Networks, Barriers, and Trade

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Abstract

We study a flexible class of trade models with international production networks and arbitrary wedge-like distortions like markups, tariffs, or nominal rigidities. We characterize the general equilibrium response of variables to shocks in terms of microeconomic statistics. Our results are useful for decomposing the sources of real GDP and welfare growth, and for computing counterfactuals. Using the same set of microeconomic sufficient statistics, we also characterize societal losses from increases in tariffs and iceberg trade costs and dissect the qualitative and quantitative importance of accounting for disaggregated details. Our results, which can be used to compute approximate and exact counterfactuals, provide an analytical toolbox for studying large-scale trade models and help to bridge the gap between computation and theory.

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1 Introduction

Trade economists increasingly recognize the importance of using large-scale computational general equilibrium models for quantitative policy analysis. A downside of relying on purely computational methods is that it may be hard to know which forces in the model drive specific results. On the other hand, stylized models, while transparent and parsimonious, can lead to unreliable quantitative predictions compared to large-scale models.

This paper attempts to provide a theoretical map of territory usually explored by machines. It studies real GDP and welfare in open economies with disaggregated and interconnected production structures. We address two types of questions: (i) how to measure and decompose the sources of output and welfare changes using ex-post sufficient statistics, à la Solow (1957), and (ii) how to predict the responses of output, welfare, as well as disaggregated prices and quantities, to changes in technologies or wedges using ex-ante sufficient statistics, à la Jones (1965). Our analysis is fairly general (for example, nesting most Armington-style models) and helps to isolate the common forces and sufficient statistics necessary to answer these questions without committing to specific functional forms. We use these results to show how accounting for the details of the production structure can theoretically and quantitatively change answers to a broad range of questions in open-economy settings.

Our framework allows for arbitrary distorting wedges (like taxes, markups, or sticky prices), and we derive comparative statics with respect to both wedges and technologies in terms of primitives. We derive how every equilibrium price and quantity responds to changes in technologies and wedges as a function of the input-output matrix, elasticities of substitution, and wedges in the initial equilibrium.

Since our focus is on real GDP and welfare, we begin by showing that changes in real GDP and welfare can be decomposed, to a first-order approximation, into a direct technology effect of the shock, holding fixed the allocation of resources, and a pure reallocation component. For real GDP, reallocation effects are irrelevant if the initial allocation is efficient. If the initial allocation is inefficient, then reallocations can boost real GDP by reallocating resources away from low marginal value firms towards high marginal value ones. Furthermore, we show that these reallocations can be tracked using the change in factor income shares in the domestic economy. For welfare, reallocation effects are non-zero even when the equilibrium is efficient. Furthermore, we show that the reallocation effects for welfare depend on what we call the *factoral* terms-of-trade, which depend on international factor income shares.[1] Our decompositions of welfare and real GDP can be applied ex-post

[1] We borrow the term "factoral terms-of-trade" from Viner (1937), though our formal definition coincides with his only in very simple environments.

to decompose the sources of welfare and output growth over time, or used as an intermediate step to answer ex-ante counterfactual questions.

To answer how welfare and real GDP respond to a counterfactual shock, we need to know both the direct effect of the shock and the indirect (reallocative) effect of the shock. To a first-order approximation, the direct effects of shocks are simple to understand and rely only on input-output shares and wedges in the initial (pre-shock) equilibrium. Reallocation effects, on the other hand, are more complex, even to a first-order, and depend on general equilibrium movements of factor income shares. We characterize the response of factor income shares to exogenous shocks as a function of the input-output network, the elasticities of substitution in production and consumption, returns-to-scale, and initial wedges. Once in possession of changes in factor prices, then it is simple to calculate how reallocation effects affect welfare and GDP to a first-order.

We also provide second-order approximations with respect to technology and wedges for the world as a whole, and the real GDP of each country. These results show that losses from tariffs or other distortions are approximately equal to a sales-weighted sum of deadweight loss (Harberger) triangles. We provide explicit formulas for these Harberger triangles in terms of microeconomic primitives (the input-output network, elasticities of substitution, and returns to scale).

Using a series of pen-and-paper examples, we show how microeconomic details, like the presence of input-output linkages, complementarities in the domestic economy, frictions to factor mobility across sectors, and nominal rigidities magnify the welfare losses from negative trade shocks. For example, we show that a negative trade shock is much more costly if domestic sectors are complements and domestic sectors have decreasing returns to scale. This is especially relevant for thinking about disruptions in, for example, the supply of energy as studied by Bachmann et al. (2022). We also show how nominal rigidities can help to explain why, in the short-run, a disruption in trade can cause domestic unemployment, as in Rodríguez-Clare et al. (2020), and result in complete pass-through of tariffs into consumer prices, as in Fajgelbaum et al. (2020).

Our comparative static results, which generalize Jones's *hat-algebra* beyond frictionless $2 \times 2 \times 2$ no input-output economies, pin down how every price and quantity responds to shocks. This means that repeated iteration on these first-order calculations also yields exact nonlinear comparative statics, providing an alternative computational method to the *exact* hat-algebra (e.g. Dekle et al., 2008) that is commonly used in the literature. Whereas exact hat-algebra requires solving a large nonlinear system of excess demand equations once, our differential approach requires solving a smaller linear system repeatedly. Computationally, for large and highly nonlinear models, this differential equation approach is significantly faster.[2] We use this method, and a quantitative multi-country, multi-sector model of the world economy with input-output connections, to show that the analytical intuitions we derive using simple examples remain valid in quantitatively more realistic environments.

The outline of the paper is as follows. In Section 2, we set up the model and define the objects of interest. In Section 3, we derive some first-order growth-accounting results useful for measurement and decompositions. In Section 4, we derive first-order comparative statics in terms of microeconomic primitives, useful for prediction. In Section 5, we apply the results in Section 4 to approximate societal losses from tariffs and technology shocks to the second order. In Section 6, we provide analytical examples showing how different mechanisms affect the transmission of trade shocks to welfare. Section 7 contains quantitative examples showing that the intuition gleaned from the analytical examples is useful in understanding larger scale models. We conclude in Section 8. Proofs are in the online appendix. Additional details can also be found in the appendix of the working paper version of this paper, Baqaee and Farhi (2019).

Related Literature. This paper connects three different literatures: the literature on the welfare effects of trade shocks, the literature on production networks, and the literature on growth accounting. We discuss each literature in turn starting with the one on the gains (or losses) from trade shocks. Our results generalize some of the results in Costinot and Rodriguez-Clare (2014) to environments with non-linear input-output connections. We generalize the input-output models emphasized in Caliendo and Parro (2015), Caliendo et al. (2017), Morrow and Trefler (2017), Fally and Sayre (2018), and Bernard et al. (2019). Our paper is also related to contemporaneous work by Huo et al. (2020), who decompose bilateral GDP comovement into shock transmission and shock correlation.

A vast and active branch of the literature uses large-scale computational general equilibrium (CGE) models for policy analysis. We refer readers to the CGE handbook, Dixon and Jorgenson (2012), as well as to Corong et al. (2017), who provide a detailed overview of the Global Trade Analysis Project, a standardized database and CGE modeling platform for policy analysis. The analytical results in this paper complement the quantitative approach of this literature, and the welfare and GDP decompositions we provide can be used to help interpret the output from large-scale models.

Our results about the effects of trade in distorted economies also relate to Berthou et al. (2018) and Bai et al. (2018). Our results also relate to complementary work with non-

[2] We provide flexible Matlab code for performing these loglinearizations and numerically integrating the results. Our computational approach, which, instead of solving a nonlinear system of equations, numerically integrates derivatives, is similar to the way computational general equilibrium (CGE) models are sometimes solved (for a survey, see Dixon et al., 2013).

parametric or semi-parametric models of trade like Adao et al. (2017) and Allen et al. (2014). These papers study reduced-form general equilibrium demand systems under assumptions that ensure this demand system is invertible and invariant to shocks. Our results show how to construct these general equilibrium objects from microeconomic primitives, building an explicit bridge from disaggregated microeconomic information to aggregate objects. Our characterization of how factor shares and prices respond to shocks is related to a large literature, for example, Trefler and Zhu (2010), Davis and Weinstein (2008), Feenstra and Sasahara (2017), Dix-Carneiro (2014), Galle et al. (2017), among others.

The literature on production networks has primarily been concerned with the propagation of shocks in closed economies, typically assuming a representative agent. For instance, Long and Plosser (1983), Acemoglu et al. (2012), Atalay (2017), Carvalho et al. (2016), Baqaee and Farhi (2017a,b), Baqaee (2018), Carvalho and Tahbaz-Salehi (2018), Liu (2017), among others. A recent focus of the literature, particularly in the context of open economies, has been to model the formation of firm-to-firm links. This strand of the literature takes discreteness seriously, for example, Chaney (2014), Lim (2017), Tintelnot et al. (2018), and Kikkawa et al. (2018). Our approach is different: rather than modeling the formation of links as a discrete decision, we assume a differentiable form of adjustment where the presence and strength of links is determined by cost minimization subject to a smooth production technology. This means that we can only handle the extensive margin via choke prices. In exchange for this simplification, we provide a fairly general local characterization of the equilibrium.

Our growth accounting results are related to closed-economy results like Solow (1957), Hulten (1978), as well as to the literature extending growth-accounting to open economies, including Kehoe and Ruhl (2008) and Burstein and Cravino (2015). Perhaps closest to us are Diewert and Morrison (1985) and Kohli (2004) who introduce output indices that account for terms-of-trade changes. Our real income and welfare-accounting measures share their goal, though our decomposition into pure productivity changes and reallocation effects is different. In explicitly accounting for the existence of intermediate inputs, our approach also speaks to how one can circumvent the double-counting problem and spillovers arising from differences in gross and value-added trade, issues studied by Johnson and Noguera (2012) and Koopman et al. (2014). Relative to these other papers, our approach has the bonus of easily handling inefficiencies and wedges.

Our approach is general, and relies on duality, along the lines of Dixit and Norman (1980). We differ from the classic analysis, however, in that, we state our comparative static results in terms of observable microeconomic sufficient statistics: input-output shares, changes in shares, and (microeconomic) elasticities of substitution. Our approach relies heavily on the notion of the allocation matrix, which helps give a physical (primal) interpretation to the theorems, and is convenient for analyzing inefficient economies. In inefficient economies, the absence of macro-level envelope conditions mean that the abstract approach, like Dixit and Norman (1980) and Chipman (2008), runs into problems. However, our results readily extend to inefficient economies.

2 Framework

In this section, we set up the model and define the statistics of interest.

2.1 Model Environment

There is a set of countries C, a set of producers N producing different goods, and a set of factors F. Each producer and each factor is assigned to be within the borders of one of the countries in C. The sets of producers and factors inside country c are N_c and F_c. The set F_c of factors physically located in country c may be owned by any household, and not necessarily the households in country c. To streamline the exposition, we assume that there is a representative consumer in each country.[3]

Distortions. Since tax-like wedges can implement any feasible allocation of resources in our model, including inefficient allocations, we use wedges to represent distortions. These tax wedges may be explicit, like tariffs, or they may be implicit, like markups, sticky prices, or financial frictions. For ease of notation, to represent a wedge on i's purchases of inputs from k, we introduce a fictitious middleman k' that buys from k and sells to i at a "markup" $\mu_{k'}$. The revenues collected by these markups/wedges are rebated back to the households in a way we specify below.[4]

Producers. Every good $i \in N$ belongs to some country $c \in C$ and is produced using a constant-returns-to-scale production function

$$y_i = A_i F_i \left(\{x_{ik}\}_{k \in N}, \{l_{if}\}_{f \in F_c} \right),$$

[3]See Appendix L in the NBER working paper version of this paper for a discussion of how to extend the results to heterogeneous households within countries.

[4]These fictitious middlemen are convenient for writing compact formulas, but adding them to the model explicitly is computationally inefficient. In the computational appendix, Appendix D, we discuss these issues in more detail.

where y_i is the total quantity of good i produced, x_{ik} is intermediate inputs from k, l_{if} is factor inputs from f, and A_i is an exogenous Hicks-neutral productivity shifter.[5,6] Producer i chooses inputs to minimize costs and sets prices equal to marginal cost times a wedge $p_i = \mu_i \times mc_i$. We capture bilateral wedges between say i and j by adding a fictional intermediary that buys from i and sells to j at some markup.

Factors. Households earn income from primary factors and revenues generated by wedges. A primary factor is a non-produced good whose supply is, for now, taken to be exogenous.[7] To model revenues earned by wedges, for each country $c \in C$, we introduce a "fictitious" factor that collects the markup/wedge revenue accruing to residents of country c. We denote the set of true primary factors by F and the set of true and fictitious factors by F^*. (We will not use fictitious factors to define the equilibrium, but will refer to them in our comparative statics). The $C \times (N + F)$ matrix Φ is the ownership matrix, where Φ_{ci} is the share of i's value-added (sales minus costs) that goes to households in country c.

Households. The representative household in country c has homothetic preferences[8]

$$W_c = \mathcal{W}_c(\{c_{ci}\}_{i \in N}),$$

and faces a budget constraint given by

$$\sum_{i \in N} p_i c_{ci} = \sum_{f \in F} \Phi_{cf} w_f L_f + \sum_{i \in N} \Phi_{ci} \left(1 - 1/\mu_i\right) p_i y_i + T_c,$$

where c_{ci} is the quantity of the good i consumed by household c, w_f and L_f is the wage and quantity of factor f, p_i is the price and y_i is the quantity of good i, and T_c is an exogenous

[5]This is more general than it might appear. First, production has constant returns to scale without loss of generality, because non-constant returns can be captured via fixed factors. Second, the assumption that each producer produces only one output good is also without loss of generality. A multi-output production function is a single output production function where all but one of the outputs enter as negative inputs. Finally, productivity shifters are Hicks-neutral without loss of generality. To represent input-augmenting technical change for i's use of input k, introduce a fictitious producer buying from k and selling to i, and hit this fictitious producer with a Hicks-neutral shock.

[6]We rule out fixed costs in our analysis. Our results accommodate an extensive margin of product entry-exit, but only if it operates according to a choke-price, rather than a fixed cost. For an analysis of general equilibrium models with fixed costs see Baqaee and Farhi (2020).

[7]In Section 4.3, we endogenize factor supply using a labor-leisure tradeoff. In Appendix K of the working paper version of this paper, we discuss how to endogenize factor supply by using a Roy model and discuss the connection of our results with those in Galle et al. (2017).

[8]In mapping our model to data, we interpret domestic "households" as any agent which consumes resources without producing resources to be used by other agents. Specifically, this means that we include domestic investment and government expenditures in our definition of "households".

lump-sum transfer. The right-hand side is consumer c's income: the first summand is income earned by primary factors, the second summand is income earned from wedges (the "fictitious" factor for c), and the final summand is net transfers.

Iceberg Trade Costs. We capture changes in iceberg trade costs as Hicks-neutral productivity changes to specialized importers or exporters whose production functions represent the trading technology. The decision of where trading technologies should be located is ambiguous since they generate no income. It is possible to place them in the exporting country or the importing country, and this would make no difference in terms of the welfare of agents or the allocation of resources.[9]

Equilibrium. Given productivities A_i, wedges μ_i, and a vector of transfers satisfying $\sum_{c\in C} T_c = 0$, a general equilibrium is a set of prices p_i, intermediate input choices x_{ij}, factor input choices l_{if}, outputs y_i, and consumption choices c_{ci}, such that: (i) each producer chooses inputs to minimize costs taking prices as given; (ii) the price of each good is equal to the wedge on that good times its marginal cost; (iii) each household maximizes utility subject to its budget constraint taking prices as given; and, (iv) the markets for all goods and factors clear so that $y_i = \sum_{c\in C} c_{ci} + \sum_{j\in N} x_{ji}$ for all $i \in N$ and $L_f = \sum_{j\in N} l_{jf}$ for all $f \in F$.

2.2 Definitions and Notation

In this subsection, we define the statistics of interest and introduce useful notation.

Nominal Output and Expenditure. Nominal output or Gross Domestic Product (GDP) for country c is the total final value of the goods produced in the country. It coincides with the total value-added earned by the producers located in the country:

$$GDP_c = \sum_{i\in N} p_i q_{ci} = \underbrace{\sum_{f\in F_c} w_f L_f}_{\substack{\text{income from factors} \\ \text{in country } c}} + \underbrace{\sum_{i\in N_c} (1 - 1/\mu_i)\, p_i y_i}_{\substack{\text{income from wedges} \\ \text{in country } c}},$$

where $q_{ci} = y_i 1_{\{i\in N_c\}} - \sum_{j\in N_c} x_{ji}$ is the "final" or net quantity of good $i \in N$ produced by country c. Note that q_{ci} is negative for imported intermediate goods.

[9] We do not need to take a precise stand at this stage, but we note that this will matter for our conclusions regarding country-level real GDP changes (as pointed out by Burstein and Cravino, 2015).

Nominal Gross National Expenditure (GNE) for country c, also known as domestic absorption, is the total final expenditures of the residents of the country. In our model, it coincides with nominal Gross National Income (GNI), which is the total income earned by the factors owned by a country's residents adjusted for international transfers:

$$GNE_c = \sum_{i \in N} p_i c_{ci} = \underbrace{\sum_{f \in F} \Phi_{cf} w_f L_f}_{\substack{\text{income from factors} \\ \text{owned by household } c}} + \underbrace{\sum_{i \in N} \Phi_{ci} (1 - 1/\mu_i) \, p_i y_i}_{\substack{\text{income from wedges} \\ \text{accruing to household } c}} + \underbrace{T_c}_{\substack{\text{transfers} \\ \text{to household } c}} .$$

The right-hand side is just consumer c's budget constraint.

To denote variables for the world, we drop the country-level subscripts. Nominal GDP and nominal GNE are *not* the same at the country level, but they are the same at the world level:

$$GDP = GNE = \sum_{f \in F} w_f L_f + \sum_{f \in N} (1 - 1/\mu_i) p_i y_i = \sum_{i \in N} p_i q_i = \sum_{i \in N} p_i c_i,$$

where, for the world, final consumption coincides with net output $c_i = q_i$ because $c_i = \sum_{c \in C} c_{ci} = \sum_{c \in C} q_{ci} = q_i$, and net transfers are zero, $T = 0$, because $T = \sum_{c \in C} T_c$. Let world GDP be the numeraire, so that $GDP = GNE = 1$. Hence, unless otherwise stated, all prices and transfers are expressed in units of this numeraire.

Real Output and Expenditure. To convert nominal variables into real variables, as in the data, we use Divisia indices throughout. To a first-order, the change in the real GDP of country c and the corresponding GDP deflator are defined to be

$$\mathrm{d}\log Y_c = \sum_{i \in N} \Omega_{Y_c,i} \, \mathrm{d}\log q_{ci}, \quad \mathrm{d}\log P_{Y_c} = \sum_{i \in N} \Omega_{Y_c,i} \, \mathrm{d}\log p_i, \tag{1}$$

where $\Omega_{Y_c,i} = p_i q_{ci} / GDP_c$ is good i's share in the final output of country c.[10] Throughout the paper, for any variable x, we define $\mathrm{d}\log x = \mathrm{d}\, x / x$. This is an abuse of notation, but it allows us to write $\mathrm{d}\log x$ even when x is a negative number.

The change in real GNE of country c and the corresponding deflator are

$$\mathrm{d}\log W_c = \sum_{i \in N} \Omega_{W_c,i} \, \mathrm{d}\log c_{ci}, \quad \mathrm{d}\log P_{W_c} = \sum_{i \in N} \Omega_{W_c,i} \, \mathrm{d}\log p_i, \tag{2}$$

where $\Omega_{W_c,i} = p_i c_{ci} / GNE_c$ is good i's share in country c's consumption basket. By Shephard's lemma, changes in real GNE are equal to changes in welfare for every country.

[10]Our definition of real GDP coincides with the double-deflation approach to measuring real GDP, where the change in real GDP is defined to be the sum of changes in real value-added for domestic producers.

Discrete changes in real GDP and real GNE are given by integrating equations (1) and (2). We denote the corresponding discrete changes by $\Delta \log Y$, $\Delta \log Y_c$, $\Delta \log W$, and $\Delta \log W_c$. In the case of GDP, this is how these objects are typically measured in the data, and in the case of GNE, this integral coincides with the nonlinear change in the welfare of each agent c as measured by a money-metric (since preferences are homothetic).

As with the nominal variables, real GDP and real GNE are *not* the same at the country level. However, these differences vanish at the world level so that, for the world, $\mathrm{d}\log Y = \mathrm{d}\log W$ and $\mathrm{d}\log P_Y = \mathrm{d}\log P_W$.[11] Conveniently, changes in country real GDP and real GNE aggregate up to their world counterparts.[12]

Input-Output Matrices. The Heterogenous-Agent Input-Output (HAIO) matrix is the $(C + N + F) \times (C + N + F)$ matrix Ω whose ijth element is equal to i's expenditures on inputs from j as a share of its total revenues/income

$$\Omega_{ij} = \frac{p_j x_{ij}}{p_i y_i} \mathbf{1}_{\{i \in N\}} + \frac{p_j c_{ij}}{GNE_i} \mathbf{1}_{\{i \in C\}}.$$

The HAIO matrix Ω includes the factors of production and the households, where factors consume no resources (zero rows), while households produce no resources (zero columns). The Leontief inverse matrix is

$$\Psi = (I - \Omega)^{-1} = I + \Omega + \Omega^2 + \dots.$$

Whereas the input-output matrix Ω records the *direct* link from one agent or producer to another, the Leontief inverse matrix Ψ records the *direct and indirect* exposures through the production network.

Denote the diagonal matrix of wedges by μ (where non-taxed quantities have wedge $\mu_i = 1$) and define the *cost-based* HAIO matrix and Leontief inverse to be

$$\tilde{\Omega} = \mu \Omega, \qquad \tilde{\Psi} = (I - \tilde{\Omega})^{-1}.$$

It will sometimes be convenient to treat goods and factors together and index them by $k \in N + F$ where the plus symbol denotes the union of sets. To this effect, we slightly extend our definitions. We interchangeably write y_k and p_k for the quantity L_k and wage w_k of factor $k \in F$.

[11] Real GDP and real GNE for the world are defined by aggregating across all countries, so $\mathrm{d}\log Y = \sum_{i \in N} (p_i q_i / GDP) \, \mathrm{d}\log q_i$, $\mathrm{d}\log P_Y = \sum_{i \in N} (p_i q_i / GDP) \, \mathrm{d}\log p_i$, $\mathrm{d}\log W = \sum_{i \in N} (p_i c_i / GNE) \, \mathrm{d}\log c_i$, and $\mathrm{d}\log P_W = \sum_{i \in N} (p_i c_i / GNE) \, \mathrm{d}\log p_i$,.

[12] Namely, $\mathrm{d}\log Y = \sum_{c \in C} (GDP_c / GDP) \, \mathrm{d}\log Y_c$ and $\mathrm{d}\log W = \sum_{c \in C} (GNE_c / GNE) \, \mathrm{d}\log W_c$.

Input-Output Exposures. Each $i \in C+N+F$ is *exposed* to each $j \in C+N+F$ through revenues Ψ_{ij} and through costs $\tilde{\Psi}_{ij}$. Intuitively, Ψ_{ij} measures how expenditures on i affect the sales of j (due to backward linkages), whereas $\tilde{\Psi}_{ij}$ measures how the price of j affects the marginal cost of i (due to forward linkages). In the absence of wedges, $\mu_i = 1$ for every i, these two objects coincide.

When i is a household, we use special notation to denote *backward* and *forward* exposure. In particular, let household c's exposures to k be

$$\lambda_k^{W_c} = \Psi_{c,k} = \sum_{i\in N} \Omega_{c,i}\Psi_{ik}, \quad \tilde{\lambda}_k^{W_c} = \tilde{\Psi}_{c,k} = \sum_{i\in N} \tilde{\Omega}_{c,i}\tilde{\Psi}_{ik}.$$

In words, c's exposure to k is the expenditure share weighted average of the exposure of c's suppliers to k.

By analogy, the forward and backward exposure of country c's GDP (as opposed to welfare) is defined as

$$\lambda_k^{Y_c} = \sum_{i\in N} \Omega_{Y_c,i}\Psi_{ik}, \quad \tilde{\lambda}_k^{Y_c} = \sum_{i\in N} \Omega_{Y_c,i}\tilde{\Psi}_{ik},$$

where recall that $\Omega_{Y_c,i} = p_i q_{ci}/GDP_c$ is the share of a good i in GDP. As usual, the world-level backward and forward exposure to k are denoted by suppressing the country subscript: that is, λ_k^Y and $\tilde{\lambda}_k^Y$ respectively.

We sometimes denote exposure to factors with capital letters, Λ or $\tilde{\Lambda}$, to distinguish them from non-factor producers, lower-case λ or $\tilde{\lambda}$. In other words, when $f \in F$, we write $\Lambda_f^{Y_c} = \lambda_f^{Y_c}$, $\Lambda_f^{W_c} = \lambda_f^{W_c}$, and $\tilde{\Lambda}_f^{W_c} = \tilde{\lambda}_f^{W_c}$ to emphasize that f is a factor.

Sales and Income Shares. Exposures of GDP to a good or factor k at the country and world levels have a direct connection to the sales of k:

$$\lambda_k^{Y_c} = \mathbf{1}_{\{k\in N_c+F_c\}} \frac{p_k y_k}{GDP_c}, \quad \lambda_k = \frac{p_k y_k}{GDP},$$

where $\mathbf{1}$ is an indicator function. Hence, the exposure of world GDP λ_k^Y to k is just the sales share (or *Domar weight*) of k in world output $\lambda_k = p_k y_k/GDP$. Similarly, the exposure of country c's GDP to k is the *local Domar weight* of k in country c, that is $\lambda_k^{Y_c} = \mathbf{1}_{\{k\in N_c+F_c\}}(GDP/GDP_c)\lambda_k$.

We also define *factor income shares*: the share of factor $f \in F^*$ in the income of country c is denoted by

$$\Lambda_f^c = \mathbf{1}_{\{f\in F\}} \frac{\Phi_{cf} w_f L_f}{GNE_c} + \mathbf{1}_{\{f\in F^*-F\}} \sum_{i\in N} \frac{\Phi_{ci}(1-\frac{1}{\mu_i}) p_i y_i}{GNE_c},$$

recalling that $f \in F^* - F$ is a fictitious factor that simply collects wedge revenue but is not used in production. The share of each factor in world income is Λ_f, where we suppress the c superscript.

3 Comparative Statics: Ex-Post Sufficient Statistics

In this section, we characterize the response of real GDP and welfare to shocks. We state our results in terms of changes in endogenous, but observable, sufficient statistics. In the next section, we solve for changes in these endogenous variables in terms of microeconomic primitives.

Allocation Matrix. To better understand the intuition for the results, we introduce the allocation matrix, which helps give a physical (primal) interpretation of the theorems. Following Baqaee and Farhi (2017b), define the *allocation matrix* $\mathcal{X}$ as follows: let $\mathcal{X}_{ij} = x_{ij}/y_j$ be the share of good j used by i, where i and j index households, factors, and producers. Every feasible allocation is defined by a feasible allocation matrix $\mathcal{X}$, a vector of productivities A, and a vector of factor supplies L. In particular, the equilibrium allocation gives rise to an allocation matrix $\mathcal{X}(A, L, \mu, T)$ which, together with A, and L, completely describes the equilibrium.[13]

Given an allocation matrix, we decompose changes in any quantity, say welfare W_c of country c, into changes due to the technological environment, for a given allocation matrix, and changes in the allocation matrix, for given technology. In vector notation, this is

$$d\log W_c = \underbrace{\frac{\partial \log W_c}{\partial \log A}\, d\log A + \frac{\partial \log W_c}{\partial \log L}\, d\log L}_{\Delta \text{ technology}} + \underbrace{\frac{\partial \log W_c}{\partial \mathcal{X}}\, d\,\mathcal{X}}_{\Delta \text{ allocation}}.$$

Real GDP. We start by considering how real GDP responds to shocks, stated in terms of country c variables. To state the result, we introduce special notation for the exposures of domestic production to imported intermediate inputs. Define country c's input-output matix Ω^c to be the $N_c \times N_c$ sub-matrix of the global input-output matrix Ω corresponding to producers in country c with associated Leontief inverse $\Psi^c = (I - \Omega^c)^{-1}$. Define the country-level cost-based matrices $\tilde{\Omega}^c$ and $\tilde{\Psi}^c$ in a similar way. When k is an imported inter-

[13] Since there may be multiple equilibria, technically, $\mathcal{X}(A, L, \mu, T)$ is a correspondence. In this case, we restrict attention to perturbations of isolated equilibria. As shown by Debreu (1970), equilibria are generically locally isolated.

mediate input ($k \in N - N_c$), with some abuse of notation, define the following variables

$$\Lambda_k^{Y_c} = \sum_{i \in N_c} \sum_{j \in N_c} \Omega_{Y_c,i} \Psi_{ij}^c \Omega_{jk} = -\frac{p_k q_{ck}}{GDP_c}, \quad \text{and} \quad \tilde{\Lambda}_k^{Y_c} = \sum_{i \in N_c} \sum_{j \in N_c} \Omega_{Y_c,i} \tilde{\Psi}_{ij}^c \tilde{\Omega}_{jk}.$$

Note that $\Lambda_k^{Y_c}$ is equal to the value of imports k divided by GDP. It is important that the summations in the expressions above run over *only* domestic goods N_c and not all goods N. That is, these variables are partial exposures of GDP to intermediate input k, only accounting for how domestic producers are exposed to k but not accounting for the fact that the value of k is subtracted from GDP. Theorem 1 decomposes real GDP changes into direct technology effects (due to changes in domestic productivity, domestic factors, and imported materials) and reallocation effects (due to reshuffling of resources across domestic producers holding fixed domestic productivity, factors, and imported materials).

