1 Exchange

Exercise 1.1

(a) The contrapositive of $(*)$ is $(**)$: "if $x \notin T$, then $x \notin S$". $\mathbf{Z} \subset \mathbf{R}$ is verified (i) using $(*)$ since if x is an integer, then it is real and (ii) using $(**)$ since if x is not real, then it is not an integer.

(b) The converse of $(*)$ is the assertion "if $x \in T$, then $x \in S$." The converse of the assertion demonstrated in (a) is "if x is real, then x is an integer" which is false.

Exercise 1.2

Clearly the right hand side is a subset of the left since $0 \in [-\alpha, \alpha]$ for all $\alpha \geq 0$. To prove that the left hand side is a subset of the right, we establish the contrapositive:

$$\text{if} \quad x \neq 0, \quad \text{then} \quad x \notin \bigcap_{\alpha \in \mathbf{R}_+} [-\alpha, \alpha].$$

But if $x \neq 0$, then there exists an $\alpha' > 0$ such that $x \notin [-\alpha', \alpha']$.

Exercise 1.3

(a) Bijective.

(b) Neither injective nor surjective, hence not bijective. $f\colon \{2\} \to \{2\}$ is bijective with inverse $f^{-1}\colon \{2\} \to \{2\}$ mapping $2 \mapsto 2$.

(c) Surjective but not injective, hence not bijective (*Hint:* Investigate the behavior of the derivative of f). $f\colon [1, \infty) \to [0, \infty)$ is bijective with inverse the solution of $y = x^3 - x$ for x as a function of y.

(d) Injective but not surjective, hence not bijective. $f\colon [0, 1] \to [0, .5]$ is bijective with inverse $f^{-1}\colon [0, .5] \to [0, 1]$ mapping $y \mapsto y/(1 - y)$.

Verification that $f \circ f^{-1} = \text{id}_Y$ and $f^{-1} \circ f = \text{id}_X$ is left to the reader.

Exercise 1.4

(a) A line passing through the origin and the point $(0, 1, 1)$.

(b) A plane passing through the origin and the two points $(0,1,1)$ and $(1,0,1)$.

(c) $\mathbf{R}^3$.

Exercise 1.5

See Figure 1.

Exercise 1.6

Let f and g be vectors in L': i.e., linear functionals on L. Then $f+g \in L'$ since $f+g$ is also a linear functional on L as we now verify:

(i) For any $x, x' \in L$,

$$\begin{aligned}(f+g)(x+x') &= f(x+x') + g(x+x') \\ &= f(x) + f(x') + g(x) + g(x') \\ &= (f+g)(x) + (f+g)(x')\end{aligned}$$

(ii) For any $x \in L$ and for any $\alpha \in \mathbf{R}$,

$$\begin{aligned}(f+g)(\alpha x) &= f(\alpha x) + g(\alpha x) \\ &= \alpha f(x) + \alpha g(x) \\ &= \alpha(f+g)(x)\end{aligned}$$

Since $f(x)+g(x) = g(x)+f(x)$, the vectors $f+g$ and $g+f$ represent the same linear functional on L. Hence, $f+g = g+f$. Similarly, we conclude that $f+(g+h) = (f+g)+h$ for any vectors $f, g, h \in L'$. The 0 vector in L' is the function $0\colon L \to \mathbf{R}$ which everywhere equals zero. It is easy to check that this is indeed a **linear** functional, and clearly $f+0 = f$ for every $f \in L'$. To each $f \in L'$, we define $-f$ to be the functional mapping $x \mapsto -f(x)$. Again, it is easy to verify that this functional is linear and that $f+(-f) = 0$ for every $f \in L'$.

For any scalar $\alpha \in \mathbf{R}$ and vector $f \in L'$, the function $\alpha f\colon L \to \mathbf{R}$ which maps $x \mapsto \alpha f(x)$ is a linear functional and hence belongs to L':

(i) For any $x, x' \in L$,

$$\begin{aligned}(\alpha f)(x+x') &= \alpha[f(x+x')] \\ &= (\alpha f)(x) + (\alpha f)(x')\end{aligned}$$

(ii) For any $\beta \in \mathbf{R}$ and for any $x \in L$,

$$\begin{aligned}(\alpha f)(\beta x) &= \alpha[f(\beta x)] \\ &= \beta(\alpha f)(x)\end{aligned}$$

It is then easy to verify for any $f, g \in L'$ and for any $\alpha, \beta \in \mathbf{R}$ that

$$\begin{aligned}1f &= f \\ \alpha(f+g) &= \alpha f + \alpha g \\ (\alpha+\beta)f &= \alpha f + \beta f\end{aligned}$$

by verifying that the left and right hand sides of each expression define the same linear functional on L.

Exercise 1.7

Suppose that for some $x \in M$ we have $x = m_1 + m_2 = m_1' + m_2'$ where $m_1, m_1' \in M_1$ and $m_2, m_2' \in M_2$. Then $m_1 - m_1' = m_2' - m_2 := y \in M_1 \cap M_2$ and, hence, $y = 0$. Therefore, $m_1 = m_1'$ and $m_2 = m_2'$.

Exercise 1.8

(a) Since $L = M_1 \oplus \{\, y \in L \mid y = \lambda z,\ \lambda \in \mathbf{R}\,\}$, we can express $x = m_1(x) + m_2(x) = m_1(x) + p(x)z$ uniquely by the definition of the direct sum.

(b) By definition of the direct sum, any vectors $x, y \in L$ have unique representations of the form $x = m_1(x) + p(x)z$ and $y = m_1(y) + p(y)z$. Therefore, for any $\alpha, \beta \in \mathbf{R}$,

$$\alpha x + \beta y = \alpha m_1(x) + \beta m_1(y) + (\alpha p(x) + \beta p(y))z. \qquad (*)$$

But we also have the representation

$$\alpha x + \beta y = m_1(\alpha x + \beta y) + p(\alpha x + \beta y)z. \qquad (**)$$

By uniqueness of the direct sum representation, $(*)$ and $(**)$ must be equivalent, implying that $p(\alpha x + \beta y) = \alpha p(x) + \beta p(y)$. Finally, note that $H(p, 0) := p^{-1}(\{0\}) = M_1$.

Exercise 1.9

Clearly M_1 and M_2 are linear subspaces, and M_2 has dimension one. We first establish that $M_1 + M_2 = L$. For any $x \in L$, define $\alpha(x) = p \cdot x \,/\, p \cdot z$. Then $m_1(x) := x - \alpha(x)z \in M_1$, $\alpha(x)z \in M_2$, and so $x = (x - \alpha(x)z) + \alpha(x)z \in M_1 + M_2$. Next we show that $M_1 \cap M_2 = \{0\}$. Suppose that $x \in M_1 \cap M_2$. Then $x \in M_1$ implies that $p \cdot x = 0$ and $x \in M_2$ implies that $x = \lambda z$ for some $\lambda \in \mathbf{R}$. Therefore, $p(\lambda z) = \lambda\, p \cdot z = p \cdot x = 0$. Since $p \cdot z \neq 0$, we conclude that $\lambda = 0$ and, hence, that $x = 0$.

Exercise 1.10

$X_i = \mathbf{R}_+ \times \mathbf{Z}_+$.

Exercise 1.11 $A + B = \{\,(1,1),(2,3),(0,-1),(1,1)\,\}$, $2A = \{\,(2,2),(4,6)\,\}$, $2A + B = \{\,(2,2),(4,6),(1,0),(3,4)\,\}$.

Exercise 1.12

If $A = \{\,(0,0),(1,0),(0,1)\,\}$, then $2A = \{\,(0,0),(2,0),(0,2)\,\}$ but $A+A = \{\,(0,0),(1,0),(0,1),(2,0),(0,2),(1,1)\,\}$.

Exercise 1.13

If $(x,p) \in \mathrm{WE}(\mathcal{E})$, then $\Delta_w P_i(x_i) \subset H_o^+(p,0)$ for all $i \in I$. Assume that $z \in \sum_{i \in S} \Delta_w P_i(x_i)$ for some $S \subset I$; i.e., $z = \sum_{i\in S} y_i$ where $y_i \in \Delta_w P_i(x_i)$ for all $i \in S$. Because $x \in \mathrm{WE}(\mathcal{E})$ and $y_i \in \Delta_w P_i(x_i)$ for all $i \in S$, $p \cdot y_i > 0$ for all $i \in S$. Therefore, $\sum_{i\in S} p \cdot y_i > 0$ and, hence, $z \neq 0$. Since z can be any arbitrary element of $\sum_{i\in S} \Delta_w P_i(x_i)$, we conclude that $0 \notin \sum_{i\in S} \Delta_w P_i(x_i)$ for all $S \subset I$. By Lemma 1.11, this implies that $x \in \mathrm{C}(\mathcal{E})$.

Exercise 1.14

(a) See Figure 2.

(b) See Figure 3.

Exercise 1.15

See Figure 4.

Exercise 1.16

(a) Set the marginal rate of substitution for consumer i equal to the ratio of prices,

$$\mathrm{MRS}_i(x_i) = \frac{4 - x_{i2}}{-x_{i1}} = \frac{p_1}{p_2},$$

and substitute into the budget equation, $p_1 x_{i1} + p_2 x_{i2} = p \cdot w_i$, to obtain the demand equations.

(b) Equate the marginal rates of substitution for the two consumers,

$$\frac{4 - x_{12}}{-x_{11}} = \frac{4 - x_{22}}{-x_{21}},$$

and combine with the feasibility constraints,

$$x_{11} + x_{21} = 5 \quad \text{and} \quad x_{12} + x_{22} = 3,$$

to obtain $x_{11} + x_{12} = 4$. The Pareto optimal set also includes the allocations on the boundary of the Edgeworth box for which either

$$x_1 = (c, 3) \quad \text{and} \quad x_2 = (5 - c, 0) \quad \text{for} \quad c \in [0, 1]$$

or

$$x_1 = (0, 4 + c) \quad \text{and} \quad x_2 = (3, 1 - c) \quad \text{for} \quad c \in [0, 1].$$

(c) To solve for the endpoints of the core, set $u_1(x_1) = u_1(w_1)$ to obtain $x_{11}(4 - x_{12}) = 4$ and use the fact that Pareto optimality requires $4 - x_{12} = x_{11}$ to conclude that $x_1 = (2, 2)$ is one of the endpoints. Similarly, set $u_2(x_2) = u_2(w_2)$ to obtain $x_{21}(4 - x_{22}) = 4$ and use the Pareto optimality requirement $4 - x_{22} = x_{21}$ to conclude that $x_2 = (2, 2)$ is the other endpoint.

(d) Note that, using the normalization $p_1 = 1$, we have $p \cdot w_1 = 4 + 3p_2$ and $p \cdot w_2 = 1$. Clearing the market for commodity one,

$$\phi_{11}(p) + \phi_{21}(p) = [(4 - p_2)/2] + [(1 - 4p_2)/2] = 5$$

yields $p_2 = -1$.

(e) $x_1 = x_2 = (5/2, 3/2)$.

(f) $\Delta_w x_1 = (-3/2, -3/2)$ and $\Delta_w x_2 = (3/2, 3/2)$. See Figure 5.

(g) The endowment function becomes

$$w_i = \begin{cases} (4, 0) & \text{if } i = 1; \\ (1, 3) & \text{if } i = 2. \end{cases}$$

The set of Pareto optimal allocations is, of course, unchanged. But since now $u_1(w_1) = 16$ and $u_2(w_2) = 1$ the endpoints of the core are given by $x_1 = (4, 0)$ and $x_2 = (1, 3)$ so that the core has shrunk to a single point (the endowment point). Using the normalization $p_1 = 1$, we have $p \cdot w_1 = 4$ and $p \cdot w_2 = 1 + 3p_2$. Clearing the market for commodity one yields

$$\phi_{11}(p) + \phi_{21}(p) = [(4 - 4p_2)/2] + [(1 - p_2)/2] = 5$$

which implies once again that $p_2 = -1$. (Actually, we have a corner solution, but equating MRS's to the price ratio happens to work.) The equilibrium allocation becomes $x_1 = (4, 0)$ and $x_2 = (1, 3)$ so that there is no trade.

(h) The endowment function and hence the Walrasian equilibrium and the core are undefined (at least in the "economic" game presented here): the result will depend on "politics," "power," or other noneconomic considerations.

Exercise 1.17

See Figure 6.

Exercise 1.18

(a) See Figure 7.

(b) See Figure 7.

(c) The equilibrium price functional is $p = (.5, .5)$ and the two equilibrium net trade allocations are

$$\Delta_w x_1 = (.5, -.5),\ \Delta_w x_2 = (-.5, .5) \quad \text{and}$$
$$\Delta_w y_1 = (-.5, .5),\ \Delta_w y_2 = (.5, -.5)$$

See Figure 7.

(d) Whatever price functional you choose, the net trades never sum to zero.

(e) Now there are Walrasian equilibria corresponding to the same price functional found in part (c): assign any two consumers the net trade $(.5, -.5)$ and the other two the net trade $(-.5, .5)$.

(f) Equilibria exist when the number of consumers is even, but there are no equilibria if the number of consumers is odd.

Exercise 1.19

Letting I_1 denote the set of consumers of the first type and I_2 the set of consumers of the second type, we have

$$\phi_i(p) = \begin{cases} (\alpha\, p \cdot w_i\,/p_1, (1-\alpha)\, p \cdot w_i\,/p_2) & i \in I_1; \\ (\beta\, p \cdot w_i\,/p_1, (1-\beta)\, p \cdot w_i\,/p_2) & i \in I_2 \end{cases}$$

where

$$p \cdot w_i = \begin{cases} p_1 a_1 + p_2 b_1 & \text{if } i \in I_1; \\ p_1 a_2 + p_2 b_2 & \text{if } i \in I_2. \end{cases}$$

Clearing the market for commodity one, we obtain

$$\sum_{i \in I_1} \frac{\alpha\, p \cdot w_i}{p_1} + \sum_{i \in I_2} \frac{\beta\, p \cdot w_i}{p_1} = \sum_{i \in I_1} a_1 + \sum_{i \in I_2} a_2 = r(a_1 + a_2)$$

which reduces to

$$r\alpha \left(\frac{p_1 a_1 + p_2 b_1}{p_1} \right) + r\beta \left(\frac{p_1 a_2 + p_2 b_2}{p_1} \right) = r(a_1 + a_2)$$

and, finally, to

$$\alpha \left(\frac{p_1 a_1 + p_2 b_1}{p_1} \right) + \beta \left(\frac{p_1 a_2 + p_2 b_2}{p_1} \right) = a_1 + a_2$$

which is identical to equation (1.1). Similarly, the market clearing equation for commodity two reduces to (1.2). Therefore, the equilibrium price functional and the equilibrium allocation received by consumers of each type is identical to that for the economy with one consumer of each type.

Exercise 1.20

The argument parallels that in the text:

$$\Delta_w x_i = \begin{cases} (1.2, -.8) & \text{for } i = 1; \\ (-1.2, .8) & \text{for } i = 2. \end{cases}$$

If we assign the net trade allocation

$$\Delta_w y_i = \begin{cases} (1+\epsilon)(1.2, -.8) & \text{for consumers of type 1;} \\ (1-\epsilon)(-1.2, .8) & \text{for consumers of type 2,} \end{cases}$$

then the feasibility condition requires

$$(r-1)(1+\epsilon)(1.2,-.8)+r(1-\epsilon)(-1.2,.8)=0$$

which reduces to

$$\epsilon=\frac{1}{2r-1}$$

as before. Therefore, as r increases, ϵ approaches 0 so that, given enough replication, we can find a net trade allocation which lies in the preferred sets of each consumer.

Exercise 1.21

Left to reader.

Exercise 1.22

The Edgeworth box diagrams are misleading because each represents a two-dimensional cross section of $\mathbf{R}^4$. Since $p \cdot (x_i - w_i) = 0$ for $i = 1, 2$, the net trades of both consumers do lie on a hyperplane (in $\mathbf{R}^4$) passing through the origin.

Exercise 1.23

Since the same difficulty arises in both proofs, we discuss only the proof of Theorem 1.12. The problem arises when using the linearity of p to conclude that

$$p \cdot \sum_{i \in I} \Delta_w y_i = \sum_{i \in I} p \cdot \Delta_w y_i$$

which need not be true if the sum is infinite. In the example discussed in the text, the left hand side of this expression equals zero while the right hand side is infinite.

