this indefinitely until this procedure converges, at which point we have our upper-bound.

As mentioned above, the lower-bound algorithm provided by Blundell et al. (2003) has a typographical error. We provide an amended version below.

Amended Algorithm B (Lower-bound). To recover a lower-bound for $e(p_s, v(p_t, I_t))$, start by defining $q^* = q_t(I_t)$ and let T be the set of periods for which we have data.

(1) Set $i = 0$, and let $F^{(i)} = \{I_s^i : p_t \cdot q_s(I_s^i) = I_t\}_{s \in T}$.

(2) Set $F^{(i+1)} = \{\max_{I_k \in F^{(i)}} \{I_s^{i+1} : I_k = p_k \cdot q_s(I_s^{i+1})\}\}_{s \in T}$.

(3) If $F^{(i+1)} = F^{(i)}$, then set $Q_W(q^*) = \{q_s(I_s^i)\}_{s \in T}$ and stop. Else set $i = i + 1$ and go to step (2).

We have that $\max_{q_s \in Q_W(q^*)} p_s \cdot q_s \leq e(p_s, v(p_t, I_t))$. For the income levels I_t for which $F^{(0)}$ is empty for $s \neq t$ (because there are no households at s whose consumption bundle costs I_t at t prices), we cannot calculate a lower-bound.

Intuitively, in step (1), for each period s, we find the income level I_s^0 such that $p_t \cdot q_s(I_s^0) = I_t$. The bundle $q_s(I_s^0)$ was affordable at t but was not purchased. Hence, the true cost-of-living in period s must be greater than I_s^0. The collection of income levels constructed in this step is $F^{(0)}$ and all are less than the true cost-of-living. In step (2), for each period s, we search over I_k^i and find the maximum level of income I_s^{i+1} such that $I_k^i = p_k \cdot q_s(I_s^{i+1})$ is satisfied. The new I_s^{i+1} is weakly greater than I_s^i but we still know that I_s^{i+1} is less than the true cost-of-living. We continue this indefinitely until this procedure converges, at which point we have our lower-bound.

A.7.2 Results with UK Data

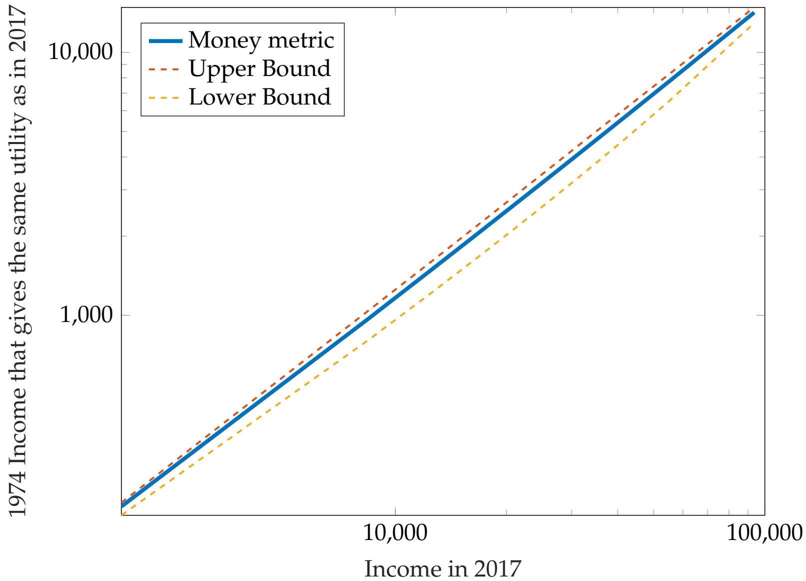


Figure A.6: Upper- and lower-bound using the amended Blundell et al. (2003) algorithm for the UK data in Section 4. Our algorithm produces the blue line. We can obtain bounds using the Blundell et al. (2003) algorithm for all households in the 2017 sample except for the top 1 percentile and the bottom 0.1 percentile.