Theorem 1 (Real GDP)**.** *The change in real GDP of country c in response to productivity shocks, factor supply shocks, transfer shocks, and shocks to wedges is, to a first-order,*[14]

$$\mathrm{d}\log Y_c = \underbrace{\sum_{i \in N_c} \tilde{\lambda}_i^{Y_c} \mathrm{d}\log A_i + \sum_{f \in F_c} \tilde{\Lambda}_f^{Y_c} \mathrm{d}\log L_f + \sum_{k \in N - N_c} \left(\tilde{\Lambda}_k^{Y_c} - \Lambda_k^{Y_c} \right) \mathrm{d}\log(q_{ck})}_{\Delta \text{ technology}}$$
$$\underbrace{- \sum_{i \in N_c} \tilde{\lambda}_i^{Y_c} \mathrm{d}\log \mu_i - \sum_{f \in F_c}^{F} \tilde{\Lambda}_f^{Y_c} \mathrm{d}\log \Lambda_f^{Y_c} + \sum_{k \in N - N_c} \left(\Lambda_k^{Y_c} - \tilde{\Lambda}_k^{Y_c} \right) \mathrm{d}\log \Lambda_k^{Y_c}}_{\Delta \text{ allocation}}. \quad (3)$$

The change in world real GDP $\mathrm{d}\log Y$ *can be obtained by simply suppressing the country index c. That is,*

$$\mathrm{d}\log Y = \underbrace{\sum_{i \in N} \tilde{\lambda}_i^{Y} \mathrm{d}\log A_i + \sum_{f \in F} \tilde{\Lambda}_f^{Y} \mathrm{d}\log L_f}_{\Delta \text{ technology}} \underbrace{- \sum_{i \in N} \tilde{\lambda}_i^{Y} \mathrm{d}\log \mu_i - \sum_{f \in F}^{F} \tilde{\Lambda}_f^{Y} \mathrm{d}\log \Lambda_f^{Y}}_{\Delta \text{ allocation}}.$$

Theorem 1 generalizes Proposition 1 from Burstein and Cravino (2015) to economies with arbitrary input-output linkages and distortions. To understand equation (3), we consider a series of simple cases. First, consider the case where there are no wedges in the initial equilibrium. Then forward and backward exposures are the same $\tilde{\Lambda}_i^{Y_c} = \Lambda_i^{Y_c}$. Furthermore, since revenues generated by wedges exactly offset the reduction in primary fac-

[14] Transfer shocks do not directly affect real GDP, but they can influence real GDP through the other terms in (3).

tor income shares $\sum_{i\in N_c}\tilde{\lambda}_i^{Y_c}\,\mathrm{d}\log\mu_i = -\sum_{f\in F_c}^{F}\Lambda_f^{Y_c}\,\mathrm{d}\log\Lambda_f^{Y_c} = -\sum_{f\in F_c}^{F}\tilde{\Lambda}_f^{Y_c}\,\mathrm{d}\log\Lambda_f^{Y_c}$, there are no reallocation effects. Therefore, Theorem 1 simplifies to the following corollary.

Corollary 1 (Real GDP without Initial Wedges)**.** *In the absence of domestic wedges in the initial equilibrium, Theorem 1 simplifies to*

$$\mathrm{d}\log Y_c = \sum_{i\in N_c}\lambda_i^{Y_c}\,\mathrm{d}\log A_i + \sum_{f\in F_c}\Lambda_f^{Y_c}\,\mathrm{d}\log L_f. \tag{4}$$

When there are no initial (domestic) wedges, country c's real GDP is equal to a Domar-weighted sum of *domestic* productivity and *domestic* factor endowment shocks. In this case, changes in the allocation matrix do not affect real GDP. Intuitively, when there are no domestic wedges, there is an envelope theorem for real GDP (the competitive equilibrium maximizes the joint profits of all domestic firms for given prices). Hence, without wedges, reallocations cannot affect real GDP to a first-order. Furthermore, in the absence of wedges, foreign shocks, like shocks to iceberg costs outside c's borders, do not affect real GDP. This is because productive efficiency ensures that the marginal revenue product of foreign inputs is exactly equal to their cost. Hence, an increase in imported materials raises domestic production and imports by exactly the same offsetting amount.[15]

If there are pre-existing wedges, there are some major changes. First, there is a new term on the first line of equation (3), adding to technology effects (holding fixed the distribution of resources). Second, there are now reallocation effects. To understand the presence of the new "technology" term involving total imported intermediates, consider the following special case which eliminates reallocation effects.

Corollary 2 (Real GDP with a Representative Firm)**.** *Consider a domestic economy with a single representative firm, indexed by* 1*, that uses domestic labor, L_c, and foreign materials, M_c, has productivity shifter A_c, and charges a markup μ_c. Then Theorem 1 simplifies to*

$$d\log Y_c = \lambda_1^{Y_c} d\log A_c + \mu_c\Lambda_L^{Y_c} d\log L_c + (\mu_c - 1)\frac{p_{M_c}M_c}{GDP_c} d\log M_c.$$

The first two terms are just the pure technology effects as in (4), the only difference being that now there is a gap between the revenue-based $\Lambda_L^{Y_c}$ and cost-based $\tilde{\Lambda}_L^{Y_c}$ exposure to labor. The final term, involving imported materials, is new and reflects the fact that imported intermediates are netted out of GDP using their cost rather than their marginal revenue product. In this simple example, this gap is just $(\mu_c - 1)$. If $\mu_c > 1$, then an

[15]Since discrete changes in real GDP are obtained by integration of infinitesimal changes, as long as efficiency is maintained, we conclude that even large foreign shocks do not affect domestic real GDP holding fixed domestic technology and factor supply.

increase in imported materials will raise domestic production (at constant prices) by more than imports (at constant prices), and hence an increase in M_c raises real GDP. Note that for this example, the allocation of resources across domestic producers is, by construction, efficient and unchanging since there is only one producer in the domestic economy.[16]

Having understood the first line of (3), now focus on the second line capturing reallocations. The second line of (3) implies that, ceteris paribus, a reduction in primary factor income shares and spending on imported materials boosts real GDP. Intuitively, this is because a reduction in primary factor income shares and expenditures on imported materials signals a reallocation of resources towards producers with relatively high markups/wedges. These producers are inefficiently too small to begin with, so such reallocations boost real GDP (and profits) but reduce spending on primary factors and imported materials. These reallocations have first-order effects on real GDP even holding fixed microeconomic productivities, factor endowments, and the total quantity of imported materials.

Welfare. We now turn our attention to changes in welfare (real GNE).

Theorem 2 (Welfare)**.** *The change in welfare of country c in response to productivity shocks, factor supply shocks, and transfer shocks can be written as:*

$$\begin{aligned} \mathrm{d}\log W_c = & \underbrace{\sum_{i\in N} \tilde{\lambda}_i^{W_c}\, \mathrm{d}\log A_i + \sum_{f\in F} \tilde{\Lambda}_f^{W_c}\, \mathrm{d}\log L_f}_{\Delta\ technology} \\ & \underbrace{- \sum_{i\in N} \tilde{\lambda}_i^{W_c}\, \mathrm{d}\log \mu_i + \sum_{f\in F^*} \left(\Lambda_f^c - \tilde{\Lambda}_f^{W_c}\right) \mathrm{d}\log \Lambda_f + \frac{\mathrm{d}\,T_c}{GNE_c}}_{\Delta\ allocation}, \end{aligned} \tag{5}$$

where $\tilde{\Lambda}_f^{W_c} = 0$ whenever f is a fictitious factor. The change $\mathrm{d}\log W$ of world real GNE is obtained by suppressing the country index c. That is,

$$\mathrm{d}\log W = \underbrace{\sum_{i\in N} \tilde{\lambda}_i^{W}\, \mathrm{d}\log A_i + \sum_{f\in F} \tilde{\Lambda}_f^{W}\, \mathrm{d}\log L_f}_{\Delta\ technology} \underbrace{- \sum_{i\in N} \tilde{\lambda}_i^{W}\, \mathrm{d}\log \mu_i - \sum_{f\in F} \tilde{\Lambda}_f^{W}\, \mathrm{d}\log \Lambda_f}_{\Delta\ allocation}.$$

As with real GDP, changes in welfare can be broken into technological effects (holding fixed the distribution of resources) and reallocation effects (holding fixed technology). However, unlike real GDP, reallocation effects are first-order even when there are no wedges.

[16]This effect means that when there are markups, aggregate TFP (measured by the Solow residual) responds to external shocks even in the absence of cross-sectional misallocation. See Gopinath and Neiman (2014) for an example.

This is because, unlike real GDP, even in the absence of wedges, there is no envelope theorem for the welfare of a given country. We discuss the intuition for the technology and reallocation effects in turn.

The direct technology effect of a shock depends on each household's exposure to the technology shock. Since households consume foreign goods, either directly or indirectly through supply chains, this means that technology shocks outside of a country's borders affect the household in that country holding fixed the allocation matrix.

The second line in Theorem 2 captures reallocation effects. These reallocation effects bundle together three different forces, each of which corresponds to one of the summands on the second line of (5). The first term is the direct effect of changes in wedges on consumer prices: an increased wedge $d \log \mu_i$ raises the price of the consumption basket by $\tilde{\lambda}_i^{W_c} d \log \mu_i$, holding fixed factor prices. The second reallocation term in (5) captures how changes in factor rewards affect household c. These terms are related to Viner's *factoral terms-of-trade* and capture household c's net exposure to each factor's price. Recall that Λ_f^c is the share of country c's income from factor f, whereas $\tilde{\Lambda}_f^{W_c}$ is the share of country c's consumption costs that depend on factor f. The consumption exposure $\tilde{\Lambda}_f^{W_c}$ captures the total reliance of household c on f, taking into account direct and indirect exposures through supply chains. The factoral terms-of-trade effects consider, for each factor f, how the income earned by the factor changes $\mathrm{d} \log \Lambda_f$, and whether household c is a net seller $\Lambda_f^c - \tilde{\Lambda}_f^{W_c} > 0$ or a net buyer $\Lambda_f^c - \tilde{\Lambda}_f^{W_c} < 0$.[17] Since the summation runs over F^*, this means that income earned by wedge revenues are included here. However, even without wedges, factoral terms-of-trade terms are generally non-zero since they reallocate resources across households. The final term in (5) is simply the change in net transfers.

Once we aggregate to the level of the world, if there are no pre-existing wedges, the reallocation effects will be zero. That is, starting at an efficient equilibrium, reallocation effects are zero-sum distributive changes only and have no aggregate consequences. However, when there are pre-existing wedges, reallocation effects are no longer zero-sum, since they can make everyone better or worse off by changing the efficiency of resource allocation. Although Theorem 1 and Theorem 2 are different country by country, they coincide when applied to the whole world.

Difference Between Welfare and Output. To see the difference between Theorems 1 and 2, consider a productivity shock $\mathrm{d} \log A_i$ to a foreign producer $i \notin N_c$. Suppose there

[17] Formally, $\sum_{f \in F} (\Lambda_f^c - \tilde{\Lambda}_f^{W_c}) d \log w_f$ generalizes the "double factoral terms-of-trade" in Viner (1937). When factor supply is fixed, $d \log L_f = 0$, there are no transfers or wedges, $dT = d \log \mu = 0$, then the reallocation effect in (5) is the same as this factoral terms-of-trade (because $d \log \Lambda_f = d \log w_f$ for every factor f).

are no wedges and all production and utility functions are Cobb-Douglas. Since there are no wedges, Theorem 1 implies that domestic real GDP does not respond to the foreign productivity shock $\mathrm{d}\log Y_c = 0$. The change in welfare, according to Theorem 2, is $\mathrm{d}\log W_c = \lambda_i^{W_c}\,\mathrm{d}\log A_i \neq 0$. Intuitively, even though there are no reallocation effects (because of the Cobb-Douglas assumption), an increase in foreign productivity increases the overall amount of goods the world economy can produce and this increases the welfare of country c to the extent that the consumption basket of country c relies on i (directly and indirectly through global supply chains).[18] This, however, does not affect the real GDP of country c.

Comparison to Terms-of-Trade Decomposition. Theorem 2 should be contrasted with a more common decomposition of welfare (e.g. Dixit and Norman, 1980), which frames welfare changes as arising due to changes in domestic production (real GDP) and deviations of absorption from production (i.e changes in net payments and the terms of trade):

$$\mathrm{d}\log W_c = \underbrace{\kappa_c\,\mathrm{d}\log Y_c}_{\Delta\text{Real GDP}} + \underbrace{\kappa_c\,\mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}}_{\Delta\text{Terms of Trade}} + \underbrace{\frac{\mathrm{d}\,T_c}{GNE_c} + \sum_{f\in F}(\Lambda_f^c - \kappa_c\Lambda_f^{Y_c})\,\mathrm{d}\log\Lambda_f}_{\Delta\text{Transfers and Net Factor Payments}}, \quad (6)$$

where κ_c is GDP_c/GNE_c.[19] To make the comparison between (6) and Theorem 2 more straightforward, assume there are no transfers or net factor payments. In this case, both decompositions split welfare into a component representing production and a component representing relative price changes. In the case of Theorem 2, we look at relative factor prices whereas (6) depends on relative goods prices. However, as shown in the empirical application in Section 7.1, the factoral terms-of-trade need not be the same sign or magnitude as the standard terms-of-trade.

[18] Theorems 1 and 2 suggest that the elasticities of substitution generically matter for real GDP and welfare. This is because these elasticities of substitution discipline changes in factor income shares, and through these, reallocations. In a closed-economy with one consumer and one primary factor, Liu (2017) provides conditions under which the elasticities of substitution are irrelevant for welfare. This irrelevance does not extend to our setup since we have multiple factors, multiple consumers, and distorting wedges are not offset by non-pecuniary costs.

[19] Using the definitions in (1) and (2), the terms-of-trade term in (6) can equivalently be written as

$$\kappa_c\,\mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c} = \sum_{i\in N}\frac{p_i\,nx_{ic}}{GNE}d\log p_i,$$

where nx_{ic} is the quantity of net exports by country c of each good i. That is, for domestically produced goods, nx_{ic} is the export quantity, and for foreign goods, nx_{ic} is the total quantity imported for final consumption and intermediates. Domestically produced and consumed goods prices cancel since they appear in both the GDP deflator and the GNE deflator. Hence, the expression for the terms-of-trade in (6) is a measure of the price of net exports.

While both are useful, Theorem 2 does have some advantages over (6). First, the decomposition in Theorem 2 is not sensitive to "irrelevant" changes in how producers are assigned to countries. For example, assuming that iceberg trade costs are logged in the country that imports a good or the country that exports it has no bearing on equilibrium allocations or welfare. However, this choice affects real GDP, and by extension, the terms-of-trade since the sum of the two effects must equal the change in welfare. Similarly, if a firm changes the country where it books its profits, this affects the decomposition in (6) but not the one in Theorem 2. Second, even in inefficient environments, the breakdown between production and reallocation in Theorem 2 is maintained. However, if there are domestic distortions, real GDP is no longer purely a measure of physical productivity and itself will contain reallocative effects caused by wedges.

4 Comparative Statics: Ex-Ante Sufficient Statistics

Section 3 shows that the response of welfare and real GDP to shocks depend on changes in ex-post and endogenous sufficient statistics (like changes in factor income shares). In this section we characterize these ex-post sufficient statistics in terms of microeconomic primitives: the HAIO matrix and elasticities of substitution in production and consumption (ex-ante sufficient statistics). The results of this section can then be combined with Theorems 1 and 2 to answer counterfactual questions about welfare and real GDP. We focus on two types of shocks: productivity shocks, which nest iceberg shocks, and wedge shocks, which nest tariff changes.

4.1 Setup

To clarify exposition, we specialize production and consumption functions to be nested-CES aggregators, with an arbitrary number of nests and elasticities. This is for clarity, not tractability. Appendix E, in the working paper, shows that it is very straightforward to generalize the rest of the results in the paper to non-nested-CES economies.

Nested CES economies can be written in many different equivalent ways. As in Baqaee and Farhi (2017a), we adopt the following *standard-form* representation. We treat every CES aggregator as a separate producer and rewrite the input-output matrix accordingly, so that each producer has a single elasticity of substitution associated with it; the representative household in each country c consumes a single specialized good which, with some abuse of notation, we also denote by c. Importantly, note that this procedure changes the set of producers, which, with some abuse of notation we still denote by N.[20] In other words, every $k \in C + N$ has an associated cost function

$$p_k = \frac{\mu_k}{A_k}\left(\sum_{j \in N+F_c} \tilde{\Omega}_{kj} p_j^{1-\theta_k}\right)^{\frac{1}{1-\theta_k}},$$

where θ_k is the elasticity of substitution.

For nested-CES economies, the input-output covariance turns out to be a central object.

Input-Output Covariance. We use the following matrix notation throughout. For a matrix X, we define $X^{(i)}$ to be its ith row and $X_{(j)}$ to be its jth column. We define the *input-output covariance operator* to be

$$Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)}, \Psi_{(j)}) = \sum_{l \in N+F} \tilde{\Omega}_{kl} \Psi_{li} \Psi_{lj} - \left(\sum_{l \in N+F} \tilde{\Omega}_{kl} \Psi_{li}\right)\left(\sum_{l \in N+F} \tilde{\Omega}_{kl} \Psi_{lj}\right).$$

This is the covariance between the ith and jth columns of the Leontief inverse using the kth row of $\tilde{\Omega}$ as the probability distribution.

4.2 Comparative Statics

Sales Shares and Prices. The following characterizes how prices and sales shares, including factor income shares, respond to perturbations in an open-economy.[21]

Theorem 3 (Prices and Sales Shares)**.** *For a vector of perturbations to productivity* $d \log A$ *and wedges* $d \log \mu$*, the change in the price of a good or factor* $i \in N + F$ *is, to a first-order,*

$$d \log p_i = \sum_{k \in N} \tilde{\Psi}_{ik} (d \log \mu_k - d \log A_k) + \sum_{f \in F} \tilde{\Psi}_{if} \, d \log \Lambda_f. \tag{7}$$

To a first-order, the change in the sales share of a good or factor $i \in N + F$ *is*

$$\begin{aligned} d \log \lambda_i = & \sum_{k \in N+F} \left(\mathbf{1}_{\{i=k\}} - \frac{\lambda_k}{\lambda_i} \Psi_{ki}\right) d \log \mu_k + \sum_{k \in N} \frac{\lambda_k}{\lambda_i} \mu_k^{-1} (1 - \theta_k) Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)}, d \log p) \\ & + \sum_{g \in F^*} \sum_{c \in C} \frac{\lambda_i^{W_c} - \lambda_i}{\lambda_i} \Phi_{cg} \Lambda_g \, d \log \Lambda_g, \end{aligned} \tag{8}$$

[20] See Appendix C.1 for a worked-out example showing how to map a specific nested-CES economy in standard-form.

[21] Theorem 3 generalizes Propositions 2 and 3 from Baqaee and Farhi (2017b) to open-economies.

where $\mathrm{d}\log p$ *is the* $(N+F)\times 1$ *vector of price changes in* (7). *The change in wedge income accruing to household c (represented by a fictitious factor) is*

$$d\log\Lambda_c = \sum_i \frac{\Phi_{ci}\lambda_i}{\Lambda_c}\left(\mu_i^{-1}d\log\mu_i + (1-\mu_i^{-1})d\log\lambda_i\right). \tag{9}$$

Recall that for every factor $i \in F$, we interchangeably use λ_i or Λ_i to denote its Domar weight. This means that (8) pins down the change in primary factor income shares and (9) pins down changes in "fictitious" factor income shares. Therefore, substituting the vector of price changes (7) into (8) results in an $F^* \times F^*$ linear system in factor income shares $\mathrm{d}\log\Lambda$. The solution to this linear system gives the equilibrium changes in factor shares, which can be plugged back into equations (7) and (8) to get the change in the sales shares and prices for every (non-factor) good, and into Theorems 1 and 2 to get real GDP and welfare.

We discuss the intuition in detail below, but at a high level, equation (7) captures *forward propagation* of shocks — shocks to suppliers change the prices of their downstream consumers. On the other hand, equation (8) captures *backward propagation* of shocks — shocks to consumers change the sales of their upstream suppliers. Each term in these equations has a clear interpretation.

To see this intuition, start by considering the forward propagation equations (7): the first set of summands shows that a change in the price of k, caused either by wedges $\mathrm{d}\log\mu_k$ or productivity $\mathrm{d}\log A_k$, affects the price of i via its direct and indirect exposures $\tilde{\Psi}_{ik}$ through supply chains. The second set of summands in (7) capture how changes in factor prices, which are measured by changes in factor income shares, also propagate through supply chains to affect the price of i. These expressions use the cost-based HAIO matrix $\tilde{\Omega}$, instead of the revenue-based HAIO matrix Ω, because Shephard's lemma implies that the elasticity of the price of i to the price of one of its inputs k is given by $\tilde{\Omega}_{ik}$ and not Ω_{ik}.

For the intuition of backward propagation equations (8), we proceed term by term. The first term captures how an increase in a downstream wedge $\mathrm{d}\log\mu_k$ reduces expenditures on suppliers i. If μ_k increases, then for each dollar k earns, relatively less of it makes it to i, and this reduces the sales of i.

The second term captures the fact that when relative prices change $\mathrm{d}\log p \neq 0$, then every producer k will substitute across its inputs in response to this change. Suppose that $\theta_k > 1$ so that producer k substitutes (in expenditure shares) *towards* those inputs that have become cheaper. If those inputs that became cheap are also heavily reliant on i, then $Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)}, \mathrm{d}\log p) < 0$. Hence, substitution by k towards cheaper inputs will increase demand for i. These substitutions, which happen at the level of each producer k, must be summed across all producers.

The last set of summands, on the second line of (8), captures the fact that changes in factor prices change the distribution of income across households in different countries. This affects the demand for i if the different households are differently exposed, directly and indirectly, to i. The overall effect can be found by summing over countries c the increase in c's share of aggregate income $\sum_{g\in F^*} \Phi_{cg}\Lambda_g \,\mathrm{d}\log\Lambda_g$ multiplied by the relative welfare exposure $(\lambda_i^{W_c} - \lambda_i)/\lambda_i$ to i. If every household has the same consumption basket, the last term disappears.

Two-Country Example. This example uses the forward and backward propagation equations in Theorem 3 to linearize a two-country economy. Each country has one factor, so $C = F = 2$. Denote foreign variables by an asterisk and let L index the home factor and L_* the foreign factor. Assume that there are no wedges so that $\Omega = \tilde{\Omega}$, and consider a productivity shock $d\log A_j$ to some producer j. Substituting (7) into (8) gives the following change in the domestic factor share

$$\frac{d\log\Lambda_L}{d\log A_j} = \frac{\sum_k(\theta_k - 1)\lambda_k Cov_{\Omega^{(k)}}\left(\Psi_{(j)}, \frac{\Psi_{(L)}}{\Lambda_L}\right)}{1 + \frac{\Lambda_L}{(1-\Lambda_L)}\sum_k(\theta_k - 1)\lambda_k Var_{\Omega^{(k)}}\left(\frac{\Psi_{(L)}}{\Lambda_L}\right) - \left(\Lambda_L^W - \Lambda_L^{W_*}\right)}.$$

The numerator is a partial equilibrium effect and captures the way $d\log A_j$ redirects expenditures towards (or away) from L due to expenditure-switching (holding fixed relative factor wages). Note that it is a sum over all producers k, and the kth term is positive if $d\log A_j$ causes k to redirect its spending towards the home factor L. This happens if k's inputs are substitutes $\theta_k > 1$ and exposure to j and L positively covary $Cov_{\Omega^{(k)}}(\Psi_{(j)}, \Psi_{(L)}) > 0$. In this case, as k substitutes to use inputs most heavily exposed to j, it boosts demand for the home factor L and raises its income share.

The feedback from general equilibrium (i.e. factor markets clearing) is the denominator. The terms involving the elasticities of substitution in the denominator capture the fact that the partial equilibrium effect, by changing factor prices, triggers its own substitution effects. If inputs are substitutes $\theta_k > 1$ and k is heterogeneously exposed to the home factor through its suppliers, $Var_{\Omega^{(k)}}(\Psi_{(L)}) > 0$, then the endogenous increase in the price of L will cause k to substitute away from L. This mitigates the partial equilibrium effect in the numerator if $\theta_k > 1$ and amplifies it if $\theta_k < 1$. The final term in the denominator reflects factoral home-bias. An increase in the price of L redistributes income towards the home consumer who, in all likelihood, has home-bias for the domestic factor ($\Lambda_L^W > \Lambda_L^{W_*}$) and this effect magnifies the partial equilibrium effect.

Quantities, Real GDP, and Welfare. Since Theorem 3 pins down how prices and expenditures respond to shocks, it can also be used to derive how individual quantities respond to shocks.

Corollary 3. *(Quantities) The changes in the quantity of a good or factor i in response to a productivity shock to i is given by:*

$$\mathrm{d}\log y_i = \mathrm{d}\log \lambda_i - \mathrm{d}\log p_i,$$

where $\mathrm{d}\log \lambda$ *and* $\mathrm{d}\log p$ *are given in Theorem 3.*

Among other things, Corollary 3 can be used to predict how changes in imported intermediates respond to exogenous shocks, which is a necessary input for predicting the response of real GDP, per Theorem 1, if the initial equilibrium has wedges.

4.3 Extensions of Theorem 3

We describe some simple extensions of Theorem 3, and take advantage of them for the analytical and quantitative applications in Sections 6 and 7.

Endogenous factor supply. Theorem 3 takes changes in factor supplies as exogenous. Theorem 3 can easily be extended to account for endogenous factor supply. For example, suppose that labor in each country depends on real wages and real income $L_f = G(w_f/P_{W_c}, W_c)$. Let $\zeta_f = \partial \log G_f / \partial \log w_f$ and $\gamma_f = -\partial \log G_f / \partial \log W_c$ be the price and income elasticity of supply. The results so far assumed that $\gamma_f = \zeta_f = 0$ for all factors. More generally, equilibrium in the factor market implies that

$$d\log L_f = \frac{\zeta_f}{(1+\zeta_f)} d\log \Lambda_f + \frac{\zeta_f - \gamma_f}{(1+\zeta_f)} d\log W_c. \tag{10}$$

Equation (10) can be combined with Theorem 3 to determine all equilibrium outcomes. Equation (10) itself can be derived as a consequence of a standard labor-leisure choice problem where ζ_f and γ_f are determined by preferences.

Sticky wages. Nominal rigidities, like sticky wages, are a mainstay of business cycle analysis but have received comparably less attention from trade economists with some recent and notable exceptions like Rodríguez-Clare et al. (2020).[22] In principle, trade policy is

[22] Rodríguez-Clare et al. (2020) show that sticky wages are important for understanding the regional effects of the China shock in the US.

persistent and its effects operate at horizons where nominal rigidities do not matter. In practice, a major political consideration for trade policy is its effect on employment. For example, both the recent US tariffs against China and Germany's resistance to a trade embargo on Russia were justified, at least by politicians, on the grounds that such a policy would boost or harm domestic employment. Nominal rigidities, such as sticky wages, provide a natural explanation for why this might be the case in the short run.