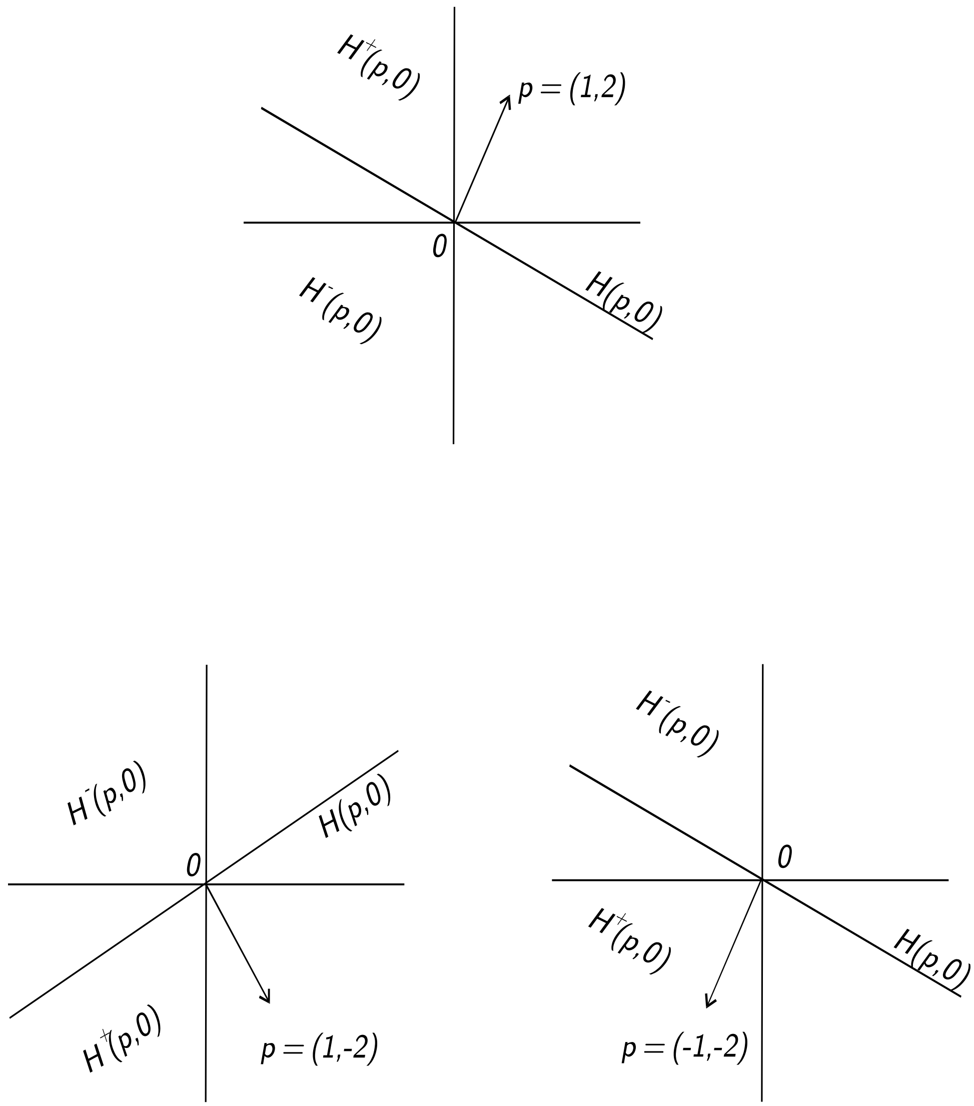


Figure 1: Hyperplanes and halfspaces

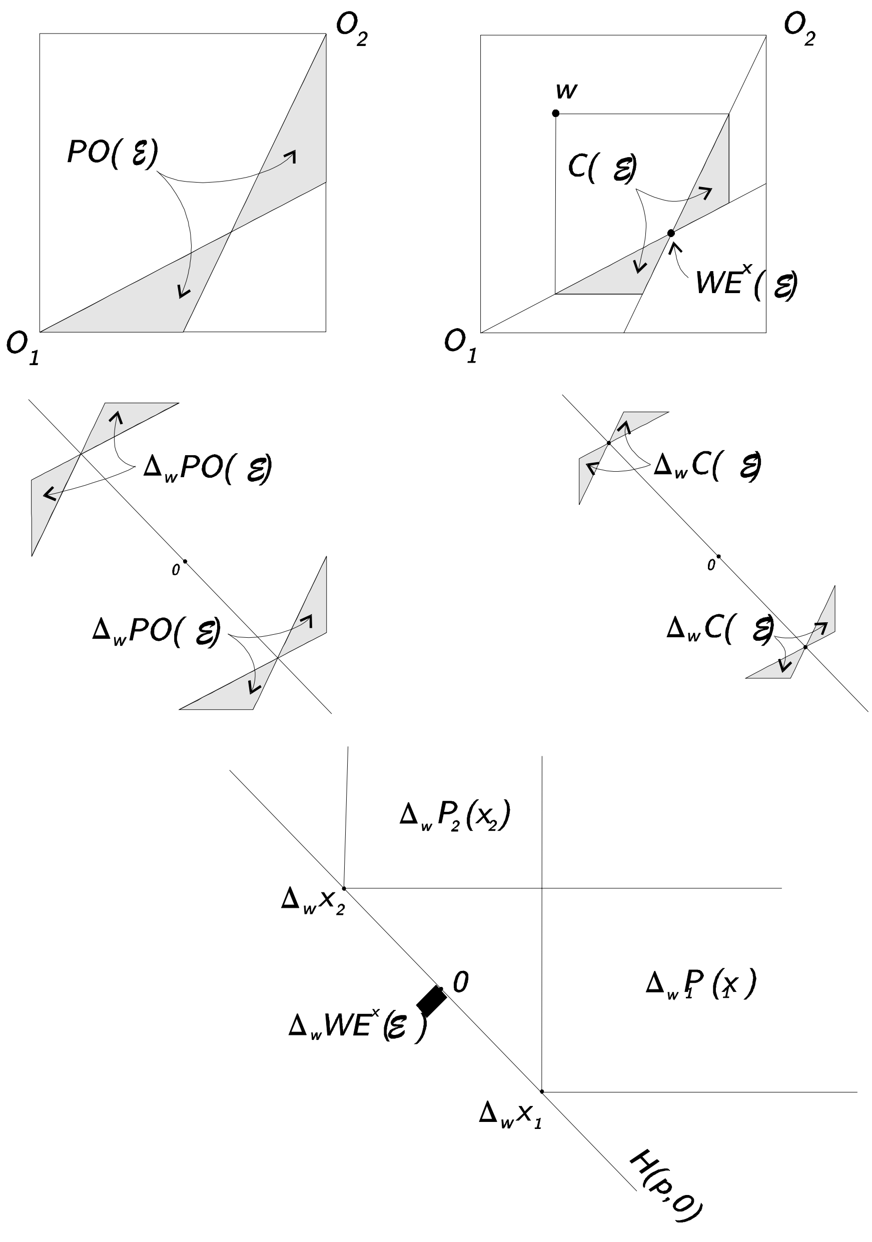


Figure 2: Perfect complements (a)

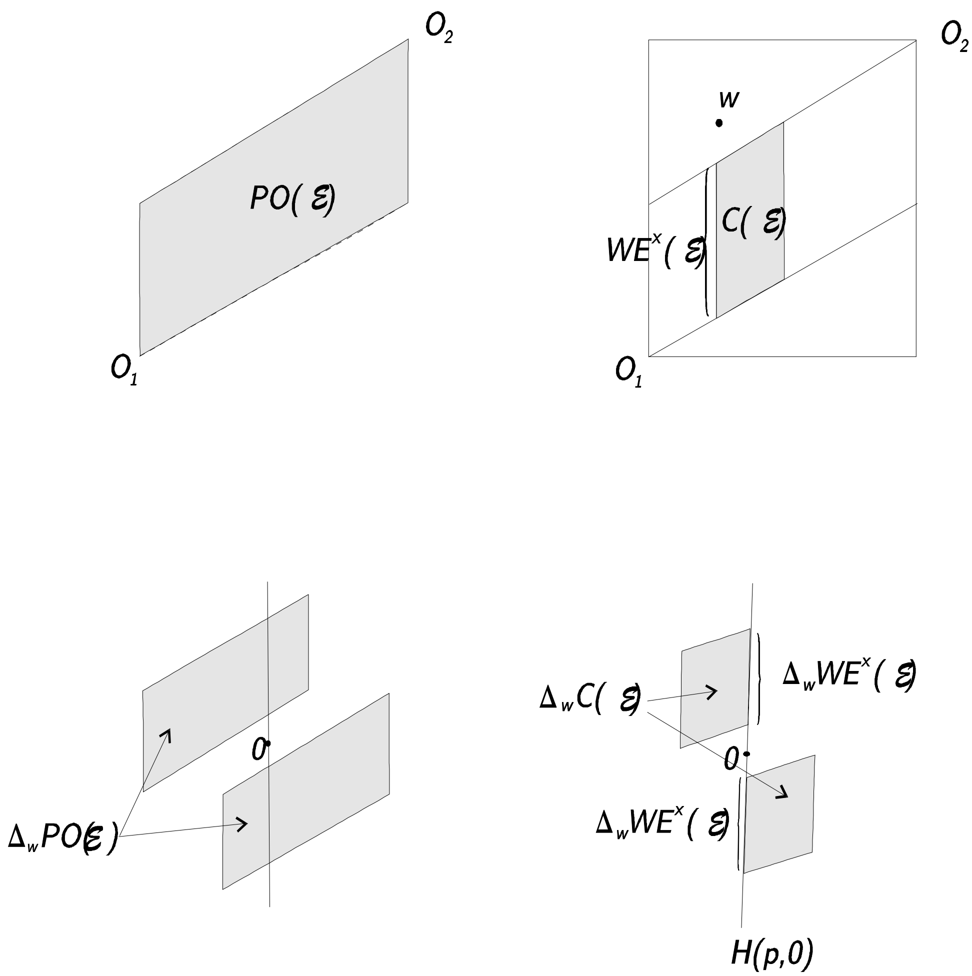


Figure 3: Perfect complements (b)

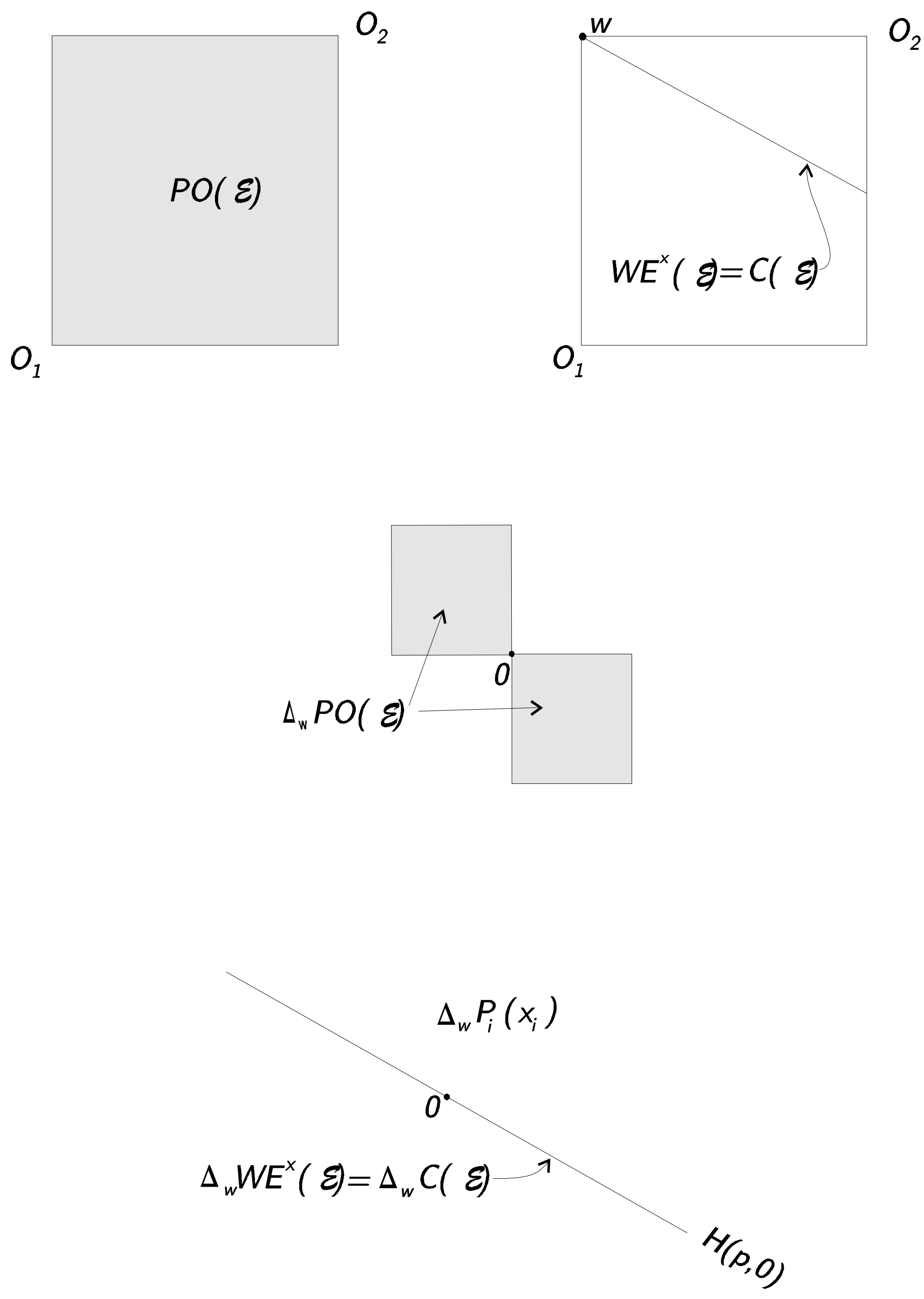
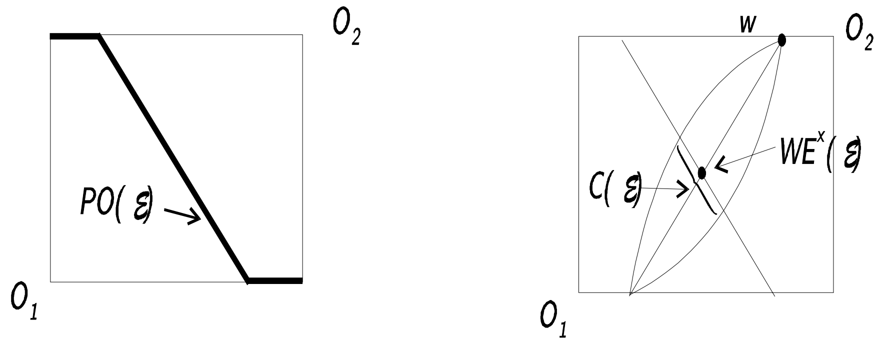