A.8 Comparison with Jaravel & Lashkari (2022)

In this appendix, we apply the first-order and second-order algorithms described in Jaravel and Lashkari (2022) (JL) to some artificial examples and compare the performance with our method.[4] We start with the example in Section 3.3, where both methods perform well. We then provide other examples where the errors in their methodology are very large. These examples are selected to contrast the mathematical properties of our two methodologies when the support of the cross-sectional distribution of utilities changes over time.

We compute the errors for each method relative to the truth for the entire range over which each method produces estimates. We do this because identifying the set of households over which the money metric can be reliably estimated (without extrapolation) is a contribution of our methodology. The JL method purports to estimate the money metric for all households in the sample and does not provide a way to know if they are performing out-of-sample extrapolations, so we calculate the error accordingly.

Table A.4 shows that both methodologies perform very well for the simple example in Section 3.3, even though the support of the cross-section distribution of utilities is not constant over time. However, if we change parameter values, then the two methods can perform very differently.

[4]By setting the base year in the Jaravel and Lashkari (2022) algorithm to t_0, their definition of *real consumption* (which differs from our definition of real consumption) matches our money metric. Our method only requires repeated cross-sections. However, the second-order JL method requires a panel to construct household-specific inflation indexes. Therefore, to apply their method we create panels by the most disaggregated income quantile possible (i.e. if we have N households per period, then we form panels based on income N-quantiles). Finally, for the polynomial fitting stage of the Jaravel and Lashkari (2022) method, we use Matlab's polyfit function because it gives lower errors than a naive OLS regression.

Table A.4: Comparison of errors for simple example in Section 3.3

Jaravel and Lashkari (2022) method:

	Infinity Norm		Root Mean Square Error	
K	First Order	Second Order	First Order	Second Order
1	0.03	0.03	8.7×10^{-3}	8.3×10^{-3}
2	0.02	0.02	1.6×10^{-3}	1.4×10^{-3}
4	0.01	0.01	1.2×10^{-3}	1.0×10^{-3}
6	7.8×10^{-4}	4.0×10^{-4}	6.4×10^{-4}	1.5×10^{-4}
8	3.6×10^{-3}	4.4×10^{-3}	6.6×10^{-4}	2.5×10^{-4}
12	1.1×10^{-3}	7.0×10^{-4}	6.4×10^{-4}	1.5×10^{-4}

Baqaee, Burstein, Koike-Mori method:

Infinity Norm		Root Mean Square Error	
Iterative	Recursive	Iterative	Recursive
7.8×10^{-3}	1.5×10^{-4}	5.0×10^{-3}	7.7×10^{-6}

Notes: The Jaravel and Lashkari (2022) methodology is applied to the artificial example in Section 3.3. We report two different norms (infinity norm and root mean square error) of the percentage difference between the true money metric and the estimate in the final period (e.g. 0.03 stands for 3% difference). The first column is their "first-order" algorithm and the second column is their "second-order" algorithm. The parameter K is the order of the polynomial used. The sample has 1000 households and annual data.

One example is provided in Table A.5. Our method, which tracks the boundary of overlapping support, does not produce any numbers for this example because there is no overlap in the support of the utility distribution between t_0 and T. However, the Jaravel and Lashkari (2022) algorithm does produce estimates and they are very inaccurate. Furthermore, these estimates do not improve as we increase the sample size or frequency of observation. Importantly, the Jaravel and Lashkari (2022) methodology does not provide a way to know whether their estimates are reliable (like in Table A.4) or unreliable like in (Table A.5). On the other hand, our methodology does not produce estimates that are not guaranteed to be reliable (given our assumptions).

Table A.5: Errors in Jaravel and Lashkari (2022) method with different parameters

	Infinity Norm		Root Mean Square Error	
K	First Order	Second Order	First Order	Second Order
1	0.27	0.27	0.25	0.24
2	0.47	0.43	0.44	0.40
4	0.38	0.41	0.34	0.36
6	1.5×10^{104}	Polyfit error	4.6×10^{102}	Polyfit error
8	1.08	1.17	0.97	1.06
12	Polyfit error	Polyfit error	Polyfit error	Polyfit error