Theorem 3 can easily be used to study models with sticky wages. To do so, we must introduce nominal variables into the model. We have so far treated world nominal GDP as the numeraire. We re-express all prices in a new numeraire, called dollars, and define e^c to be the nominal exchange rate between dollars and country c's currency. By definition, the change in the nominal wage of factor f in country c's currency, denoted by w^c_f, is

$$d\log w^c_f = d\log\Lambda_f + d\log GDP - d\log L_f + d\log e_c,$$

where $d\log\Lambda_f$ is the share of aggregate spending on factor f, GDP is world nominal GDP in dollars, L_f is the quantity of factor f, and e_c is the nominal exchange rate. If the wage of factor f is rigid in local currency, then $d\log w^c_f = 0$. Substituting this into the previous equation yields the change in employment of factor of f

$$d\log L_f = d\log\Lambda_f + d\log GDP + d\log e_c. \tag{11}$$

Hence, changes in employment are given by changes in nominal spending (in local currency) on f. Equation (11) determines employment for factor f as a function of changes in factor income shares, determined by Theorem 3, and C new nominal variables: $C-1$ nominal exchange rates and world GDP in dollars.

The behavior of these nominal variables is determined by the conduct of monetary policy. Following Woodford (2011), we can close the model by assuming the central bank in each country can directly target nominal variables in local currency.[23] For example, each country's central bank stabilizes a weighted average of domestic inflation and the nominal exchange rate:

$$\alpha_c d\log\left(p_c e_c GDP\right) + \beta_c d\log e_c = 0, \tag{12}$$

[23] Although not necessary to compute comparative statics, we can imagine that to implement its target, say (12), each country's central bank adjusts money supply. To see this, assume that a cash-in-advance constraint connects money supply to nominal spending (see, e.g., Galí, 2015). That is, consumer c's spending in local currency must equal local money-supply m_c. The change in consumer c's spending, expressed in dollars, is $\sum_f \Lambda^c_f d\log\Lambda_f + d\log GDP$. Hence, the cash-in-advance constraint dictates that $d\log m_c = \sum_f \Lambda^c_f d\log\Lambda_f + d\log GDP - d\log e_c$, where m_c is an exogenous variable controlled by the central bank. By choosing m_c, the central bank can choose e_c, and hence can implement (12).

where α_c and β_c are parameters and $d \log p_c$ is the price of the domestic consumption basket (relative to world nominal GDP) given by Theorem 3. The central bank targets zero domestic inflation if $\alpha_c > 0$ and $\beta_c = 0$, and it stabilizes the exchange rate if $\beta_c > 0$ and $\alpha_c = 0$.

Theorem 3, combined with (11) and (12), pin down all equilibrium outcomes. Theorems 1 and 2 can then be used, without modification, to derive the real GDP and real GNE effects (if there is disutility of labor, then welfare and real GNE no longer coincide). We provide a worked-out example in Section 6.

Sticky prices. Similarly, Theorem 3 can also be used to study economies with sticky prices, since a sticky price is just a wedge between price and marginal cost. Specifically, for every producer i whose prices are sticky in terms of country c's currency, we create a fictitious sticky-price intermediary, denoted by $\hat{i}$, who sells good i on behalf of i. The change in the wedge charged by $\hat{i}$ is endogenously determined by $d \log \mu_{\hat{i}} = -(1-\delta_i)(d \log p_i + d \log GDP + d \log e_c)$, where $d \log e_c$ is the nominal dollar exchange rate and $d \log GDP$ is the change in world nominal GDP in dollars. The parameter $\delta_i \in [0, 1]$, called the Calvo parameter, controls how sticky the price of i is. If $\delta_i = 0$, then the price of i is completely rigid in currency c, and if $\delta_i = 1$, then the price of i is flexible.[24] As above, to close the model and pin down nominal variables, we need to specify monetary policy as in (12).

Differential Exact-Hat Algebra. Theorem 3, which is a generalization of hat-algebra (Jones, 1965), is useful for studying small shocks and gaining intuition. For large shocks, the trade literature instead relies on exact-hat algebra (e.g. Dekle et al., 2008; Costinot and Rodriguez-Clare, 2014), which requires solving the non-linear system of supply and demand relationships. Theorem 3 provides an alternative way to make hat-algebra exact by "chaining" together infinitesimal effects. This amounts to viewing Theorem 3 as a system of differential equations that can be solved by iterative means (e.g. Euler's method). In our quantitative exercises in Section 7, we find that the differential approach is significantly faster than using state-of-the-art nonlinear solvers to perform exact hat-algebra. The improvement is larger when the number of variables increases and production functions become more non-log-linear. Furthermore, Theorem 3 can be generalized to non-CES production and consumption functions. See Appendices E and F in the working paper for more details about this computational approach.

[24] To see this, note that the price charged by $\hat{i}$ in local currency, denoted $p_{\hat{i}}^c$, is $d \log p_{\hat{i}}^c = d \log(p_{\hat{i}} GDP e_c) = d \log \mu_{\hat{i}} + d \log p_i + d \log GDP + d \log e_c = \delta_i (d \log p_i + d \log GDP + d \log e_c)$. When $\delta_i = 0$, the local price of i is rigid. When $\delta_i = 1$, the local price of i is flexible (i.e. reflects marginal cost). For more information, see Rubbo (2022), who uses a similar methodology to model and calibrate a closed economy with sticky prices and input-output networks.

Other Uses of Theorem 3. Theorem 3 can also be used to characterize other statistics of interest like factor demand and trade elasticities. We pursue some examples in the working paper version of this paper. For example, Appendix H provides the elasticity of the international factor demand system with respect to factor prices and iceberg shocks as a linear combination of microeconomic elasticities of substitution with weights that depend on the input-output table. This relates to insights from Adao et al. (2017), who show that the factor demand system is sufficient for performing certain counterfactuals. Appendix I of the working paper writes trade elasticities at any level of aggregation as a linear combination of underlying microeconomic elasticities of substitution with weights that depend on the input-output table.

5 Comparative Statics: Nonlinearities

The previous sections show how welfare and real GDP respond to changes in technologies and wedges to a first-order approximation. In this section, we extend these results to a second-order approximation for real GDP (for each country and the world) and world welfare around efficient allocations.[25]

Before stating our results, we begin by defining world welfare. To measure world welfare, we use a simple Bergson-Samuelson (BS) social welfare function

$$W^{BS}(W_1, \ldots, W_C) = \sum_c \overline{\chi}_c^W \log W_c,$$

where $\overline{\chi}_c^W$ is the initial income share of country c at the efficient equilibrium.[26] These welfare weights are chosen so that there is no incentive to redistribute across agents at the initial equilibrium. To a first-order approximation, world welfare is the same as world GDP. However, differences arise starting at the second-order.

To measure the effect of a shock on world welfare, we use consumption equivalents: what fraction of consumption would society be prepared to give up to avoid the shock.

[25]We do not provide second-order approximations far from efficiency. We also do not provide second-order approximations for country-level welfare (except in symmetric cases where country and world welfare coincide). The reason is that a second-order approximation of country-level welfare, or real GDP away from efficiency, involves second-derivatives of factor shares and goes beyond what can be characterized using Theorem 3. Such results would require using super-elasticities of substitution (elasticities of elasticities of substitution). We leave this analysis for future work.

[26]We introduce this welfare function because at the world level, non-infinitesimal changes in real GDP (or real GNE) do not coincide with a well-defined social welfare function. This is because individual household preferences across all countries are generally non-aggregable (see, for example, Baqaee and Burstein, 2021).

Formally, we measure changes in welfare by $\Delta \log \delta$, where δ solves the equation

$$W^{BS}(\delta \overline{W}_1, \ldots, \delta \overline{W}_C) = W^{BS}(W_1, \ldots, W_C),$$

where $\overline{W}_c$ and W_c are the values at the initial and final equilibrium.

Theorem 4 (World Welfare)**.** *Starting at an efficient equilibrium in response to changes in wedges or technologies, changes in world welfare are given up to the second-order by*

$$\Delta \log \delta \approx \Delta \log Y + Cov_{\Omega_{\chi^W}} \left(\Delta \log \chi_c^W, \Delta \log P_{W_c} \right).$$

Here, $\Delta \log \chi_c^W$ and $\Delta \log P_{W_c}$ are the change in country c's nominal GNE and consumer price index respectively.

In words, the change in world welfare is the sum of the change in world real GDP and a redistributive term. This redistributive term depends on the covariance of two first-order approximations: changes in expenditures by each country and changes in the price of each country's consumption basket. The redistributive term in Theorem 4 is positive whenever the covariance between the changes in household income shares and the changes in consumption price deflators is positive. It captures a familiar deviation from perfect risk-sharing. It would be zero if households could engage in perfect ex-ante risk-sharing.

Since we only need to know $\Delta \log \chi_c^W$ and $\Delta \log P_{W_c}$ to a first order, we can express the redistributive term in terms of primitives using Theorem 3. To do this, note that the change in consumer c's income is $\Delta \log \chi_c^W \approx \sum_{g \in F} \Phi_{cg} \Lambda_g \Delta \log \Lambda_g + \sum_{i \in N} \Phi_{ci} \lambda_i \Delta \log \mu_i$, and the change in the consumer price index of country c is $\Delta \log P_{W_c} \approx \sum_{i \in N} \lambda_i^{W_c} \Delta \log \mu_i + \sum_{g \in F} \Lambda_g^{W_c} \Delta \log \Lambda_g$. Hence, to express world welfare in terms of microeconomic primitives, it remains to understand the change in real GDP to a second-order. Hence, we now discuss how each country's GDP, as well as world GDP (and by virtue of Theorem 4 world welfare) are affected, to a second-order, by productivity and wedge shocks. We start with productivity shocks and then turn to wedge shocks.

5.1 Productivity/Iceberg Changes

For productivity changes, like iceberg shocks, we can use an idea similar to Baqaee and Farhi (2017a). Absent wedges, Domar weights give the first-order response of real GDP to productivity shocks (as in Corollary 1). Hence, changes in Domar weights capture, in equilibrium, the effect of nonlinearities on real GDP. Therefore, we have the following.

Corollary 4 (Real GDP Response to Technology Shocks)**.** *In the absence of wedges, the response of real GDP for each country c to productivity, factor endowment, and wedge shocks is, to a second-order approximation,*

$$\Delta \log Y_c \approx \sum_{i \in N_c} \lambda_i^{Y_c} \Delta \log A_i + \sum_{f \in F_c} \Lambda_f^{Y_c} \Delta \log L_i + \frac{1}{2} \sum_{i \in N_c} \Delta \lambda_i^{Y_c} \Delta \log A_i + \frac{1}{2} \sum_{f \in F_c} \Delta \Lambda_f^{Y_c} \Delta \log L_f.$$

For world GDP, suppress the country subscript c.

Corollary 4 implies that, to a second-order approximation, the microeconomic details of production matter only in so far as they affect the change in the sales shares of the goods experiencing shocks. For example, fragilities in supply chains amplify the negative effect of a shock to some producer j only to the extent that they increase the sales shares of j in equilibrium. Corollary 4 can be expressed in terms of microeconomic primitives (the HAIO matrix and microeconomic elasticities of substitution) using the following relationship

$$\frac{\mathrm{d}\, \lambda_j^{Y_c}}{\mathrm{d} \log A_i} = \lambda_j^{Y_c} \left(\frac{\mathrm{d} \log \lambda_j}{\mathrm{d} \log A_i} - \sum_{f \in N_c} \Lambda_f^{Y_c} \frac{\mathrm{d} \log \Lambda_f}{\mathrm{d} \log A_i} \right),$$

where $\mathrm{d}\, \lambda_j / \mathrm{d} \log A_i$ and $\mathrm{d} \log \Lambda_f / \mathrm{d} \log A_i$ are given by Theorem 3.

5.2 Tariffs/Wedge Changes

The way tariffs and other wedge-like distortions affect output is more subtle. We provide approximations for small wedges $\Delta \log \mu_i$ around the efficient equilibrium, $\log \mu = 0$. Throughout this section, the HAIO matrix can be evaluated at the no-distortion point or at the point with small distortions, since both are valid second-order approximations.[27] The former is relevant for approximating how introducing small wedges affects output, whereas the latter is relevant for approximating how eliminating existing wedges affects output.

We start by showing that losses due to wedges are approximately equal to a Domar-weighted sum of deadweight-loss triangles. We then express these deadweight-loss triangles in terms of microeconomic primitives.

[27] Formally, consider output as a function of wedges. Up to a second-order approximation in $\log \mu$ the distance to the efficient outcome is

$$\log \frac{Y(\log \mu)}{Y(0)} \approx \frac{1}{2} \Delta \log \mu' \frac{\partial^2 \log Y(0)}{\partial \log \mu^2} \Delta \log \mu \approx \frac{1}{2} \Delta \log \mu' \frac{\partial^2 \log Y(\Delta \log \mu)}{\partial \log \mu^2} \Delta \log \mu,$$

where the derivatives involve the HAIO matrix and elasticities of substitution at either the undistorted point or the point with small distortions.

Theorem 5 (Real GDP)**.** *Starting at an efficient equilibrium, up to the second-order, in response to the introduction of small tariffs or other distortions, changes in the real GDP of country c are given by*

$$\Delta \log Y_c \approx \frac{1}{2} \sum_{i \in N_c} \lambda_i^{Y_c} \Delta \log y_i \Delta \log \mu_i.$$

Changes in world real GDP (and real GNE) are given by suppressing the country subscript.

Hence, for both the world and for each country, the reduction in real GDP from tariffs and other distortions is given by the sum of all the deadweight-loss triangles $1/2\Delta \log y_i \Delta \log \mu_i$ weighted by their corresponding local Domar weights.[28,29]

That is, the way output changes when tariffs change is *only* a function of three statistics: the Domar weight of taxed goods, the size of the tax, and the change in the quantity of taxed goods. All other details (e.g. elasticities of substitution, returns to scale, input-output linkages, non-taxed goods production, etc.) matter only in so far as they play a role in determining the equilibrium value of these sufficient statistics.

Starting at an efficient equilibrium, the introduction of tariffs or other distortions leads to changes $\Delta \log y_i$ in the quantities of goods $i \in N_c$ in country c and to changes in the wedges $\Delta \log \mu_i$ between prices and marginal costs. The price-cost margin $p_i \Delta \log \mu_i$ measures the wedge between the marginal contribution to country real GDP and the marginal cost to real GDP of increasing the quantity of good i by one unit. Hence, $\lambda_i^{Y_c} \Delta \log \mu_i$ is the marginal proportional increase in real GDP from a proportional increase in the output of good i. Integrating from the initial efficient point to the final distorted point, we find that $(1/2)\lambda_i^{Y_c} \Delta \log y_i \Delta \log \mu_i$ is the contribution of good i to the change in real GDP. Production networks can magnify losses from tariffs both because they can make the triangles $1/2\Delta \log y_i \Delta \log \mu_i$ larger, and because they raise $\lambda_i^{Y_c}$, sales relative to GDP, used to weigh each triangle.

We now re-express Theorem 5 in terms of primitives: microeconomic elasticities of substitution and the HAIO matrix. To do this, we combine Theorem 5 with Theorem 3 and Corollary 3.[30]

[28] Theorem 5 holds in general equilibrium, but it has a more familiar partial equilibrium counterpart (Feenstra, 2015). For a small open economy operating in a perfectly competitive world market, import tariffs reduce the welfare by $\Delta W \approx (1/2) \sum_i \lambda_i \Delta \log y_i \Delta \log \mu_i$, where μ_i is the ith gross tariff (no tariff is $\mu_i = 1$), y_i is the quantity of the ith import, and λ_i is the corresponding Domar weight (see Appendix J in the working paper for details). Theorem 5 shows that this type of intuition can be applied (to real GDP) in general equilibrium as well.

[29] Harberger (1964) argues that an equation like the one in Theorem 5 can be used to measure welfare as long as there are compensating transfers to keep the distribution of income across households fixed. Theorem 5 shows that a similar formula can be used for changes in real GDP, even in the absence of compensating transfers. Theorem 4 shows that Harberger's formula must be altered for aggregate welfare in the absence of compensating transfers.

[30] Whereas Theorem 5 does not have a counterpart in Baqaee and Farhi (2017b), Theorem 6 generalizes

Theorem 6 (Real GDP)**.** *Starting at an equilibrium without distortions, in response to the introduction of small tariffs or other distortions, the change in real GDP of country c is*

$$\begin{aligned}\Delta \log Y_c \approx -\frac{1}{2}\sum_{l\in N_c}\sum_{k\in N}\Delta\log\mu_k\Delta\log\mu_l\sum_{j\in N}\lambda_j^{Y_c}\theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)})\\ -\frac{1}{2}\sum_{l\in N_c}\sum_{g\in F}\Delta\log\Lambda_g\Delta\log\mu_l\sum_{j\in N}\lambda_j^{Y_c}\theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)})\\ +\frac{1}{2}\sum_{l\in N_c}\sum_{c\in C}\chi_c^W\Delta\log\chi_c^W\Delta\log\mu_l(\lambda_l^{W_c}-\lambda_l)/\chi_c^Y,\end{aligned}$$

Changes in world real GDP/GNE are similar if we suppress the c subscript.

First, all the terms scale with the square of the tariffs or other distortions $\Delta\log\mu$. There is therefore a sense in which misallocation increases with the tariffs and other distortions. Second, all the terms scale with the elasticities of substitution θ of the different producers. There is therefore a sense in which elasticities of substitution magnify the costs of these tariffs and other distortions. Third, all the terms also scale with the sales shares λ of the different producers and with the square of the Leontief inverse matrix Ψ. There is therefore also a sense in which accounting for intermediate inputs magnifies the costs of tariffs and other distortions. Fourth, all the terms mix the wedges, the elasticities of substitution, and the properties of the network.

For a given producer $l \in N$, there are terms in $\Delta\log\mu_l$ on the three lines. Taken together, these terms sum up to the Harberger triangle $(1/2)\lambda_l\Delta\log\mu_l\Delta\log y_l$ corresponding to good l in terms of microeconomic primitives. The three lines break it down into three components, corresponding to three different effects responsible for the change in the quantity $\Delta\log y_l$ of good l.

The term $-\sum_{k\in N}\Delta\log\mu_k\sum_{j\in N}\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)})$ on the first line corresponds to the change $\Delta\log y_l$ in the quantity of good l coming from *substitutions* by all producers j in response to changes in all tariffs and other distortions $\Delta\log\mu_k$, holding factor wages constant.

The term $\sum_{g\in F}\Delta\log\Lambda_g\sum_{j\in N}\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)})$ on the second line corresponds to the change $\Delta\log y_l$ in the quantity of good l coming from *substitutions* by all producers j in response to the endogenous changes in factor wages $\Delta\log w_g = \Delta\log\Lambda_g$ brought about by all the changes in tariffs and other distortions.

The term $\sum_{c\in C}\chi_c^W\Delta\log\chi_c^W(\lambda_l^{W_c}-\lambda_l)$ on the third line corresponds to the change $\Delta\log y_l$ in the quantity of good l coming from *redistribution* across agents with different spending

Proposition 5 from that paper to open-economies.

patterns, in response to the endogenous changes in factor wages brought about by all the changes in tariffs and other distortions.

6 Analytical Examples

In this section, we consider stylized examples to hone intuition and illustrate questions our framework can be used to answer. In each example, we consider a trade shock, either an iceberg or tariff shock, and discuss how different assumptions affect the answer. We consider the role that input-output linkages, domestic complementarities, returns to scale, and nominal rigidities play in affecting the way welfare responds to trade shocks. We revisit some of these issues in the next section, Section 7, using a calibrated quantitative model with non-symmetric countries and show that the intuitions derived from the simple examples are useful in understanding the quantitative results.

Example I: input-output networks. This example shows how input-output connections amplify the losses from iceberg trade costs and tariffs. Consider the example depicted in Figure 1. The two countries are symmetric, Ω is imports as a share of sales at the initial equilibrium, and θ is the elasticity of substitution between intermediates and labor. To map this example economy into the framework in Section 2, note that each country has one consumer, one producer, and one factor. Hence, the HAIO matrix has six rows and columns.

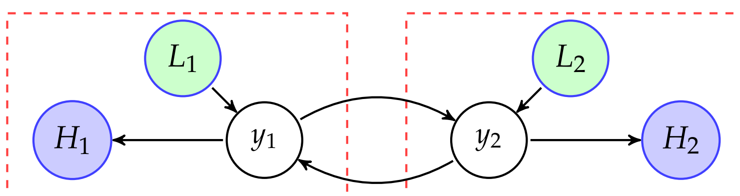


Figure 1: Solid lines show the flow of goods. Green, purple, and white nodes are factors, households, and goods. Boundaries of countries are represented by dashed boxes.

Suppose that we raise iceberg trade costs in both countries by $\Delta \log \tau$. By symmetry, changes in country real output, country welfare, world real output, and world welfare are all the same. Corollary 4 implies that to a second-order approximation:

$$\begin{aligned}\Delta \log W &\approx -(\lambda_{12} + \lambda_{21})\Delta \log \tau - \frac{1}{2}(\Delta\lambda_{12} + \Delta\lambda_{21})(\Delta \log \tau)^2 \\ &\approx -\frac{\Omega}{(1-\Omega)}\Delta \log \tau - \frac{1}{2}\frac{(1-\theta)\Omega^2}{(1-\Omega)}(\Delta \log \tau)^2,\end{aligned}$$

where λ_{ij} is the sales share of country j to country i. The second line uses Theorem 3 to write the welfare change in terms of primitives, using the fact that, by symmetry, $\lambda_{12} = \lambda_{21}$. This expression shows that a higher intermediate input share raises both the first-order and the second-order effect. Losses are increasing in Ω for two reasons. First, a higher Ω means that goods effectively cross the border more times and this inflates the expenditure share on imports relative to GDP at the initial equilibrium $\lambda_{12} = \lambda_{21} = \Omega/[2(1-\Omega)]$. Second, a higher Ω also implies that a given iceberg cost is paid many times as the good recrosses the border, and this increases the relative price of imports more, given the iceberg shock, leading to a larger change in the expenditure share of traded goods. Losses are decreasing in the elasticity of substitution because the sales share of traded goods rises by less in response to the shock when the elasticity of substitution is high.

Now consider a symmetric tariff, $\Delta \log \mu$, instead. Theorems 5 and 6 imply that up to a second-order approximation, the reduction in real GDP and welfare are

$$\Delta \log W = \Delta \log Y \approx -\frac{1}{2}(\lambda_{12}\Delta \log y_{12} \Delta \log \mu + \lambda_{21}\Delta \log y_{21} \Delta \log \mu) \approx -\theta \frac{\Omega}{2(1-\Omega)^2}(\Delta \log \mu)^2,$$

where y_{ij} is the quantity of imports from country j by country i, λ_{ij} is the corresponding sales share, and by symmetry $y_{12} = y_{21}$. There are some similarities but also major differences compared to the iceberg shock. First, unlike iceberg shocks, there are no first-order effects, since starting at a point with no wedges, reallocations are zero-sum to a first-order. Second, unlike iceberg shocks, the losses are increasing in the elasticity of substitution θ. This is because a given tariff causes a bigger change in quantities when price elasticities are higher. Formally, the change in quantity is $-\Delta \log y_{12} = -\Delta \log y_{21} = [\theta/(1-\Omega)]\Delta \log \mu$. However, similar to iceberg shocks, losses are increasing in the intermediate input share Ω. The reasons are also similar. First, a higher Ω raises the expenditure share on imports relative to GDP at the initial equilibrium. Second, a higher Ω also implies that a given tariff must be paid many times as the good recrosses the border, and this increases the relative price of imports more, for a given tax, leading to a larger reduction in quantities. In other words, more input-output linkages enlarge each Harberger triangle and raise the Domar weights used to aggregate the triangles.

Example II: complementarities and factor mobility. Arkolakis et al. (2012) show that, in a broad range of one-sector economies, the welfare costs of trade shocks depend on import shares and trade elasticities. We use a simple example to show how these costs also depend on features of the domestic economy like sectoral complementarities and factor mobility across domestic industries. Indeed, complementarities and factor mobility can strongly interact with one another to make trade shocks more costly. For example, a disruption in energy imports is much more costly if energy is a strong complement to other goods *and* if the importing economy is incapable of expanding production in domestic energy generation by reallocating factors.

Consider a symmetric two country model. Households consume non-traded "services" and traded "commodities". The elasticity of substitution between varieties of commodities is θ and the elasticity of substitution between services and commodities is $\sigma < \theta$. The initial (pre-shock) household budget share of commodities is β, and the share of domestic commodities as a share of global commodities is Ω. We adopt the Ricardo-Viner assumption that every good is produced using a Cobb-Douglas composite of two factors: generic labor that can move between commodities and services and sector-specific labor that cannot. The expenditure share on generic and sector-specific factor is α and $1-\alpha$.[31]

Corollary 5. *For this example, the change in welfare of country c due to a universal iceberg shock,* $\Delta \log \tau$, *is*

$$\Delta \log W_c \approx -\beta(1-\Omega)\Delta \log \tau - \frac{1}{2}\beta(1-\Omega)\left[\frac{(1-\sigma)(1-\beta)(1-\Omega)}{1-(1-\sigma)(1-\alpha)} + (1-\theta)\Omega\right]\Delta \log \tau^2, \quad (13)$$

to a second-order approximation.

The first term in (13) is the first-order effect and the second term is the second-order effect. We obtain the second-order effect since world and country-level welfare coincide in this example. To obtain Corollary 5, note that Theorem 2 shows that to a first-order approximation, the change in welfare is

$$d \log W_c = -\lambda_T^{W_c} d \log \tau + \sum_{f \in F} \left(\Lambda_f^c - \Lambda_f^{W_c}\right) d \log \Lambda_f,$$

where $\lambda_T^{W_c}$ is the exposure to the traded good. The first term captures the "mechanical" effect of the iceberg shock, holding fixed the allocation of resources, and the remaining terms capture reallocation effects due to changes in relative factor rewards.

Since this example is symmetric and efficient, reallocation effects always sum to zero, so the change in welfare, to a first-order approximation, is just

$$\Delta \log W_c \approx -\lambda_T^{W_c} \Delta \log \tau = -\beta(1-\Omega)\Delta \log \tau.$$

[31] The sector-specific factor assumption, popularized by Jones (1971, 1975), is usually used to understand the distributional effects of trade (e.g. Kovak, 2013). Here, our focus is on the aggregate consequences of this assumption.

This is just the import share of consumption times the iceberg shock. Unsurprisingly, the higher the share, the more costly is the iceberg shock.

To derive the nonlinear part, we note it is given by the *change* in the trade share (since the trade share is the first-order effect). Theorem 3 determines this change. To understand the intuition for the nonlinear part, consider three extreme cases. First, suppose there is only one sector ($\sigma = \theta$) and one factor ($\alpha = 1$). This matches the simplest environment considered by Arkolakis et al. (2012). In this case, the cost of an iceberg shock, to second-order, is

$$\Delta \log W_c \approx -\lambda_T^{W_c} \Delta \log \tau - \frac{1}{2}(1 - \lambda_T^{W_c})\lambda_T^{W_c}(1-\theta)\Delta \log \tau^2.$$

Conditional on the import share $\lambda_T^{W_c}$, the iceberg shock is more costly the lower is the trade elasticity θ, exactly as in Arkolakis et al. (2012).

Now suppose that there are two separate sectors ($\sigma < \theta$) but factors are still fully mobile across commodities and services ($\alpha = 1$). In this case, (13) becomes

$$\Delta \log W_c \approx -\lambda_T^{W_c} \Delta \log \tau - \frac{1}{2}\beta(1-\Omega)\left[(1-\sigma)(1-\beta)(1-\Omega) + (1-\theta)\Omega\right]\Delta \log \tau^2. \quad (14)$$

We now also have to consider the elasticity of substitution between commodities and services σ. In particular, if $\sigma < 1$, then this amplifies the cost of the iceberg trade shock relative to the first-order approximation. In other words, complementarities in the domestic economy can amplify the negative consequences of the iceberg shock.