Figure 4: Perfect substitutes



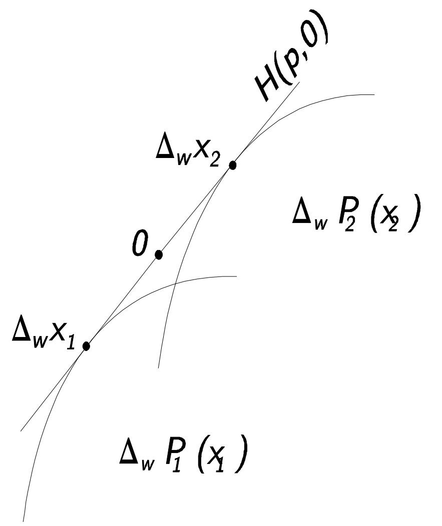


Figure 5: An economy with a bad

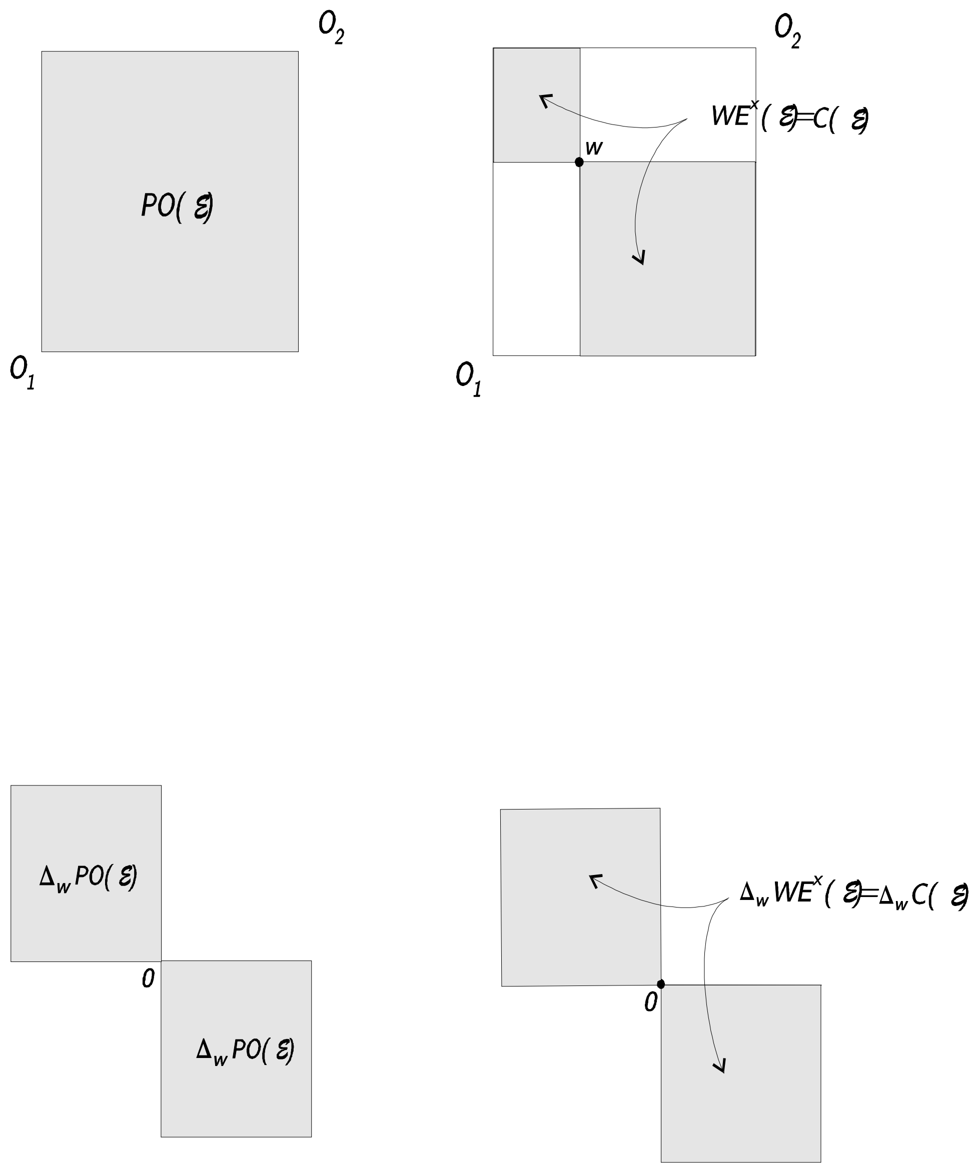


Figure 6: Vector-ordering preferences

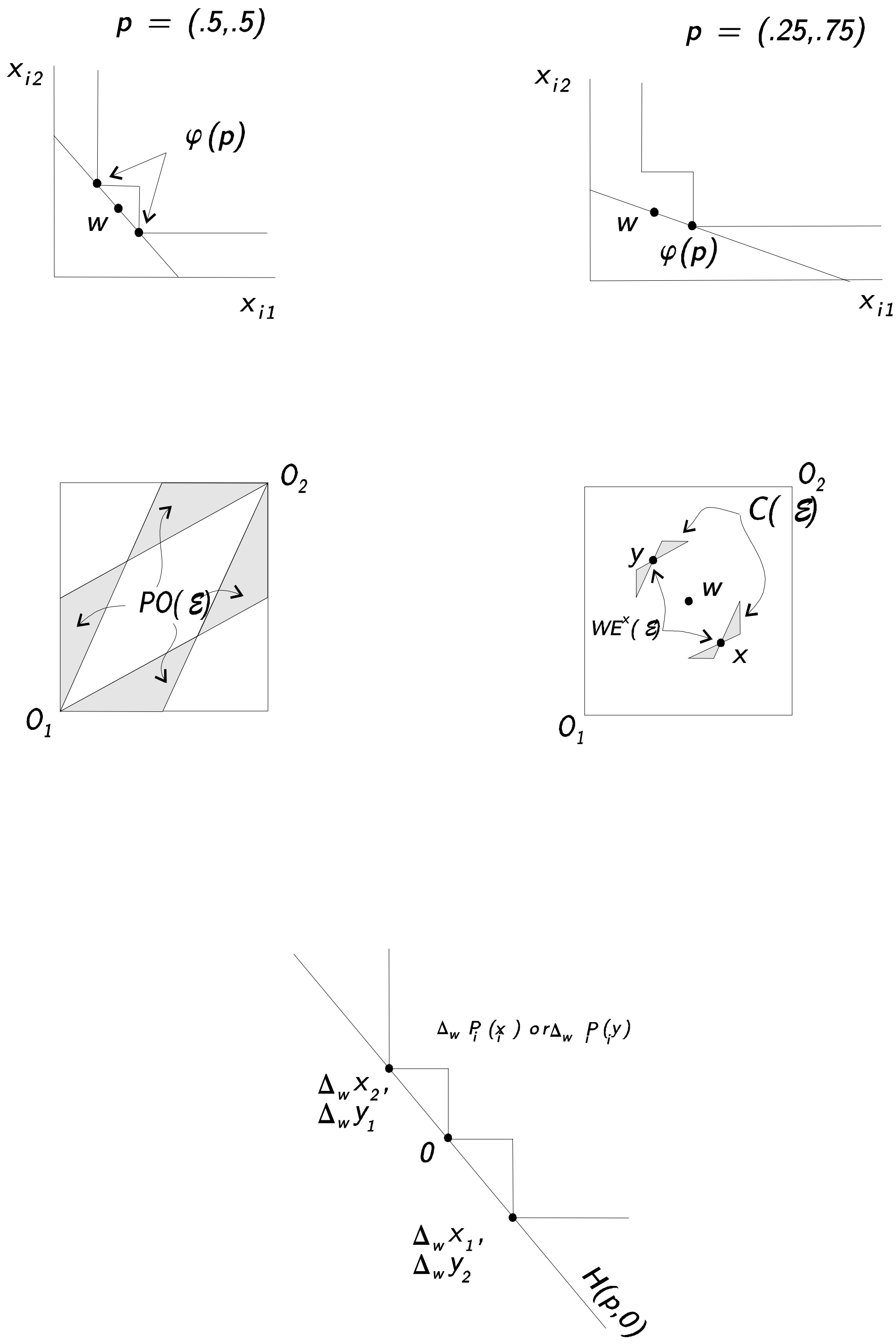


Figure 7: Stair step preferences

2 Production

Exercise 2.1

(a) $\operatorname{sp} S = \operatorname{aff} S$, the line passing through the origin and the point $(1, 1)$; $\operatorname{co} S = \{ (1, 1) \}$.

(b) Again $\operatorname{sp} S = \operatorname{aff} S$, the same set as in part (a), but $\operatorname{co} S$ is now the line segment with endpoints $(1, 1)$ and $(2, 2)$.

(c) $\operatorname{sp} S = \mathbf{R}^2$ while $\operatorname{aff} S$ is the line passing through the points $(1, 1)$ and $(2, 3)$. $\operatorname{co} S$ is the line segment with $(1, 1)$ and $(2, 3)$ as endpoints.

(d) $\operatorname{sp} S = \mathbf{R}^2$. $\operatorname{aff} S$ is the line passing through the three points in S while $\operatorname{co} S$ is the line segment with $(0, -1)$ and $(2, 3)$ as endpoints.

(e) $\operatorname{sp} S = \operatorname{aff} S = \mathbf{R}^2$ while $\operatorname{co} S$ is the triangle having the three points in S as vertices.

Diagrams left to reader.

Exercise 2.2

B is convex because it is the intersection of convex sets, and it must be the smallest because any other convex set containing S will be one of the sets in $\mathcal{C}$. Since clearly $F \subset \operatorname{co} F$, $E \subset F$ implies $E \subset \operatorname{co} F$. Therefore, since $\operatorname{co} F$ is convex, all convex combinations of elements of E will be contained in $\operatorname{co} F$, implying $\operatorname{co} E \subset \operatorname{co} F$. Since $S \subset B$, we conclude that $\operatorname{co} S \subset \operatorname{co} B = B$, the latter equality following since B is convex. However, we also know that $B \subset \operatorname{co} S$ since B is the intersection of all convex sets such as $\operatorname{co} S$ which contain S. Therefore, $\operatorname{co} S \subset B$ and $B \subset \operatorname{co} S$ which implies that $\operatorname{co} S = B$.

Exercise 2.3

Translating the block notation implicit in this expression gives

$$\alpha x + (1 - \alpha) y \in S \quad \forall x, y \in S \quad \text{and} \quad \forall \alpha \in [0, 1]$$

which is precisely the definition for S to be convex.

Exercise 2.4

$\alpha = 1$: the line segment C_1;

$\alpha = .75$: the rectangle with vertices

$$\{ (0,0), (.75,0), (.75,.25), (0,.25) \};$$

$\alpha = .5$: the rectangle with vertices

$$\{ (0,0), (.5,0), (.5,.5), (0,.5) \};$$

$\alpha = .25$: the rectangle with vertices

$$\{ (0,0), (.25,0), (.25,.75), (0,.75) \};$$

$\alpha = 0$: the line segment C_2.

Diagrams are left to reader.

Exercise 2.5

Clearly $S - S \subset \mathbf{R^n}$. To establish the reverse inclusion, consider an arbitrary element $x \in \mathbf{R^n}$. Define

$$x^+ := (\max\{ 0, x_1 \}, \ldots, \max\{ 0, x_n \})$$

and

$$x^- := (\max\{ 0, -x_1 \}, \ldots, \max\{ 0, -x_n \}).$$

Then $x^+, x^- \in S$ and $x^+ - x^- = x$.

Exercise 2.6

The only possible candidate for the zero "vector" is the set $\{0\}$. But it is not true in general that for every nonempty convex subset C there exists an inverse $-C$ such that $C + (-C) = \{0\}$; consider, for example, $[1,2] \subset \mathbf{R}$.

Exercise 2.7

Left to the reader (the proofs are almost identical to that for convexity).

Exercise 2.8

Left to reader.

Exercise 2.9

Suggestion: try the transformations defined by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Exercise 2.10

I give the proof for the case of affine subspaces, leaving the (almost identical) proof for convex subsets to the reader. Note first that, by definition, there exists a linear transformation $C\colon X \to Y$ and a vector $c \in Y$ such that $Ax = Cx + c \quad \text{for all} \quad x \in X$.

(a) For any $y, y' \in AS$ and $\alpha \in \mathbf{R}$, we want to show that $\alpha y + (1-\alpha)y' \in AS$. By definition, $y = Cx + c$ and $y' = Cx' + c$ for some $x, x' \in S$. Therefore,

$$\begin{aligned} \alpha y + (1-\alpha)y' &= \alpha(Cx + c) + (1-\alpha)(Cx' + c) \\ &= C(\alpha x + (1-\alpha)x') + c \in AS \end{aligned}$$

since $\alpha x + (1-\alpha)x' \in S$.

(b) For any $x, x' \in A^{-1}S$ and $\alpha \in \mathbf{R}$, we have $y =: Cx + c \in S$ and $y' =: Cx' + c \in S$. Because S is an affine subspace, $\alpha y + (1-\alpha)y' \in S$ and so

$$\alpha(Cx + c) + (1-\alpha)(Cx' + c) = C(\alpha x + (1-\alpha)x') + c \in S.$$

Therefore, $\alpha x + (1-\alpha)x' \in A^{-1}S$.

Exercise 2.11

This follows directly from the definition: $y \geq x$ iff $y_i \geq x_i$ for all $i = 1, \ldots, n$. Because $\lambda x_i \geq 0$ if $x_i \geq 0$ and $\lambda > 0$, $\lambda x \geq 0$ for all $x \in \mathbf{R}^{\mathbf{n}}_{+}$ provided that $\lambda > 0$ which proves that $\mathbf{R}^{\mathbf{n}}_{+}$ is a cone. If $x_i \geq 0$ and $y_i \geq 0$ for all $i \in I$, then

$$\alpha x_i + (1-\alpha)y_i > 0 \qquad \forall i \in I \quad \text{and} \quad \forall \alpha \in [0,1]$$

so

$$x, y \in \mathbf{R}^{\mathbf{n}}_{+} \quad \text{and} \quad \alpha \in [0,1] \quad \text{implies} \quad \alpha x + (1-\alpha)y \in \mathbf{R}^{\mathbf{n}}_{+}$$

which proves that $\mathbf{R}^{\mathbf{n}}_{+}$ is convex. If $x \in \mathbf{R}^{\mathbf{n}}_{+} \cap (-\mathbf{R}^{\mathbf{n}}_{+})$, then $x_i \geq 0$ and $-x_i \geq 0$ for all $i \in I$ so that $x = 0$ which proves that $\mathbf{R}^{\mathbf{n}}_{+}$ is proper. Since clearly $0 \in \mathbf{R}^{\mathbf{n}}_{+}$, the proper, convex cone $\mathbf{R}^{\mathbf{n}}_{+}$ is pointed.

Verification of the analogous properties for $\mathbf{R}^{\mathbf{n}}_{-}$ follows along the same lines.

Exercise 2.12

This follows directly from the definition: $y \gg x$ iff $y_i > x_i$ for $i = 1, \ldots, n$. Details of the verification are left to the reader. The cones are not pointed.

Exercise 2.13

- reflexive: $S \supset S$ for any set S;
- transitive: $S_1 \supset S_2$ and $S_2 \supset S_3$ implies $S_1 \supset S_3$ for any sets S_1, S_2 and S_3;
- antisymmetric: if $S \supset T$ and $T \supset S$ then $S = T$.

To show that this partial ordering need not be complete, consider any two sets which intersect with neither one a subset of the other: e.g.,

$$S = [0, 2] \quad \text{and} \quad T = [1, 3].$$

Exercise 2.14

- reflexive: $x \geq x$ for all $x \in \mathbf{R^n_+}$;
- transitive: $x \geq y$ and $y \geq z$ implies $x \geq z$ for any $x, y, z \in \mathbf{R^n_+}$;
- antisymmetric: $x \geq y$ and $y \geq x$ implies $x = y$ for any $x, y \in \mathbf{R^n_+}$.

The fact that $\geq$ satisfies conditions (a) and (b) of the definition for an ordered vector space is immediate.

Exercise 2.15

- reflexive: $f(t) \geq f(t)$ for all $t \in T$, so $f \geq f$ for any $f \in \mathbf{R}^T$;
- transitive: if $f(t) \geq g(t)$ and $g(t) \geq h(t)$ for all $t \in T$, then $f(t) \geq h(t)$ for all $t \in T$;
- antisymmetric: if $f(t) \geq g(t)$ and $g(t) \geq f(t)$ for all $t \in T$, then $f(t) = g(t)$ for all $t \in T$.

Again verification that $\geq$ satisfies conditions (a) and (b) of the definition for an ordered vector space is immediate.

$$L_+ = \{ f \in \mathbf{R}^T \mid f(t) \geq 0 \; \forall t \in T \}$$

and

$$L_- = \{ f \in \mathbf{R}^T \mid f(t) \leq 0 \; \forall t \in T \}.$$

Exercise 2.16

See Figure 8.

(a) $K^o = \mathbf{R}^2_-$;

(b) $K^o = \{ x \in \mathbf{R}^2 \mid x \in \mathbf{R}^2_+ \quad \text{or} \quad (x_1 \leq 0 \quad \text{and} \quad x_2 \geq -x_1) \}$.

Exercise 2.17

The activity $y^1 = (-1, 1, 0, 1)$ is possible for the factory operating alone and the activity $y^2 = (-1, 0, 1, 0)$ for the laundry operating alone, but $y^1 + y^2 = (-2, 1, 1, 1)$ is not possible for both operating simultaneously (given the presence of one unit of soot, the laundry requires an extra unit of labor to produce a unit of clean shirts).

Exercise 2.18

The demand set for consumer 2 is computed just as in the text: set the MRS for consumer 2 equal to the price ratio,

$$\frac{x_{22}}{3(x_{21} + 4)} = \frac{p_1}{p_2},$$

and solve simultaneously with the budget constraint $p_1 x_{21} + p_2 x_{22} = 0$ to obtain

$$\phi_2(p) = \left(-1, \frac{p_1}{p_2} \right)$$

as before. For consumer 1, set the MRS equal to the price ratio,

$$\frac{3x_{12}}{x_{11} + 4} = \frac{p_1}{p_2},$$

and solve simultaneously with the *new* budget constraint $p_1 x_{11} + p_2 x_{12} = p \cdot w_1 = p_2$ to obtain

$$\phi_1(p) = \left(\frac{p_2}{4p_1} - 3, \frac{3}{4} + \frac{3p_1}{p_2} \right).$$

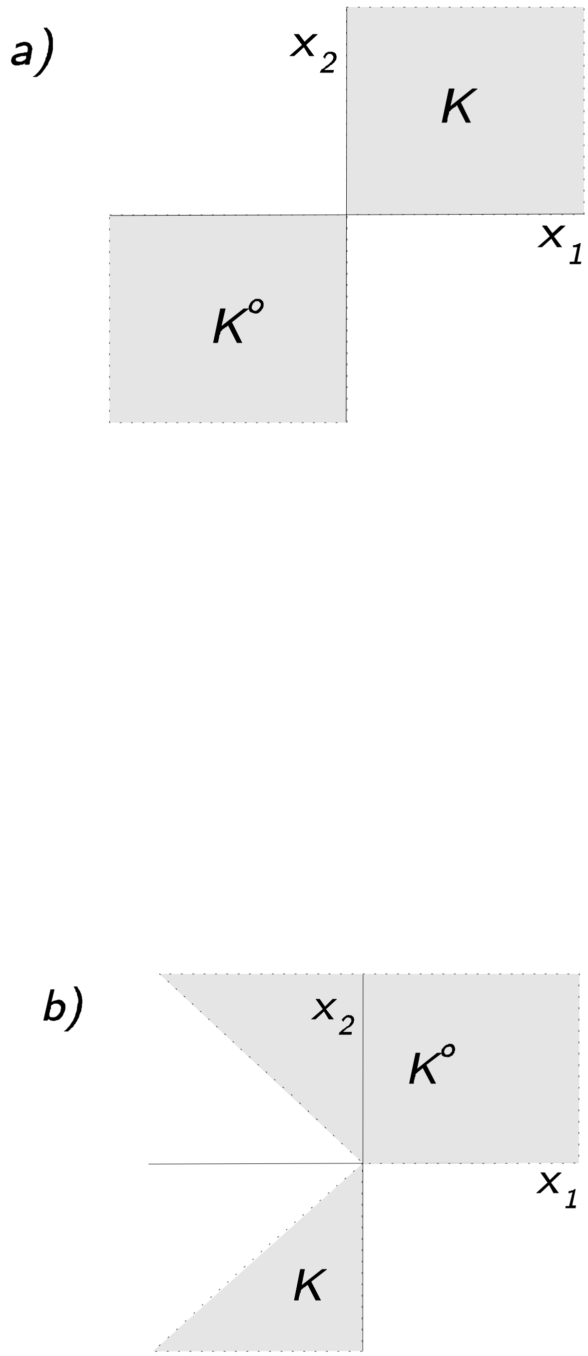


Figure 8: Polar cones.

Since a positive amount of commodity two will be produced in equilibrium, the equilibrium hyperplane once again coincides with the diagonal boundary of the aggregate technology set, yielding the equilibrium price functional $p = (.5, .5)$. At those prices, the equilibrium allocation is

$$x_i = \begin{cases} (-11/4, 15/4) & \text{for } i = 1; \\ (-1, 1) & \text{for } i = 2 \end{cases}$$

with the equilibrium net trade allocation

$$\Delta_w x_i = \begin{cases} (-11/4, 11/4) & \text{for } i = 1; \\ (-1, 1) & \text{for } i = 2. \end{cases}$$

Exercise 2.19

(a) Lemma 2.24: $x \in \mathrm{Dom}(\mathcal{E}, I)$

$$\begin{aligned} &\text{iff} \quad \exists\, x' \in \mathrm{F}(\mathcal{E}, I) \ni x'_i \in P_i(x_i)\ \forall i \in I \\ &\text{iff} \quad \exists\, x' \colon I \to L \ni \sum_{i \in I} \Delta_w x'_i \in Y \\ &\qquad\qquad \text{and} \quad \Delta_w x'_i \in \Delta_w P_i(x_i)\ \forall\, i \in I \\ &\text{iff} \quad Y \cap \sum_{i \in I} \Delta_w P_i(x_i) = \emptyset. \end{aligned}$$

Lemma 2.25: By definition, $\mathrm{PO}(\mathcal{E}) = \mathrm{F}(\mathcal{E}, I) \backslash \mathrm{Dom}(\mathcal{E}, I)$. Therefore, by Lemma 2.24,

$$x \in \mathrm{PO}(\mathcal{E}) \quad \text{iff} \quad Y \cap \sum_{i \in I} \Delta_w P_i(x_i) = \emptyset.$$

(b) Lemma 2.24: an allocation x is dominated if it is possible to find net trades $\Delta_w x'_i$ lying in each consumer's preferred net trade set which sum to a vector which can be produced (i.e., which lies in Y).

Lemma 2.25: an allocation x is Pareto optimal if it is *not* possible to find net trades lying in each consumer's preferred net trade set which sum to a vector which can be produced.

(c) If $Y = \{0\}$, then

$$Y \cap \sum_{i \in I} \Delta_w P_i(x_i) \neq \emptyset \quad \text{and} \quad 0 \in \sum_{i \in I} \Delta_w P_i(x_i)$$

are equivalent.

(d) If $(x,p) \in \mathrm{WE}(\mathcal{E})$, then $\Delta_w P_i(x_i) \subset H_o^+(p,0)$ for all $i \in I$. But

$$z \in \sum_{i\in I} \Delta_w P_i(x_i) \quad \text{implies} \quad z = \sum_{i\in I} y_i$$

where $y_i \in \Delta_w P_i(x_i)$ for all $i \in I$. Therefore, $p \cdot y_i > 0$ for all $i \in I$ and so $p \cdot z > 0$. But, if $z \in Y$, then $p \cdot z \leq 0$. We conclude that

$$Y \cap \Delta_w \sum_{i\in I} P_i(x_i) = \emptyset$$

which implies that $x \in \mathrm{PO}(\mathcal{E})$.