Notes: This table shows the accuracy of the Jaravel and Lashkari (2022) algorithm for different values of K (polynomial degree), as defined in the notes for Table A.4. The expenditure function is $e(p, U) = \left(\sum_i \omega_i U^{\varepsilon_i(1-\gamma)} p_i^{1-\gamma}\right)^{1/(1-\gamma)}$ where $(\gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (5, 0.3, 1, 2)$ and ω is all 1. There are 1000 households uniformly distributed in the income distribution over $[1, 1.1]$. Average nominal income is the numeraire and the income distribution does not change over time. There are 40 periods and the price of the three goods rise (relative to income) at a constant rate from $(1, 1, 1)$ to $(2, 3, 4)$. If Matlab fails to find a unique polynomial due to (numerical) multi-collinearity, we write “Polyfit error.” Although we do not report the numbers, the errors in these cases are large. Quadrupling the number of households and doubling the frequency of observation does not appreciably change the results in this table.

In Table A.5, there is no overlapping support, so our method produces no estimates. In the next example, the distribution of money metric values in the final period is, by construction, a subset of the one in the initial period. This means that our method produces estimates for every household in the sample. That is, we compare the performance of our method to JL for the same set of households (since all households in the final period are in a region of overlapping support). The results are reported in Table A.6. Once again, increasing the frequency of observation and number of households do not appreciably change the estimates.

Table A.6: Comparison of errors for non-homothetic CES example with different parameters

Jaravel and Lashkari (2022) method:

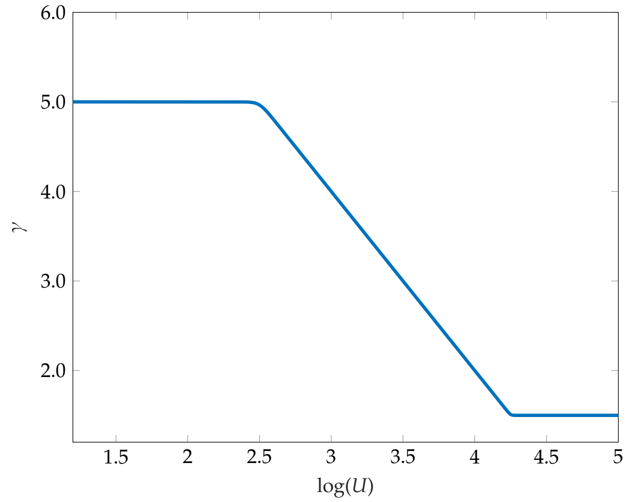
	Infinity Norm		Root Mean Square Error	
K	First Order	Second Order	First Order	Second Order
1	0.15	0.15	0.08	0.07
2	0.17	0.13	0.05	0.05
4	1.2×10^{91}	Not converged	1.8×10^{89}	Not converged
6	4.9×10^{124}	Polyfit error	7.0×10^{122}	Polyfit error
8	NaN	Polyfit error	NaN	Polyfit error
12	Polyfit error	Polyfit error	Polyfit error	Polyfit error

Baqaee, Burstein, Koike-Mori method:

Infinity Norm		Root Mean Square Error	
Iterative	Recursive	Iterative	Recursive
1.4×10^{-3}	2.5×10^{-6}	1.3×10^{-3}	1.7×10^{-6}

Notes: This table shows the accuracy of the Jaravel and Lashkari (2022) algorithm for different values of K (polynomial degree), as defined in the notes for Table A.4. The expenditure function is $e(p, U) = \left(\sum_i \omega_i U^{(1-\gamma)\varepsilon_i} p_i^{1-\gamma}\right)^{1/(1-\gamma)}$ where $(\gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (5, 1.6, 2, 3.3)$ and $\omega = (1, 1, 1)$. There are 5000 households equally distributed in the income distribution and 100 periods. The initial income distribution is $[0.8, 1.4]$. Between period 1 and 50, the income distribution uniformly and linearly changes to $[0.003, 34.4]$. Between period 51 and 75, the income distribution uniformly and linearly changes to $[0.5, 8.2]$. Between period 76 and 100, the income distribution uniformly and linearly changes to $[2.7, 2.9]$. The price vector changes from $(1, 1, 1)$ to $(2, 3, 4)$. If the second-order algorithm does not converge within 100 iterations, we write "Not converged." If the estimated values of the money metric explode, we write "NaN" for not a number. If we fail to find a unique polynomial (due to numerical multi-collinearity), we write "Polyfit error." Although we do not report the numbers, the errors in these cases are large. Results are similar for higher order polynomials, if we quadruple the number of households, or double the frequency of observations.