Finally, suppose that $\alpha = 0$, so that commodities and services factors are completely immobile. In this case, we get

$$\Delta \log W_c \approx -\lambda_T^{W_c} \Delta \log \tau - \frac{1}{2}\beta(1-\Omega)\left[(1/\sigma - 1)(1-\beta)(1-\Omega) + (1-\theta)\Omega\right]\Delta \log \tau^2.$$

As before, complementarity in the domestic economy $\sigma < 1$ amplifies the negative consequences of the iceberg shock. However, this effect is much more potent than (14) when σ is close to zero. When $\sigma < 1$, if factors are mobile across sectors, reduced trade in commodities causes factors to move into producing commodities to maintain consumption. If factors are immobile across sectors, the reduction in welfare from reduced trade is much greater since the domestic economy cannot reorganize itself to maintain consumption of commodities. This amplification effect depends on both complementarity across sectors in the domestic economy ($\sigma < 1$) and factor specificity ($\alpha < 1$). If commodities and services are neither complements nor substitutes ($\sigma = 1$), then whether or not factors are mobile across sectors is irrelevant, since even if factors could be moved from one sector to another, they would not. Similarly, the effects of the complementarity are much milder if factors

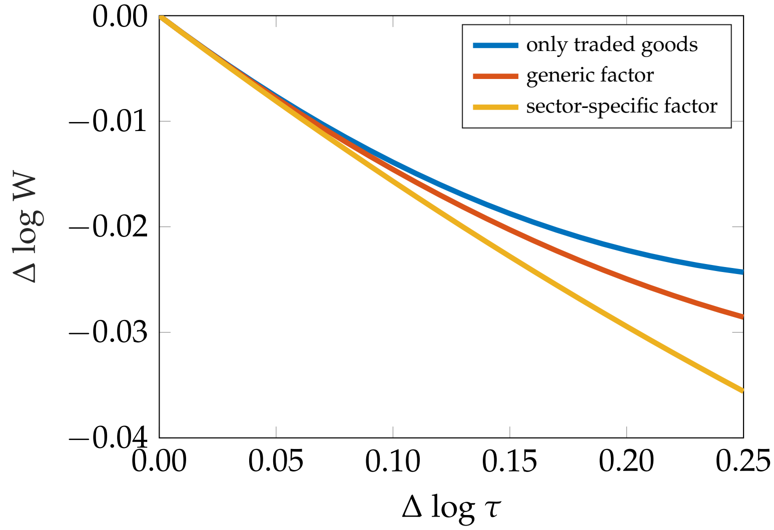


Figure 2: The change in welfare implied by (13) for the case with no non-traded goods ($\sigma = \theta$), generic factor only ($\alpha = 1$), and sector-specific factors only ($\alpha = 0$). In all cases, the import share of consumption is kept constant at $\lambda_T^{W_c} = 1/6$, so the different specifications are all first-order equivalent. The elasticity of substitution across traded goods is $\theta = 5$ and across sectors is $\sigma = 0.1$.

can freely move across sectors to reinforce production of traded goods. Figure 2 numerically illustrates these three cases. We supplement this intuitive example with a quantitative exercise in Section 7.

Example III: sticky wages. To see how nominal rigidities can raise the costs of trade shocks, suppose countries are symmetric and that each country has an endowment of capital and labor. Assume all producers have the same capital-labor intensity. The wage paid to labor is rigid in domestic currency, but the rental rate of capital is flexible. Consider a universal increase in iceberg trade costs $d \log \tau$. Theorem 2 implies that the change in welfare of each country c is

$$\begin{aligned} d \log W_c &= -\sum_{i \in N} \lambda_i^{W_c} \, \mathrm{d} \log \tau + \sum_{f \in N} \Lambda_f^{W_c} \, \mathrm{d} \log L_f + \sum_{f \in F} \left(\Lambda_f^c - \Lambda_f^{W_c} \right) \mathrm{d} \log \Lambda_f \\ &= -\sum_{i \in N} \lambda_i^{W_c} d \log \tau + \sum_{f \in F} \Lambda_f^{W_c} \, \mathrm{d} \log L_f. \end{aligned} \tag{15}$$

The second line follows from the absence of factoral terms-of-trade movements, which is a consequence of symmetry. Intuitively, welfare falls for two reasons: (i) the mechanical effect of the iceberg shock on domestic consumers, and (ii) the endogenous reduction in employment due to sticky wages. Assume that every central bank targets zero domestic inflation. Using (11) and (12), the change in employment of labor in each country is

$$d \log L_{\text{labor}} = -\frac{\sum_{k \in N} \lambda_k^{W_c} d \log \tau_k}{1 - \Lambda_{\text{labor}}^{Y_c}}.$$

In words, employment falls more the bigger is the mechanical effect of the iceberg shock on consumer prices. Furthermore, the reduction in employment is greater when labor's share of income is higher. Intuitively, the central bank combats the inflationary impulse of the iceberg shock by reducing nominal spending, and this reduction in nominal spending reduces the rental price of capital and helps stabilize the price level (since nominal wages are rigid). The smaller is capital's share of income, the more the price of capital has to fall to stabilize inflation, and the larger is the necessary reduction in nominal spending. These reductions in nominal spending reduce employment one-for-one since nominal wages are fixed. Substituting this into (15) implies that the welfare effect of the iceberg shock is

$$d \log W_c = -\frac{\sum_{i \in N} \lambda_i^{W_c} d \log \tau_i}{1 - \Lambda_{\text{labor}}}. \tag{16}$$

When the sticky factor's share of income is zero, $\Lambda_{\text{labor}} = 0$, welfare responds only to the direct effect of the iceberg shock. As we increase the sticky factor's share of income, the losses in welfare become larger because of the reduction in employment.

Example IV: protectionism with and without nominal rigidities. So far, we have focused on symmetric examples where income redistribution, through factoral terms-of-trade, does not play a role. We end this section by considering a non-symmetric example of protectionism inspired by Fajgelbaum et al. (2020) who document complete pass-through of US tariffs on China into US consumer prices. This finding is at odds with a typical full-employment neoclassical model since an American tariff, by reducing demand for Chinese labor, should depress Chinese wages and hence lower the before-duty prices of Chinese goods.[32] This example shows that sticky-wages and a managed exchange rate can rationalize the complete pass-through result of Fajgelbaum et al. (2020). This example also shows that these ingredients qualitatively change the welfare consequences of the tariff.

For this example, consider a two country economy each with a single factor (labor) in free trade. Suppose that the domestic country (US) imposes a vector of good-specific taxes

[32] The Fajgelbaum et al. (2020) result is robust to the inclusion of different combinations of fixed-effects. Specifically, they find complete pass-through of the tax into US prices even in the absence of country-origin $\times$ time fixed effects. In other words, they do not find evidence that Chinese wages fell in response to the tariff. See Table A.13 of their paper. Amiti et al. (2019) also study this episode, though their empirical specifications always include country-origin $\times$ time fixed effects.

$d \log \mu$ and collects revenues that are rebated to the domestic household lump-sum. With some abuse of notation, for any variable x, denote the home variable by x and the foreign counterpart by x_*. We start by discussing the flexible wage economy before turning our attention to the sticky wage economy.

Flexible wages: As usual, according to Theorem 2, the change in domestic welfare is

$$d \log W = \sum_i \left(\lambda_i^Y - \lambda_i^W \right) d \log \mu_i + \left(1 - \Lambda_L^{W_c} \right) \left(d \log \Lambda_L - d \log \Lambda_{L_*} \right). \tag{17}$$

The first term in (17) captures the mechanical increases in income and prices caused by the tariffs and the second term captures the change in the factoral terms-of-trade for factors L and L_* induced by the tariffs. This can be further be simplified to

$$d \log W_c = \sum_i \left(\lambda_i^Y - \lambda_i^W \right) d \log \mu_i + \frac{1 - \Lambda_L^W}{1 - \Lambda_L} \left(d \log \Lambda_L + \sum_i \lambda_i d \log \mu_i \right). \tag{18}$$

Home welfare can increase because of the first summand: tariffs could generate income in excess of the increase in consumer prices, holding fixed primary factor rewards; or the second summand: tariffs can raise the home wage relative to the foreign wage.

Appendix C.2 uses Theorem 3 to re-express (18) in terms of microeconomic primitives and discusses the intuition. In the main text, for brevity, assume all elasticities of substitution θ_i are equal to one. Then, Theorem 3 implies that

$$\Lambda_L d \log \Lambda_L = \frac{- \sum_k \lambda_k \Psi_{kL} d \log \mu_k + (\Lambda_L^W - \Lambda_L^{W_*}) \sum_k \lambda_k d \log \mu_k}{1 - (\Lambda_L^W - \Lambda_L^{W_*})}.$$

The first term in the numerator is the direct effect of the tax on k, which reduces spending on American labor to the extent that k directly or indirectly uses American labor (Ψ_{kL}). If a Chinese firm k does not indirectly use American labor, then $\Psi_{kL} = 0$ and a tariff on k will not mechanically reduce demand for American labor. That is, if the tariff is well-designed, then this term should be small. The second term in the numerator captures how the tax, by generating tariff revenues for American consumers, can change demand for American labor through income redistribution. The second term is positive as long as there is factoral home bias ($\Lambda_L^W > \Lambda_L^{W_*}$). The denominator is a general equilibrium feedback — redistribution towards American households raises American wages, which further tilts demand in favor of American labor, which further raises American wages, and so on.

To summarize, if the tariff is well-designed, then Chinese wages fall relative to American wages ($d \log \Lambda_{L^*} < d \log \Lambda_L$), and this factoral terms-of-trade manipulation results in incomplete pass-through of the tariff into US prices. That is, even if the taxed goods are exclusively consumed by Americans (i.e. $\lambda_i^Y = \lambda_i^W$), the tariff can improve American welfare by manipulating the factoral terms-of-trade.

Downward rigid wages: Now consider the same economy as above but suppose that wages are downwardly rigid in both countries in terms of local currency. Furthermore, suppose that the foreign country pegs their nominal exchange rate to the home country while the home country implements an inflation target of zero. Downward wage rigidity implies that $d \log w_f = \max\{0, d \log \Lambda_f + d \log GDP\}$ and $d \log L_f = \min\{0, d \log \Lambda_f + d \log GDP\}$ for both the foreign and domestic factor.[33] If a vector of tariffs successfully lowers Chinese wages relative to US wages in the flexible equilibrium, then the same tariff in an economy with sticky wages changes welfare by

$$d \log W = \sum_i \left(\lambda_i^Y - \lambda_i^W \right) d \log \mu_i. \tag{19}$$

The positive term captures the income American consumers earn from the tax whereas the negative term captures the fact that the taxes raise consumer prices by consumers' exposure to these prices. Unlike (17), changes in relative factor rewards no longer appear. Hence, the gains to the Americans are smaller than (17) under the reasonable case where the tariff improves the factoral terms-of-trade. Sticky wages, and the consequent absence of beneficial changes in the factoral terms-of-trade, also help explain why tariffs on foreign consumption goods are passed through to domestic consumer prices one-for-one. The expression in (19) is positive when the items being taxed are mostly being re-exported, in which case $\lambda_i^Y > \lambda_i^W$. In the other extreme, when the taxed quantities are exclusively used for domestic consumption ($\lambda_i^Y = \lambda_i^W$), the change in welfare from the imposition of the tariff are, to a first-order, equal to zero. In this case, the increase in revenues exactly offsets the increase in prices faced by domestic consumers.

7 Quantitative Results

In this section, we provide some quantitative illustrations of our results. In Section 7.1, we use the ex-post results in Section 3 to decompose the sources of welfare growth in different countries and contrast our welfare decomposition to the more typical terms-of-trade decomposition. In Section 7.2, we revisit some of the examples in Section 6 using a quantitative model. In both Sections 7.1 and 7.2, we rely on the World Input-Output Database (WIOD) (see Timmer et al., 2015), which has 40 countries as well as a "rest-of-

[33] This is an extreme case of endogenous factor supply described in (10), where $d \log L_f = \min\{0, d \log w_f\}$ and $d \log w_f = d \log \Lambda_f + d \log GDP - d \log L_f$.

the-world" composite country. Each country has four factors of production: high-skilled, medium-skilled, low-skilled labor, and capital; and 30 industries. Since tariffs are quite low during our sample, for simplicity, we abstract from initial tariffs.[34] Appendix A contains additional details about how the model is mapped to the data.

7.1 Ex-Post Growth Accounting

In this section, we compare decompositions of real GNE according to Theorem 2 against the more typical terms-of-trade decomposition in (6). In the absence of wedges and net factor payments, these two decompositions are

$$\mathrm{d}\log W_c = \underbrace{\sum_{f\in F}\Lambda_f^{W_c}\,\mathrm{d}\log L_f + \sum_{i\in N}\lambda_i^{W_c}\,\mathrm{d}\log A_i}_{\Delta\text{ technology}} + \underbrace{\sum_{f\in F}\left(\Lambda_f^c - \Lambda_f^{W_c}\right)\mathrm{d}\log\Lambda_f}_{\Delta\text{Factoral Terms of Trade}} + \underbrace{\frac{\mathrm{d}\,T_c}{GNE_c}}_{\Delta\text{Transfers}},$$

and

$$\mathrm{d}\log W_c = \underbrace{\kappa_c\left(\sum_{f\in F_c}\Lambda_f^{Y_c}\,\mathrm{d}\log L_f + \sum_{i\in N_c}\lambda_i^{Y_c}\,\mathrm{d}\log A_i\right)}_{\Delta\text{Real GDP}} + \underbrace{\kappa_c\,\mathrm{d}\log P_{Y_c} - \mathrm{d}\log P_{W_c}}_{\Delta\text{Terms of Trade}} + \underbrace{\frac{\mathrm{d}\,T_c}{GNE_c}}_{\Delta\text{Transfers}},$$

where $\kappa_c = GDP_c/GNE_c$ and we have substituted in Corollary 1 for real GDP (see Appendix A for more on data construction).

Figure 3 shows both decompositions using data for the United States and Italy (assuming away net factor payments and capturing trade imbalances using transfers). These countries are chosen because they illustrate how the two decompositions can be similar or different.[35] The left panel displays the standard terms-of-trade decomposition and the right one the factoral terms-of-trade decomposition.

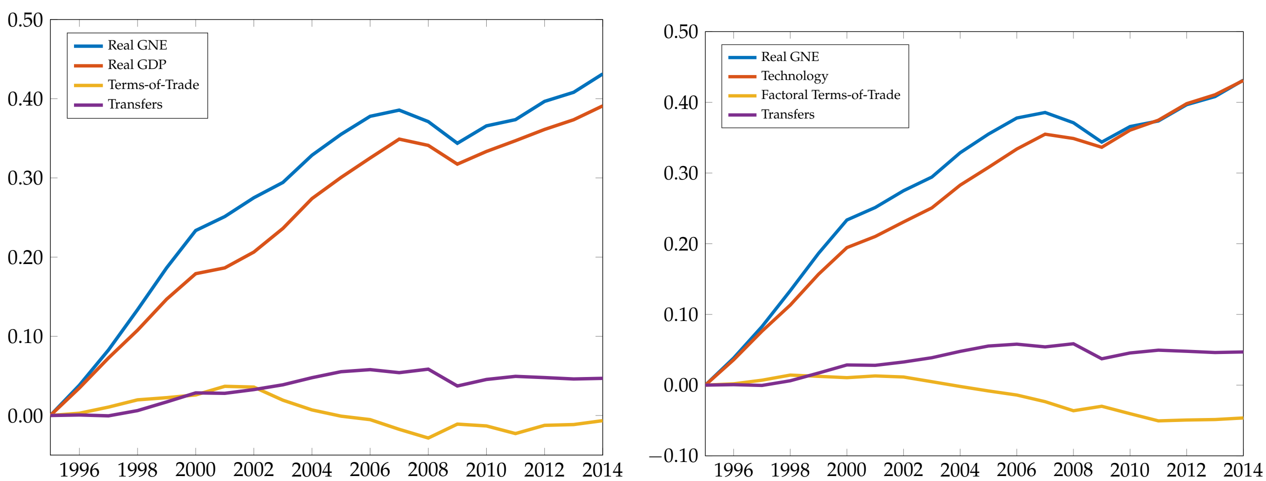
For some countries, like the United States, the factoral and goods terms-of-trade decompositions tell a similar story. In Figure 3a, the yellow lines in both panels are similar, implying that changes in the terms-of-trade and factoral terms-of-trade are similar. Since the sum of the red, yellow, and purple lines must add up to the change in real GNE in both pictures, and since the net transfers are the same, the similarity of the yellow lines in the two figures implies that growth in real GDP in the left panel must be similar to the pure technology term in the right panel. In other words, technology for goods the United States produces (real GDP) grew in line with technology for goods the US consumes ("Technol-

[34] Results are similar with initial tariffs, since these tariffs are small, and are available upon request.

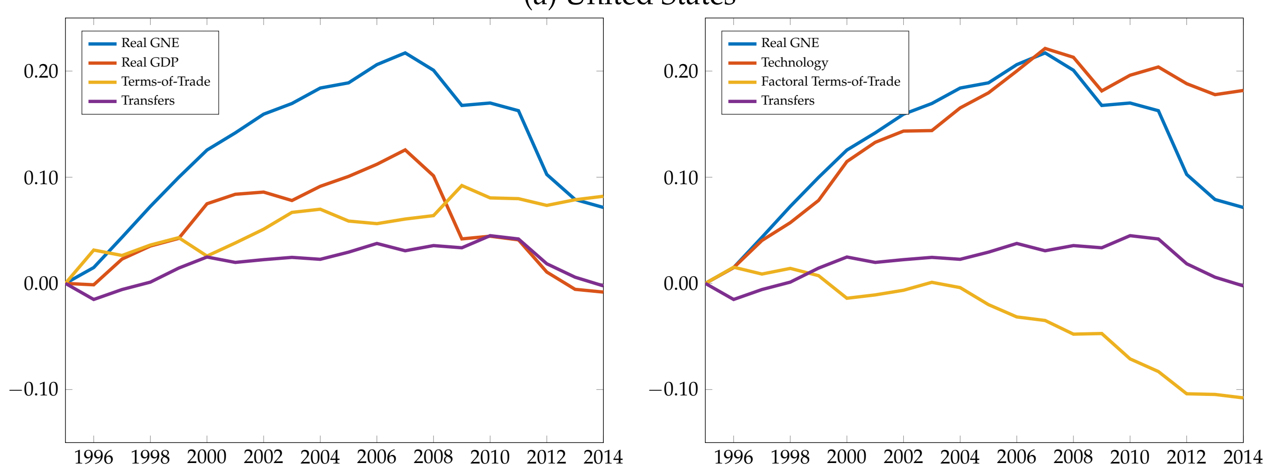
[35] Appendix M in the working paper contains the breakdown for all countries.

ogy" in the right panel), with only a relatively minor role for reallocation.

However, for other countries, like Italy, the two pictures are quite different. According to the left panel of Figure 3b, Italian real GDP grew far more slowly than Italian real GNE. The left panel attributes this gap mostly to an improvement in the terms-of-trade, meaning that the price of foreign goods Italians consume fell more than the price of goods Italy exports. The right panel provides a different narrative: Italy's consumption grew more slowly than technology for those goods that Italians consume.[36] This difference is explained by a deterioration in the factoral terms-of-trade (reallocation excluding transfers). Intuitively, the right panel tells us that foreign factor rewards outpaced Italy's factor rewards, and this implies that Italy is consuming a smaller share of a bigger global pie.



(a) United States



(b) Italy

Figure 3: The left and right panels show a cumulative decomposition of real GNE using the terms-of-trade and factoral terms-of-trade decompositions.

[36] For these exercises, technology includes changes in factor endowments.

7.2 Ex-Ante Counterfactuals

In this section, we use a calibrated production network model to show the importance of the HAIO matrix and elasticities of substitution. We use the quantitative model to computationally revisit the issues studied using pen-and-paper examples in Section 6.

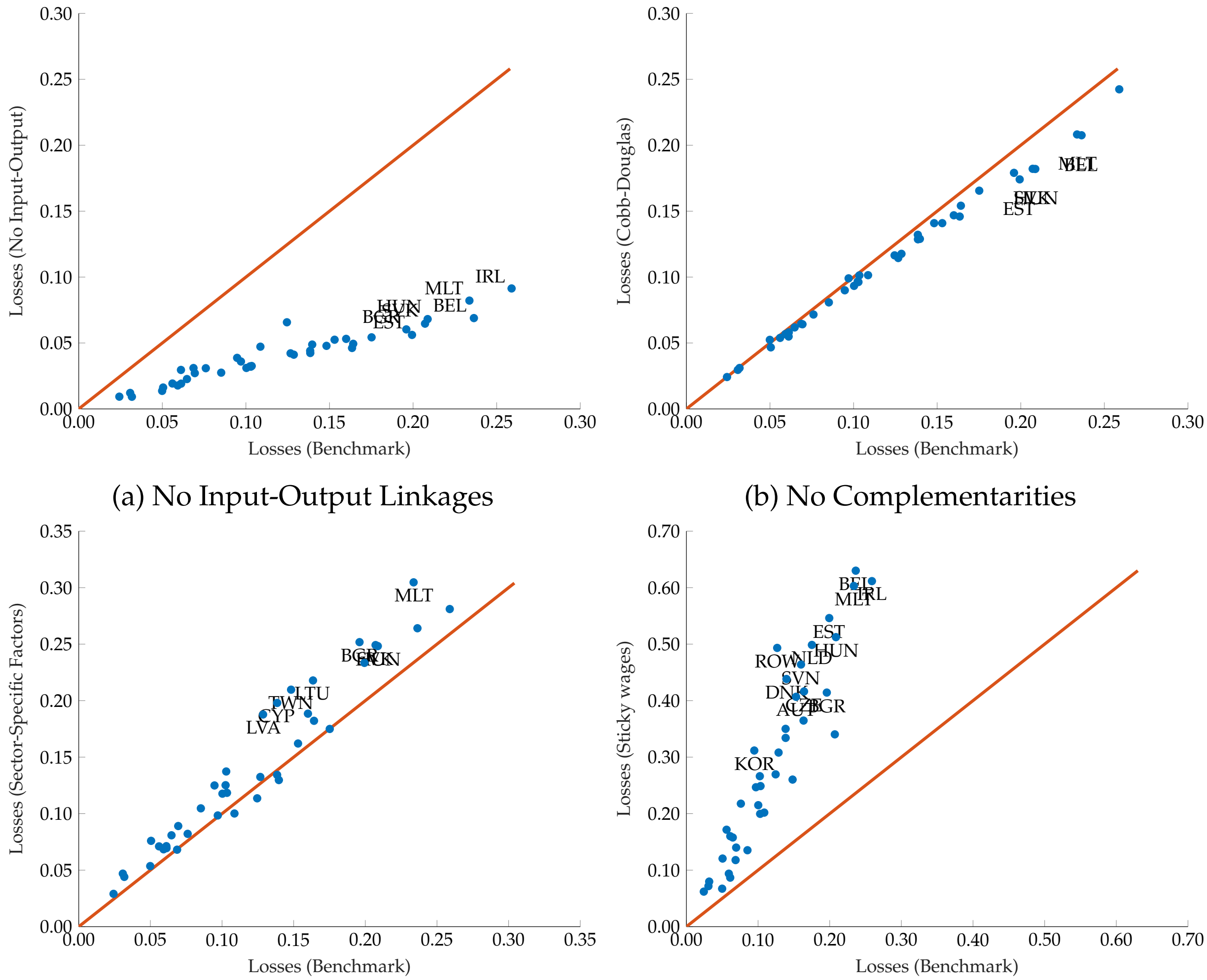
Unlike the growth-accounting exercise in Figure 3, for counterfactual questions, we have to take a stance on elasticities of substitution. We assume production and consumption have a nested-CES structure. Each industry produces output by combining its value-added (consisting of the four domestic factors) with intermediate goods (from other industries). The elasticity of substitution across intermediates is θ_1, between factors and intermediate inputs is θ_2, across different primary factors is θ_3, and the elasticity of substitution of household consumption across industries is θ_0. When a producer or the household in country c purchases inputs from industry j, it consumes a CES aggregate of goods from this industry sourced from various countries with elasticity of substitution $\varepsilon_j + 1$.

We use estimates from Caliendo and Parro (2015) to calibrate $\varepsilon_i + 1$, the elasticity of substitution between traded and domestic varieties of each industry. We set the domestic elasticities of substitution $(\theta_0, \theta_1, \theta_2, \theta_3) = (0.9, 0.2, 0.5, 1)$, following Atalay (2017) who estimates them at annual frequency. The exact values of these elasticities are not so important for our purposes. Our aim is to show how counterfactual predictions depend on the values of these elasticities. To do this, we consider how results change if all these elasticities are set equal to one. We calibrate initial expenditure shares to match the WIOD in 2008.

Using the calibrated model, we compute the change in welfare for each country in response to a reversal of globalization. Specifically, we raise all iceberg costs by 60%. In the benchmark model, this reduces the sales share of traded goods from an initial value of 30% of GDP to the 1960s value of 8% of world GDP. The reductions in welfare by country are shown in Figure 4 under different assumptions. We discuss each panel in turn.

Remark. To solve the model, we repeatedly iterate on Theorems 2 and 3 and numerically integrate the result. We provide code, detailed in Appendix D, that loglinearizes arbitrary general equilibrium models of the type studied in this paper, and computes global comparative statics. This approach is faster and more numerically stable than traditional methods, especially for very large and nonlinear models. Appendix G in the working paper details the computational performance of differential exact-hat algebra and the accuracy of first-order approximations.

Panel 4a plots, for each country, the reduction in welfare under the benchmark calibration (x-axis) against a calibration that ignores input-output linkages (y-axis). The no input-output calibration follows Arkolakis et al. (2012) and assumes the sales of every producer to each destination are the same as in the data. This calibration preserves trade as



(a) No Input-Output Linkages

(b) No Complementarities

(c) No Factor Mobility

(d) Sticky Wages

Figure 4: Log reduction in welfare by country in response to a 60% increase in iceberg trade costs. The x-axis is the reduction implied by the benchmark model and the y-axis is the reduction under alternative assumptions. Countries with the largest deviation from the 45-degree line are labelled. If a country is above the 45-degree line, then the response of welfare is stronger relative to the benchmark model. Luxembourg has been removed for readability since it is an outlier.

a share of sales, rather than GDP. Every dot is below the 45-degree line meaning that IO linkages raise the importance of trade shocks. This is a consequence of the intermediate input multiplier mentioned in Example I in Section 6. The elasticity of world welfare to iceberg shocks is just trade as a share of GDP, and this is lower in a calibration that ignores input-output linkages by a factor of approximately two. (Trade over GDP is equal to the product of sales over GDP and trade over sales, and sales over GDP is around two). Since this is a first-order effect, it affects all countries regardless of how open they are.

Panel 4b compares the benchmark model with complementarities to a model where sectoral production and consumption functions are Cobb-Douglas ($\theta_0 = \theta_1 = \theta_2 = \theta_3 = 1$) and trade elasticities are unchanged). Most countries are below the 45-degree line. This is consistent with the second example in Section 6 and Figure 2 which show that domestic complementarities raise the costs of trade shocks. The differences are more pronounced for more open economies because the trade shock to these countries is larger, and domestic complementarities only become relevant for large trade shocks (as in Figure 2). Nevertheless, the effects are relatively mild since the shock under consideration is far from autarky (complementarities in the domestic economy would play a much more important role for larger shocks that take the economy closer to autarky).

Panel 4c shows how limiting factor mobility across sectors affects losses. This can be considered a shorter-run scenario where factors cannot move across sectors. Most points are above the 45-degree line, meaning that this makes the trade disruption more costly. For intuition, consult Figure 2, which shows that limited factor mobility raises the costs of iceberg shocks if there are domestic complementarities. The effect is largest for more open economies and for countries with unbalanced domestic economies (e.g. Malta, Eastern European countries, and Taiwan) who rely on their large neighbors for much of their imports in specific sectors. These countries are more affected by a breakdown in trade since they cannot maintain domestic production in import-intensive goods by reallocating domestic factors of production towards those goods. As with complementarities, these effects become more pronounced when the shock to the domestic economy is large. This requires that the domestic economy be sufficiently open, sufficiently imbalanced, and that the iceberg shock is sufficiently large.

Finally, Panel 4d shows how sticky wages affect outcomes. For illustration, we assume exchange rates are floating and monetary policy in each country targets zero-percent inflation. All countries are above the 45-degree line showing that nominal rigidities amplify the costs of the shock. Intuitively, the trade shock raises the price of consumption, and inflation-targeting requires that nominal expenditures shrink to limit the increase in inflation. This reduction in nominal demand, caused by monetary policy, induces unemployment in each country which dramatically increases the welfare losses from the iceberg shocks. Unlike complementarities and factor immobility, this is a first-order effect that appears even for relatively small shocks. Quantitatively, the effect of the shock is roughly doubled, in line with the example in equation (16).

8 Conclusion

This paper establishes a unified framework and provides a flexible toolbox for studying output and welfare in open and potentially distorted economies. We provide ex-post sufficient statistics for measurement and ex-ante sufficient statistics for counterfactuals that can be used to answer many disparate questions in macroeconomics and trade. We use these results to study how input-output linkages, domestic complementarities, limited factor mobility, and nominal rigidities can act to amplify welfare losses from trade disruptions.