Exercise 2.20

(a) Lemma 2.27: $x \in \mathrm{Dom}(\mathcal{E}, S)$

$$\begin{aligned}
&\text{iff} \quad \exists\, x' \in \mathrm{F}(\mathcal{E}, S) \ni x'_i \in P_i(x_i)\ \forall i \in S \\
&\text{iff} \quad \exists\, x' \colon I \to L \ni \sum_{i\in S} \Delta_w x'_i \in Y_S \\
&\qquad\qquad \text{and} \quad \Delta_w x'_i \in \Delta_w P_i(x_i)\ \forall\, i \in S \\
&\text{iff} \quad Y_S \cap \sum_{i\in S} \Delta_w P_i(x_i) = \emptyset.
\end{aligned}$$

Lemma 2.28: By definition,

$$\mathrm{C}(\mathcal{E}) = \mathrm{F}(\mathcal{E}, I) \setminus \bigcup_{S\subset I} \mathrm{Dom}(\mathcal{E}, S).$$

Therefore, by Lemma 2.8,

$$x \in \mathrm{C}(\mathcal{E}) \quad \text{iff} \quad Y_S \cap \sum_{i\in S} \Delta_w P_i(x_i) = \emptyset.$$

(b) If $Y_S = \{0\}$, then

$$Y_S \cap \sum_{i\in S} \Delta_w P_i(x_i) \neq \emptyset \quad \text{and} \quad 0 \in \sum_{i\in S} \Delta_w P_i(x_i)$$

are equivalent.

Exercise 2.21

As usual, the inclusion $\mathrm{C}(\mathcal{E}) \subset \mathrm{WE}(\mathcal{E})$ is immediate from the definitions. To establish the first inclusion, suppose that $(x,p) \in \mathrm{WE}(\mathcal{E})$ but $x \notin \mathrm{C}(\mathcal{E})$. By the definition of the core, $x \in \mathrm{Dom}(\mathcal{E}, S)$ for some nonempty $S \subset I$ and so there exists an allocation $y \in \mathrm{F}(\mathcal{E}, S)$ such that $y \succ_S x$. But then $\Delta_w y_i \in \Delta_w P_i(x_i)$ for all $i \in S$. By the definition of Walrasian equilibrium, this implies that $p\Delta_w y_i > 0$ for all $i \in S$ and hence that

$$p \cdot \sum_{i \in S} \Delta_w y_i = \sum_{i \in S} p \cdot \Delta_w y_i > 0$$

However, $y \in \mathrm{F}(\mathcal{E}, S)$ implies

$$\sum_{i \in S} \Delta_w y_i \in Y_S \subset Y$$

so that, by the definition of Walrasian equilibrium,

$$p \cdot \sum_{i \in S} \Delta_w y_i \leq 0,$$

a contradiction.

Exercise 2.22

Left to reader.

Exercise 2.23

Left to reader.

Exercise 2.24

Left to reader.

Exercise 2.25

$p_1 = .4$, $p_2 = .6$, $q = 5$, and $x_{11} = x_{22} = 5$. See Figure 9.

Exercise 2.26

Let $Y = \{\, y \in \mathbf{R}^3 \mid y = \lambda(b_1, b_2, -1),\ \lambda \geq 0 \,\}$. The zero profit condition then implies that $p \cdot y = p_1 \lambda b_1 + p_2 \lambda b_2 - \lambda = 0$ so that, if $\lambda > 0$, then $b_1 p_1 + b_2 p_2 = 1$. Letting q denote the number of sheep slaughtered, we have

$$\frac{a_1}{2p_1} = b_1 q, \quad \frac{a_2}{2p_2} = b_2 q, \quad \text{and so} \quad \frac{a_1 + a_2}{2} = q(b_1 p_1 + b_2 p_2) = q.$$

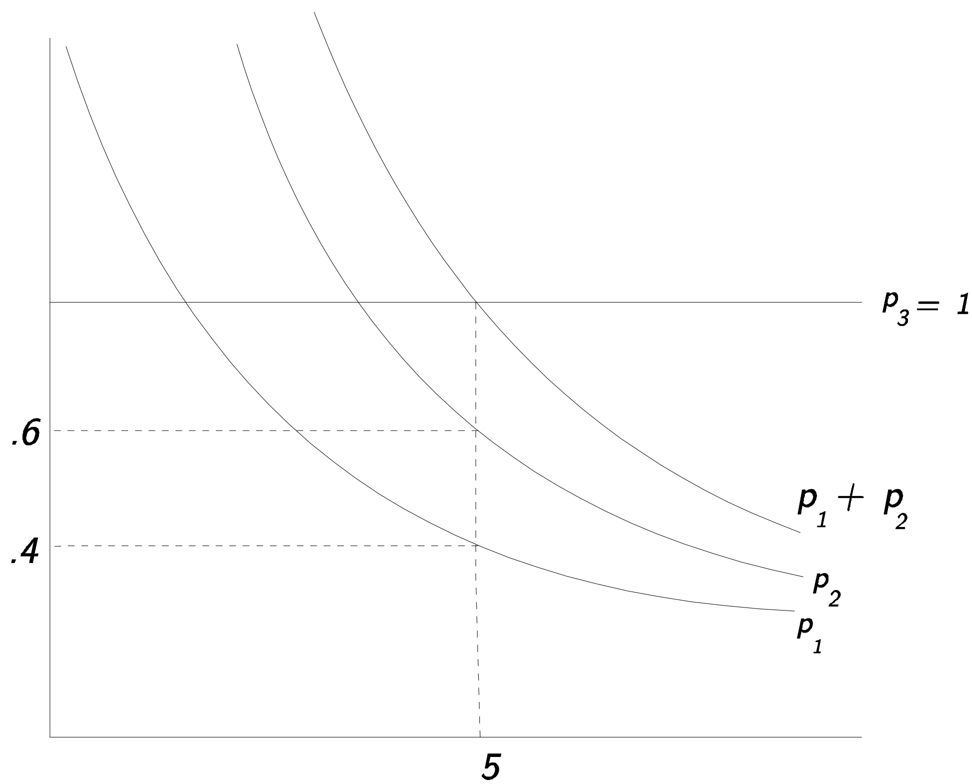
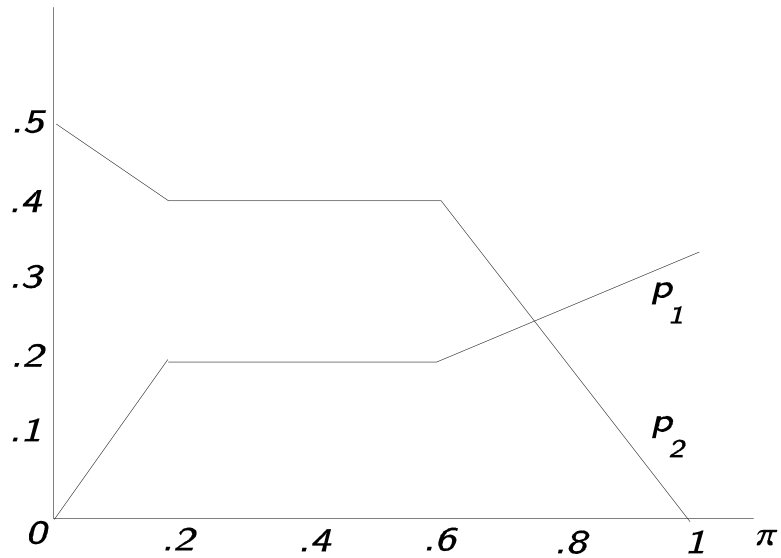


Figure 9: Marshallian joint supply.



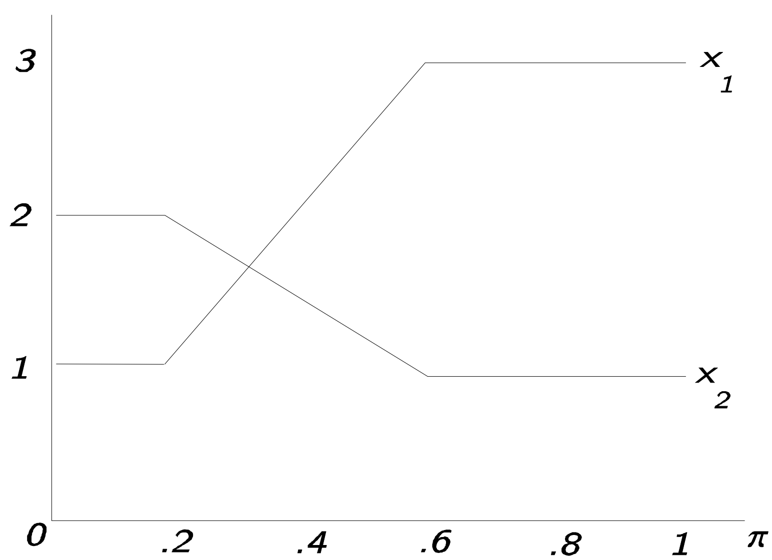


Figure 10: Contingent commodities.

Therefore,

$$p_1 = \frac{a_1}{2b_1 q} = \frac{a_1}{b_1(a_1 + a_2)} \quad \text{and} \quad p_2 = \frac{a_2}{2b_2 q} = \frac{a_2}{b_2(a_1 + a_2)}.$$

Exercise 2.27

(a) $x = (3, 1, -1)$ and $p = (\pi/3, 1 - \pi, 1)$.

(b) Case A: If $\pi > 3/5$, then $p = (\pi/3, 1 - \pi, 1)$ and $x = (3, 1, -1)$. Case B: If $\pi < 1/5$, then $p = (\pi, (1 - \pi)/2, 1)$ and $x = (1, 2, -1)$. Case C: If $1/5 \leq \pi \leq 3/5$, then $p = (1/5, 2/5, 1)$ and $x = (5\pi, (5 - 5\pi)/2, -1)$.

See Figure 10.

Exercise 2.28

See Figure 11.

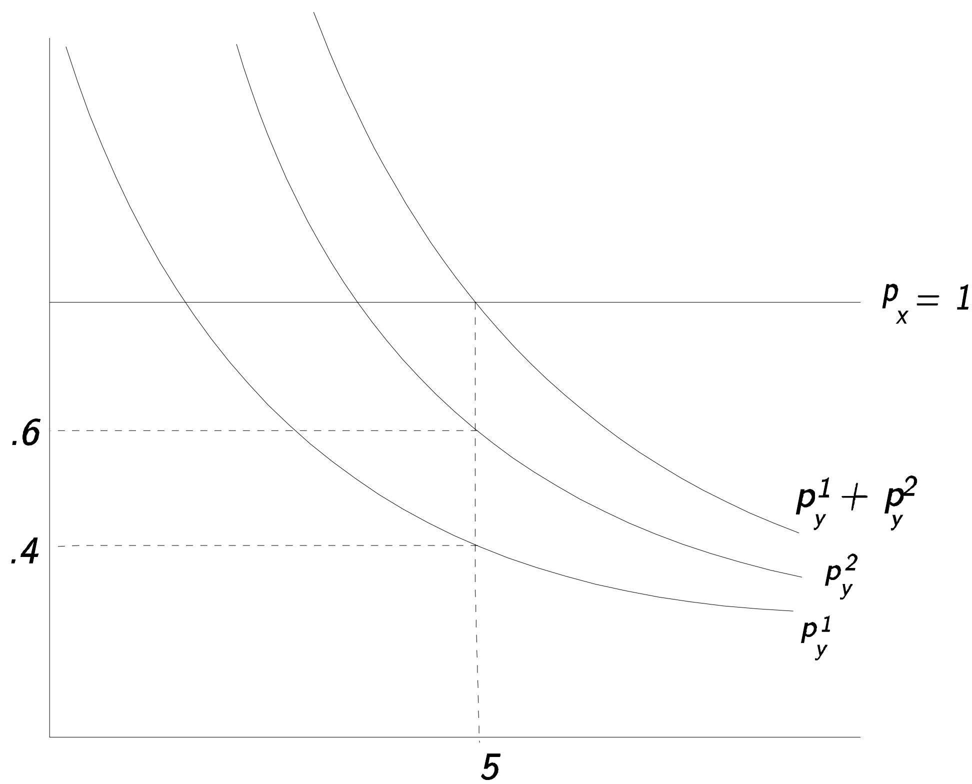


Figure 11: Lindahl diagram.

3 Aumann's model

Exercise 3.1

The per capita demand for commodity one is given by

$$\begin{aligned}\int_0^1 \frac{p \cdot w_{i1}}{2p_1}\,di &= \int_0^1 \frac{p_1(2+20i)+p_2(10-8i)}{2p_1}\,di \\ &= 6+\frac{3p_2}{p_1}\end{aligned}$$

and the per capita endowment by

$$\int_0^1 w_{i1}\,di = \int_0^1 (2+20i)\,di = 12.$$

Clearing the market for commodity one, we have

$$6+\frac{3p_2}{p_1} = 12.$$

Using the normalization $p_1+p_2=1$ yields the equilibrium price functional $p=(1/3,2/3)$.

Therefore, the equilibrium wealth function is

$$p \cdot w_i = \frac{1}{3}(2+20i)+\frac{2}{3}(10-8i) = \frac{22+4i}{3}$$

and the equilibrium allocation

$$x_{i1} = 11+2i \quad \text{and} \quad x_{i2} = 5.5+i$$

to consumer i.

Exercise 3.2

Using $p \cdot w_i = 4i(1-i)$, we obtain for the per capita demand and endowment of the first commodity

$$\int_0^1 \frac{i^2\, p \cdot w_i}{p_1}\,di = \frac{1}{5p_1} \quad \text{and} \quad \int_0^1 w_{i1}\,di = \frac{2}{3}$$

respectively, yielding $p=(.3,.7)$ for the equilibrium price functional (using the normalization $p_1+p_2=1$) and

$$x_i = \left(\frac{40}{3}i^3(1-i), \frac{40}{7}(i-i^2-i^3+i^4)\right)$$

for the equilibrium allocation to consumer i.

Exercise 3.3

Using the normalization $p_1 + p_2 + p_3 = 1$,

$$p \cdot w_i = \sum_{j=1}^{3} p_j (4i - 4i^2) = 4i - 4i^2.$$

For commodity 1 or 2, per capita demand and endowment are given by

$$\int_0^1 \frac{p \cdot w_i}{4p_j}\, di = \frac{1}{6p_j} \quad \text{and} \quad \int_0^1 w_{ij}\, di = \frac{2}{3}$$

respectively, yielding equilibrium prices $p = (.25, .25, .5)$. The equilibrium allocation gives

$$x_{ij} = 4i - 4i^2 \quad \text{for} \quad j = 1, 2, 3$$

to consumer i.

Exercise 3.4

If $y = \lambda(-.5, 0, 1)$ for some $\lambda > 0$, then $p \cdot y = \lambda(-.5p_1 + p_3) = 0$ implies $p_3 = .5p_1$. Using

$$\int_0^1 \frac{p \cdot w_i}{3p_2}\, di = \frac{k}{6p_2} \quad \text{and} \quad \int_0^1 w_{i2}\, di = \frac{k}{2}$$

we obtain the equilibrium price functional $p = (2/3, 1/3, 1/3)$. The equilibrium allocation gives $x_i = (ki/2, ki, ki)$ to consumer i.

Exercise 3.5

Using the price normalization $p_1 + p_2 = 1$,

$$p \cdot w_i = p_1 k_i + p_2 k_i = k_i(p_1 + p_2) = k_i.$$

Therefore,

$$\int_I x_1 = \int_I \frac{\alpha_i\, p \cdot w_i}{p_1}\, di = \frac{1}{p_1} \int_I \alpha k.$$

Equating this expression to $\int_I w_1 = \int_I k$ yields the equilibrium price

$$p_1 = \frac{\int_I \alpha k}{\int_I k}$$

and corresponding equilibrium allocation

$$x_{i1} = \frac{\alpha_i k_i \int_I k}{\int_I \alpha k} \quad \text{and} \quad x_{i2} = \frac{(1-\alpha_i)k_i(1-\int_I k)}{\int_I \alpha k}.$$

Exercise 3.6

The per capita demand for commodity 1 is

$$\int_I x_1 = \sum_{t\in T} \int_{S_t} \frac{\alpha_t(p_1 a_t + p_2 b_t)}{p_1} = \sum_{t\in T} \frac{\alpha_t(p_1 a_t + p_2 b_t)}{p_1} \lambda(S_t)$$

and the per capita endowment is

$$\int_I w_1 = \sum_{t\in T} \int_{S_t} a_t = \sum_{t\in T} a_t \lambda(S_t).$$

Clearing the market for commodity one yields

$$\sum_{t\in T} \frac{\alpha_t(p_1 a_t + p_2 b_t)}{p_1} \lambda(S_t) = \sum_{t\in T} a_t \lambda(S_t).$$

Using the normalization $p_1 + p_2 = 1$ and solving for p_1, we conclude that in equilibrium

$$p_1 = \frac{\sum_{t\in T} a_t b_t \lambda(S_t)}{\sum_{t\in T} \left[(1-\alpha_t)a_t + \alpha_t b_t\right] \lambda(S_t).}$$

Exercise 3.7

Substituting into the utility function $u_i(x_i) = \sqrt{x_{i1}^2 + (i x_{i2})^2}$ yields

$$u_i(1,0) = 1 \quad \text{and} \quad u_i(0,1/i) = 1.$$

However, evaluating the utility of the commodity bundle $.5(1,0)+.5(0,1/i) = (.5,.5/i)$ gives $u_i(.5,.5/i) = 1/\sqrt{2}$ which is less than 1.

Exercise 3.8

In the pure exchange model, the inequality $p \cdot w_i > 3(p \cdot w_i - p_2)$ becomes $i < .15p_2/p_1$ while the inequality $3(p \cdot w_i - p_2) > 5(p \cdot w_i - 2p_2)$ becomes $i < .35p_2/p_1$. The per capita demand for automobiles is then

$$\int_I x_2 = \int_{.15p_2/p_1}^{.35p_2/p_1} 1 + \int_{.35p_2/p_1}^{1} 2 = 2 - .5\frac{p_2}{p_1}$$

while the per capita endowment, of course, remains the same: $\int_I w_2 = 1$. Clearing the market for automobiles and using the normalization $p_1 + p_2 = 1$ yields the equilibrium price functional $p = (1/3, 2/3)$. The equilibrium allocation is unchanged.