Figure A.7: The elasticity of substitution as a function of utility for the example in Table A.7



Our final example uses a more nonlinear demand system. Let preferences be defined by

$$e(p, U) = \left(\sum_i \omega_i \left(U^{\varepsilon_i} p_i \right)^{1-\gamma(U)} \right)^{1/\left(1-\gamma(U)\right)}, \tag{1}$$

where we allow the elasticity of substitution γ to depend on utility, as in Fally (2022). To keep the preferences well behaved, we constrain the elasticity of substitution to be between a lower- and upper-bound value. For example, the most straightforward way to do this is to set

$$\gamma(U) = \max\left\{ \min\left\{ \underline{\gamma}, \gamma_0 - \eta \log U \right\}, \bar{\gamma} \right\}. \tag{2}$$

The Jaravel and Lashkari (2022) propositions require smoothness, so we instead use the following functional form

$$\gamma(U) = \left(\overline{\gamma}^{\chi_1 - 1} + \left(\left[\underline{\gamma}^{\frac{\chi_2 - 1}{\chi_2}} + \left(\gamma_0 - \eta \log(U) \right)^{\frac{\chi_2 - 1}{\chi_2}} \right]^{\frac{\chi_2}{\chi_2 - 1}} \right)^{\chi_1 - 1} \right)^{\frac{1}{\chi_1 - 1}}, \tag{3}$$

where we set $\chi_1 = 100$ and $\chi_2 = 0.01$. This function is plotted in Figure A.7 and smoothly approximates the maximum and minimum functions. In practice, the errors are similarly large whether we use (2) or (3).

We simulate artificial data using this demand system and report the results in Table A.7. The Jaravel and Lashkari (2022) methodology has substantially larger errors and does not seem to converge as we increase the number of parameters in the polynomial approximation or the sample size. Our methodology, in contrast, produces very small errors.

Table A.7: Comparison of Jaravel and Lashkari (2022) and Baqaee, Burstein, Koike-Mori errors for more complex example

Jaravel and Lashkari (2022) method:

	Infinity Norm		Root Mean Square Error	
K	First Order	Second Order	First Order	Second Order
1	0.17	0.17	0.11	0.10
2	0.25	0.25	0.16	0.15
4	14	Not converged	0.53	Not converged
6	1.9×10^{205}	Not converged	6.0×10^{203}	Not converged
8	3.1×10^{92}	Polyfit error	9.9×10^{90}	Polyfit error
12	Polyfit error	Polyfit error	Polyfit error	Polyfit error

Baqaee, Burstein, Koike-Mori method:

Infinity Norm		Root Mean Square Error	
Iterative	Recursive	Iterative	Recursive
7.4×10^{-3}	1.7×10^{-5}	4.6×10^{-3}	1.1×10^{-5}

Notes: This table shows the accuracy of the Jaravel and Lashkari (2022) algorithm for different values of K (polynomial degree), as defined in the notes for Table A.4. The expenditure function is (1) with $\varepsilon = [0.2, 1, 1.65]$ and ω_i calibrated so that the budget share of each good for the median household in the first period is the same. The parameters in (3) are $\gamma_0 = 10$, $\underline{\gamma} = 1.5$ $\bar{\gamma} = 5$, $\eta = 2$, $\chi_1 = 100$ and $\chi_2 = 0.01$. The income distribution starts as a uniform distribution between $[2, 50]$ and grows uniformly by a factor of 14 over 40 periods. The price vector changes from $(1, 1, 1)$ to $(7, 5, 3)$. If the second-order algorithm does not converge, we write "Not converged." If Matlab fails to find a unique polynomial (due to numerical multi-collinearity), we write "Polyfit error." Although we do not report the numbers, the errors in these cases are large. Results are similar for higher order polynomials, if we quadruple the number of households, or double the frequency of observations.

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