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Online Appendix to *Networks, Barriers, and Trade* Additional Supplementary Materials in NBER Working Paper

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A Data Appendix

To conduct the counterfactual exercises in Section 7, we use the World Input-Output Database (Timmer et al., 2015). We use the 2013 release of the data for the final year which has no-missing data — that is 2008. We use the 2013 release because it has more detailed information on the factor usage by industry. We aggregate the 35 industries in the database to get 30 industries to eliminate missing values, and zero domestic production shares, from the data. In Table 1, we list our aggregation scheme, as well as the elasticity of substitution, based on Caliendo and Parro (2015) and taken from Costinot and Rodriguez-Clare (2014) associated with each industry. We calibrate the model to match the input-output tables and the socio-economic accounts tables in terms of expenditure shares in steady-state (before the shock).

For the growth accounting exercise in Section 7.1, we use both the 2013 and the 2016 release of the WIOD data. When we combine this data, we are able to cover a larger number of years. We compute our growth accounting decompositions for each release of the data separately, and then paste the resulting decompositions together starting with the year of overlap. To construct the consumer price index and the GDP deflator for each country, we use the final consumption weights or GDP weights of each country in each year to sum up the log price changes of each good. To arrive at the price of each good, we use the gross output prices from the socio-economic accounts tables which are reported at the (country of origin, industry) level into US dollars using the contemporaneous exchange rate, and then take log differences. This means that we assume that the log-change in the price of each good at the (origin, destination, industry of supply, industry of use) level is the same as (origin, industry of supply) level. If there are differential (changing) transportation costs over time, then this assumption is violated.

To arrive at the contemporaneous exchange rate, we use the measures of nominal GDP in the socioeconomic accounts for each year (reported in local currency) to nominal GDP in the world input-output database (reported in US dollars).

	WIOD Sector	Aggregated sector	Trade Elasticity
1	Agriculture, Hunting, Forestry and Fishing	1	8.11
2	Mining and Quarrying	2	15.72
3	Food, Beverages and Tobacco	3	2.55
4	Textiles and Textile Products	4	5.56
5	Leather, Leather and Footwear	4	5.56
6	Wood and Products of Wood and Cork	5	10.83
7	Pulp, Paper, Paper , Printing and Publishing	6	9.07
8	Coke, Refined Petroleum and Nuclear Fuel	7	51.08
9	Chemicals and Chemical Products	8	4.75
10	Rubber and Plastics	8	4.75
11	Other Non-Metallic Mineral	9	2.76
12	Basic Metals and Fabricated Metal	10	7.99
13	Machinery, Enc	11	1.52
14	Electrical and Optical Equipment	12	10.6
15	Transport Equipment	13	0.37
16	Manufacturing, Enc; Recycling	14	5
17	Electricity, Gas and Water Supply	15	5
18	Construction	16	5
19	Sale, Maintenance and Repair of Motor Vehicles...	17	5
20	Wholesale Trade and Commission Trade, ...	17	5
21	Retail Trade, Except of Motor Vehicles and...	18	5
22	Hotels and Restaurants	19	5
23	Inland Transport	20	5
24	Water Transport	21	5
25	Air Transport	22	5
26	Other Supporting and Auxiliary Transport....	23	5
27	Post and Telecommunications	24	5
28	Financial Intermediation	25	5
29	Real Estate Activities	26	5
30	Renting of M&Req and Other Business Activities	27	5
31	Public Admin/Defence; Compulsory Social Security	28	5
32	Education	29	5
33	Health and Social Work	30	5
34	Other Community, Social and Personal Services	30	5
35	Private Households with Employed Persons	30	5

Table 1: The sectors in the 2013 release of the WIOD data, and the aggregated sectors in our data.

B Proofs

Throughout the proofs, let χ_c be the share of total world income accruing to country c.

Proof of Theorem 1. Nominal GDP is equal to

$$P_{Y_c} Y_c = \sum_{i \in N_c} (1 - 1/\mu_i) p_i y_i + \sum_{f \in F_c} w_f L_f$$

Hence

$$\begin{aligned}
d \log P_{Y_c} + d \log Y_c &= \sum_{i \in N_c} (1 - 1/\mu_i) \lambda_i^{Y_c} d \log \left((1 - 1/\mu_i) \lambda_i^{Y_c} \right) \\
&\quad + \sum_{f \in F_c} \Lambda_f^{Y_c} \left(d \log w_f + d \log L_f \right) \\
d \log Y_c &= \sum_{i \in N_c} (1 - 1/\mu_i) \lambda_i^{Y_c} d \log \left((1 - 1/\mu_i) \lambda_i^{Y_c} \right) \\
&\quad + \sum_{f \in F_c} \Lambda_f^{Y_c} \left(d \log w_f + d \log L_f \right) - d \log P^{Y_c}.
\end{aligned}$$

The price of domestic goods is given by

$$d \log p_i = d \log \mu_i - d \log A_i + \sum_{j \in N_c} \tilde{\Omega}_{ij} d \log p_j + \sum_{j \notin N_c} \tilde{\Omega}_{ij} d \log p_j,$$

which implies that

$$d \log p = (I - \tilde{\Omega}^c)^{-1} \left(d \log \mu_i - d \log A_i + \tilde{\Omega}^F \left(d \log \Lambda - d \log L \right) + \tilde{\Omega}^M d \log p^M \right),$$

where $\tilde{\Omega}^c$ is the cost-based domestic IO table, $\tilde{\Omega}^F$ are cost-based factor shares, and $\tilde{\Omega}^M$ are cost-based intermediate import shares, and $d \log p^M$ represents the change in the price of imported intermediate goods. Use the fact that

$$\begin{aligned}
d \log P_{Y_c} &= \sum_{i \in N_c} \Omega_{Y_c, i} d \log p_i - \sum_{i \in N - N_c} \Lambda_i^{Y_c} d \log p_i \\
&= \sum_{i \in N_c} \tilde{\lambda}_i^{Y_c} \left(d \log \mu_i - d \log A_i \right) + \sum_{f \in F_c} \tilde{\Lambda}_f^{Y_c} \left(d \log \Lambda_f - d \log L_f \right) \\
&\quad + \sum_{i \in N - N_c} \tilde{\Lambda}_i^{Y_c} d \log p_i - \sum_{i \in N - N_c} \Lambda_i^{Y_c} d \log p_i.
\end{aligned}$$

For an imported intermediate

$$d\log p_i = d\log\Lambda_i^{Y_c} - d\log q_i + d\log GDP$$

Substitute this back to get

$$\begin{aligned}
d\log Y_c &= \sum_{i\in N_c}(1-1/\mu_i)\lambda_i^{Y_c} d\log\left((1-1/\mu_i)\lambda_i^{Y_c}\right) + \sum_{f\in F_c}\Lambda_f^{Y_c}\left(d\log w_f + d\log L_f\right)\\
&- \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c}\left(d\log\mu_i - d\log A_i\right) - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\left(d\log\Lambda_f - d\log L_f\right)\\
&- \sum_{i\in N-N_c}\tilde{\Lambda}_i^{Y_c} d\log p_i + \sum_{i\in N-N_c}\Lambda_i^{Y_c} d\log p_i\\
&= \sum_{f\in F_c^*}\Lambda_f^{Y_c} d\log\Lambda_f - \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c}\left(d\log\mu_i - d\log A_i\right) - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\left(d\log\Lambda_f - d\log L_f\right)\\
&- \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c} - d\log q_i + d\log GDP\right)\\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log A_i + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right) d\log q_i\\
&+ \sum_{f\in F_c^*}\Lambda_f^{Y_c}\left(d\log\Lambda_f^{Y_c} + d\log GDP_c\right) - \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log\mu_i - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\left(d\log\Lambda_f^{Y_c} + d\log GDP_c\right)\\
&- \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c} + d\log GDP\right)\\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log A_i + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right) d\log q_i\\
&- \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log\mu_i - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log\Lambda_f^{Y_c} - \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c}\right)\\
&+ \left[1 - \left(\sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\right) - \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\right] d\log GDP_c\\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log A_i + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right) d\log q_i\\
&- \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log\mu_i - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log\Lambda_f^{Y_c} - \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c}\right).
\end{aligned}$$

The last line follows from the fact that

$$\sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} + \sum_{i\in N-N_c}\tilde{\Lambda}_i^{Y_c} = \left[1 + \sum_{i\in N-N_c}\Lambda_i^{Y_c}\right].$$

■

Proof of Theorem 2. Note that welfare is given by

$$W_c = \frac{\sum_{f \in F^*} \Phi_{cf} w_f L_f + T_c}{P^{W_c}}.$$

Hence, letting world GDP be the numeraire,

$$\mathrm{d}\log W_c = \sum_f \Lambda_f^c \left(\mathrm{d}\log \Lambda_f\right) + \frac{dT}{GNE_c} - \left(\tilde{\Omega}_{(W_c)}\right)' \mathrm{d}\log p.$$

Use the fact that

$$\mathrm{d}\log p_i = \sum_{j \in N} \tilde{\Psi}_{ij} \,\mathrm{d}\log A_j + \sum_{f \in F} \tilde{\Psi}_{if} \left(\mathrm{d}\log \Lambda_f - \mathrm{d}\log L_f\right)$$

to complete the proof. ■

Proof of Theorem 3. For each good,

$$\lambda_i = \sum_c \Omega_{W_c,i} \chi_c + \sum_i \Omega_{ji} \lambda_j,$$

where χ_c is the share of total income accruing to country c and $\Omega_{W_c,i}$ is the share of income household c spends on good i. This means

$$\lambda_i \,\mathrm{d}\log \lambda_i = \sum_c \chi_c \Omega_{W_c,i} \,\mathrm{d}\log \Omega_{W_c,i} + \sum_j \Omega_{ji} \lambda_j \,\mathrm{d}\log \Omega_{ji} + \sum_j \Omega_{ji} \,\mathrm{d}\,\lambda_j + \sum_c \Omega_{W_c,i} \chi_c \,\mathrm{d}\log \chi_c.$$

Now, note that

$$\mathrm{d}\log \Omega_{W_c,i} = (1-\theta_c)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_{y_c}\right)$$

$$\mathrm{d}\log \Omega_{ji} = (1-\theta_j)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_j + \mathrm{d}\log \mu_j\right) - \mathrm{d}\log \mu_j$$

$$\mathrm{d}\log \chi_c = \sum_{f \in F_c^*} \frac{\Lambda_f}{\chi_c} \,\mathrm{d}\log \Lambda_f + \sum_{i \in c} \frac{\lambda_i}{\mu_i} \,\mathrm{d}\log \mu_i.$$

$$\mathrm{d}\log p_i = \tilde{\Psi}\left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + \tilde{\Psi}\tilde{\alpha} \,\mathrm{d}\log \Lambda.$$

$$\mathrm{d}\log P_{y_c} = b'\tilde{\Psi}\left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + b'\tilde{\Psi}\tilde{\alpha} \,\mathrm{d}\log \Lambda.$$

For shock $\mathrm{d}\log \mu_k$, we have

$$\mathrm{d}\log \Omega_{W_c,i} = (1-\theta_c)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \,\mathrm{d}\log \Lambda_f - \sum_j \Omega_{W_c,j}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf} \,\mathrm{d}\log \Lambda_f\right)\right).$$

$$\mathrm{d}\log\Omega_{ji} = (1-\theta_j)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) - \theta_j\,\mathrm{d}\log\mu_j.$$

Putting this altogether gives

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_i\sum_c (1-\theta_c)\chi_c\Omega_{W_c,i}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \sum_j \Omega_{W_c,j}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right)\Psi_{il} \\
& + \sum_i\sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} \\
& - \theta_k\lambda_k\sum_i \Omega_{ki}\Psi_{il} + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

Simplify this to

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_c (1-\theta_c)\chi_c\left[\sum_i \Omega_{W_c,i}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il}\right. \\
& \left. - \left(\sum_i \Omega_{W_c,i}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right)\left(\sum_i \Omega_{W_c,i}\Psi_{il}\right)\right] \\
& + \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\sum_i \tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} - \left(\sum_i \tilde{\Omega}_{ji}\Psi_{il}\right)\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) \\
& - \theta_k\lambda_k\left(\Psi_{kl} - \mathbf{1}(l=k)\right) + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

Simplify this further to get

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_c (1-\theta_c)\chi_c Cov_{b^{(c)}}\left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\right) \\
& + \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\sum_i \tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} \\
& - \left(\sum_i \tilde{\Omega}_{ji}\Psi_{il}\right)\left(\sum_i \tilde{\Omega}_{ji}\tilde{\Psi}_{ik} + \sum_i \tilde{\Omega}_{ji}\sum_f \Psi_{if}\,\mathrm{d}\log\Lambda_f\right) \\
& - \theta_k\lambda_k\left(\Psi_{kl} - \mathbf{1}(l=k)\right) + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c,
\end{aligned}$$

Using the input-output covariance notation, write

$$\mathrm{d}\,\lambda_l = \sum_c (1-\theta_c)\chi_c Cov_{\Omega_{(W_c)}}\left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\right)$$

$$+\sum_j (1-\theta_j)\lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}\left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\, \mathrm{d}\log\Lambda_f, \Psi_{(l)}\right)$$

$$-(1-\theta_k)\lambda_k(\Psi_{kl} - \mathbf{1}(l=k)) - \theta_k\lambda_k\left(\Psi_{kl} - \mathbf{1}(l=k)\right) + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\, \mathrm{d}\log\chi_c,$$

This then simplifies to give from the fact that $\sum_i \Omega_{W_c,i}\Psi_{il} = \lambda_l^{W_c}$:

$$\lambda_l\, \mathrm{d}\log\lambda_l = \sum_{j\in N,C} (1-\theta_j)\lambda_j\mu_j^{-1} Cov(\tilde{\Psi}_{(k)} + \sum_f^F \mathrm{d}\log\Lambda_f, \Psi_{(l)})$$
$$- \lambda_k\left(\Psi_{kl} - \mathbf{1}(k=l)\right) + \sum_c \chi_c \lambda_l^{W_c}\, \mathrm{d}\log\chi_c.$$

To complete the proof, note that

$$P_{y_c}Y_c = \sum_f w_f L_f + \sum_{i\in N_c}\left(1 - \frac{1}{\mu_i}\right) p_i y_i.$$

Hence,

$$\mathrm{d}(P_{y_c}Y_c) = \sum_{f\in c} w_f L_f\, \mathrm{d}\log w_f + \sum_{i\in c}\left(1-\frac{1}{\mu_i}\right) p_i y_i\, \mathrm{d}\log(p_i y_i) + \sum_{i\in c}\frac{\mathrm{d}\left(1-\frac{1}{\mu_i}\right)}{\mathrm{d}\log\mu_i} p_i y_i\, \mathrm{d}\log\mu_i.$$

In other words, since $P_y Y = 1$, we have

$$\mathrm{d}\chi_c = \sum_{f\in c}\Lambda_f\, \mathrm{d}\log w_f + \sum_{i\in c}\left(1-\frac{1}{\mu_i}\right)\lambda_i\, \mathrm{d}\log\lambda_i + \sum_{i\in c}\frac{\mathrm{d}\left(1-\frac{1}{\mu_i}\right)}{\mathrm{d}\log\mu_i}\lambda_i\, \mathrm{d}\log\mu_i.$$

Hence,

$$\mathrm{d}\log\chi_c = \sum_{f\in F_c^*}\frac{\Lambda_f}{\chi_c}\, \mathrm{d}\log\Lambda_f + \sum_{i\in c}\frac{\lambda_i}{\chi_c}\, \mathrm{d}\log\mu_i.$$

■

Proof of Theorem 5. Proof of Part(1):

The expression for $\mathrm{d}^2\log Y$ follows from applying part (2) to the whole world. The equality of real GNE and real GDP at the world level completes the proof.

Proof of Part (2):

Denote the set of imports into country c by M_c. Then, we can write:

$$\frac{d\log Y_c}{d\log \mu_i} = \sum_{f\in F_c} \Lambda_f^{Y_c} \frac{d\log \Lambda_f}{d\log \mu_i} + \sum_j \frac{d\,\lambda_j}{d\log \mu_i} \frac{\left(1-\frac{1}{\mu_j}\right)}{P_{Y_c}Y_c} + \frac{\lambda_i^{Y_c}}{\mu_i} - \frac{d\log P_{Y_c}}{d\log \mu_i},$$

where

$$\frac{d\log P_{Y_c}}{d\log \mu_i} = \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} \frac{d\log \Lambda_f}{d\log \mu_i} + \sum_{m\in M_c} \tilde{\lambda}_m^{Y_c} \frac{d\log p_m}{d\log \mu_i} - \tilde{\lambda}_i^{Y_c} - \sum_{m\in M_c} \Lambda_m^{Y_c} \frac{d\log p_m}{d\log \mu_i},$$

and

$$\tilde{\lambda}_i^{Y_c} = \sum_j \Omega_{Y_c,j} \tilde{\Psi}_{ji}.$$

Combining these expressions, we get

$$\begin{aligned}\frac{d\log Y_c}{d\log \mu_i} &= \sum_{f\in F_c} \left(\Lambda_f^{Y_c} - \tilde{\Lambda}_f^{Y_c}\right) \frac{d\log \Lambda_f}{d\log \mu_i} + \sum_{m\in M_c} \left(\lambda_m^{Y_c} - \tilde{\lambda}_m^{Y_c}\right) \frac{d\log p_m}{d\log \mu_i} \\ &\quad + \sum_{j\in N_c} \lambda_j^{Y_c} \frac{d\log \lambda_j}{d\log \mu_i} \left(1 - \frac{1}{\mu_j}\right) + \frac{\lambda_i^{Y_c}}{\mu_i} - \tilde{\lambda}_i^{Y_c}.\end{aligned}$$

At the efficient point,

$$\begin{aligned}\frac{d^2 \log Y_c}{d\log \mu_i\, d\log \mu_k} &= \sum_{f\in F_c} \left(\frac{d\,\Lambda_f^{Y_c}}{d\log \mu_i} - \frac{d\,\tilde{\Lambda}_f^{Y_c}}{d\log \mu_i}\right) \frac{d\log \Lambda_f}{d\log \mu_k} \\ &\quad + \sum_{m\in M_c} \left(\frac{d\,\lambda_m^{Y_c}}{d\log \mu_i} - \frac{d\,\tilde{\lambda}_m^{Y_c}}{d\log \mu_i}\right) \frac{d\log p_m}{d\log \mu_k} - \frac{d\,\tilde{\lambda}_k^{Y_c}}{d\log \mu_i} \\ &\quad + \lambda_k^{Y_c} \left(\frac{d\log \lambda_k^{Y_c}}{d\log \mu_i} - \delta_{ki}\right) + \frac{1}{P_{Y_c}Y_c} \frac{d\,\lambda_i^{Y_c}}{d\log \mu_k},\end{aligned}$$

where δ_{ki} is the a Kronecker delta.

Using Lemma 8,

$$\begin{aligned}\frac{d^2 \log Y_c}{d\log \mu_i\, d\log \mu_k} &= -\sum_{f\in F_c} \lambda_i^{Y_c} \Psi_{if} \frac{d\log \Lambda_f}{d\log \mu_k} - \sum_{m\in M_c} \lambda_i^{Y_c} \Psi_{im} \frac{d\log p_m}{d\log \mu_k} - \lambda_i^{Y_c} \left(\Psi_{ik} - \delta_{ik}\right) \\ &\quad - \lambda_k^{Y_c} \delta_{ik} + \frac{d\,\lambda_i}{d\log \mu_k} \frac{1}{P_{Y_c}Y_c}, \\ &= -\sum_{f\in F_c} \lambda_i^{Y_c} \Psi_{if} \frac{d\log \Lambda_f}{d\log \mu_k} - \sum_{m\in M_c} \lambda_i^{Y_c} \Psi_{im} \frac{d\log p_m}{d\log \mu_k} - \lambda_i^{Y_c} \Psi_{ik}\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_i^{Y_c}\left(\frac{\mathrm{d}\log p_i}{\mathrm{d}\log\mu_k}+\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}\right),\\
&= \lambda_i^{Y_c}\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}.
\end{aligned}
$$

■

Lemma 7. *Let χ_h be the income share of country h at the initial equilibrium. Then*

$$
\frac{\mathrm{d}\lambda_j}{\mathrm{d}\log\mu_k}-\sum_h\overline{\chi}_h\frac{\mathrm{d}\log\tilde{\lambda}_j^{W_h}}{\mathrm{d}\log\mu_k}=\sum_h\frac{\mathrm{d}\chi_h}{\mathrm{d}\log\mu_i}\lambda_j^{W_h}-\lambda_i\left(\Psi_{ij}-\delta_{ij}\right).
$$

Proof. Let μ be the diagonal matrix of μ_i and I_{μ_k} be a matrix of all zeros except μ_k for its kth diagonal element. Then

$$
\overline{\chi}'\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\log\mu_k}=\chi'\frac{d\tilde{\Omega}_{(W)}}{\mathrm{d}\log\mu_k}+\chi'\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\log\mu_k}\mu\Omega+\chi'\tilde{\lambda}I_{\mu_k}\Omega+\chi'\tilde{\lambda}\mu\frac{\mathrm{d}\Omega}{\mathrm{d}\log\mu_k},
$$

where $\tilde{\Omega}_{(W)}$ is a matrix whose cith element is household c's expenditure share $\tilde{\Omega}_{W_c,i}$ on good i.

On the other hand,

$$
\lambda=\chi'\tilde{\Omega}_{(W)}+\lambda\Omega.
$$

Form this, we have

$$
\frac{\mathrm{d}\lambda}{\mathrm{d}\log\mu_k}=\frac{\mathrm{d}\chi'}{\mathrm{d}\log\mu_k}\tilde{\Omega}_{(W)}+\chi'\frac{d\tilde{\Omega}_{(W)}}{\mathrm{d}\log\mu_k}+\lambda\frac{\mathrm{d}\Omega}{\mathrm{d}\log\mu_k}+\frac{\mathrm{d}\lambda}{\mathrm{d}\log\mu_k}\Omega.
$$

Combining these two expressions

$$
\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\log\mu_k}-\overline{\chi}'\frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right)=\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\log\mu_k}-\overline{\chi}'\frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right)\Omega+\frac{\mathrm{d}\chi}{\mathrm{d}\log\mu_k}\tilde{\Omega}_{(W)}-\chi'\tilde{\lambda}^{(h)}I_{\mu_k}\Omega.
$$

Rearrange this to get

$$
\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\log\mu_k}-\overline{\chi}'\frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right)=\frac{\mathrm{d}\chi}{\mathrm{d}\log\mu_k}\tilde{\Omega}_{(W)}\Psi-\chi'\tilde{\lambda}^{(h)}I_{\mu_k}(\Psi-I),
$$

or

$$
\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\log\mu_k}-\overline{\chi}'\frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right)=\frac{\mathrm{d}\chi}{\mathrm{d}\log\mu_k}\tilde{\Omega}_{(W)}\Psi-\lambda I_{\mu_k}(\Psi-I).
$$

■

Lemma 8. *At the efficient steady-state*

$$\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\,\tilde{\lambda}_j^{Y_c}}{\mathrm{d}\log\mu_k}=-\lambda_k^{Y_c}\left(\Psi_{kj}-\delta_{kj}\right).$$

Proof. Start from the relations

$$\lambda_j^{Y_c}=\chi_j^{Y_c}+\sum_i\lambda_i^{Y_c}\Omega_{ij},$$

and

$$\tilde{\lambda}_j^{Y_c}=\chi_j^{Y_c}+\sum_i\tilde{\lambda}_i^{Y_c}\mu_i\Omega_{ij}.$$

Differentiate both to get

$$\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\,\tilde{\lambda}_j^{Y_c}}{\mathrm{d}\log\mu_k}=\sum_i\left(\frac{\mathrm{d}\,\lambda_j^{Y_c}}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\,\tilde{\lambda}_j^{Y_c}}{\mathrm{d}\log\mu_k}\right)\Omega_{ij}-\lambda_k^{Y_c}\Omega_{ki}.$$

Rearrange this to get the desired result. ∎

Proof of Corollary 4. Let $\overline{\chi}_h^W$ be the elasticity of social welfare with respect to the consumption of country h (i.e. log Pareto weight). Then

$$\frac{\mathrm{d}\log W^{BS}}{\mathrm{d}\log\mu_k}=\sum_{h\in H}\overline{\chi}_h^W\frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k}=\sum_h\overline{\chi}_h^W\left(\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}\right).$$

$$\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_k}=\sum_{f\in F_c}\frac{\Lambda_f}{\chi_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}+\sum_{i\in N_h}\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{(1-\frac{1}{\mu_i})}{\chi_h}.$$

$$\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}=\sum_{f\in F}\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}+\tilde{\lambda}_k^{W_h}.$$

Hence, assuming the normalization $P_YY=1$ gives

$$\begin{aligned}\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i}=&\sum_h\overline{\chi}_h^W\Bigg(\sum_f\frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{1}{\chi_h^W}+\sum_f\frac{\Lambda_f}{\chi_h^W}\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}\\&-\sum_f\frac{\Lambda_f}{\chi_h^W}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i}+\frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i}\frac{1}{\chi_h^W\mu_k}-\frac{\lambda_k}{\chi_h^W\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i}-\frac{\lambda_k}{\chi_h^W\mu_k}\delta_{ki}\\&\sum_i\frac{\mathrm{d}^2\,\lambda_j}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h}+\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{1}{\mu_i\chi_h^W}+\sum_j\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h^W}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i}\end{aligned}$$

$$- \sum_f \frac{\mathrm{d}\,\tilde{\Lambda}_f^{W_h}}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \sum_f \tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_k^{W_h}}{\mathrm{d}\log\mu_i}\Bigg).$$

At the efficient point, this simplifies to

$$\begin{aligned}\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} = &\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i} - \sum_h \overline{\chi_h^W}\frac{\mathrm{d}\,\tilde{\Lambda}_f^{W_h}}{\mathrm{d}\log\mu_i}\right)\\ &+ \frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i} - \sum_h \overline{\chi_h^W}\frac{\mathrm{d}\,\tilde{\lambda}_k^{W_h}}{\mathrm{d}\log\mu_i} - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i}\\ &- \lambda_k\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} - \lambda_k\delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}.\end{aligned}$$

By Lemma 7, at the efficient point,

$$\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_i} - \sum_h \overline{\chi}_h^W\frac{\mathrm{d}\,\tilde{\lambda}_j^{W_h}}{\mathrm{d}\log\mu_i} = \sum_h \frac{\mathrm{d}\,\chi_h^W}{\mathrm{d}\log\mu_i}\tilde{\lambda}_j^{W_h} - \lambda_i\left(\Psi_{ij} - \delta_{ij}\right).$$

Whence, we can further simplify the previous expression to

$$\begin{aligned}\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} &= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h^W}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right)\\ &+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i(\Psi_{ik} - \delta_{ik}) - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\\ &- \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i - \lambda_k\delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k},\\ &= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right)\\ &+ \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i\Psi_{ik} - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\\ &- \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k},\end{aligned}$$

and using $\mathrm{d}\log\lambda_i = \mathrm{d}\log p_i + \mathrm{d}\log y_i$,

$$= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right)$$

$$
\begin{aligned}
&+\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \\
&- \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i + \lambda_i \frac{\mathrm{d}\log p_i}{\mathrm{d}\log\mu_k} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}, \\
&= \sum_{f,h} \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \lambda_i \sum_f \Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \\
&+ \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \\
&- \lambda_k \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} \\
&+ \lambda_i \left(\sum_f \Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \Psi_{ik}\right), \\
&= \sum_{f,h} \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\chi_h\tilde{\Lambda}_f^{W_h} - \Lambda_f\right) \\
&+ \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_k \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\left(\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \tilde{\lambda}_k^{W_h}\right) \\
&- \sum_{f,h} \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f - \lambda_k \sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k} \\
&- \left(\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f\right)\left(\sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\right) - \lambda_k \sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + Cov_\chi\left(\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}\right),
\end{aligned}
$$

since

$$
-\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f = -\sum_f \frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_k} = \frac{\mathrm{d}\left(1 - \sum_j \lambda_j(1-\frac{1}{\mu_j})\right)}{\mathrm{d}\log\mu_k} = -\lambda_k
$$

at the efficient point, and

$$
\sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} = 0.
$$

■

Proof of Theorem 6. From Theorem 5, we have

$$\mathcal{L} = -\frac{1}{2}\sum_l (d\log \mu_l)\lambda_l d\log y_l.$$

With the maintained normalization $PY = 1$, we also have

$$d\log y_l = d\log \lambda_l - d\log p_l,$$

$$d\log p_l = \sum_f \Psi_{lf} d\log \Lambda_f + \sum_k \Psi_{lk} d\log \mu_k,$$

where, from Theorem 3,

$$\begin{aligned} d\log \lambda_l =& \sum_k (\delta_{lk} - \frac{\lambda_k}{\lambda_l}\Psi_{kl}) d\log \mu_k - \sum_j \frac{\lambda_j}{\lambda_l}(\theta_j - 1) Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log \mu_k - \sum_g \Psi_{(g)} d\log \Lambda_g, \Psi_{(l)}) \\ &+ \frac{1}{\lambda_l}\sum_{g\in F^*}\sum_c \left(\lambda_l^{W_c} - \lambda_l\right)\Phi_{cg}\Lambda_g d\log \Lambda_g, \end{aligned}$$

and

$$\begin{aligned} d\log \Lambda_f =& -\sum_k \lambda_k \frac{\Psi_{kf}}{\Lambda_f} d\log \mu_k - \sum_j \lambda_j(\theta_j - 1) Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log \mu_k - \sum_g \Psi_{(g)} d\log \Lambda_g, \frac{\Psi_{(f)}}{\Lambda_f}) \\ &+ \frac{1}{\Lambda_f}\sum_{g\in F^*}\sum_c \left(\Lambda_i^{W_c} - \Lambda_f\right)\Phi_{cg}\Lambda_g d\log \Lambda_g. \end{aligned}$$