In the production model, the zero profit requirement $p_2 = \beta p_1$ combined with the normalization $p_1 + p_2 = 1$ yields the equilibrium price functional $p = (1/(1+\beta), \beta/(1+\beta))$.

Exercise 3.9

(a) If cars are produced, then the zero profit condition implies

$$0 = p \cdot y = \lambda(p_2 - 4p_3) = \lambda(p_2 - 4) \quad \text{for some} \quad \lambda > 0$$

and so $p_2 = 4$.

(b) The maximum utility achievable by consumer i is

$$V_0 := (1 + (1)(0) + (2)(0))(p \cdot w_i / p_3) = p \cdot w_i$$

if he chooses no car,

$$V_1 := (1 + (1)(1) + (2)(0))((p \cdot w_i - p_1)/p_3) = 2(p \cdot w_i - p_1)$$

if he chooses a used car, and

$$V_2 := (1 + (1)(0) + (2)(1))((p \cdot w_i - p_2)/p_3) = 3(p \cdot w_i - p_2)$$

if he chooses a new car.

Since we are given $k_1 < .5$, the relevant income for the consumer $i = k_1$ indifferent between owning no car and a used car is $p \cdot w_i = 12i$. Therefore, $V_0 = V_1$ iff

$$p \cdot w_i = 2(p \cdot w_i - p_1) \quad \text{iff} \quad i = p_1/6$$

so that $k_1 = p_1/6$.

Since we are given $k_2 > .5$, the relevant income for the consumer $i = k_2$ indifferent between owning a used car and owning a new car is $p \cdot w_i = 12i + p_1$. Therefore, $V_1 = V_2$ iff

$$2(p \cdot w_i - p_1) = 3(p \cdot w_i - p_2) \quad \text{iff} \quad i = (p_2 - p_1)/4$$

so that $k_2 = (p_2 - p_1)/4$.

(c) Since

$$\int_I x_1 = \frac{p_2 - p_1}{4} - \frac{p_1}{6}, \quad \int_I w_1 = .5, \quad \text{and} \quad p_2 = 4,$$

we obtain $p_1 = 6/5$ as the equilibrium price of a used car.

(d) Since $p_1 = 6/5$ and $p_2 = 4$,

$$k_1 = \frac{p_1}{6} = .2 \quad \text{and} \quad k_2 = \frac{p_2 - p_1}{4} = .7.$$

Therefore, 20 per cent (the interval $[0, .2)$) get no car, 50 per cent get a used car (the interval $[.2, .7)$), and 30 per cent get a new car (the interval $[.7, 1]$).

Exercise 3.10

Left to reader.

Exercise 3.11

(a) $X_i = \{0, 1\} \times \{0, 1\} \times \mathbf{R}_+$

(b) Setting $u_i(0, 0, p \cdot w_i) = u_i(1, 0, p \cdot w_i - p_1)$ for the consumer $i = k_1$ indifferent between no car and a Honda gives $k_1 = .2p_1$. Setting $u_i(1, 0, p \cdot w_i - p_1) = u_i(0, 1, p \cdot w_i - p_2)$ for the consumer $i = k_2$ indifferent between a Honda and a Porsche gives $k_2 = .3p_2 - .2p_1$. Finally, setting $u_i(0, 1, p \cdot w_i - p_2) = u_i(1, 1, p \cdot w_i - p_1 - p_2)$ for the consumer $i = k_3$ indifferent between owning "only" a Porsche and owning both a Honda and a Porsche gives $k_3 = .2p_1 + p_2$.

(c) Per capita demand and endowment for Hondas are

$$\int_I x_1 = k_2 - k_1 + 1 - k_3 \quad \text{and} \quad \int_I w_1 = .4$$

respectively. Per capita demand and endowment for Porsches are

$$\int_I x_2 = 1 - k_2 \quad \text{and} \quad \int_I w_2 = .5.$$

Clearing the two automobile markets gives

$$k_1 - k_2 + k_3 = .6 \quad \text{and} \quad k_2 = .5.$$

Using the results of part (b) to substitute for k_1, k_2 and k_3 and simplifying, the market clearing equations become

$$3p_1 - p_2 = 3 \quad \text{and} \quad -2p_1 + 3p_2 = 5$$

which have solution $p_1 = 2$ and $p_2 = 3$.

(d) Use the equilibrium prices and the results of part (b) to compute

$$k_1 = .4, \quad k_2 = .5, \quad \text{and} \quad k_3 = .7.$$

Since

$$\int_I p \cdot w = \int_0^{.9} 10i\; di + \int_{.9}^1 (4p_1 + 5p_2 + 10i)\; di = 7.3,$$

$$\begin{aligned} \int_I x_3 &= \int_0^{k_1} p \cdot w + \int_{k_1}^{k_2} (p \cdot w - p_1) + \int_{k_2}^{k_3} (p \cdot w - p_2) \\ &\quad + \int_{k_3}^{1} (p \cdot w - p_1 - p_2) \\ &= \int_0^1 p \cdot w - \int_{k_1}^{k_2} p_1 - \int_{k_2}^{k_3} p_2 - \int_{k_3}^{1} (p_1 + p_2) = 5 \end{aligned}$$

But

$$\int_I w_3 = \int_0^1 10i\; di = 5$$

so the market for the divisible commodity does clear.

(e) Left to reader.

Exercise 3.12

(a) For an activity vector $y = \lambda(1, 1, -\beta)$, the zero profit condition requires $p \cdot y = \lambda(p_1 + p_2 - \beta) = 0$. If $\lambda > 0$, then $p_1 + p_2 = \beta$.

(b) Integrating consumer demand gives

$$\int_I x_1 = (p_1)^{-1} \int_I \alpha_1 k \quad \text{and} \quad \int_I x_2 = (p_2)^{-1} \int_I \alpha_1 k.$$

Letting

$$q = \int_I x_1 = \int_I x_2$$

denote the per capita quantities of mutton and hides, we conclude that

$$\beta = p_1 + p_2 = q^{-1} \int_I (\alpha_1 + \alpha_2) k$$

and hence

$$q = \beta^{-1} \int_I (\alpha_1 + \alpha_2) k$$

and

$$p_1 = \beta \frac{\int_I \alpha_1 k}{\int_I (\alpha_1 + \alpha_2) k} \quad \text{and} \quad p_2 = \beta \frac{\int_I \alpha_2 k}{\int_I (\alpha_1 + \alpha_2) k}.$$

(c) $q = 1$, $p_1 = \lambda(S_1)$, $p_2 = \lambda(S - 2)$, and

$$x_i = \begin{cases} (1/\lambda(S_1), 0, 1) & \text{if } i \in S_1 \\ (0, 1/\lambda(S_2), 1) & \text{if } i \in S_2. \end{cases}$$

Exercise 3.13

(a) $\lambda(I) = \ell(I) = 1$.

(b) If $0 < x < 1$,

$$\lambda(\{x\}) = \lambda(I) - \lambda([0, x)) - \lambda((x, 1]) = 1 - x - (1 - x) = 0.$$

If $x = 0$,

$$\lambda(\{x\}) = \lambda(I) - \lambda((0, 1]) = 1 - 1 = 0.$$

If $x = 1$,

$$\lambda(\{x\}) = \lambda(I) - \lambda([0, 1)) = 1 - 1 = 0.$$

(c) Letting $T = \mathbf{Q} \cap [0, 1]$ and using the fact that $\mathbf{Q}$ (and hence T) is countable,

$$\lambda(S_1) = \lambda(\cup_{x \in T} \{x\}) = \sum_{x \in T} \lambda(\{x\}) = 0.$$

(d) $\lambda(S_2) = \lambda(I) - \lambda(S_1) = 1 - 0 = 1$.

Exercise 3.14

We check Definition 3.1:

$$f^{-1}((a, \infty)) = \begin{cases} \emptyset & \text{if } a > 1; \\ \mathbf{Q} \cap (0, 1] & \text{if } 0 \leq a \leq 1; \end{cases}$$

Therefore, f is measurable since each of the sets $\emptyset$, $\mathbf{Q}$, and $(0,1]$ is measurable.

Exercise 3.15

Since I is finite, any collection $\{ S_t \in \mathcal{C} \mid t \in T \}$ of disjoint subsets of I is necessarily finite. It suffices, therefore, to prove finite additivity. For counting measure,

$$\mu(\cup_{t\in T} S_t) = \#(\cup_{t\in T} S_t) = \sum_{t\in T} \# S_t = \sum_{t\in T} \mu(S_t).$$

For normalized counting measure,

$$\mu(\cup_{t\in T} S_t) = \frac{\# \cup_{t\in T} S_t}{\# I} = \frac{\sum_{t\in T} \# S_t}{\# I} = \sum_{t\in T} \mu(S_t).$$

Neither is a nonatomic measure. Normalized counting measure is a probability measure.

Exercise 3.16

THEOREM: Let $\mathcal{E} = (\{ X_i, P_i, w_i \mid i \in I \}, \{ Y(S) \mid S \subset I \})$ be an Aumann economy with $(I, \mathcal{B}(I), \lambda)$ a nonatomic measure space of consumers and $Y(S)$ a nonempty, pointed, convex cone with $Y(S) \subset Y(I)$ for all $S \subset I$. Then $\mathrm{WE}^x(\mathcal{E}) \subset \mathrm{C}(\mathcal{E}) \subset \mathrm{PO}(\mathcal{E})$.

PROOF: The inclusion $\mathrm{C}(\mathcal{E}) \subset \mathrm{PO}(\mathcal{E})$ is immediate. To show that $\mathrm{WE}^x(\mathcal{E}) \subset \mathrm{PO}(\mathcal{E})$, suppose not. Then there exists an allocation $x \in \mathrm{WE}^x(\mathcal{E})$ but $x \notin \mathrm{C}(\mathcal{E})$: i.e., there is a coalition $S \in \mathcal{B}(I)$ with $\lambda(S) > 0$ and an allocation $x' \in \mathrm{F}(\mathcal{E}, S)$ such that $x'_i \succ_i x_i$ for a.e. $i \in S$. But then $p \cdot x'_i > p \cdot w_i$ a.e. $i \in S$ and so $\int_S p \cdot x' > \int_S p \cdot w$ or

$$\int_S p \cdot (x' - w) = p \cdot \int_S (x' - w) =: p \cdot y' > 0,$$

contradicting the hypothesis that $x' \in \mathrm{F}(\mathcal{E}, S)$ and hence that $p \cdot y' = p \cdot \int_S (x' - w) \le p \cdot 0 = 0$.

Exercise 3.17

Letting $t_1 = (\alpha_1, (a_1, b_1))$ and $t_2 = (\alpha_2, (a_2, b_2))$, we have

$$\tau = .25\delta_{t_1} + .75\delta_{t_2}.$$

In the example, $x_1 = (3, 9/5)$ and $x_2 = (7/3, 7/5)$. Letting $f_1 = (x_1, t_1)$ and $f_2 = (x_2, t_2)$, we have

$$\mu = .25\delta_{f_1} + .75\delta_{f_2}.$$

Exercise 3.18

Let $\succ$ denote the stairstep preference common to all consumers, and $w = (1.5, 1.5)$ the common endowment vector. Letting $t = (\succ, w)$, we have $\tau = \delta_t$.

Letting $f_1 = (x_1, t_1)$ and $f_2 = (x_2, t_2)$ where $x_1 = (.5, 2.5)$ and $x_2 = (2.5, .5)$, we have

$$\mu = .5\delta_{t_1} + .5\delta_{t_2}.$$

Exercise 3.19

For $\omega \in [0, 3)$,

$$x(\emptyset) = 0, x(\{k\}) = 0, x(\{c\} = \omega, \quad \text{and} \quad x(\{\, k, c \,\}) = \omega.$$

For $\omega \in [3, 7)$,

$$x(\emptyset) = 0, x(\{k\}) = 1, x(\{c\} = \omega - 2, \quad \text{and} \quad x(\{\, k, c \,\}) = \omega - 1.$$

For $\omega \in [3, 10]$,

$$x(\emptyset) = 0, x(\{k\}) = 2, x(\{c\} = \omega - 4, \quad \text{and} \quad x(\{\, k, c \,\}) = \omega - 2.$$

Since by definition, $x \in M(J)$ takes the form $x = x_c\delta_c + x_k\delta_k$, the vectors $\{\, \delta_c, \delta_k \,\}$ span $M(J)$. Applying the measure $\alpha\delta_c + \beta\delta_k$ to the set $\{c\}$ gives

$$(\alpha\delta_c + \beta\delta_k)\{c\} = \alpha$$

while applying it to $\{k\}$ gives

$$(\alpha\delta_c + \beta\delta_k)\{k\} = \beta.$$

Therefore,

$$\alpha\delta_c + \beta\delta_k = 0 \quad \text{implies} \quad \alpha = \beta = 0,$$

which proves that δ_c and δ_k are independent.

Exercise 3.20

(a) $V^k = (\omega - \beta k^2)(1 + \alpha k^2)$. Therefore,

$$\frac{dV^k}{dk} = (\omega - \beta k^2)(2\alpha k) + (1 + \alpha k^2)(-2\beta k) = 0$$

implies the optimum

$$k^* = \sqrt{\frac{\alpha\omega - \beta}{2\alpha\beta}}.$$

If $\alpha = 1$, $\beta = 2$, and $\omega = 10$, $k^* = \sqrt{2}$.

(b) $k(\omega) = \sqrt{(\omega - 2)/4}$ which has support $[1, \sqrt{3}]$. The distribution is clearly not uniform.

Exercise 3.21

We take as parameters the values given on page 137 of the text: $\alpha = 1$, $\beta = 2$, and $\bar{k} \geq 3$. In the convexified economy, a consumer with endowment $w = \omega\delta_c$ will choose a jurisdiction offering a local public good of quality

$$k = \frac{\omega - 2}{4};$$

i.e.,

$$k = \begin{cases} 1.0 & \text{if } \omega = 6; \\ 1.5 & \text{if } \omega = 8; \\ 2.0 & \text{if } \omega = 10; \\ 2.5 & \text{if } \omega = 12; \\ 3.0 & \text{if } \omega = 14. \end{cases}$$

The corresponding Rosen diagram is shown in Figure 12.

Since each of the five types has measure .2 while scale economies are exhausted for jurisdictions serving .1, the Tiebout equilibrium for the convexified economy will also be an equilibrium for the original economy. However, if economies of scale are exhausted only for jurisdictions serving at least a quarter of the population, then there is no Tiebout equilibrium for the original economy.

Figure 12: Rosen diagram

4 Topology

Exercise 4.1

(a) Let (X, τ) be a topological space. Note first that the definitions of T_4 and T_3 explicitly assume that (X, τ) is T_1. Consequently, singtleton sets are closed. To prove this, consider any point $y \neq x$. Since (X, τ) is T_1, there is an open neighborhood G_y of y disjoint from $\{x\}$. Because

$$X \backslash \{x\} = \bigcup_{y \in X \backslash \{x\}} \{y\},$$

we conclude that

$$X \backslash \{x\} \subset \bigcup_{y \in X \backslash \{x\}} G_y \quad \text{and so} \quad X \backslash \{x\} = \bigcup_{y \in X \backslash \{x\}} G_y.$$

Since the right hand side is a union of open sets, it is open. Thus, the complement of $\{x\}$ is open and hence $\{x\}$ is closed.

$T_4 \Rightarrow T_3$: For any $x \in X$, the singleton set $\{x\}$ is closed. Let F be a closed set such that $x \notin F$. Because (X, τ) is T_4, the closed sets $\{x\}$ and F are contained in open neighborhoods G_x and G_F which are disjoint and so (X, τ) is T_3.

$T_3 \Rightarrow T_2$: Given $x, y \in X$, the singleton sets $\{x\}$ and $\{y\}$ are closed. If $x \neq y$, then the fact that (X, τ) is T_3 implies the existence of open neighborhoods G_x and G_y of $\{x\}$ and $\{y\}$ which are disjoint. Therefore, (X, τ) is T_2.

$T_2 \Rightarrow T_1$: Note that here, in contrast to the previous two steps, we do not **assume** (X, τ) is T_1. Given $x, y \in X$ such that $x \neq y$, the fact that (X, τ) is T_2 implies the existence of open neighborhoods G_x and G_y of x and y which are disjoint. Consequently, $y \notin G_x$ which shows that (X, τ) is T_1.

(b) If (X, τ) has the discrete topology, every subset of X is open. Let F and F' be any pair of disjoint subsets of X. Because their complements are open sets (recall that under the discrete topology **every** subset is open), F and F' are closed sets. But they are also open sets (because every subset is an open subset): i.e., they are "clopen." Define $G_F = F$ and $G_{F'} = F'$. These neighborhoods of F and F' are open and disjoint. We conclude that (X, τ) is T_4.

(c) For any $x, y \in X$ such that $x \neq y$, the only open set other than $\emptyset$ is X. But if $G_x = X$, then clearly $y \in G_x$. Therefore, (X, τ) is not T_1.

To show that singleton sets are not closed, note that $\{x\}$ is closed iff $X \backslash \{x\}$ is open. But if X contains more than one point, then $X \backslash \{x\} \neq X$ and $X \backslash \{x\} \neq \emptyset$. Therefore, $X \backslash \{x\}$ is not an open set and hence $\{x\}$ is not closed.

Exercise 4.2

Let β represent the set of open intervals in $\mathbf{R}$.

- For any $x \in X$ and any $\epsilon > 0$, $x \in (x - \epsilon, x + \epsilon) \in \beta$.
- Let $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$ be any two intervals and $x \in B_1 \cap B_2$. Then $B_3 := B_1 \cap B_2$ is an open interval and $x \in B_3 \subset B_1 \cap B_2$. (What happens if B_1 and B_2 are disjoint? Then the statement is true "by vacuous implication"!)

We conclude that β satisfies the two requirements of a basis.

Exercise 4.3

(a) Because $(-\infty, -\alpha)$ and (α, ∞) are open, their union is open. Therefore, for any $\alpha \geq 0$, the set $[-\alpha, \alpha] = \mathbf{R} \backslash \{ (-\infty, -\alpha) \cup (\alpha, \infty) \}$ is closed. Consequently, $\cap_{\alpha \in \mathbf{R}_+} [-\alpha, \alpha]$ is closed because it is the intersection of a collection of closed sets.