We will now use these expressions to replace in formula for the second-order loss function. We get

$$\begin{aligned} \mathcal{L} =& -\frac{1}{2}\sum_l\sum_k (\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k})\lambda_k\lambda_l d\log \mu_k d\log \mu_l + \frac{1}{2}\sum_l \lambda_l d\log \mu_l \sum_f \Psi_{lf} d\log \Lambda_f \\ &+ \frac{1}{2}\sum_l\sum_j (d\log \mu_l)\lambda_j(\theta_j - 1) Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log \mu_k - \sum_g \Psi_{(g)} d\log \Lambda_g, \Psi_{(l)}) \\ &- \frac{1}{2}\sum_l d\log \mu_l \left(\sum_g\sum_c \left(\lambda_l^{W_c} - \lambda_l\right)\Phi_{cg}\Lambda_g d\log \Lambda_g\right) \end{aligned}$$

$$\mathcal{L} = -\frac{1}{2}\sum_l\sum_k (\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k})\lambda_k\lambda_l d\log \mu_k d\log \mu_l + \frac{1}{2}\sum_l \lambda_l d\log \mu_l \sum_f \Psi_{lf} d\log \Lambda_f$$

$$+\frac{1}{2}\sum_{l}\sum_{j}(d\log\mu_l)\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\sum_{k}\Psi_{(k)}d\log\mu_k-\sum_{g}\Psi_{(g)}d\log\Lambda_g,\Psi_{(l)})$$
$$-\frac{1}{2}\sum_{l}\left(\sum_{c}\left(\lambda_l^{W_c}-\lambda_l\right)\chi_c d\log\chi_c\right)d\log\mu_l$$

We can rewrite this expression as

$$\mathcal{L}=\mathcal{L}_I+\mathcal{L}_X+\mathcal{L}_H$$

where

$$\mathcal{L}_I=\frac{1}{2}\sum_{k}\sum_{l}[\frac{\Psi_{kl}-\delta_{kl}}{\lambda_l}+\frac{\Psi_{lk}-\delta_{lk}}{\lambda_k}+\frac{\delta_{kl}}{\lambda_l}-1]\lambda_k\lambda_l d\log\mu_k d\log\mu_l$$
$$+\frac{1}{2}\sum_{k}\sum_{l}\sum_{j}d\log\mu_k d\log\mu_l\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)}),$$

$$\mathcal{L}_X=\frac{1}{2}\sum_{l}\sum_{f}(\frac{\Psi_{lf}}{\Lambda_f}-1)\lambda_l\Lambda_f d\log\mu_l d\log\Lambda_f$$
$$-\frac{1}{2}\sum_{l}\sum_{g}d\log\mu_l d\log\Lambda_g\sum_{j}\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}),$$

$$\mathcal{L}_H=-\frac{1}{2}\sum_{l}\left(\sum_{c}\left(\lambda_l^{W_c}-\lambda_l\right)\chi_c d\log\chi_c\right)d\log\mu_l,$$

where $d\log\Lambda$ is given by the usual expression.[1] Finally, using Lemma 10, we can write

$$\mathcal{L}_{\mathcal{I}}=\frac{1}{2}\sum_{l}\sum_{k}(d\log\mu_l)(d\log\mu_k)\sum_{j}\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)}).$$

and

$$\mathcal{L}_X=-\frac{1}{2}\sum_{l}\sum_{g}d\log\mu_l d\log\Lambda_g\sum_{j}\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}).$$

[1] We have used the intermediate step

$$\mathcal{L}_X=\frac{1}{2}\sum_{l}\sum_{k}\lambda_k\lambda_l d\log\mu_k d\log\mu_l+\frac{1}{2}\sum_{l}\sum_{f}d\log\mu_l d\log\Lambda_f\lambda_l\Psi_{lf}$$
$$-\frac{1}{2}\sum_{l}\sum_{g}d\log\mu_l d\log\Lambda_g\sum_{j}\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}).$$

■

Lemma 9. *The following identity holds*

$$\sum_j \lambda_j \left(\tilde{\Psi}_{jk}\Psi_{jl} - \sum_m \Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml} \right) = \tilde{\lambda}_k \lambda_l.$$

Proof. Write Ω so that it contains all the producers, all the households, and all the factors *as well as* a new row (indexed by 0) where $\Omega_{0i} = \chi_i$ if $i \in C$ and 0 otherwise. then, letting e_0 be the standard basis vector corresponding to the 0th row, we can write

$$\lambda' = e_0' + \lambda'\Omega,$$

or equivalently

$$\lambda'(I - \Omega) = e_0'.$$

Let X^{kl} be the vector where $X^{kl}_m = \tilde{\Psi}_{mk}\Psi_{ml}$. Then

$$\begin{aligned}\sum_j \lambda_j \left(\tilde{\Psi}_{jk}\Psi_{jl} - \sum_m \Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml} \right) &= \lambda'(I - \Omega)X^{kl}, \\ &= e_0'(I - \Omega)^{-1}(I - \Omega)X^{kl}, \quad = e_0'X^{kl} = \tilde{\Psi}_{0k}\Psi_{0l} = \tilde{\lambda}_k\lambda_l.\end{aligned}$$

■

Lemma 10. *The following identity holds*

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \lambda_l \lambda_k [\frac{\tilde{\Psi}_{lk} - \delta_{lk}}{\lambda_k} + \frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\delta_{lk}}{\lambda_k} - \frac{\tilde{\lambda}_k}{\lambda_k}].$$

Proof. We have

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$

$$\sum_j \lambda_j \mu_j^{-1} \left[\sum_m \tilde{\Omega}_{jm}\tilde{\Psi}_{mk}\Psi_{ml} - \left(\sum_m \tilde{\Omega}_{jm}\tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm}\Psi_{ml} \right) \right],$$

or

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$

$$\sum_j \lambda_j \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right),$$

or

$$\begin{aligned}
\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \\
&\sum_j \lambda_j \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} \\
&\quad + \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right),
\end{aligned}$$

or using, Lemma 9

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = -\tilde{\lambda}_k \lambda_l + \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} - \sum_j \lambda_j \left(\tilde{\Psi}_{jk} - \delta_{jk} \right) (\Psi_{jl} - \delta_{jl}),$$

and finally

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \lambda_l \lambda_k [\frac{\tilde{\Psi}_{lk} - \delta_{lk}}{\lambda_k} + \frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\delta_{lk}}{\lambda_k} - \frac{\tilde{\lambda}_k}{\lambda_k}].$$

■

Proposition 1 (Structural Output Loss)**.** *Starting at an efficient equilibrium in response to the introduction of small tariffs or other distortions,*

$$\begin{aligned}
\Delta \log Y \approx -\frac{1}{2} \sum_{l \in N} \sum_{k \in N} \Delta \log \mu_k \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}) \\
- \frac{1}{2} \sum_{l \in N} \sum_{g \in F} \Delta \log \Lambda_g \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}) \\
+ \frac{1}{2} \sum_{l \in N} \sum_{c \in C} \chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c} - \lambda_l).
\end{aligned}$$

Proof. The proof follows along the same lines as Theorem 6. ■

C Additional Examples

C.1 Writing an economy in standard form

We use a two-country example to show how to map a specific nested-CES model into *standard-form* required by Theorem 3. Suppose there are n industries at home and foreign. The utility function of home and foreign consumers is

$$W = \prod_{i=1}^{n} (x_{0i})^{\Omega_{0i}}, \qquad W_{*} = \prod_{i=1}^{n} (x_{0i}^{*})^{\Omega_{0i}},$$

where x_{0i} and x_{0i}^{*} are home and foreign consumption of goods from industry i. The production function of industry i (at home or foreign) is a Cobb-Douglas aggregate of intermediates and the local factor

$$y_i = L_{ij}^{\Omega_{iL}} \prod_{i=1}^{n} x_{ij}^{\Omega_{ij}}.$$

Suppose that the intermediate good x_{ij} is a CES combination of domestic and foreign varieties of j, with initial home share Ω_j and foreign share $\Omega_j^{*} = 1 - \Omega_j$, and elasticity of substitution $\varepsilon_j + 1$. Since the market share of home and foreign in industry j does not vary by consumer i, this means there is no home-bias.

In standard-form, this economy has $N = 3n$ producers: the first n are industries at home, the second n are industries in foreign, and the last n are CES aggregates of domestic and foreign varieties that every other industry buys. The HAIO matrix for this economy, in standard-form, is $(2 + 3n + 2) \times (2 + 3n + 2)$:

$$\Omega = \left[\begin{array}{c|c|c|c|cc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\ \Omega_{0i}\ \right]_{i=1}^{n} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\ \Omega_{0i}\ \right]_{i=1}^{n} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\ \Omega_{ij}\ \right]_{i,j=1}^{n} & \left[\ \Omega_{iL}\ \right]_{i=1}^{n} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\ \Omega_{ij}\ \right]_{i,j=1}^{n} & \mathbf{0} & \left[\ \Omega_{iL}\ \right]_{i=1}^{n} \\
\hline
\mathbf{0} & \begin{matrix} \Omega_1 & \cdots & 0 \\ & \ddots & \\ 0 & & \Omega_n \end{matrix} & \begin{matrix} \Omega_1^{*} & \cdots & 0 \\ & \ddots & \\ 0 & & \Omega_n^{*} \end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right].$$

The first two rows and columns correspond to the households, the next $2n$ rows and columns correspond to home industries and foreign industries respectively. The next n rows and columns correspond to bundles of home and foreign varieties. The last two rows and columns correspond to the home and foreign factor. The vector elasticities of substitution θ for this economy is a vector with $2+3n$ elements $\theta = (1, \cdots, 1, \varepsilon_1 + 1, \cdots, \varepsilon_n + 1)$, where ε_i is the trade elasticity in industry i.

Now that we have written this economy in standard-form, we can use Theorem 3 to study the change in home's share of income following a productivity shock $d \log A_j$ to some *domestic* producer j:

$$\frac{d \log \Lambda_L}{d \log A_j} = \frac{\lambda_j}{\Lambda_L} \frac{\varepsilon_j \Omega_j^* \Omega_{jL}}{1 + \sum_i \varepsilon_i \frac{\lambda_i \Omega_{iL}}{\Lambda_L} \frac{\Omega_{iL}}{1-\Lambda_L} \Omega_i^*} \geq 0,$$

which is positive as long as domestic and foreign varieties are substitutes $\varepsilon_j > 0$ for every j. The numerator captures the fact that a shock to j will increase demand for the home factor if j uses the home factor $\Omega_{jL} > 0$. The denominator captures the fact that an increase in the price of the home factor attenuates the increase in demand for the home factor by bidding up the price of home goods.

The positive productivity shock to j will therefore shrink the market share of every other domestic producer, a phenomenon known as Dutch disease. To see this, apply Theorem 3 to some domestic producer $i \neq j$ to get

$$\frac{d \log \lambda_i}{d \log A_j} = -\varepsilon_i \Omega_i^* \frac{\Omega_{iL}}{1 - \Lambda_L} \frac{d \log \Lambda_L}{d \log A_j} < 0.$$

In words, the shock to j boosts the price of the home factor, which makes i less competitive in the world market if i relies on the home factor $\Omega_{iL} > 0$. Hence, if $\varepsilon_j > 0$ for every j, a domestic productivity shock to one sector will cause Dutch disease and shrink the market share of other domestic producers by bidding up home wages.

C.2 More details on Example IV from Section 6

First, the forward propagation equations (7) from Theorem 3 imply that the change in the price of each good is

$$d \log p = \sum_{k \in N} \Psi_{(k)} d \log \mu_k + \frac{\Psi_{(L)}}{\Lambda_L} d\Lambda_L - \frac{(1 - \Psi_{(L)})}{1 - \Lambda_L} \left[d\Lambda_L + \sum_i \lambda_i d \log \mu_i \right].$$

The first-term captures the direct effect of the tariff on the price of each good, the second term captures the effect of the change in the wage of domestic workers, and the last term captures the effect of changes in the foreign wage. Here, we use the fact that the change in the foreign wage relative to world GDP is the negative of the change in the home wage and the tax revenues collected (the expression in square brackets).

Substituting the expression for prices into the backward propagation equations from Theorem 3 yields the following expression for the home factor's change in aggregate income:

$$d\Lambda_L = \frac{-d\log\mu_L + \sum_{k\in N}\lambda_k(1-\theta_k)Cov_{\Omega^{(k)}}(\Psi_{(L)}, \Psi_{(M)}d\log\mu + \Psi_{(L)}\frac{d\Lambda_R}{1-\Lambda_L}) + (\Lambda_L^{W_L} - \Lambda_L^{W_{L*}})d\Lambda_R}{1 - \frac{1}{\Lambda_L(1-\Lambda_L)}\sum_{k\in N}\lambda_k(1-\theta_k)Var_{\Omega^{(k)}}(\Psi_{(L)}) - (\Lambda_L^{W} - \Lambda_L^{W_*})}, \tag{20}$$

where $d\log\mu_L = \sum_k \lambda_k\Psi_{kL}d\log\mu_k$ and $\Psi_{(M)}d\log\mu = \sum_{k\in N}\Psi_{(k)}d\log\mu_k$. The tariff revenues are $d\Lambda_R = \sum_k \lambda_k d\log\mu_k$. Each term in (20) is intuitive: the numerator is the effect of the tax in partial equilibrium, holding fixed factor prices in terms of world GDP. The denominator is the general equilibrium effect capturing the endogenous substitution and income redistribution effects triggered by changes in factor prices. This comes from solving the fixed point for factor shares $d\log\Lambda$ in Theorem 3.

To understand the intuition, consider the numerator, which consists of three effects. The first summand in the numerator is the direct incidence of the tax on the home labor, taking into account supply chains. The second term, involving the covariance, is how the tax causes substitution by changing relative prices of goods, and the covariance captures whether or not goods whose relative prices rise tend to be reliant on home labor. The final term in the numerator captures the fact that the tariff revenues, by redistributing income between home and foreign, change demand for the domestic factor. The denominator then accounts for the fact that the partial equilibrium change in factor prices result in additional rounds of expenditure-switching due to substitution and income redistribution.

From home's perspective, the ideal tariff, which raises home wages relative to foreign wages, is one which is imposed on goods that do not directly or indirectly use the domestic factor. For such goods, $d\log\mu_L = 0$. Furthermore, if substitution elasticities are greater than one, $\theta_k \geq 1$, then the ideal tariff should be levied on goods which negatively correlate with domestic factor usage, in which case $Cov_{\Omega^{(k)}}(\Psi_{(L)}, \Psi_{(M)}d\log\mu) < 0$. In other words, if a good is heavily exposed to the tax, then it should also be heavily exposed to foreign (rather than domestic) labor.

D Computational Appendix

This appendix describes our computational procedure, as well as the Matlab code in our replication files. Before running the code, customize your folder directory in the code accordingly. Notice that the description below is based on the generic version of the code under flexible wages stored in "Generic" folder.

Writing nested-CES economies in standard-form is useful for intuition, but it is computationally inefficient since it greatly expands the size of the input-output matrix. Therefore, for computational efficiency, we instead use the generalization in Appendix E to directly linearize the nested-CES production functions without first putting them into standard form.

Overview

First, we provide an overview of the different files before providing an in depth description of each.

1. **main_load_data_rev.m**: Function that calculates expenditure shares from data.
2. **main_dlogW_org.m**: Main code that loads inputs and calls functions to iterate.
3. **AES_func.m**: Function that calculates Allen-Uzawa elasticities of substitution.
4. **Nested_CES_linear_final_rev.m**: Function that solves the system of linear equations described in Theorem 3.
5. **Nested_CES_linear_result_final.m**: Function that calculates derivatives that are used to derive welfare changes or iterate for large shocks.

While 1. and 3. are specific to our quantitative application, 2., 4. and 5. are general purpose functions that can be used to derive comparative statics and solve any model in the class we study. We now describe each part of the code in some detail.

1. Function code that loads data

The data used here is based on 2013 release of World Input-Output Database in year 2008. According to Appendix A, there are $C = 41$ countries including ROW (rest-of-world), and $N = 30$ sectors in each country. The code is flexible in terms of which countries to be included in the analysis by `keep_c` input variable. Any countries not included in `keep_c` are put into an aggregate rest-of-the-world composite country. The order of countries are in `countries` variable in the main code. Notice that we always exclude 35th country for ROW for `keep_c` input and put ROW in the last for welfare output. For example, this is why USA is 41st country for input, and 40th country for output.

Code: main_load_data_rev.m

Data input:

1. Trade elasticity when a country imports or buys inputs in each sector from different destinations (`trade_elast`: N by 1 vector)

2. Input-output matrix across country and sectors (`Omega_tilde`: CN by CN matrix, (i, j) element: expenditure share of sector i on sector j)

3. Household expenditure share on heterogenous goods (`beta`: CN by C matrix, (i, c) element: expenditure share of household c on sector i)

4. Value-added share (`alpha`: CN by 1 vector, $(i, 1)$ element: value-added share of sector i), Primary Factor share (`alpha_VA`: CN by F matrix, (i, f) element: expenditure share of sector i on factor f out of factor usage)

5. GNE of each country relative to world GNE (`GNE_weights`: C by 1 vector)

6. (Optional) If there are initial tariffs:

   (a) Tariff matrix when household (column) buys goods (row) – `Tariff_cons_matrix_new`: CN by C matrix ((i, c) element: tariff rate of household c, destination, on sector i, origin)

   (b) Tariff matrix when a sector (row) buys goods (column) –`Tariff_matrix_new`: CN by CN matrix ((i, j) element: tariff rate of sector i, destination, on sector j, origin)

User input:

1. `keep_c` controls which countries to be included. For example,
   Command `keep_c = (1:41); keep_c(35) = [];` include all 41 countries in the data.

2. If the economy does not have initial tariff, `initial_tariff_index`= 1. Otherwise, if the economy has initial tariff, =2.

3. If factors are country-specific (4 factors per country), `factor_index`= 1. Otherwise, if factors are country-sector-specific (N factors per country), =2.

Outputs:

1. `data`, `shock` struct

From the inputs, the code automatically calculates input shares (`beta_s`, `beta_disagg`, `Omega_s`, `Omega_disagg`, `Omega_total_C`, `Omega_total_N`) and the input-output matrix (`Omega_total_tilde`, `Omega_total`). These variables are used to calculate Allen-Uzawa elasticities of substitution and solve system of linear equations.

2. Main code that loads inputs and calls functions

Code: main_dlogW_org.m

Data input:

1. `data`, `shock` struct from **main_load_data_rev.m**

User input:

1. Elasticity of substitution parameters for nested CES structure: Elasticity of substitution (1) across sectors in consumption (`sigma`), (2) across composite of value-added and intermediates (`theta`), (3) across primary factors (`gamma`), and (4) across intermediate inputs (`epsilon`). In the text of the paper, these elasticities are relabeled as $(\sigma, \epsilon, \theta, \gamma) = (\theta_0, \theta_1, \theta_2, \theta_3)$.
2. If the economy gets universal iceberg trade cost shock, `shock_index` = 1. Otherwise, if the economy gets universal tariff shock, = 2.
3. When intensity of shock is x%, `intensity` = x.
4. When shock is discretized by x/y% and model cumulates the effect of shocks y times, `ngrid` = y.
5. Ownership structure
   (a) Ownership structure of factor (`Phi_F`: C by CF matrix, (c, f) element: Factor income share of factor f owned by household c)
   (b) Ownership structure of tariff revenue (`Phi_T`: C+CN by CN+CF by C 3-D matrix, (i, j, c) element: Tariff revenue share owned by household c when household/sector i buys from sector/factor j)
6. (Optional) Technical details about how to customize iceberg trade cost shock matrix $dlog\tau$ and tariff shock matrix $dlogt$ are described in **Nested_CES_linear_final_rev.m**

Output:

1. `dlogW` (C by `ngrid` matrix) collects change in real income of each country for each iteration of discretized shocks
2. `dlogW_sum` (C by 1 vector) shows change in real income of each country from linearized system by summing up `dlogW`
3. `dlogW_world` (1 by `ngrid` vector) is change in real income of world for each iteration of discretized shocks
4. `dlogR` (C by `ngrid` matrix) collects reallocation terms of each country for each iteration of discretized shocks
5. `dlogR_sum` (C by 1 vector) shows reallocation terms of each country from linearized system by summing up `dlogR`

3. Allen-Uzawa Elasticity of Substitution (AES)

This code computes Allen-Uzawa elasticities of substitution for each sector. These are then used following Appendix E.

Code: AES_func.m

Inputs:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)
2. Elasticity of substitution parameters for nested CES structure: That is, $(\sigma, \epsilon, \theta, \gamma) = (\theta_0, \theta_1, \theta_2, \theta_3)$.
3. Trade elasticity when a country imports or buys domestic product (`trade_elast`: N by 1 vector).
4. Value-added share (`alpha`: CN by 1 vector, $(i, 1)$ element: value-added share of sector i).
5. Expenditure shares:
    (a) b_{ic}(`beta_s` : C by N matrix, (c, i) element: How much household c consumes sector i good).

(b) ω_j^{ic}(`Omega_s`: CN by N matrix, (ic, j) element: How much sector i in country c uses sector j good).

(c) $\tilde{\Omega}_{jm}^{0c}$(`Omega_total_C` : C by CN matrix, (c, jm) element: How much household c buys from sector j in country m).

(d) $\tilde{\Omega}_{jm}^{ic}$(`Omega_total_N` : CN by CN+CF matrix, (ic, jm) element: How much sector i in country c buys from good/factor j in country m).

Outputs:

1. $\theta_{0c}(ic', jm)$ (`AES_C_Mat`: CN by CN by C 3-D matrix, (ic', jm, c) element: AES of household in country c that substitutes good i in country c' and good j in country m)

2. $\theta_{kc}(ic', jm)$ (`AES_N_Mat`: CN by CN+CF by CN 3-D matrix, (ic', jm, kc) element: AES of producer of sector k in country c that substitutes good i in country c' and good/factor j in country m)

3. $\theta_{kc}(fc, jm)$ (`AES_F_Mat`: CF by CN+CF by CN 3-D matrix, (fc, jm, kc) element: AES of producer of sector k in country c that substitutes factor f in country c and good j in country m)

To describe how this code functions, we introduce the following notation.

Notation:

Let $p_{ic'}^{kc}$ be the bilateral price when industry or household k in country c buys from industry i in country c'. That is

$$p_{ic'}^{kc} = \tau_{ic'}^{kc} t_{ic'}^{kc} p_{ic'},$$

where $\tau_{ic'}^{kc}$ is an iceberg cost on kc purchasing goods from ic' and $t_{ic'}^{kc}$ is a tariff on kc purchasing goods from ic', and where $p_{ic'}$ is the marginal cost of producer i in country c'. Define

$$\Omega_{jm}^{ic} = \frac{p_{jm} x_{jm}^{ic}}{p_{ic} y_{ic}}, \quad \tilde{\Omega}_{jm}^{ic} = \frac{t_{jm}^{ic} p_{jm} x_{jm}^{ic}}{p_{ic} y_{ic}},$$

where $p_{jm} x_{jm}^{ic}$ is expenditures of ic on jm not including the import tariff. Notice that every row of $\tilde{\Omega}_{jm}^{ic}$ should always sum up to 1. Also, assume that C is a set of countries, and F_c is the factors owned by Household in country c. Then,

Households: The price of final consumption in country c

$$P_{0c} = \left(\sum_i b_{ic} \left(P_i^{0c} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

where $b_{ic} = \sum_{m \in C} \tilde{\Omega}_{im}^{0c}$. The price of consumption good from industry i in country c

$$P_i^{0c} = \left(\sum_{m \in C} \delta_m^{0c} \left(t_{im}^{0c} \tau_{im}^{0c} p_{im} \right)^{1-\varepsilon_i} \right)^{\frac{1}{1-\varepsilon_i}},$$

where $\varepsilon_i + 1$ is the trade elasticity for industry i and $\delta_m^{0c} = \tilde{\Omega}_{im}^{0c} / \left(\sum_{v \in C} \tilde{\Omega}_{iv}^{0c} \right)$.

Producers: The marginal cost of good i produced by country c

$$p_{ic} = \left(\alpha_{ic} P_{w_{ic}}^{1-\theta} + (1 - \alpha_{ic}) P_{M_{ic}}^{1-\theta} \right)^{\frac{1}{1-\theta}}$$

where $\alpha_{ic} = \sum_{f \in F_c} \tilde{\Omega}_{fc}^{ic}$. The price of value-added bundled used by producer i in country c

$$p_{w_{ic}} = \left(\sum_{f \in F_c} \alpha_f^{ic} w_{fc}^{1-\gamma} \right)^{\frac{1}{1-\gamma}},$$

where $\alpha_f^{ic} = \tilde{\Omega}_{fc}^{ic} / \left(\sum_{d \in F_c} \tilde{\Omega}_{dc}^{ic} \right)$. The price of intermediate bundle used by producer i in country c

$$p_{M_{ic}} = \left(\sum_j \omega_j^{ic} \left(q_j^{ic} \right)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}},$$

where $\omega_j^{ic} = \left(\sum_{m \in C} \tilde{\Omega}_{jm}^{ic} \right) / (1 - \alpha_{ic})$. The price of intermediate bundle good j used by producer i in country c

$$q_j^{ic} = \left(\sum_{m \in C} \delta_{jm}^{ic} \left(\tau_{jm}^{ic} t_{jm}^{ic} p_{jm} \right)^{1-\varepsilon_i} \right)^{\frac{1}{1-\varepsilon_i}},$$

where $\varepsilon_i + 1$ is the trade elasticity for good i and $\delta_{jm}^{ic} = \tilde{\Omega}_{jm}^{ic} / \left(\sum_{v \in C} \tilde{\Omega}_{iv}^{ic} \right)$.

Deriving Allen-Uzawa elasticities for nested-CES models is straightforward. To do so, we proceed as follows:

Derivation:

(1) $\theta_{0c}(ic', jm)$ Household demand in country c for good i from c' is

$$x_{ic'}^{0c} = \tilde{\Omega}_{ic'}^{0c} \left(\frac{p_{ic'}^{0c}}{P_i^{0c}}\right)^{-\varepsilon_i} \left(\frac{P_i^{0c}}{P^{0c}}\right)^{-\sigma} C_c$$

Hence

$$\theta_{0c}(ic', jm) = \frac{1}{\tilde{\Omega}_{jm}^{0c}} \frac{\partial \log x_{ic'}^{0c}}{\partial \log p_{jm}^{0c}} = -\varepsilon_i \frac{\left(\mathbf{1}(jm = ic') - \mathbf{1}(j = i)\delta_{jm}^{0c}\right)}{\tilde{\Omega}_{jm}^{0c}} - \frac{\sigma\left(\mathbf{1}(j = i)\delta_{jm}^{0c} - \tilde{\Omega}_{jm}^{0c}\right)}{\tilde{\Omega}_{jm}^{0c}}.$$

This can be simplified as

$$\theta_{0c}(ic', jm) = \frac{\varepsilon_i}{\sum_{v \in C} \tilde{\Omega}_{iv}^{0c}} + \sigma\left(1 - \frac{1}{\sum_{v \in C} \tilde{\Omega}_{iv}^{0c}}\right) = \frac{\varepsilon_i}{b_{ic}} + \sigma\left(1 - \frac{1}{b_{ic}}\right) \text{ when } i = j \ \& \ ic' \neq jm,$$

$$\theta_{0c}(ic', jm) = -\frac{\varepsilon_i}{\tilde{\Omega}_{jm}^{0c}} + \frac{\theta_i}{b_{ic}} + \sigma\left(1 - \frac{1}{b_{ic}}\right) \text{ when } ic' = jm.$$

Otherwise, $\theta_{0c}(ic', jm) = \sigma$.

(2) $\theta_{kc}(ic', jm)$ When k is not a household, demand by k in country c for good i from c' is

$$x_{ic'}^{kc} = \tilde{\Omega}_{ic'}^{kc} \left(\frac{p_{ic'}^{kc}}{P_i^{kc}}\right)^{-\varepsilon_i} \left(\frac{P_i^{kc}}{P_M^{kc}}\right)^{-\epsilon} \left(\frac{P_M^{kc}}{p_{kc}}\right)^{-\theta} Y_{kc}.$$

Hence

$$\theta_{kc}(ic', jm) = \frac{1}{\tilde{\Omega}_{jm}^{kc}} \frac{\partial \log x_{ic'}^{kc}}{\partial \log p_{jm}^{kc}} = -\varepsilon_i \frac{\left(\mathbf{1}(jm = ic') - \mathbf{1}(j = i)\delta_{jm}^{kc}\right)}{\tilde{\Omega}_{jm}^{kc}} - \frac{\epsilon\left(\mathbf{1}(j = i)\delta_{jm}^{kc} - \mathbf{1}(j \notin F)\delta_{jm}^{kc}\omega_j^{kc}\right)}{\tilde{\Omega}_{jm}^{kc}}$$
$$- \frac{\theta\left(\mathbf{1}(j \notin F)\delta_{jm}^{kc}\omega_j^{kc} - \tilde{\Omega}_{jm}^{kc}\right)}{\tilde{\Omega}_{jm}^{kc}}.$$

This can be simplified as

$$\theta_{kc}(ic', jm) = \frac{\varepsilon_i}{(1 - \alpha_{kc})\omega_j^{kc}} + \epsilon\left(\frac{1}{1 - \alpha_{kc}} - \frac{1}{(1 - \alpha_{kc})\omega_j^{kc}}\right)$$
$$+ \theta\left(1 - \frac{1}{1 - \alpha_{kc}}\right) \text{ when } i = j \in N \ \& \ ic' \neq jm,$$

$$\theta_{kc}(ic', jm) = -\frac{\varepsilon_i}{\tilde{\Omega}^{kc}_{jm}} + \frac{\varepsilon_i}{(1-\alpha_{kc})\omega^{kc}_j} + \epsilon\left(\frac{1}{1-\alpha_{kc}} - \frac{1}{(1-\alpha_{kc})\omega^{kc}_j}\right) + \theta\left(1 - \frac{1}{1-\alpha_{kc}}\right) \text{ when } ic' = jm,$$

$$\theta_{kc}(ic', jm) = \frac{\epsilon}{1-\alpha_{kc}} + \theta\left(1 - \frac{1}{1-\alpha_{ic}}\right) \text{ when } i \neq j \in N,$$

and when $j \in F$, $\theta_{kc}(ic', jm) = \theta$.

(3) $\theta_{kc}(fc, jm)$ Lastly, when k is not a household, demand by k in country c for factor f is

$$x^{kc}_{fc} = \tilde{\Omega}^{kc}_{fc}\left(\frac{p_{fc}}{p_{w_{kc}}}\right)^{-\gamma}\left(\frac{p_{w_{kc}}}{p^{kc}}\right)^{-\theta} Y_{kc}.$$

Hence,

$$\theta_{kc}(fc, jm) = \frac{1}{\tilde{\Omega}^{kc}_{jm}}\frac{\partial \log x^{kc}_{fc}}{\partial \log p^{kc}_{jm}} = -\gamma\frac{\left(\mathbf{1}(jm = fc) - \mathbf{1}(jm \in F_c)\alpha^{ic}_j\right)}{\tilde{\Omega}^{kc}_{jm}} - \theta\frac{\left(\mathbf{1}(jm \in F_c)\alpha^{ic}_j - \tilde{\Omega}^{kc}_{jm}\right)}{\tilde{\Omega}^{kc}_{jm}}.$$

Notice that $\theta_{kc}(fc, jm) = \theta$ if $j \in N$. Also,

$$\theta_{kc}(fc, jc) = \frac{\gamma}{\sum_{g\in F_c}\tilde{\Omega}^{kc}_{gc}} + \theta\left(1 - \frac{1}{\sum_{g\in F_c}\tilde{\Omega}^{kc}_{gc}}\right) = \frac{\gamma}{\alpha_{kc}} + \theta\left(1 - \frac{1}{\alpha_{kc}}\right) \text{ when } j \in F \;\&\; m = c,$$

$$\theta_{kc}(fc, jc) = -\frac{\gamma}{\tilde{\Omega}^{kc}_{fc}} + \frac{\gamma}{\alpha_{kc}} + \theta\left(1 - \frac{1}{\alpha_{kc}}\right) \text{ when } fc = jm.$$

4. Solving system of linear equations

This code takes the following inputs, forms the linear system of market clearing conditions in factor markets in Theorem 3 and computes the change in factor shares in equilibrium.

Code: Nested_CES_linear_final_rev.m

Input:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)

2. Allen-Uzawa elasticities of substitution:

   (a) $\theta_{0c}(ic', jm)$ (`AES_C_Mat`: CN by CN by C 3-D matrix)

   (b) $\theta_{kc}(ic', jm)$ (`AES_N_Mat`: CN by CN+CF by CN 3-D matrix)

(c) $\theta_{kc}(fc, jm)$ (`AES_F_Mat` CF by CN+CF by CN 3-D matrix)

3. Input-output matrix and Leontief inverse

   (a) $\tilde{\Omega}^{ic}_{jm}$ (`Omega_total_tilde`: C+CN+CF by C+CN+CF matrix) : Standard form of Cost-based IO matrix

   (b) Ω^{ic}_{jm} (`Omega_total`: C+CN+CF by C+CN+CF matrix) : Standard form of Revenue-based IO matrix

   (c) $\tilde{\Psi}^{ic}_{jm}$ (`Psi_total_tilde`) : Leontief inverse of $\tilde{\Omega}^{ic}_{jm}$

   (d) Ψ^{ic}_{jm} (`Psi_total`) : Leontief inverse of Ω^{ic}_{jm}

4. Initial sales share λ_{CN} (`lambda_CN`: C+CN by 1 vector) and factor income Λ_F (`lambda_F`: CF by 1 vector)

5. Ownership structure of factor (`Phi_F`: C by CF matrix) and tariff revenue (`Phi_T`: C+CN by CN by C 3-D matrix) defined in **main_dlogW_org.m**

6. If factors are country-specific (4 factors per country), `factor_index`= 1. Otherwise, if factors are country-sector-specific (N factors per country), =2.

7. (Optional) If economy has initial tariff, initial tariff matrix (`init_t`: C+CN by CN matrix) defined in **main_load_data_rev.m**

Current version of code simulates universal iceberg trade cost or tariff shock. If the user wants to specify the shocks, customize

1. universal iceberg trade cost shock matrix (`dlogtau`: C+CN by CN+CF matrix, (i, j) element: log change in iceberg trade cost when household/sector i buys from sector/factor j) or

2. tariff shock matrix (`dlogt`: C+CN by CN+CF matrix, (i, j) element: log change in tariff when household/sector i buys from sector/factor j).

Output:

Let $d\Lambda_F$ be the vector of changes in the sales of primary factors and

$$d\Lambda_{F,c',*} = \sum_{ic}\sum_{jm} \Phi_{c',ic,jm}\Omega^{ic}_{jm}(t^{ic}_{jm} - 1)d\lambda_{ic}$$

be the change in wedge-revenues of household c' due to changes in sales shares, where $\Phi_{c',ic,jm}$ is the share of tax revenues on ic's purchases of jm that go to household c'. The linear system in Theorem 3 can be written as:

$$\begin{bmatrix} d\Lambda_F \\ d\Lambda_{F_*} \end{bmatrix} = A \begin{bmatrix} d\Lambda_F \\ d\Lambda_{F_*} \end{bmatrix} + B$$

This code outputs:

1. `A` (C+CF by C+CF matrix) and `B` (C+CF by 1 vector).

Using these outputs, the code inverts the system and solves for $d\Lambda_F$(`dlambda_F`) and $d\Lambda_{F_*}$ (`dlambda_F_star`), which are used to obtain derivatives calculated by **Nested_CES_linear_result_final.m**. It updates $\tilde{\Omega}$ and other variables which are used in the next iteration.

5. Calculate derivatives

This code takes the equilibrium factor market response calculated in the previous step and uses these to update all endogenous variables so that the whole process can be repeated.

Code: Nested_CES_linear_result_final.m

Input:

All inputs used in **Nested_CES_linear_final_rev.m** are also used in this code. Additionally, it requires

1. `GNE_weights` (C by 1 vector): A ratio of GNE of each country to world GNE
2. $d\Lambda_F$(`dlambda_F`) and $d\Lambda_{F^*}$(`dlambda_F_star`) : Solutions from **Nested_CES_linear_final_rev.m**

Output:

1. $d\lambda$ (`dlambda_result`: C+CN+CF by 1 vector): Change in sales shares;
2. $d\chi$ (`dchi_std`: C+CN+CF by 1 vector): Change in household income shares;
3. $dlogP$ (`dlogP_Vec`: C+CN+CF by 1 vector): Change in either the price index (household), marginal cost (sector), or factor price;
4. $d\tilde{\Omega}^{ic}_{jm}$ (`dOmega_total_tilde`: C+CN+CF by C+CN+CF matrix) : Change in Cost-based IO matrix;

5. $d\Omega^{ic}_{jm}$ (`dOmega_total`: C+CN+CF by C+CN+CF matrix) : Change in Revenue-based IO matrix.

For each iteration, change in real income of country c is

$$d \log W_c = d \log \chi_c - d \log P_c$$

where $d \log P_c$ is change in price index of household c. Meanwhile, outputs are used to update $\lambda, \chi, \Omega, \tilde{\Omega}$, which are used as a simulated data with discretized shock in next iteration.

Supplementary Materials to *Networks, Barriers, and Trade*

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E Beyond CES

In this appendix, we show how to generalize the results in the paper beyond nested-CES functional forms.

E.1 Generalizing Sections 4 and 5 and Appendix F

In a similar vein to Baqaee and Farhi (2017a), we can extend the results in Sections 4 and 5 to arbitrary neoclassical production functions simply by replacing the input-output covariance operator with the *input-output substitution operator* instead.

For a producer k with cost function $\mathbf{C}_k$, the Allen-Uzawa elasticity of substitution between inputs x and y is

$$\theta_k(x,y) = \frac{\mathbf{C}_k d^2\mathbf{C}_k/(dp_x dp_y)}{(d\mathbf{C}_k/dp_x)(d\mathbf{C}_k/dp_y)} = \frac{\epsilon_k(x,y)}{\Omega_{ky}},$$

where $\epsilon_k(x,y)$ is the elasticity of the demand by producer k for input x with respect to the price p_y of input y, and $\tilde{\Omega}_{ky}$ is the expenditure share in cost of input y. We also use this definition for final demand aggregators.

The *input-output substitution operator* for producer k is defined as

$$\begin{aligned}\Phi_k(\Psi_{(i)},\Psi_{(j)}) &= -\sum_{x,y\in N+F} \tilde{\Omega}_{kx}[\delta_{xy} + \tilde{\Omega}_{ky}(\theta_k(x,y)-1)]\Psi_{xi}\Psi_{yj},\\ &= \frac{1}{2}E_{\Omega^{(k)}}\left((\theta_k(x,y)-1)(\Psi_i(x)-\Psi_i(y))(\Psi_j(x)-\Psi_j(y))\right),\end{aligned}$$

where δ_{xy} is the Kronecker delta, $\Psi_i(x) = \Psi_{xi}$ and $\Psi_j(x) = \Psi_{xj}$, and the expectation on the second line is over x and y.

In the CES case with elasticity θ_k, all the cross Allen-Uzawa elasticities are identical with $\theta_k(x,y) = \theta_k$ if $x \neq y$, and the own Allen-Uzawa elasticities are given by $\theta_k(x,x) = -\theta_k(1-\tilde{\Omega}_{kx})/\tilde{\Omega}_{kx}$. It is easy to verify that when $\mathbf{C}_k$ has a CES form we recover the input-output covariance operator:

$$\Phi_k(\Psi_{(i)},\Psi_{(j)}) = (\theta_k - 1)Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)},\Psi_{(j)}).$$

Even outside the CES case, the input-output substitution operator shares many properties with the input-output covariance operator. For example, it is immediate to verify, that: $\Phi_k(\Psi_{(i)},\Psi_{(j)})$ is bilinear in $\Psi_{(i)}$ and $\Psi_{(j)}$; $\Phi_k(\Psi_{(i)},\Psi_{(j)})$ is symmetric in $\Psi_{(i)}$ and $\Psi_{(j)}$; and $\Phi_k(\Psi_{(i)},\Psi_{(j)}) = 0$ whenever $\Psi_{(i)}$ or $\Psi_{(j)}$ is a constant.

All the structural results in the paper can be extended to general non-CES economies by simply replacing terms of the form $(\theta_k - 1)Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)}, \Psi_{(j)})$ by $\Phi_k(\Psi_{(i)}, \Psi_{(j)})$.

For example, when generalized beyond nested CES functional forms, Theorem 3 becomes the following.

Theorem 11. For a vector of perturbations to productivity $\mathrm{d}\log A$ and wedges $\mathrm{d}\log \mu$, the change in the price of a good or factor $i \in N + F$ is the same as (7). The change in the sales share of a good or factor $i \in N + F$ is

$$\mathrm{d}\log \lambda_i = \sum_{k \in N+F} \left(\mathbf{1}_{\{i=k\}} - \frac{\lambda_k}{\lambda_i} \Psi_{ki} \right) \mathrm{d}\log \mu_k + \sum_{k \in N} \frac{\lambda_k}{\lambda_i} \mu_k^{-1} \Phi_k(\Psi_{(i)}, \mathrm{d}\log p) + \sum_{g \in F^*} \sum_{c \in C} \frac{\lambda_i^{W_c} - \lambda_i}{\lambda_i} \Phi_{cg} \Lambda_g \,\mathrm{d}\log \Lambda_g,$$

where $\mathrm{d}\log p$ is the $(N + F) \times 1$ vector of price changes in (7). The change in wedge income accruing to household c (represented by a fictitious factor) is the same as (9).

F Differential Exact-Hat Algebra

We can conduct global comparative statics by viewing Theorem 3 as a system of differential equations that can be solved by iterative means (e.g. Euler's method or Runge-Kutta). The endogenous terms in Equations (7) and (8) depend only on HAIO and Leontief matrices $(\tilde{\Omega}, \Omega, \tilde{\Psi}, \Psi)$. However, a similar logic to (8) can be used to derive changes in these matrices. In particular, the change in the HAIO matrix $\tilde{\Omega}$ is

$$d\tilde{\Omega}_{ij} = (1 - \theta_i) Cov_{\tilde{\Omega}^{(i)}} \left(d\log p, I_{(j)} \right),$$

where $I_{(j)}$ is the jth column of the identity matrix. The change in the Leontief inverse is

$$d\tilde{\Psi}_{ij} = \sum_{k \in N} \tilde{\Psi}_{ik} (1 - \theta_i) Cov_{\tilde{\Omega}^{(k)}} \left(d\log p, \tilde{\Psi}_{(j)} \right).$$

Similarly, changes in Ω are

$$d\Omega_{ij} = \mu_i^{-1} d\tilde{\Omega}_{ij} - d\log \mu_i$$

and changes in Ψ are

$$d\Psi_{ij} = \sum_{k \in N} \Psi_{ik} \mu_k^{-1} (1 - \theta_k) Cov_{\tilde{\Omega}^{(k)}} (d\log p, \Psi_{(j)}) - \sum_{k} \Psi_{ik} (\Psi_{kj} - \mathbf{1}_{\{k=j\}}) d\log \mu_k.$$

As explained in Appendix D, this means that we can conduct global comparative statics by repeatedly solving a $(C+F) \times (C+F)$ linear system and cumulating the results, instead of solving a system of $(C+N+F) \times (C+N+F)$ nonlinear equations. A similar approach is sometimes used in the CGE literature, for example Dixon et al. (1982), to solve high-dimensional models because exact-hat algebra is computationally impracticable for large models.[1] For the quantitative model in Section 7, the differential approach is faster than using state-of-the-art nonlinear solvers to perform exact hat-algebra (see Appendix G)

G Numerical Accuracy and Efficiency

We provide flexible Matlab code, detailed in Appendix D, that loglinearizes arbitrary general equilibrium models of the type studied in this paper and computes local comparative statics. In this section, we investigate the accuracy and computational efficiency of this approach.

Accuracy of Loglinearization. Figure 5 displays the numerical accuracy of the first-order approximation for universal iceberg and tariff shocks of different sizes. Note that this Figure 5 is not relevant for differential exact-hat algebra (as performed in Section 7) because once we iterate on the first-order approximation, it becomes exact. The left and right panels show the root-mean-squared-error in log welfare, using the benchmark model, using dollar-weighting and country-weighting. As expected, the error is larger for bigger shocks, and the dollar-weighted error is smaller since nonlinearities are smaller for larger countries and less open countries.

Computational Efficiency. By repeated iteration on the loglinear solution, the code can also compute exact nonlinear responses to shocks. We refer to this way of solving the nonlinear model as "differential exact hat-algebra." We compare the computational efficiency of differential exact hat-algebra with exact hat-algebra using Matlab's built-in *fmincon* nonlinear solver as well as a state-of-the-art industrial numerical solver Artelys *Knitro*. We provide the nonlinear solvers with analytical expressions of the Jacobian, which significantly boosts their performance. Figure 6 shows how long each solver takes to solve the model for a 60% universal increase in iceberg shocks using the benchmark elasticities. On the x-axis we vary the number of variables by varying the number of countries in descending order of country GDP. For example, when there are two countries, we only have the US and an

[1] In the CGE literature, supply and demand relationships are log-linearized and then integrated numerically by Euler's method.

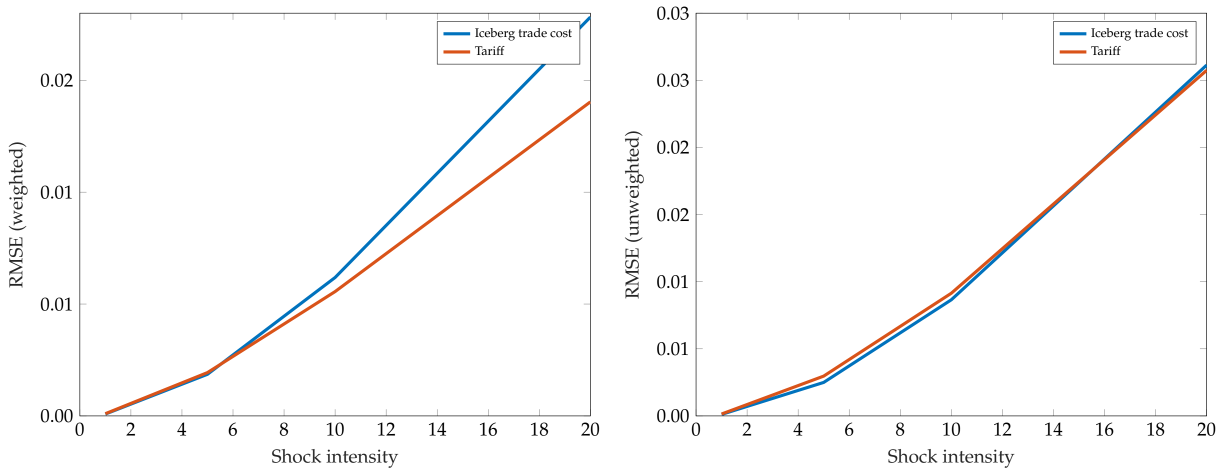


Figure 5: Error of the first-order approximation for a universal shock to trade barriers.

aggregate composite "rest-of-the-world" country.[2] We increase the number of variables by disaggregating the rest-of-the-world. Figure 6 shows that differential exact hat-algebra is much faster than *fmincon* and even *Knitro*, especially as the number of countries increases.[3]

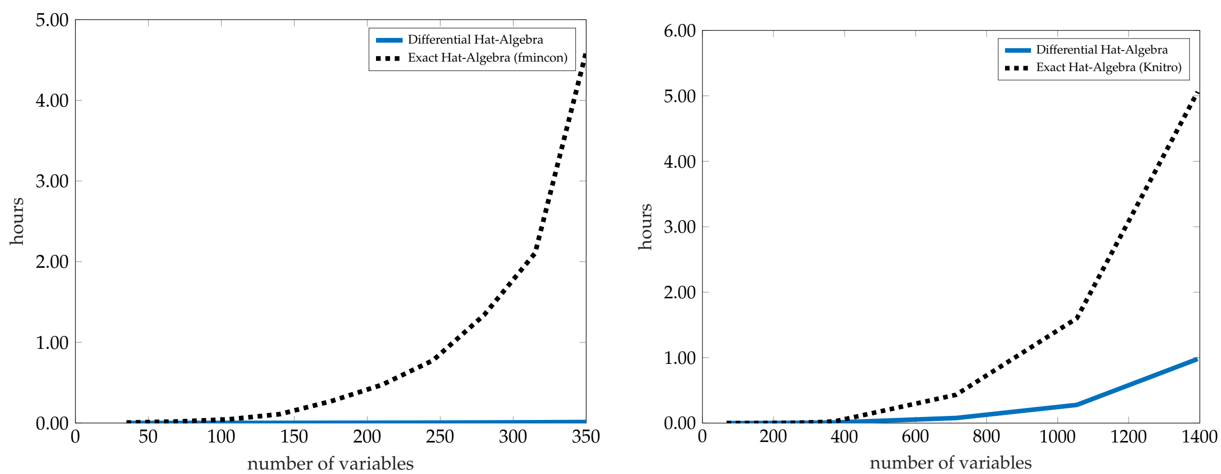


Figure 6: Time taken to solve the nonlinear model for a 60% universal iceberg shock as a function of the number of countries.

An additional virtue of the differential exact-hat algebra over standard exact-hat algebra is that when the model becomes highly nonlinear, for example when intersectoral elasticities of susbtitution are close to zero, nonlinear solvers take longer and when the domestic elasticities of substitution (θ_0, θ_1, θ_2, θ_3, θ_4) are lowered to below 0.2, *fmincon* and *Knitro* fail to find a solution at all. On the other hand, differential exact hat-algebra always

[2]Each additional country increases the number of variables by 34 — four factor and thirty goods prices.

[3]For example, the computer we used cannot solve the factor-specific version of the model using exact-hat algebra due to insufficient memory.

works regardless of the elasticities of substitution. This is particularly useful for large-scale applications where strong complementarities are important. For example, this algorithm is used by Bachmann et al. (2022) to study how an embargo of Russian goods would affect Germany in the short-run. This application would not have been numerically feasible using the nonlinear solvers mentioned here.

H Factor Demand System

Adao et al. (2017) show that trading economies can be represented as if only factors are traded within and across borders, and households have preferences over factors directly. Theorem 3 can be used to flesh out this representation by locally characterizing its associated reduced-form Marshallian demand for factors in terms of sufficient-statistic microeconomic primitives. For example, in the absence of wedges, the expenditure share of household c on factor f under the "trade-in-factors" representation is given by Ψ_{cf}; the elasticities $\partial \log \Psi_{cf} / \partial \log A_i$ holding factor prices constant then characterize its Marshallian price elasticities as well as its Marshallian elasticities with respect to iceberg trade shocks:

$$\frac{\partial \log \Psi_{cf}}{\partial \log A_i} = \sum_{k \in N} \frac{\Psi_{ck}}{\Psi_{cf}} (\theta_k - 1) Cov_{\Omega^{(k)}}(\Psi_{(f)}, \Psi_{(i)}).$$

Similarly, by Theorem 3, we know that the elasticity of the factor income share of some factor j with respect to the price of another factor i, holding fixed all other factor prices, is given by

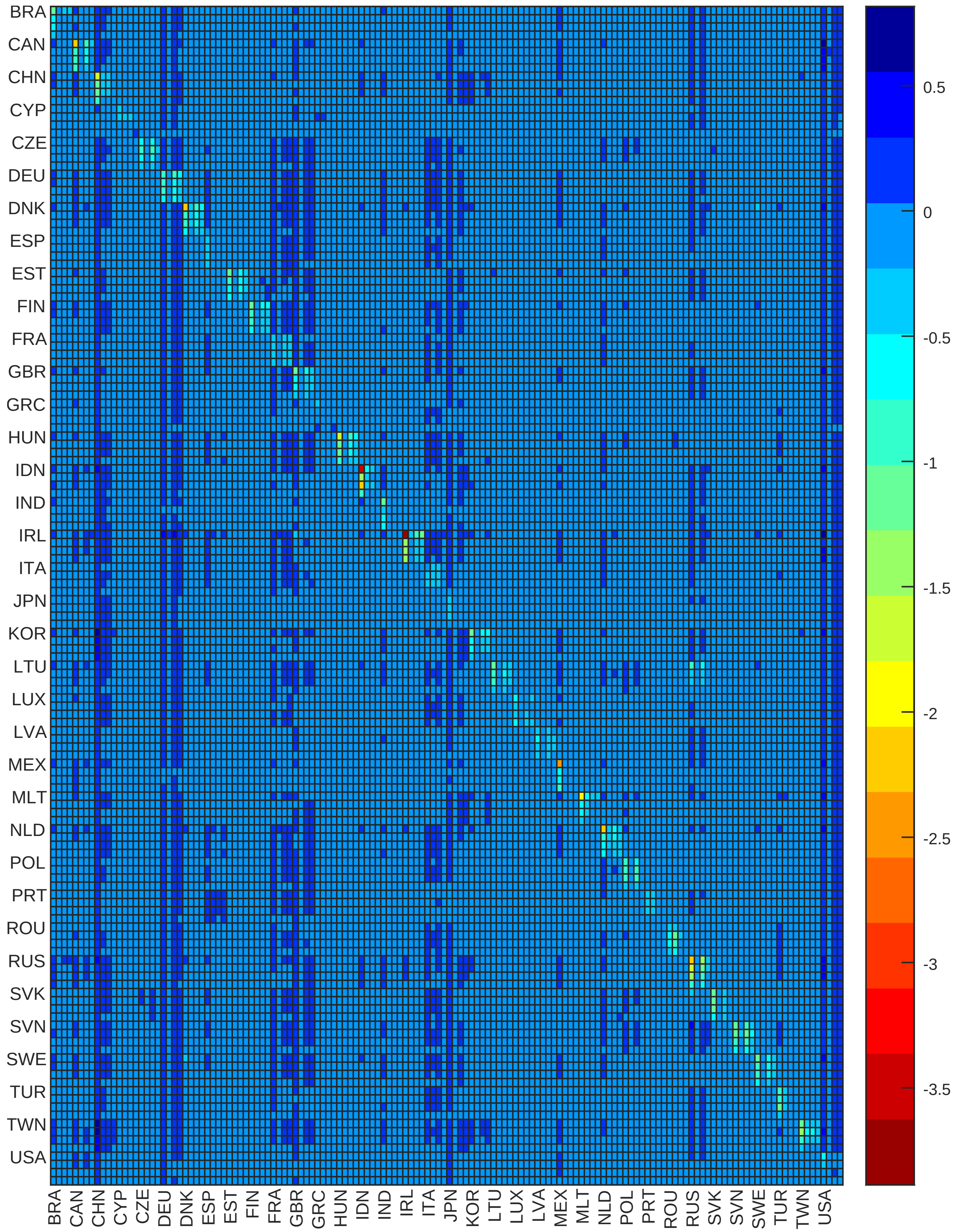
$$\frac{\partial \log \Lambda_j}{\partial \log w_i} = \sum_{k \in N} (1 - \theta_k) \frac{\lambda_k}{\Lambda_j} Cov_{\Omega^{(k)}}(\Psi_{(i)}, \Psi_{(j)}) + \sum_{c \in C} (\Lambda_j^{W_c} / \Lambda_j - 1) \Phi_{ci} \Lambda_i, \tag{21}$$

recalling that for factors $f \in F$, we interchangeably write Λ_f or λ_f to refer to their Domar weight. Figure 7 illustrates these elasticities of the factor demand system for a selection of the countries using the benchmark calibration. The ijth element gives the elasticity of j's world income share with respect to the price of i (holding fixed all other factor prices). Each country has four factors: capital, low, medium, and high skilled labor. Some interesting patterns emerge:

1. There are dark blue columns corresponding to factors in major countries like China, Germany, Britain, Japan, and the USA. For these factors, an increase in their price strongly raises the share of world income going to the rest (low-skilled labor in these countries does not obey this pattern).