Since $\cap_{\alpha \in \mathbf{R}_+} [-\alpha, \alpha] = \{0\}$ (cf. Exercise 1.2), you can also reach the same conclusion by noting that the singleton set $\{0\}$ is closed.

(b) Although $(-\alpha, \alpha)$ if an open set for each $\alpha > 0$, the intersection $\cap_{\alpha \in \mathbf{R}_{++}} (-\alpha, \alpha)$ is not a **finite** intersection of open sets and hence need not be open. In fact, it is not because $\cap_{\alpha \in \mathbf{R}_{++}} (-\alpha, \alpha) = \{0\}$ (prove this following along the lines of Exercise 1.2) and $\{0\}$ is closed but not open. Since $\cap_{\alpha \in \mathbf{R}_{++}} (-\alpha, \alpha)$ is an intersection of open sets, it is a Borel set. However, because the index set R_{++} is not countable, this set need not be a G_δ. Nevertheless, in this case it is a G_δ set because

$$\bigcap_{\alpha \in \mathbf{R}_{++}} (-\alpha, \alpha) = \bigcap_{\alpha \in \mathbf{Z}_{++}} (-\alpha, \alpha) \quad (= \{0\})$$

(which is easy to prove) and $\mathbf{Z}_{++}$ is a countable set.

Exercise 4.4

Let $B = \prod_{\alpha \in A} B_\alpha$ be a typical basis element for the product topology where $B_\alpha = X_\alpha$ except for $\alpha \in A'$ where $A' := \{\alpha_1, \ldots, \alpha_n\}$ is a finite subset of A. Define

$$B'_i = \prod_{\alpha \in A} \tilde{B}_\alpha$$

where

$$\tilde{B}_\alpha = \begin{cases} B_\alpha & \text{if } \alpha = \alpha_i; \\ X_\alpha & \text{otherwise.} \end{cases}$$

Then $B = \cap_{i=1}^n B'_i$ as was to be shown.

Let B_1 and B_2 be open intervals in $\mathbf{R}$ so that $B_1 \times B_2$ is an open rectangle. (Note: in general, any set of the form $G_1 \times G_2$, where G_1 and G_2 are open subsets but not necessarily open intervals of $\mathbf{R}$, is conventionally called an "open rectangle" of $\mathbf{R}^2$. Here we are considering a special case.) Letting

$$B'_1 = B_1 \times \mathbf{R} \quad \text{and} \quad B'_2 = \mathbf{R} \times B_2,$$

we have

$$B_1 \times B_2 = B'_1 \cap B'_2.$$

Exercise 4.5

Given a ball $B(x, \epsilon)$ with $\epsilon > 0$, choose a rational number ϵ' such that $0 < \epsilon' < \epsilon$. (This exploits the fact that every open interval contains a rational. In fact, we can choose an integer $n \in \mathbf{Z}_{++}$ such that $0 < 1/n < \epsilon$.) Then

$$B(x, \epsilon') \subset B(x, \epsilon).$$

On the other hand, if $B(x, \epsilon')$ is a ball with (strictly positive) rational radius ϵ', then we can certainly find a number $\epsilon \in \mathbf{R}_+$ such that $0 < \epsilon < \epsilon'$. Therefore,

$$B(x, \epsilon) \subset B(x, \epsilon').$$

We conclude that both collections of open balls, those with arbitrary (strictly positive) radii and those with (strictly positive) rational radii, are bases for the same topology.

Exercise 4.6

(a) Reflexive: $S \subset S$ for all $S \in \mathcal{S}$.

Transitive: $R \subset S$ and $S \subset T$ implies $R \subset T$ for all $R, S, T \in \mathcal{S}$.

Antisymmetric: $S \subset T$ and $T \subset S$ implies $S = T$ for all $S, T \in \mathcal{S}$.

(b) Let $X = \{ a, b \}$ and $\mathcal{S} = 2^X$. If $a \neq b$, then $\{a\} \not\subset \{b\}$ and $\{b\} \not\subset \{a\}$. Therefore, $\subset$ does not totally order $\mathcal{S}$.

Let $X = \{ a, b, c \}$ and $\mathcal{S} = \{ \emptyset, \{a\}, \{ a, b \}, X \}$. Then $\emptyset \subset \{a\} \subset \{ a, b \} \subset X$ and so $\mathcal{S}$ is totally ordered.

Exercise 4.7

(a) ($\Rightarrow$): Consider a net $x^\alpha \to x$. Since the net converges, for any open neighborhood G_α there exists an α^* such that $x^\alpha \in G_x$ for all $\alpha \succeq \alpha^*$. However, with the discrete topology, $G_x := \{x\}$ is itself an open neighborhood of $\{x\}$ and hence $x^\alpha \in G_x$ implies $x^\alpha = x$. Therefore, $x^\alpha = x$ for all $\alpha \succeq \alpha^*$ as claimed.

($\Leftarrow$): Consider a net $\langle x^\alpha \mid \alpha \in A \rangle$ and an $\alpha^* \in A$ such that $x^\alpha = x$ for all $\alpha \succeq \alpha^*$. Any open neighborhood G_x of x contains x, and so $x^\alpha \in G_x$ for all $\alpha \succeq \alpha^*$. Therefore, $x^* \to x$.

(b) Let $\langle x^\alpha \mid \alpha \in A \rangle$ be a net in (X, τ). With the trivial (indiscrete) topology, the only neighborhood containing x is X itself. For $G_x = X$, $x^\alpha \in G_x$ for **all** $\alpha \in A$. Therefore, letting α^* be any element of A, we conclude that $x^\alpha \in G_x$ for all $\alpha \succeq \alpha^*$ and hence that $x^\alpha \to x$.

Exercise 4.8

(a) By definition, $\operatorname{int} S$ is the union of all the open sets contained in S, and any union of open sets is open. Similarly, $\operatorname{cl} S$ is the intersection of all closed sets containing S, and any intersection of closed sets is closed.

(b) (i) ($\Rightarrow$): If S is closed, then S is one of the closed sets which contains S and is clearly contained in any closed set which contains S. Thus, the intersection of closed sets containing S equals S: i.e., $\operatorname{cl} S = S$.

(⇐): If $S = \mathrm{cl}\, S$, then S equals the intersection of a collection of closed sets and hence is itself closed.

(*ii*) (⇒): If S is open, then S is one of the open sets contained in S and clearly it contains any other open set contained in S. Thus, the union of the open sets contained in S equals S: i.e., $\mathrm{int}\, S = S$.

(⇐): If $S = \mathrm{int}\, S$, then S equals the union of a collection of open sets and hence is itself open.

(c) Since $S \subset \mathrm{cl}\, S$ and $\mathrm{acc}\, S \subset S$, we have $S \cup \mathrm{acc}\, S \subset \mathrm{cl}\, S$. To show the reverse inclusion, suppose that $x \in \mathrm{cl}\, S$ but $x \notin S$. By Theorem 4.15, $x \in \mathrm{cl}\, S$ iff $G_x \cap S \neq \emptyset$ for every open neighborhood G_x of x. Since $x \notin S$, this means that $G_x \cap (S \backslash \{x\}) \neq \emptyset$ for every such G_x and hence that

$$x \in \mathrm{acc}\, S := \{\, x \in X \mid G_x \cap (S \backslash \{x\}) \neq \emptyset \quad \forall G_x \in \tau \ \ni x \in G_x \,\}.$$

Exercise 4.9

(a) Because x is isolated, there is no net in $X \backslash \{x\}$ which converges to x. Therefore, according to Theorem 4.15, $x \in \mathrm{int}\{x\}$ which implies that $\{x\}$ is open. However, since x is isolated, $\mathrm{cl}\{x\} = \{x\}$ and so $\{x\}$ is also closed.

(b) $x \in \mathrm{acc}\, S$ iff there exists a net in $S \backslash \{x\}$ which converges to x. Since x is isolated, there is no such net.

Exercise 4.10

This is largely left to your imagination. Here is one example: Let

$$S = \{\, (x_1, x_2) \in \mathbf{R}^2 \mid -1 < x_1 \leq 1 \ \& \ -1 < x_2 \leq 1 \,\},$$

a rectangle which contains its upper and its right bounding line but not its left or lower bounding line. Then

$$\mathrm{int}\, S = \{\, (x_1, x_2) \in \mathbf{R}^2 \mid -1 < x_1 < 1 \ \& \ -1 < x_2 < 1 \,\},$$

the rectangle with all bounding lines excluded;

$$\mathrm{cl}\, S = \{\, (x_1, x_2) \in \mathbf{R}^2 \mid -1 \leq x_1 \leq 1 \ \& \ -1 \leq x_2 \leq 1 \,\},$$

the rectangle with all bounding lines included;

$$\operatorname{bd} S = \operatorname{cl} S \setminus \operatorname{int} S,$$

the boundary lines of the rectangle; and

$$\operatorname{acc} S = \operatorname{cl} S,$$

once again the rectangle with all bounding lines included.

Exercise 4.11

We want to show that $\operatorname{cl} \mathbf{Q} = \mathbf{R}$. Suppose not. If $x \in \mathbf{R}$ but $x \notin \operatorname{cl} \mathbf{Q}$, then by Theorem 4.15 there is an open neighborhood G_x of x such that $G_x \cap \mathbf{Q} = \emptyset$. Since the open intervals constitute a basis for the standard topology on $\mathbf{R}$, there exists an open interval I such that $x \in I \subset G_x$. However, since $G_x \cap \mathbf{Q} = \emptyset$, we conclude that $I \cap \mathbf{Q} = \emptyset$ which is impossible because every open interval contains a rational number.

Exercise 4.12

The intervals $(-\infty, a]$ and $[a, \infty)$ are closed subsets of $\mathbf{R}$. Therefore, if f is continuous, then according to Theorem 4.17(c) the inverse images $f^{-1}(-\infty, a]$ and $f^{-1}[a, \infty)$ must be closed.

To establish the converse, first note that for any function $f\colon X \to Y$ and any subset $S \subset Y$,

$$f^{-1}(Y \setminus S) = X \setminus f^{-1}(S).$$

We remarked following Theorem 4.17 that a function $f\colon \mathbf{R} \to \mathbf{R}$ is continuous iff the inverse images $f^{-1}(-\infty, a)$ and $f^{-1}(a, \infty)$ are open for all $a \in \mathbf{R}$. Since $f^{-1}(-\infty, a) = \mathbf{R} \setminus f^{-1}[a, \infty)$, the inverse image $f^{-1}(-\infty, a)$ is open iff the inverse image $f^{-1}[a, \infty)$ is closed. Similarly, since $f^{-1}(a, \infty) = \mathbf{R} \setminus f^{-1}(-\infty, a]$, the inverse image $f^{-1}(a, \infty)$ is open iff the inverse image $f^{-1}(-\infty, a]$ is closed. We conclude that f is continuous iff the inverse images $f^{-1}(-\infty, a]$ and $f^{-1}[a, \infty)$ are closed for all $a \in \mathbf{R}$.

Applying this criterion to the function $f\colon x \mapsto x^2$, we have

$$f^{-1}(-\infty, a] = \begin{cases} [-\sqrt{a}, \sqrt{a}] & \text{if } a \geq 0; \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$f^{-1}[a, \infty) = \begin{cases} (-\infty, -\sqrt{a}] \cup [\sqrt{a}, \infty] & \text{if } a \geq 0; \\ \mathbf{R} & \text{otherwise} \end{cases}$$

which are closed sets for all $a \in \mathbf{R}$. Therefore, the function f is continuous.

Exercise 4.13

Note first that for any $G \subset Z$,

$$x \in f^{-1}(g^{-1}(G)) \iff f(x) \in g^{-1}(G) \iff g(f(x)) \in G$$
$$\iff g \circ f(x) \in G \iff x \in (g \circ f)^{-1}(G)$$

so that

$$f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G) \quad \text{for all} \quad G \subset Z.$$

Let G be any open subset of Z and suppose that the functions f and g are continuous. Continuity of g implies that $g^{-1}(G)$ is open. Continuity of f implies that $f^{-1}(g^{-1}(G))$ is open. Therefore, $(g \circ f)^{-1}(G)$ is open for any open set $G \subset Z$ which establishes the continuity of the function $g \circ f$.

Exercise 4.14

(a) A constant function, $f\colon \mathbf{R} \to \mathbf{R}$ mapping $x \mapsto c \in \mathbf{R}$, is continuous and closed but not open: $f(S) = \{c\}$, a closed set, for every set S (open or not) in the domain of f.

The function

$$f\colon \mathbf{R} \to \mathbf{R}, \qquad x \mapsto \frac{x}{1 + |x|}$$

is continuous and open but not closed: $f(\mathbf{R}) = (-1, 1)$, an open set.

(b) Let f be open and $F \subset X$ closed. Then $F = X \backslash G$ where G is open and so

$$f(F) = f(X) \backslash f(G) = Y \backslash f(G)$$

(this is where we use the fact that f is bijective — otherwise, we could only assert that $f(F) \subset f(X) \backslash f(G)$). Since f is open, $f(G)$ is an open set. Therefore, $f(F)$ is closed because it is the complement of an open set. Because this holds for any closed set $F \subset X$, we conclude that f is a closed map.

Similarly, suppose that f is closed and $G \subset X$ is open. Then $G = X \backslash F$ where F is closed and so, because f is bijective,

$$f(G) = f(X) \backslash f(F) = Y \backslash f(F).$$

Since f is closed, $f(F)$ is a closed set. Therefore, $f(G)$ is open because it is the complement of a closed set. Since G was arbitrary, this proves that f is an open map.

Exercise 4.15

Because the standard topology of $\mathbf{R}^2$ is T_4 and the subsets $\operatorname{cl} S_1$ and $\operatorname{cl} S_2$ are closed, we can find two disjoint sets G_1 and G_2 such that $\operatorname{cl} S_1 \subset G_1$ and $\operatorname{cl} S_2 \subset G_2$. Letting $Y = S_1 \cup S_2$, the sets $S_1 = G_1 \cap Y$ and $S_2 = G_2 \cap Y$ are open sets **in the relative topology of Y**. Therefore, Y equals the union of two disjoint open sets which means, by definition, that it is disconnected.

Exercise 4.16

($\Rightarrow$): Since one-point sets are open in the discrete topology, the collection

$$\{ G_x \subset X \mid G_x = \{x\} \ \forall x \in X \}$$

is an open cover of X: i.e.,

$$X = \bigcup_{x \in X} G_x.$$

Because X is compact, this cover has a finite subcover: i.e.,

$$X = \bigcup_{x \in X'} G_x := \{ x_1, \ldots, x_n \}$$

for some $n < \infty$. Thus, X is finite.
($\Leftarrow$): Suppose that X is finite, say $X = \{ x_1, \ldots, x_n \}$, and let $\{ G_\alpha \subset X \mid \alpha \in A \}$ be an open covering. Since each point x_i can be covered by one of the open sets in this covering, we can cover X with a finite subcover. Therefore, X is compact.

Exercise 4.17

$$X \backslash \bigcup_{\alpha \in A} S_\alpha = \bigcap_{\alpha \in A} (X \backslash S_\alpha)$$

because

$$x \in X \backslash \bigcup_{\alpha \in A} S_\alpha \iff x \in X \ \& \ x \notin S_\alpha \ \forall \alpha \in A \iff$$

$$x \in X \backslash S_\alpha \ \forall \alpha \in A \iff x \in \bigcap_{\alpha \in A} (X \backslash S_\alpha).$$

Exercise 4.18

(a) Let $f: X \to Y$ be a continuous function and $C \subset X$ a connected set. We want to prove that $f(C)$ is connected. Suppose not so that $f(C) = G_1 \cup G_2$ where G_1 and G_2 are nonempty, disjoint and open. Then $C = f^{-1}(G_1 \cup G_2) = f^{-1}(G_1) \cup g^{-1}(G_2)$. The sets $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are open in (X, τ_X) because f is continuous, disjoint because G_1 and G_2 are disjoint, and nonempty because G_1 and G_2 are nonempty. Therefore, C is the union of two nonempty, disjoint and open sets which means that C is disconnected, a contradiction. We conclude that $f(C)$ is connected.

Let $f: X \to Y$ be a continuous function and $K \subset X$ a compact set. We want to prove that $f(K)$ is compact. If

$$\Gamma = \{ G_\alpha \in \tau_Y \mid \alpha \in A \}$$

is a covering of $f(K)$ by sets open in (Y, τ_Y), then the collection

$$\{ f^{-1}(G_\alpha) \in \tau_X \mid \alpha \in A \}$$

is a covering of K by open sets (since f is continuous). Because K is compact, there is a finite subcovering, say

$$K \subset f^{-1}(G_1) \cup f^{-1}(G_2) \cup \cdots \cup f^{-1}(G_n).$$

But then

$$f(K) \subset G_1 \cup G_2 \cup \cdots G_n;$$

i.e., the open covering Γ has a finite subcover. Thus, K is compact.

(b) If $f(x) = x^2$ and $C = [1, 4]$, then $f^{-1}(C) = [-2, -1] \cup [1, 2]$ is not connected. If $f(x) = 0$ for all $x \in \mathbf{R}$ and $K = [-1, 1]$, then $f^{-1}(K) = \mathbf{R}$ is not compact.

Exercise 4.19

(a) Let $\{ (X_\alpha, \tau_\alpha) \mid \alpha \in A \}$ be a collection of compact topological spaces and let

$$X = \prod_{\alpha \in A} X_\alpha.$$

If X is given the product topology, then according to the Tychonoff Theorem X is compact. Suppose that τ' is another topology on X weaker than τ: i.e., $\tau' \subset \tau$. Consider any covering Γ of X by sets open in (X, τ'). Because $\tau' \subset \tau$, every set $G_\alpha \in \Gamma \subset \tau'$ also belongs to τ. Therefore, Γ is also a covering of X by sets open in τ. But X is compact in the product toplogy which implies that Γ has a finite subcovering. We conclude that X is compact in the topology τ'.

(b) Let $\{ (X_\alpha, \tau_\alpha) \mid \alpha \in A \}$ be a collection of compact topological spaces and let

$$X = \prod_{\alpha \in A} X_\alpha.$$

Recall that an element $x \in X$ is a function of the form

$$x\colon A \to \bigcup_{\alpha \in A} X_\alpha, \qquad \alpha \mapsto x_\alpha \in X_\alpha.$$

The open set $\tilde{B}_\alpha$ described in the text sets its α'-coordinate space $B_{\alpha'}$ equal to $X_{\alpha'}$ which insures that the α'-coordinate of x is "covered" by $\tilde{B}_{\alpha'}$: i.e., $x_{\alpha'} \in B_{\alpha'}$. Thus,

$$x \in \bigcup_{\alpha \in A} \tilde{B}_\alpha$$

for any $x \in X$ which establishes that the collection $\{ \tilde{B}_\alpha \mid \alpha \in A \}$ covers X with sets open in the box topology.

Now consider a function $x^* \in X$ with the property that $x^*_\alpha \in X_\alpha \backslash B^*_\alpha$ for every $\alpha \in A$: i.e., each of the "coordinates" x^*_α of x lies outside the corresponding $B^*_\alpha \subset\subset X_\alpha$. Since $\tilde{B}_\alpha$ is the only element of the cover for which $x^*_\alpha \in B_\alpha$, we cannot exclude $\tilde{B}_\alpha$ and still cover all of X. Because the same is true for each $\alpha \in A$, there is no finite subcover.

Exercise 4.20

Suppose that β_0 is a countable local base at 0 for a TVS. By definition, then means that this TVS is first countable. However, it need not be second countable. A basis for the topology is given by the union

$$\bigcup_{x \in X} \beta_x \quad \text{where} \quad \beta_x = \{ B_x \subset L \mid B_x = x + B_0,\ B_0 \in \beta_0 \}$$

which will contain an uncountable number of basis elements if X is uncountable.

In the case of the Euclidean topology on $\mathbf{R^n}$, the collection of balls

$$\{ B(0, 1/n) \mid n \in \mathbf{Z}_{++} \}$$

constitute a countable local base at 0. This linear topology is also second countable because we can obtain a basis for the Euclidean topology on $\mathbf{R^n}$ by considering only translations of these balls of the form

$$B_x := x + B(0, 1/n) \quad \text{where} \quad x \in \mathbf{Q^n}.$$

The union

$$\bigcup_{x \in \mathbf{Q^n}} \beta_x$$

is countable because a countable union of countable sets is countable.

Exercise 4.21

Using the Euclidean metric, a typical ball centered at 0 takes the form

$$B(0, \epsilon) = \{ y \in \mathbf{R}^2 \mid \sqrt{y_1^2 + y_2^2} < \epsilon \}$$

which translates to a ball centered at x of the form

$$B(x, \epsilon) = \{ y \in \mathbf{R}^2 \mid \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < \epsilon \}.$$

Using the sup metric, a typical ball centered at 0 takes the form

$$B(0, \epsilon) = \{ y \in \mathbf{R}^2 \mid \sup\{ y_1, y_2 \} < \epsilon \}$$

which translates to a ball centered at x of the form

$$B(x, \epsilon) = \{ y \in \mathbf{R}^2 \mid \sup\{ y_1 - x_1, y_2 - x_2 \} < \epsilon \}.$$

Since the translate of a convex set is convex, it suffices to show that the typical ball $B(0, \epsilon)$ centered at 0 is convex. The key fact is that both

$$\|x\|_2 := \left[\sum_{i=1}^{n} x_i^2 \right]^{\frac{1}{2}}$$

and

$$\|x\|_\infty := \max_i \{ |x_1|, \ldots, |x_n| \}$$

are norms, and norms are convex functions. To prove the latter statement, note that by Definition 4.24, any norm $\|\cdot\|$ on a vector space L satisfies

- $\|\alpha x\| = |\alpha| \, \|x\|$ for every $x \in L$ and for every $\alpha \in \mathbf{R}$; and

- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in L$.

Therefore, for all $x, y \in L$ and for all $\alpha \in [0, 1]$,

$$\|\alpha x + (1-\alpha)y\| \leq \|\alpha x\| + \|(1-\alpha)y\| = \alpha \|x\| + (1-\alpha) \|y\|$$

which proves that $\|\cdot\|$ is convex.

We will now prove that the balls in any normed vector space are convex. Let

$$B(0, \epsilon) := \{ x \in L \mid \|x\| < \epsilon \}.$$

If $x, y \in B(0, \epsilon)$, then by definition $\|x\| < \epsilon$ and $\|y\| < \epsilon$. Therefore, for any $x, y \in B(0, \epsilon)$ and for any $\alpha \in [0, 1]$,

$$\|\alpha x + (1-\alpha)y\| \leq \alpha \|x\| + (1-\alpha)y < \epsilon$$

and hence

$$\alpha x + (1-\alpha)y \in B(0, \epsilon).$$

We conclude that $B(0, \epsilon)$ is convex in any normed vector space and so, in particular, in the vector space $\mathbf{R^n}$ normed by $\|\cdot\|_2$ or $\|\cdot\|_\infty$.

Exercise 4.22

($\Rightarrow$): Given $a \in \operatorname{lint} A \cap S$, let $z \neq a$ be another point on the line S. Let $w := z - a$ so that $a \in [a-w, a+w] \subset S$. Since $a \in \operatorname{lint} A$ and $a + w \in L \backslash \{a\}$, there exists a $y \in (a, a+w)$ such that $[a, y) \subset A$. Similarly, there exists an $x \in (a-w, a)$ such that $(x, a] \subset A$. Therefore, $a \in (x, y) = (x, a] \cup [a, y) \subset A \cap S$ as claimed.
($\Leftarrow$): Let $a \in A$ and $z \in L \backslash \{a\}$. By hypothesis, the line passing through a and z contains a linearly open interval (x, y) such that $a \in (x, y) \subset A \cap S$, and we can clearly arrange that $y \in (a, z)$. Therefore, we have shown that there exists $y \in (a, z)$ such that $[a, y) \subset A$ as claimed.

Exercise 4.23

To be added.

Exercise 4.24

The equivalence of the assertions

- p is continuous;

- $H^+(p, \alpha)$ and $H^-(p, \alpha)$ are closed for all $\alpha \in \mathbf{R}$;

- $H_o^+(p,\alpha)$ and $H_o^-(p,\alpha)$ are open for all $\alpha \in \mathbf{R}$

follows immediately from Theorem 4.17 since

- $H^+(p,\alpha) = p^{-1}[\alpha,\infty)$ and $H^-(p,\alpha) = p^{-1}(-\infty,\alpha]$; and
- $H_o^+(p,\alpha) = p^{-1}(\alpha,\infty)$ and $H_o^-(p,\alpha) = p^{-1}(-\infty,\alpha)$.

Any of these three statements also implies

- $H(p,\alpha)$ is closed for all $\alpha \in \mathbf{R}$

since

$$H(p,\alpha) = p^{-1}(\alpha) = H^+(p,\alpha) \cap H^-(p,\alpha)$$

for any $\alpha \in \mathbf{R}$.

It remains to prove that

- $H(p,\alpha)$ is closed for all $\alpha \in \mathbf{R}$

implies that p is continuous. In fact, all that is required is that $H(p,0)$ be closed. In contrast to everything proved thus far, this claim makes use of the fact that p is a **linear** functional and hence that $H(p,0)$ is a subspace of codimension one. See H. Schaefer (1970), *Topological Vector Spaces*, pages 24–25 for a short proof.

Exercise 4.25 To avoid confusion, instead of using 0 we will let $z\colon L \to \mathbf{R}$ denote the linear functional which maps every vector $x \in L$ to 0: i.e., $z(x) = 0$ for all $x \in L$. By definition, $z \in L^*$ if z is continuous: i.e., if $z^{-1}(G)$ is open for every open set $G \subset \mathbf{R}$. Since z maps every vector $x \in L$ to 0, (i) $z^{-1}(G) = \emptyset$ if $0 \notin G$ and (ii) $z^{-1}(G) = L$ if $0 \in G$. Every topology on L includes the sets L and $\emptyset$. Therefore, $z \in L^*$. (Note: this proves that the dual space L^* is nonempty: the 0 vector, which every vector must contain, is always in L^*.)

Exercise 4.26 Let

$$\begin{aligned} A &= \{\, x \in \mathbf{R}^2 \mid x_1 > 0 \ \&\ x_2 \geq 1/x_1 \,\} \\ B &= \{\, x \in \mathbf{R}^2 \mid x_2 \leq 0 \,\} \end{aligned}$$

The only separating hyperplane is the x_1-axis which separates A and B but neither strictly nor strongly separates them.

Exercise 4.27 If $\hat{x}$ is a support point for A, then by definition it lies both on the supporting hyperplane and in A. However, if A is open, $\hat{x}$ would then be in the interior of A and consequently contained in an open neighborhood contained in A: say $\hat{x} \in G_{\hat{x}} \subset A$. But since $\hat{x}$ lies on the hyperplane, $G_{\hat{x}}$ would then intersect both open halfspaces of the hyperplane, an impossibility since the hyperplane supports A.

The support points of $\operatorname{cl} A$ are the points in $(\operatorname{cl} A) \setminus \operatorname{int} A$, the boundary of A.

5 Best response

Exercise 5.1

$=$ on $\mathbf{R}$ is: (a) reflexive: $x = x$ for all $x \in \mathbf{R}$; (b) not irreflexive since it is reflexive; (c) not complete since, e.g., neither $1 = 2$ nor $2 = 1$; (d) transitive: $x = y$ and $y = z$ implies $x = z$ for all $x, y, z \in \mathbf{R}$; (e) not negatively transitive: $1 \neq 2$ and $2 \neq 1$ but $1 = 1$; (f) symmetric: $x = y$ implies $y = x$ for all $x, y \in \mathbf{R}$; (g) not asymmetric since it is symmetric; (h) antisymmetric: $x = y$ and $y = x$ implies $x = y$ for all $x, y \in \mathbf{R}$.

$>$ on $\mathbf{R}$ is: (a) not reflexive since it is irreflexive; (b) irreflexive since $x \not> x$ for all $x \in \mathbf{R}$; (c) complete: either $x > y$ or $y > x$ for all $x, y \in \mathbf{R}$ such that $x \neq y$; (d) transitive: $x > y$ and $y > z$ implies $x > z$ for all $x, y, z \in \mathbf{R}$; (e) negatively transitive: $x \not> y$ and $y \not> z$ implies $x \not> z$; (f) not symmetric since it is asymmetric; (g) asymmetric since $x > y$ implies $y \not> x$ for all $x, y \in \mathbf{R}$; (h) antisymmetric by vacuous implication: for all $x, y \in \mathbf{R}$ such that $x > y$ and $y > x$ (there are no such x and y) it is true that $x = y$.

$\geq$ on $\mathbf{R}$ is: (a) reflexive: $x \geq x$ for all $x \in \mathbf{R}$; (b) not irreflexive since it is reflexive; (c) complete: either $x \geq y$ or $y \geq x$ for all $x, y \in \mathbf{R}$; (d) transitive: $x \geq y$ and $y \geq z$ implies $x \geq z$ for all $x, y, z \in \mathbf{R}$; (e) negatively transitive: $x \not\geq y$ and $y \not\geq z$ implies $x \not\geq z$; (f) not symmetric since, e.g., $2 \geq 1$ but $1 \not\geq 2$; (g) not asymmetric since, e.g., $1 \geq 1$ but it is not true that $1 \not\geq 1$; (h) antisymmetric: $x \geq y$ and $y \geq x$ implies $x = y$ for all $x, y \in \mathbf{R}$.

$\subset$ on 2^X is: (a) reflexive: $S \subset S$ for all $S \in 2^X$; (b) not irreflexive since it is reflexive; (c) not complete: if S and T are nonempty and disjoint, then neither $S \subset T$ nor $T \subset S$; (d) transitive: $R \subset S$ and $S \subset T$ implies $R \subset T$ for all $R, S, T \in 2^X$; (e) not negatively transitive: e.g., for subsets of $\mathbf{R}$, $\{1\} \not\subset \{2, 3\}$ and $\{2, 3\} \not\subset \{1, 2\}$ but $\{1\} \subset \{1, 2\}$; (f) not symmetric: if S is a proper subset of T then $T \not\subset S$; (g) not asymmetric: if $S \subset T$ and $S = T$ then $T \subset S$; (h) antisymmetric: $S \subset T$ and $T \subset S$ implies $S = T$ for all $S, T \in 2^X$.

Exercise 5.2

From Table 5.2, $=$ and $\geq$ on $\mathbf{R}$ and $\subset$ on 2^X are reflexive and transitive; hence, they are preorderings. $>$ on $\mathbf{R}$ is not reflexive so it is not a preordering.

Also from Table 5.2, $>$ on $\mathbf{R}$ is negatively transitive and asymmetric; hence, it is a weak ordering. $\geq$ on $\mathbf{R}$ is not a weak ordering because it is not asymmetric. $=$ on $\mathbf{R}$ and $\subset$ on 2^X are not weak orderings because neither is negatively transitive nor asymmetric.

Exercise 5.3

Proof that $\succ$ is irreflexive: Suppose not so that $x \succ x$ for some $x \in X$. Since $\succ$ is asymmetric, this implies that $x \not\succ x$, a contradiction.
Proof that $\succ$ is transitive: Let $x, y, z \in X$ with $x \succ y$ and $y \succ z$. We claim that $x \succ z$. Suppose not: i.e., $x \not\succ z$. $y \succ z$ implies $z \not\succ y$ since $\succ$ is asymmetric. $x \not\succ z$ and $z \not\succ y$ implies $x \not\succ y$ since $\succ$ is negatively transitive. But $x \not\succ y$ contradicts our hypothesis that $x \succ y$, a contradiction.

Exercise 5.4

($\Rightarrow$): Assume that $\not\succ$ is a complete preordering.

- Since $\not\succ$ is transitive,

  $$x \not\succ y \quad \text{and} \quad y \not\succ z \quad \text{implies} \quad x \not\succ z \quad \text{for all} \quad x, y, z \in X.$$

  But, by definition, this means that $\succ$ is negatively transitive.

- Since $\not\succ$ is complete, for any $x, y \in X$ either $x \not\succ y$ or $y \not\succ x$. Therefore, for any $x, y \in X$

  $$\text{if} \quad x \succ y \quad \text{then} \quad y \not\succ x$$

  which means that $\succ$ is asymmetric.

Since $\succ$ is negatively transitive and asymmetric, it is a weak ordering.
($\Leftarrow$): Assume that $\succ$ is a weak ordering.

- Suppose that $x, y \in X$ and it is not true that $x \not\succ y$. Since $\succ$ is asymmetric, if $y \succ x$ then $x \not\succ y$ contrary to our hypothesis. Therefore, $y \not\succ x$ and so $\not\succ$ is complete.

- Suppose that $x, y, z \in X$ with $x \not\succ y$ and $y \not\succ z$. Since $\succ$ is negatively transitive, this implies that $x \not\succ z$. Therefore, $\not\succ$ is transitive.

- Since $\succ$ is irreflexive, $x \not\succ x$ for all $x \in X$ which means that $\not\succ$ is reflexive.

Since $\not\succ$ is reflexive, transitive, and complete, it is a complete preordering.

Exercise 5.5

Proof of Theorem 5.8:

(a) Suppose that $I(x) = I(y)$. Since $I(x) := \{ x' \in X \mid x' \sim x \}$ and $\sim$ is reflexive, $x \in I(x)$. Similarly, $y \in I(y)$. $I(x) = I(y)$ implies that $I(x) \subset I(y)$ and so $x \in I(y)$. Therefore, since $I(y) := \{ y' \in X \mid y' \sim y \}$, we conclude that $x \sim y$.

Conversely, suppose that $x \sim y$. If $x'' \in I(x) := \{ x' \in X \mid x' \sim x \}$, then $x'' \sim x$. Since $\sim$ is transitive, $x'' \sim x$ and $x \sim y$ implies that $x'' \sim y$ and hence that $x'' \in I(y)$. Therefore, $I(x) \subset I(y)$. Similarly, we conclude that $I(y) \subset I(x)$ and hence that $I(x) = I(y)$.

(b) Since by definition $I(x) \subset X$ for all $x \in X$, we simply have to show that the collection $\{ I(x) \mid x \in X \}$ covers X. However, since $\sim$ is reflexive, $x \in I(x)$ for any $x \in X$.

Proof of Theorem 5.9: Note that if $\succ$ is a weak ordering, then $\not\succ$ and hence $\succeq$ is a complete preordering and $\sim$ is an equivalence relation.

Claim: Either $R(x) \subset R(y)$ or $R(y) \subset R(x)$ for every $x, y \in X$.

Proof: Since $\succeq$ is complete, either $x \succeq y$ or $y \succeq x$. If $x \succeq y$ and $z \in R(x)$, then $z \succeq x$ (by definition of $R(x)$). $z \succeq x$ and $x \succeq y$ implies $z \succeq y$ (because $\succeq$ is transitive) and so $z \in R(y)$ (by definition of $R(y)$). Therefore, $R(x) \subset R(y)$. Similarly, if $y \succeq x$, we conclude that $R(y) \subset R(x)$.

Claim: Either $R^{-1}(x) \subset R^{-1}(y)$ or $R^{-1}(y) \subset R^{-1}(x)$ for every $x, y \in X$.

Proof: Again, either $x \succeq y$ or $y \succeq x$. If $x \succeq y$ and $z \in R^{-1}(y)$, then $y \succeq z$ (by definition of $R^{-1}(y)$). $x \succeq y$ and $y \succeq z$ implies $x \succeq z$ (because $\succeq$ is transitive) and so $z \in R^{-1}(x)$ (by definition of $R^{-1}(x)$). Therefore, $R^{-1}(y) \subset R^{-1}(x)$. Similarly, if $y \succeq x$, we conclude that $R^{-1}(x) \subset R^{-1}(y)$.

Claim: Either $P(x) \subset P(y)$ or $P(y) \subset P(x)$ for every $x, y \in X$.

Proof: Because $\succ$ is a binary relation, either $x \succ y$ or $x \not\succ y$. If $x \succ y$ and $z \in P(x)$, then $z \succ x$ (by definition of $P(x)$), $z \succ y$ (because $\succ$ is transitive), and so $x \in P(y)$ (by definition of $P(y)$). Therefore, $P(x) \subset P(y)$. If $x \not\succ y$ but $y \succ x$, then a similar argument shows that $P(y) \subset P(x)$. Finally, if $x \not\succ y$ and $y \not\succ x$, then by definition $x \sim y$ and so $P(x) = P(y)$ (i.e., both $P(x) \subset P(y)$ and $P(y) \subset P(x)$).