2. There is a block-diagonal structure where an increase in domestic capital prices lowers both domestic labor and capital income shares. On the other hand, an increase in labor prices often raises domestic labor income and lowers domestic capital's share of world income. This is despite the fact that at the micro-level, the elasticity of substitution among domestic factors is symmetric.

Figure 7: The international factor demand system for a selection of countries



The ijth element is the elasticity of factor j with respect to the price of factor i, holding fixed other factor prices, given by equation (21).

I Aggregation and Stability of the Trade Elasticity

In this section, we characterize trade elasticities at different levels of aggregation in terms of microeconomic primitives. We also prove necessary and sufficient conditions for ensuring that the trade elasticity is constant and stable. We also relate the instability of the trade elasticity to the Cambridge Capital controversy — a mathematically similar issue that arose in capital theory in the middle of the 20th century.

I.1 Aggregating and Disaggregating Trade Elasticities

We start by defining a class of aggregate elasticities. Consider two sets of producers I and J. Let $\lambda_I = \sum_{i \in I} \lambda_i$ and $\lambda_J = \sum_{j \in J}$ be the aggregate sales shares of producers in I and J, and let $\chi_i^I = \lambda_i / \lambda_I$ and $\chi_j^J = \lambda_j / \lambda_J$. Let k be another producer. We then define the following aggregate elasticities capturing the bias towards I vs. J of a productivity shock to m as:

$$\varepsilon_{IJ,m} = \frac{\partial (\lambda_I / \lambda_J)}{\partial \log A_m},$$

where the partial derivative indicates that we allow for this elasticity to be computed holding some things constant.

To shed light on trade elasticities, we proceed as follows. Consider a set of producers $S \subseteq N_c$ in a country c. Let J be denote a set of domestic producers that sell to producers in S, and I denote a set of foreign producers that sell to producers in S. Without loss of generality, using the flexibility of network relabeling, we assume that producers in I and J are specialized in selling to producers in S so that they do not sell to producers outside of S.

Consider an iceberg trade cost modeled as a negative productivity shock $\mathrm{d}\log(1/A_m)$ to some producer m. We then define the trade elasticity as $\varepsilon_{IJ,k} = \partial(\lambda_J / \lambda_I) / \partial \log(1/A_m) = \partial(\lambda_I / \lambda_J) / \partial \log A_m$. As already mentioned, the partial derivative indicates that we allow for this elasticity to be computed holding some things constant. There are therefore different trade elasticities, depending on exactly what is held constant. Different versions of trade elasticities would be picked up by different versions of gravity equations regressions with different sorts of fixed effects and at different levels of aggregation.

There are several possibilities for what to hold constant, ranging from the most partial equilibrium to the most general equilibrium. At one extreme, we can hold constant the prices of all inputs for all the producers in I and J and the relative sales shares of all the producers in S:

$$\varepsilon_{IJ,m} = \sum_{s\in S}\sum_{i\in I}\chi_i^I(\theta_s - 1)\frac{\lambda_s}{\lambda_i}Cov_{\Omega^{(s)}}(I_{(i)}, \Omega_{(m)}) - \sum_{s\in S}\sum_{j\in J}\chi_j^J(\theta_s - 1)\frac{\lambda_s}{\lambda_j}Cov_{\Omega^{(s)}}(I_{(j)}, \Omega_{(m)}), \quad (22)$$

where $I_{(i)}$ and $I_{(j)}$ are the ith and jth columns of the identity matrix. An intermediate possibility is to hold constant the wages of all the factors in all countries:

$$\varepsilon_{IJ,k} = \sum_{i\in I}\chi_i^I\Gamma_{ik} - \sum_{j\in J}\chi_j^J\Gamma_{jk},$$

where

$$\Gamma_{ik} = \sum_{s\in C+N}(\theta_k - 1)\frac{\lambda_s}{\Lambda_f}Cov_{\Omega^{(s)}}\left(\Psi_{(i)}, \Psi_{(k)}\right),$$

And at the other extreme, we can compute the full general equilibrium:

$$\varepsilon_{IJ,m} = \sum_{i\in I}\chi_i^I\left(\Gamma_{im} - \sum_{g\in F}\Gamma_{ig}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m} + \sum_{g\in F}\Xi_{ig}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m}\right) \\ - \sum_{j\in J}\chi_j^J\left(\Gamma_{jm} - \sum_{g\in F}\Gamma_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m} + \sum_{g\in F}\Xi_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m}\right),$$

$\mathrm{d}\log\Lambda_f / \mathrm{d}\log A_m$ is given in Theorem 3 and

$$\Xi_{jg} = \sum_{c\in C}\frac{(\lambda_j^{W_c} - \lambda_j)}{\lambda_j}\Phi_{cg}\Lambda_g.$$

The trade elasticity is a linear combination of microeconomic elasticities of substitution, where the weights depend on the input-output structure. Except at the most microeconomic level where there is a single producer s in S and in the most partial-equilibrium setting where we recover $\epsilon_s - 1$, this means that the aggregate trade elasticity is typically an endogenous object, since the input-output structure is itself endogenous.[4] Furthermore, in the presence of input-output linkages, it is typically nonzero even for trade shocks that are not directly affecting the sales of I to J, except in the most partial-equilibrium setting.

[4]In Appendix I.3, we provide necessary and sufficient conditions for the trade elasticity to be constant in the way.

Example: Trade Elasticity in a Round-About World Economy

In many trade models, the trade elasticity, defined holding factor wages constant, is an invariant structural parameter. As pointed out by Yi (2003), in models with intermediate inputs, the trade elasticity can easily become an endogenous object. Consider the two-country, two-good economy depicted in Figure 1. The representative household in each country only consumes the domestic good, which is produced using domestic labor and imports with a CES production function with elasticity of substitution θ. We consider the imposition of a trade cost hitting imports by country 1 from country 2. For the sake of illustration, we assume that the trade cost does not apply to the exports of country 1 to country 2.

The trade elasticity holding factor wages and foreign input prices constant is a constant structural parameter, and given simply by

$$\theta - 1.$$

However, echoing our discussion above, the trade elasticity holding factor wages constant is different, and is given by

$$\frac{\theta - 1}{1 - \Omega_{21}\Omega_{12}},$$

where Ω_{ij} is the expenditure share of i on j, e.g. its intermediate input import share. As the intermediate input shares increase, the trade elasticity becomes larger. Simple trade models without intermediate goods are incapable of generating these kinds of patterns.

Of course, since the intermediate input shares Ω_{ij} are themselves endogenous (depending on the iceberg shock), this means that the trade elasticity varies with the iceberg shocks. In particular, if $\theta > 1$, then the trade elasticity increases (nonlinearly) as iceberg costs on imports fall in all countries since intermediate input shares rise. [5]

I.2 Necessary and Sufficient Conditions for Constant Trade Elasticity

In this section, we study conditions under which the trade elasticity (*holding fixed factor prices*) is constant. This trade elasticity between i and j with respect to shocks to k is defined as

$$\varepsilon_{ij,k} = \frac{\partial(\lambda_i/\lambda_j)}{\partial \log A_k},$$

[5] In Appendix I.3, we show that there it is possible to generate "trade re-switching" examples where the trade elasticity is non-monotonic with the trade cost (or even has the "wrong" sign) in otherwise perfectly respectable economies. These examples are analogous to the "capital re-switching" examples at the center the Cambridge Cambridge Capital controversy.

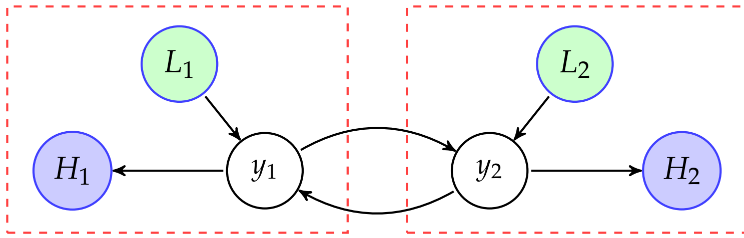


Figure 8: The solid lines show the flow of goods. Green nodes are factors, purple nodes are households, and white nodes are goods. The boundaries of each country are denoted by dashed box.

holding fixed factor prices. We say that a good m is *relevant* for $\varepsilon_{ij,k}$ if

$$\lambda_m Cov_{\Omega^{(m)}}(\Psi_{(k)}, \Psi_{(i)}/\lambda_i - \Psi_{(j)}/\lambda_j) \neq 0.$$

If m is not relevant, we say that it is irrelevant. For instance, if some producer m is exposed symmetrically to i and j through its inputs

$$\Omega_{ml}(\Psi_{li} - \Psi_{lj}) = 0 \quad (l \in N),$$

then $\varepsilon_{ij,k}$ is not a function of θ_m and m is irrelevant. Another example is if some producer $m \neq j$ is not exposed to k through its inputs

$$\Psi_{mk} = 0,$$

then $\varepsilon_{ij,k}$ is not a function of θ_m and m is irrelevant.

Corollary 6 (Constant Trade Elasticity)**.** *Consider two distinct goods i and j that are imported to some country c. Then consider the following conditions:*

(i) Both i and j are unconnected to one another in the production network: $\Psi_{ij} = \Psi_{ji} = 0$, and i is not exposed to itself $\Psi_{ii} = 1$.

(ii) The representative "world" household is irrelevant

$$Cov_{\chi}\left(\Psi_{(i)}, \frac{\Psi_{(i)}}{\lambda_i} - \frac{\Psi_{(j)}}{\lambda_j}\right) = 0,$$

which holds if both i and j are only used domestically, so that only household c is exposed to i and j. That is, $\lambda_i^{W_h} = \lambda_j^{W_h} = 0$ for all $h \neq c$. This assumption holds automatically if i and j are imports and domestic goods and there are no input-output linkages.

(iii) For every relevant producer l, the elasticity of substitution $\theta_l = \theta$.

The trade elasticity of i relative to j with respect to iceberg shocks to i is constant, and equal to

$$\varepsilon_{ij,i} = (\theta - 1).$$

if, and only if, (i)-(iii) hold.

The conditions set out in the example above, while seemingly stringent, actually represent a generalization of the conditions that hold in gravity models with constant trade elasticities. Those models oftentimes either assume away the production network, or assume that traded goods always enter via the same CES aggregator.

A noteworthy special case is when i and j are made directly from factors, without any intermediate inputs. Then, we have the following

Corollary 7. *(Network Irrelevance) If some good i and j are only made from domestic factors, then*

$$\sum_{m \in C,N} \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(i)}, \Psi_{(j)}/\lambda_i - \Psi_{(i)}/\lambda_i) = 1.$$

Hence, if all microeconomic elasticities of substitution θ_m are equal to the same value $\theta_m = \theta$ then $\varepsilon_{ij,j} = \theta$.

Suppose that i is domestic goods and j are foreign imports, both of which are made only from factors (no intermediate inputs are permitted). Then a shock to j is equivalent to an iceberg shock to transportation costs. In this case, the trade elasticity of imports j into the country producing i with respect to iceberg trade costs is a convex combination of the underlying microelasticities. Of course, whenever all micro-elasticities of substitution are the same, the weights (which have to add up to one) become irrelevant, and this is the situation in most benchmark trade models with constant trade elasticities. Specifically, this highlights the fact that having common elasticities is not enough to deliver a constant trade elasticity (holding fixed factor prices) in the presence of input-output linkages as shown in the round-about example in the previous section.

I.3 Trade Reswitching

Yi (2003) shows that the trade elasticity can be nonlinear due to vertical specialization, where the trade elasticity can increase as trade barriers are lowered. Building on this insight, we can also show that, at least in principle, the trade elasticity can even have the “wrong sign” due to these nonlinearities. This relates to a parallel set of paradoxes in capital theory.

To see how this can happen, imagine there are two ways of producing a given good: the first technique uses a domestic supply chain and the other technique uses a global value chain. Whenever the good is domestically produced, the iceberg costs of transporting the good are, at most, incurred once — when the finished good is shipped to the destination. However, when the good is made via a global value chain, the iceberg costs are incurred as many times as the good is shipped across borders. As a function of the iceberg cost parameter τ, the difference in the price of these two goods (holding factor prices fixed) is a polynomial of the form

$$B_n \tau^n - B_1 \tau, \tag{23}$$

where B_n and B_1 are some coefficients and n is the number of times the border is crossed. The nonlinearity in τ, whereby the iceberg cost's effects are compounded by crossing the border, drives the sensitivity of trade volume to trade barriers in Yi (2003). The benefits from using a global value chain are compounded if the good has to cross the border many times.

However, this discussion indicates the behavior of the trade elasticity can, in principle, be much more complicated. In fact, an interesting connection can be made between the behavior of the trade elasticity and the (closed-economy) reswitching debates of the 1950s and 60s. Specifically, equation (23) is just one special case. In general, the cost difference between producing goods using supply chains of different lengths is a polynomial in τ – and this polynomial can, in principle, have more than one root. This means that the trade elasticity can be non-monotonic as a function of the trade costs, in fact, it can even have the "wrong" sign, where the volume of trade decreases as the iceberg costs fall. This mirrors the apparent paradoxes in capital theory where the relationship between the capital stock and the return on capital can be non-monotonic, and an increase in the interest rate can cause the capital stock to increase.

To see this in the trade context, imagine two perfectly substitutable goods, one of which is produced by using 10 units of foreign labor, the other is produced by shipping 1 unit of foreign labor to the home country, back to the foreign country, and then back to the home country and combining it with 10 units of domestic labor. If we normalize both foreign and domestic wages to be unity, then the costs of producing the first good is $10(1+\tau)$, whereas the cost of producing the second good is $(1+\tau)^3 + 10$, where τ is the iceberg trade cost. When $\tau = 0$, the first good dominates and goods are only shipped once across borders. When τ is sufficiently high, the cost of crossing the border is high enough that the first good again dominates. However, when τ has an intermediate value, then it can become worthwhile to produce the second good, which causes goods to be shipped across borders many times, thereby inflating the volume of trade.

Such examples are extreme, but they illustrate the point that in the presence of input-output networks, the trade elasticity even in partial equilibrium (holding factor prices constant) can behave quite unlike any microeconomic demand elasticity, sloping upwards when, at the microeconomic level, every demand curve slopes downwards.

Non-Symmetry and Non-Triviality of Trade Elasticities

Another interesting subtlety of Equation (22) is that the aggregate trade elasticities are non-symmetric. That is, in general $\varepsilon_{ij,l} \neq \varepsilon_{ji,l}$. Furthermore, unlike the standard gravity equation, Equation (22) shows that the cross-trade elasticities are, in general, nonzero. Hence, changes in trade costs between k and l can affect the volume of trade between i and j *holding fixed* relative factor prices and incomes. This is due to the presence of global value chains, which transmit shocks in one part of the economy to another independently of the usual general equilibrium effects (which work through the price of factors).

J Partial Equilibrium Counterpart to Theorem 5

Proposition 2. *For a small open economy operating in a perfectly competitive world market, the introduction of import tariffs reduces the welfare of that country's representative household by*

$$\Delta W \approx \frac{1}{2} \sum_i \lambda_i \Delta \log y_i \Delta \log \mu_i,$$

where μ_i is the ith gross tariff (no tariff is $\mu_i = 1$), y_i is the quantity of the ith import, and λ_i is the corresponding Domar weight.

Proof. To prove this, let $e(p)W$ be the expenditure function of the household. We have $e(p)W = p \cdot q + \sum_i (\mu_i - 1) p_i y_i$. Differentiate this once to get $c \cdot \mathrm{d}\, p + e(p)\, \mathrm{d}\, W = q \cdot \mathrm{d}\, p + \mathrm{d}\, q \cdot p + \sum_i \mathrm{d}\, \mu_i p_i y_i + \sum_i (\mu_i - 1)\, \mathrm{d}(p_i y_i)$. Theorem 2 implies that this can be simplified to $e(p)\, \mathrm{d}\, W = (q - c) \cdot \mathrm{d}\, p + \sum_i \mathrm{d}\, \mu_i p_i y_i + \sum_i (\mu_i - 1)\, \mathrm{d}(p_i y_i) = \sum_i (\mu_i - 1)\, \mathrm{d}(p_i y_i)$, where the left-hand side is the equivalent variation. Now differentiate this again, and evaluate at $\mu_i = 1$ to get $\sum_i p_i\, \mathrm{d}\, y_i$. Hence the second-order Taylor approximation, at $\mu = 1$, is $\frac{1}{2} \sum_i \mathrm{d}\, \mu_i p_i\, \mathrm{d}\, y_i = \frac{1}{2} \sum_i \mathrm{d} \log \mu_i p_i y_i\, \mathrm{d} \log y_i$, and our normalization implies $p_i y_i$ is equal to its Domar weight. ■

K Extension to Roy Models

Galle et al. (2017) combine a Roy-model of labor supply with an Eaton-Kortum model of trade to study the effects of trade on different groups of workers in an economy. In this section, we show how our framework can be adapted for analyzing such models.

Suppose that H_c denotes the set of households in country c. As in Galle et al. (2017), households consume the same basket of goods, but supply labor in different ways. We assume that each household type has a fixed endowment of labor L_h, which are assigned to work in different industries according to the productivity of workers in that group and the relative wage differences offered in different industries.

As usual, let world GDP be the numeraire. Define Λ_f^h to be type h's share of income derived from earning wages f

$$\Lambda_f^h = \frac{\Phi_{hf}\Lambda_f}{\chi_h},$$

where $\chi_h = \sum_{k \in F} \Phi_{hk}\Lambda_k$. The Roy model of Galle et al. (2017) implies that

$$\frac{\chi_h}{\overline{\chi_h}} = \left(\sum_f \overline{\Lambda}_f^h \left(\frac{w_f}{\overline{w}_f} \right)^{\gamma_h} \right)^{\frac{1}{\gamma_h}} \frac{L^h}{\overline{L}^h},$$

where γ_h is the supply elasticity, variables with overlines are initial values, L^h is the stock of labor h has been endowed with (since we analyze log changes, only shocks to the endowment value are relevant). Galle et al. (2017) show that the above equations can be microfounded via a model where homogenous workers in each group type draw their ability for each job from Frechet distributions, and choose to work in the job that offers them the highest return. The Roy model generalizes the factor market, with $\gamma_h = 1$ representing the case where labor cannot be moved across markets by h. If $\gamma_h > 1$ then h can take advantage of wage differentials to redirect its labor supply and boost its income. When $\gamma \to \infty$, labor mobility implies that all wages in the economy are equalized (and the model collapses to a one-factor model).

Of course, due to the fact that factor shares Λ_f^h endogenously respond to factor prices, Theorem 3 can no longer be used to determine how these shares will change in equilibrium. Therefore, we extend those propositions here.

Proposition 3. *The response of the factor prices to a shock* $d \log A_k$ *is the solution to the following system:*

1. *Product Market Equilibrium:*

$$\Lambda_l \frac{\mathrm{d}\log \Lambda_l}{\mathrm{d}\log A_k} = \sum_{j\in\{H,N\}} \lambda_j(1-\theta_j) Cov_{\Omega^{(j)}}\left(\Psi_{(k)} + \sum_f \Psi_{(f)} \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \Psi_{(l)}\right)$$
$$+ \sum_{h\in H} (\lambda_l^{W_h} - \lambda_l)\left(\sum_{f\in F_c} \Phi_{hf}\Lambda_f \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}\right).$$

2. *Factor Market Equilibrium:*

$$\mathrm{d}\log \Lambda_f = \sum_{h\in H} E_{\Phi^{(h)}}\left[\gamma_h\left(E_{\Lambda^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \left(E_{\Lambda^{(h)}}(\mathrm{d}\log w)\right) + (\mathrm{d}\log L)\right].$$

Given this, the welfare of the hth group is

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\Lambda_s^h - \Lambda_s^{W_h}\right)\mathrm{d}\log w_s + \lambda_k^{W_h} + \mathrm{d}\log L^h.$$

The product market equilibrium conditions are exactly the same as those in Theorem 3, but now we have some additional equations from the supply-side of the factors (which are no longer endowments). Letting $\gamma_h = 1$ for every $h \in H$ recovers Theorem 3.

L Heterogenous Households Within Countries

To extend the model to allow for a set of heterogenous agents $h \in H_c$ within country $c \in C$, we proceed as follows. We denote by H the set of all households. Each household h in country c maximizes a homogenous-of-degree-one demand aggregator

$$C_h = \mathcal{W}_h(\{c_{hi}\}_{i\in N}),$$

subject to the budget constraint

$$\sum_{i\in N} p_i c_{hi} = \sum_{f\in F} \Phi_{hf} w_f L_f + T_h,$$

where c_{hi} is the quantity of the good produced by producer i and consumed by the household, p_i is the price of good i, Φ_{hf} is the fraction of factor f owned by household, w_f is the wage of factor f, and T_h is an exogenous lump-sum transfer.

We define the following country aggregates: $c_{ci} = \sum_{h\in H_c} c_{hi}$, $\Phi_{cf} = \sum_{h\in H_c} \Phi_{hf}$, and $T_c = \sum_{h\in H_c} T_h$. We also define the HAIO matrix at the household level as a $(H+N+F)\times$

$(H+N+F)$ matrix Ω and the Leontief inverse matrix as $\Psi = (I-\Omega)^{-1}$.

All the definitions in Section 2 remain the same. In addition, we introduce the corresponding household-level definitions for a household h. First, the nominal output and the nominal expenditure of the household are:

$$GDP_h = \sum_{f \in F} \Phi_{hf} w_f L_f, \quad GNE_h = \sum_{i \in N} p_i c_{hi} = \sum_{f \in F} \Phi_{hf} w_f L_f + T_h,$$

where we think of the household as a set producers intermediating the uses by the different producers of the different factor endowments of the household. Second, the changes in real output and real expenditure or welfare of the household are:

$$\mathrm{d}\log Y_h = \sum_{f \in F} \chi_f^{Y_h} \,\mathrm{d}\log L_f, \quad \mathrm{d}\log P_{Y_h} = \sum_{f \in F} \chi_f^{Y_h} \,\mathrm{d}\log w_f,$$

$$\mathrm{d}\log W_h = \sum_{i \in N} \chi_i^{W_h} \,\mathrm{d}\log c_{hi}, \quad \mathrm{d}\log P_{W_h} = \sum_{i \in N} \chi_i^{W_h} \,\mathrm{d}\log p_i,$$

with $\chi_f^{Y_h} = \Phi_{hf} w_f L_f / GDP_h$ and $\chi_i^{W_h} = p_i c_{hi} / GNE_h$. Third, the exposure to a good or factor k of the real expenditure and real output of household h is given by

$$\lambda_k^{W_h} = \sum_{i \in N} \chi_i^{W_h} \Psi_{ik}, \quad \lambda_k^{Y_h} = \sum_{f \in F} \chi_f^{Y_h} \Psi_{fk},$$

where recall that $\chi_i^{W_h} = p_i c_{hi} / GNE_h$ and $\chi_f^{Y_h} = \Phi_{hf} w_f L_f / GDP_h$. The exposure in real output to good or factor k has a direct connection to the sales of the producer:

$$\lambda_k^{Y_h} = 1_{\{k \in F\}} \frac{\Phi_{hk} p_k y_k}{GDP_h},$$

where $\lambda_k^{Y_h} = 1_{\{k \in F\}} \Phi_{hk} (GDP/GDP_h) \lambda_k$ the local Domar weight of k in household h and where $\Phi_{hk} = 0$ for $k \in N$ to capture the fact that the household endowment of the goods are zero. Fourth, the share of factor f in the income or expenditure of the household is given by

$$\Lambda_f^h = \frac{\Phi_{hf} w_f L_f}{GNE_h}.$$

The results in Section 3 go through without modification. Theorems 1 and 2 can be extended to the level of a household h by simply replacing the country index c by the household index h.

The results in Section 4 go through except the term on the second line of (8) must be replaced by

$$\sum_{h\in H} \frac{\lambda_i^{W_h} - \lambda_i}{\lambda_i} \Phi_{hf} \Lambda_f,$$

where we write λ_i and Λ_i interchangeably when $i \in F$ is a factor.

The results in Section 5 go through with the following changes. Theorem 5 goes through without modification, and extends to the household level where $\Delta \log Y_h \approx 0$. Theorem 4 goes through with some minor modifications. The world Bergson-Samuelson welfare function is now $W^{BS} = \sum_h \overline{\chi}_h^W \log W_h$, changes in world welfare are measured as $\Delta \log \delta$, where δ solves the equation $W^{BS}(\overline{W}_1, \ldots, \overline{W}_H) = W^{BS}(W_1/\delta, \ldots, W_H/\delta)$, where $\overline{W}_h$ are the values at the initial efficient equilibrium. We use a similar definition for country level welfare δ_c, and the same notation for household welfare δ_h. Changes in world welfare are given up to the second order by

$$\Delta \log \delta \approx \Delta \log W + Cov_{\chi_h^W} \left(\Delta \log \chi_h^W, \Delta \log P_{W_h} \right),$$

changes in country welfare are given up to the first order by

$$\Delta \log \delta_c \approx \Delta \log W_c \approx \Delta \log \chi_c^W - \Delta \log P_{W_c},$$

and the change in country welfare up to the first order by

$$\Delta \log \delta_h \approx \Delta \log W_h \approx \Delta \log \chi_h^W - \Delta \log P_{W_h}.$$

Theorems 6 goes through with some minor modifications. The final term on the last line must be replaced by

$$\frac{1}{2} \sum_{l\in N} \sum_{c\in H} \chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c} - \lambda_l).$$

M Growth Accounting Results

Table 2: Decomposition of real GNE growth

	GNE	GDP	ToT	Technology	Factoral ToT	Transfers
AUS	0.665	0.526	0.134	0.619	0.041	0.006
AUT	0.213	0.315	0.000	0.402	-0.087	-0.102
BEL	0.285	0.252	0.038	0.431	-0.142	-0.004
BGR	0.322	-0.217	0.354	-0.145	0.282	0.185
BRA	0.549	0.532	0.049	0.538	0.043	-0.032
CAN	0.630	0.525	0.110	0.581	0.055	-0.005
CHN	1.780	1.810	0.159	1.583	0.386	-0.188
CYP	0.322	0.275	-0.046	0.261	-0.033	0.093
CZE	0.413	0.283	0.295	0.412	0.166	-0.166
DEU	0.160	0.306	-0.013	0.428	-0.135	-0.132
DNK	0.239	0.199	0.095	0.318	-0.024	-0.056
ESP	0.330	0.280	0.003	0.346	-0.063	0.047
EST	0.793	0.125	0.661	0.351	0.435	0.008
FIN	0.347	0.432	-0.121	0.386	-0.075	0.037
FRA	0.317	0.358	-0.079	0.374	-0.095	0.038
GBR	0.437	0.358	0.058	0.465	-0.049	0.021
GRC	0.165	0.130	-0.027	0.110	-0.006	0.062
HUN	0.326	0.278	0.141	0.308	0.111	-0.092
IDN	0.633	0.660	-0.006	0.684	-0.030	-0.020
IND	1.236	1.264	-0.043	1.169	0.053	0.015
IRL	0.503	0.575	0.290	0.482	0.383	-0.361
ITA	0.072	-0.008	0.082	0.182	-0.108	-0.002
JPN	0.034	0.104	-0.102	0.187	-0.185	0.032
KOR	0.590	0.834	-0.149	0.739	-0.054	-0.094
LTU	0.739	0.515	0.187	0.423	0.278	0.038
LUX	0.605	0.162	0.979	0.581	0.561	-0.537
LVA	0.728	0.095	0.404	0.263	0.235	0.230
MEX	0.640	0.526	0.090	0.537	0.079	0.023
MLT	0.432	0.464	0.124	0.345	0.243	-0.156
NLD	0.249	0.374	0.009	0.495	-0.112	-0.134
POL	0.746	0.779	-0.039	0.638	0.101	0.006
PRT	0.096	0.041	0.040	0.131	-0.051	0.016
ROU	0.698	0.397	0.189	0.277	0.308	0.112
RUS	0.721	0.583	0.315	0.632	0.267	-0.178
SVK	0.690	0.557	0.196	0.403	0.349	-0.063
SVN	0.339	0.391	0.015	0.398	0.009	-0.067
SWE	0.360	0.413	-0.014	0.443	-0.045	-0.039
TUR	0.849	0.986	-0.232	0.794	-0.040	0.096
TWN	0.502	1.066	-0.410	0.727	-0.070	-0.155
USA	0.431	0.391	-0.007	0.431	-0.046	0.047
ROW	0.753	0.655	0.084	0.639	0.101	0.014

The sample is 1996-2014. Each row decomposes the cumulative log change in real GNE for each country. The first decomposition follows (6). Columns 2, 3 and 6 sum to column 1. The second decomposition follows (5). Columns 4, 5, and 6 sum to column 1.