Claim: Either $P^{-1}(x) \subset P^{-1}(y)$ or $P^{-1}(y) \subset P^{-1}(x)$ for every $x, y \in X$.

Proof: Again, either $x \succ y$ or $x \not\succ y$. If $x \succ y$ and $z \in P^{-1}(y)$, then $y \succ z$ (by definition of $P^{-1}(y)$), $x \succ z$ (because $\succ$ is transitive), and so $z \in P^{-1}(x)$

(by definition of $P^{-1}(x)$). Therefore, $P^{-1}(y) \subset P^{-1}(x)$. If $x \not\succ y$ but $y \succ x$, a similar argument shows that $P^{-1}(x) \subset P^{-1}(y)$. Finally, if $x \not\succ y$ and $y \not\succ x$, then $x \sim y$ and so $P^{-1}(x) = P^{-1}(y)$.

Exercise 5.6

THEOREM: Let $\succ$ be a binary relation on X and define $\succeq$ as follows: $x \succeq y$ iff $y \not\succ x$. Let $K \subset X$.

(a) $B^c(K) \subset B(K)$.

(b) If $\not\succ$ is complete on X, then $B^c(K) = B(K)$.

Proof: The proof is essentially the same as that of Theorem 5.19.

(a) If $x \in B^c(K)$, then $K \subset R^{-1}(x)$: i.e., $x \succeq y$ for all $y \in K$ or, equivalently, $y \not\succ x$ for all $y \in K$. Therefore, $P(x) \cap K = \emptyset$ and so $x \in B(K)$.

(b) It suffices to prove that $B(K) \subset B^c(K)$. If $x \in B(K)$, then $P(x) \cap K = \emptyset$. Since $\not\succ$ is complete, $R^{-1}(x) = X \backslash P(x)$ and so $K \subset R^{-1}(x)$. Therefore, $x \in B^c(K)$.

Exercise 5.7

Since $\eta(p) := \{\, y \in Y \mid H_o^+(p, p \cdot y) \cap Y \,\}$, we are implicitly regarding the firm (or the "production sector") as having preferences characterized by

$$P(y) := H_o^+(p, p \cdot y) \quad \text{for each} \quad y \in Y.$$

Note that, in contrast to the neoclassical consumer, these preferences for the firm depend on the "environment" p as well as on the activity vector y. As is the case throughout the first two sections of this chapter, we hold the environment fixed.

For fixed p, $y' \succ y$ iff $p \cdot y' > p \cdot y$. We want to establish that these preferences are asymmetric and negatively transitive.

(a) Asymmetry: If $y' \succ y$, then $p \cdot y' > p \cdot y$ and so $p \cdot y \not> p \cdot y'$; i.e., $y \not\succ y'$.

(b) Negative transitivity: If $y'' \not\succ y'$ and $y' \not\succ y$, then $p \cdot y'' \not> p \cdot y'$ and $p \cdot y' \not> p \cdot y$ and hence $p \cdot y'' \not> p \cdot y$. Therefore, $y'' \not\succ y$.

Because $\succ$ is a weak ordering, $\succeq$ is a complete preordering. We want to show that

$$R(y) := \{\, y' \in Y \mid y' \succeq y \,\} = \{\, y' \in Y \mid p \cdot y' \geq p \cdot y \,\}$$

is convex. Let $y', y'' \in R(y)$ (i.e., $p \cdot y' \geq p \cdot y$ and $p \cdot y'' \geq p \cdot y$) and, for any $\alpha \in [0, 1]$, consider $y^\alpha := \alpha y' + (1 - \alpha) y''$. Then

$$p \cdot y^\alpha = p \cdot (\alpha y' + (1 - \alpha) y'') = \alpha\, p \cdot y' + (1 - \alpha)\, p \cdot y'' \geq p \cdot y,$$

proving that $R(y)$ is convex.

Exercise 5.8

(a) The preference relation $\succeq$ is not a complete preordering (it is reflexive and transitive but not complete). Nevertheless, $\sim$ is an equivalence relation with equivalence classes the singleton sets: $\{O\}$, $\{A\}$, $\{B\}$, and $\{AB\}$. Note first that if $x \succeq y$ and $y \succeq x$, then $x = y$; therefore, $\sim$ is the same as $=$. $\sim$ is reflexive (each type can donate to itself), symmetric (since $x \sim y$ iff $x = y$, clearly $x \sim y$ implies $y \sim x$), and transitive (simply check this for all possible types x, y, z). Therefore, $\sim$ is an equivalence relation with equivalence classes the singleton sets.

(b) Let $X = \{\, O, A, B, AB \,\}$. Since O is the only blood type which can donate to all $x \in X$, $B^c(X) = \{O\}$. Since type O is the only type which can accept donations from no other type, $B(X) = \{O\}$ as well.

(c) Let $K = \{\, A, B, AB \,\}$. $B^c(K) = \emptyset$ because there is no type in K which can donate to all of the other types. However, $B(K) = \{\, A, B \,\}$ because neither of these types can accept blood from any other type.

Exercise 5.9

(a) See Figure 13.

(b) See Figure 14.

(c) See Figure 15.

Exercise 5.10

Left to reader.

Figure 13: Constraint is binding.

Figure 14: Weakly convex preferences.

Figure 15: Nonconvex preferences.

Exercise 5.11

Recall that $X \times X$ is given the product topology generated by sets of the form $G \times G'$ where G and G' are open sets in the topology for X. If $(x, y) \in \Gamma(\succ)$ and $\Gamma(\succ)$ is open in $X \times X$, there exist open neighborhoods G_x and G_y respectively such that

$$(x, y) \in G_x \times G_y \in \Gamma(\succ) := \{ (x', y') \in X \times X \mid y' \succ x' \}.$$

Therefore, $y' \succ x'$ for all $(x', y') \in G_x \times G_y$; i.e., for all $x' \in G_x$ and for all $y' \in G_y$.

Exercise 5.12

This is a bit of a trick question since the answer depends on the metric for $\mathbf{R}^2$.

(a) Let

$$D := \{ (x_1, x_2) \in \mathbf{R}^2 \mid \sqrt{x_1^2 + x_2^2} \leq 1 \}$$

denote the (closed) disk centered at $(0, 0)$ with radius 1. If we use the Euclidean metric for $\mathbf{R}^2$, then

$$B(D, 1) = \{ (x_1, x_2) \in \mathbf{R}^2 \mid \sqrt{x_1^2 + x_2^2} < 2 \}$$

and

$$\bar{B}(D, 1) = \{ (x_1, x_2) \in \mathbf{R}^2 \mid \sqrt{x_1^2 + x_2^2} \leq 2 \}.$$

Using the sup metric instead does not change the answer.

(b) Let

$$S := \{ (x_1, x_2) \in \mathbf{R}^2 \mid \sup\{ |x_1 - 2| , |x_2 - 2| \} \leq 1 \}$$

If we use the sup metric for $\mathbf{R}^2$, then

$$B(S, 1) = \{ (x_1, x_2) \in \mathbf{R}^2 \mid \sup\{ |x_1 - 2| , |x_2 - 2| \} < 2 \}$$

and

$$\bar{B}(S, 1) = \{ (x_1, x_2) \in \mathbf{R}^2 \mid \sup\{ |x_1 - 2| , |x_2 - 2| \} \leq 2 \}.$$

With the Euclidean metric, the corners of these squares are "rounded" with boundaries coinciding with circles of radius one centered at each corner of S.

Exercise 5.13

Letting A and B denote the disks centered at $(0,0)$ and $(3,4)$ respectively, we have

$$\sup_{x\in A} \text{dist}(x, B) = \sup_{y\in B} \text{dist}(y, A) = 6$$

and so $\delta(A, B) = 6$.

Exercise 5.14

(a) See Figure 16, panels (a) and (b).

(b) See Figure 16, panels (c) and (d).

Exercise 5.15

($\Rightarrow$): Since $B(e)$ is closed, for any $x \notin B(e)$ there exist open neighborhoods G_x of x and V_x of $B(e)$ such that $G_x \cap V_x = \emptyset$. [Here we exploit the fact that compact Hausdorff spaces are normal (T_4) and hence regular (T_3): see Munkres (1975), Theorem 2.4, p. 198.] Because B is uhc, there exists an open neighborhood U_e of e such that $B(e') \subset V_x$ for all $e' \in U_e$: i.e.,

$$(e, x) \in U_e \times G_x \subset (E \times X)\backslash\Gamma(B).$$

Therefore,

$$(E \times X)\backslash\Gamma(B) = \bigcup_{(e,x)\in(E\times X)\backslash\Gamma(B)} G_e \times V_x$$

which, as the union of open sets, is open. Since the complement of $\Gamma(B)$ is open, $\Gamma(B)$ is closed.

($\Leftarrow$): For any $e \in E$ let V be any open set in X such that $B(e) \subset V$. Because V is open, $F := X\backslash V$ is closed. Because $\Gamma(B)$ is closed, for any $x \in F$ there exist open neighborhoods G_e^x and G_x such that

$$(e, x) \in G_e^x \times G_x \subset (E \times X)\backslash\Gamma(B).$$

Because X is compact and F is closed, F is compact. Therefore, the open cover $\{ G_x \mid x \in F \}$ of F admits a finite subcover, say $\{ G_x \mid x \in F' \}$ where $\#F' < \infty$. Let

$$U := \bigcap_{x\in F'} G_e^x.$$

Figure 16: Sequential definition of uhc and lhc.

U is nonempty because $e \in U$, and it is open because the intersection is finite. Therefore, we have shown that

$$B(e') \subset V \quad \text{for all} \quad e' \in U;$$

i.e., B is uhc at $e \in E$. Since the same argument holds for any $e \in E$, B is uhc.

Exercise 5.16

The proof is essentially identical to that given for Exercise 5.15.
($\Rightarrow$): Since $B(e)$ is closed, for any $x \notin B(e)$ there exist open neighborhoods V of x and V' of $B(e)$ such that $V \cap V' = \emptyset$. Because B is uhc at e, there exists an open neighborhood U of e such that $B(e') \subset V'$ and hence $B(e') \cap V = \emptyset$ for all $e' \in U$: i.e.,

$$(e, x) \in U \times V \quad \text{and} \quad V \cap B(e') = \emptyset \quad \text{for all} \quad e' \in U.$$

Therefore, B is closed at e.

($\Leftarrow$): Let V be any open set in X such that $B(e) \subset V$. Because V is open, $F := X \backslash V$ is closed. Because B is closed at e, for any $x \in F$ there exist open neighborhoods U^x and V'_x such that

$$(e, x) \in U^x \times V'_x \quad \text{and} \quad V'_x \cap B(e') = \emptyset.$$

Because X is compact and F is closed, F is compact. Therefore, the open cover $\{ V'_x \mid x \in F \}$ of F admits a finite subcover, say $\{ V'_x \mid x \in F' \}$ where $\#F' < \infty$. Let

$$U := \bigcap_{x \in F'} U^x.$$

U is nonempty because $e \in U$, and it is open because the intersection is finite. Therefore, we have shown that

$$B(e') \subset V \quad \text{for all} \quad e' \in U;$$

i.e., B is uhc at $e \in E$.

Exercise 5.17

If $(e, x) \in (E \times X) \backslash \Gamma(B)$, then, since B is closed at e, there exist open sets U and V such that

$$(e, x) \in U \times V \subset (E \times X) \backslash \Gamma(B).$$

Thus, the complement of $\Gamma(B)$ is open and so B has closed graph.

Exercise 5.18

As is the case throughout the chapter, X is assumed to be Hausdorff. Consequently, B is uhc iff it has closed graph. According to Theorem 4.15, a point x is in the closure of a set S iff there exists a net in S which converges to x. Therefore, a set is closed iff every net contained in S converges to a point in S. We conclude that B is uhc iff every net in $\Gamma(B)$ converges to a point in $\Gamma(B)$: i.e., iff for any net $(e^\alpha, x^\alpha) \to (e, x)$ such that $(e^\alpha, x^\alpha) \in \Gamma(B)$ for all $\alpha \in A$ the limit point $(e, x) \in \Gamma(B)$.

Since

$$\Gamma(B) := \{ x \in X \mid x \in B(e) \},$$

we have

$$(e^\alpha, x^\alpha) \in \Gamma(B) \quad \text{iff} \quad x^\alpha \in B(e^\alpha).$$

Applying Theorems 4.17(d) and 4.19, the condition $(e^\alpha, x^\alpha) \to (e, x)$ is equivalent to $e^\alpha \to e$ and $x^\alpha \to x$. Therefore, by what was just proved in the preceding paragraph, B is uhc at e iff for any nets $e^\alpha \to e$ and $x^\alpha \to x$ such that $x^\alpha \in B(e^\alpha)$ for all $\alpha \in A$ the limit point $x \in B(e)$.

Exercise 5.19

Suppose that the problem is to choose $x \in K(e)$ to minimize the function

$$f: E \times X \to \mathbf{R}, \qquad (e, x) \mapsto f(e, x).$$

To convert this into a maximization problem, define $u: E \times X \to \mathbf{R}$ such that $u(e, x) = -f(e, x)$ for all $(e, x) \in E \times X$. As in Definition 5.49, define $y \succ^e x$ iff $u(e, y) > u(e, x)$ or, equivalently,

$$y \succ^e x \quad \text{iff} \quad f(e, y) < f(e, x).$$

Corollary 5.50 and its proof now apply to this constrained minimization problem exactly as stated.

Exercise 5.20

See Figure 17.

Exercise 5.21

(a) The statement of the exercise in the text contains a typo: the consumption set is supposed to be

$$X = \{ x \in \mathbf{R}^2_+ \mid x_1 + 2x_2 \geq 3 \}.$$

Figure 17: Uhc of best response.

Normalizing prices such that $p_1 + p_2 = 1$, the demand correspondence becomes

$$\phi(p) := \begin{cases} \left(\frac{2}{1+p_1}, \frac{1}{1+p_1}\right) & \text{if } p_1 \leq 1/3; \\ \left(\frac{1}{2-p_1}, \frac{2}{2-p_1}\right) & \text{if } p_1 > 1/3. \end{cases}$$

which is not uhc at $p_1 = 1/3$.

(b) Normalize prices such that $p_1 + p_2 = 1$. For case (i), the demand correspondence is

$$\phi(p) = \begin{cases} (1, 0) & \text{if } p_1 < 1/2; \\ \left(\frac{2p_1-1}{p_1}, 1\right) & \text{if } p_1 \geq 1/2; \end{cases}$$

which is not uhc at $p_1 = 1/2$. For case (ii), the demand correspondence is

$$\phi(p) = \begin{cases} (1, 0) & \text{if } p_1 < 1/2; \\ (1, 0) \cup (0, 1) & \text{if } p_1 = 1/2; \\ \left(\frac{2p_1-1}{p_1}, 1\right) & \text{if } p_1 > 1/2; \end{cases}$$

which is uhc. For case (iii), the demand correspondence is

$$\phi(p) = \begin{cases} (1, 0) & \text{if } p_1 < 2/3; \\ (1, 0) \cup (1/2, 1) & \text{if } p_1 = 2/3; \\ \left(\frac{2p_1-1}{p_1}, 1\right) & \text{if } p_1 > 2/3; \end{cases}$$

which is uhc.

Exercise 5.22

The Maximum Theorem does not apply because the consumption set $\mathbf{R}^2_+$ is unbounded and hence noncompact. We could alter the specification of the consumer by assuming

$$X = [0, K_1] \times [0, K_2]$$

where K_1, K_2 are large, positive constants. Then

$$\phi_i(p) = \left(\min\{\frac{\alpha b}{p_1}, K_1\}, \min\{\frac{(1-\alpha)b}{p_2}, K_2\}\right)$$

which is continuous throughout Δ.

6 Clearing markets

Exercise 6.1

$$\begin{aligned}\Delta_w \Phi(p) &= \sum_{i \in I} \Delta_w \phi_i(p) = \sum_{i \in I} (\phi_i(p) - w_i) = \sum_{i \in I} \phi_i(p) - \sum_{i \in I} w_i \\ &= \Phi(p) - \sum_{i \in I} w_i\end{aligned}$$

Exercise 6.2

$$\Delta_w \phi_i(\lambda p) = \phi_i(\lambda p) - w_i = \phi_i(p) - w_i = \Delta_w \phi_i(p).$$

Exercise 6.3

$$\Phi(\lambda p) = \sum_{i \in I} \phi_i(\lambda p) = \sum_{i \in I} \phi_i(p) = \Phi(p)$$

and

$$\Delta_w \Phi(\lambda p) = \sum_{i \in I} \Delta_w \phi_i(\lambda p) = \sum_{i \in I} \Delta_w \phi_i(p) = \Delta_w \Phi(p)$$

Exercise 6.4

Left to reader.

Exercise 6.5

(a) $x^2 = x$ iff $x^2 - x = 0$ iff $x(x-1) = 0$ iff $x = 0$ or $x = 1$. So $x = 0$ and $x = 1$ are the fixed points.

(b) $\sqrt{x} = x$ iff $x = x^2$ iff $x = 0$ or $x = 1$. So $x = 0$ and $x = 1$ are the fixed points.

(c) $1/(1+x) = x$ iff $1 = x + x^2$ iff $x^2 + x - 1 = 0$ iff

$$x = \frac{-1 \pm \sqrt{5}}{2},$$

the fixed points.

Exercise 6.6

(a) $S = [0, 1]$ and

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2); \\ 0 & \text{if } x \in [1/2, 1]. \end{cases}$$

(b) $S = [0, 1/4] \cup [3/4, 1]$ and $f(x) = 1/2$ for all $x \in S$.

(c) $S = (0, 1)$ and $f(x) = 1$ for all $x \in S$.

Exercise 6.7

See Figure 18.

Exercise 6.8

$$(.5, .5) = .5(0, 0) + .5(1, 1) \quad \text{and} \quad (.5.5) = .5(0, 1) + .5(1, 0)$$

Every point other than the midpoint is uniquely expressible as an affine combination of the vertices.

Exercise 6.9

From the bottom of page 249, the equilibrium price vector can be expressed in the following matrix form:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \alpha & (1/2)(1-\alpha) & (1/2)(1-\alpha) \\ (1/2)(1-\alpha) & \alpha & (1/2)(1-\alpha) \\ (1/2)(1-\alpha) & (1/2)(1-\alpha) & \alpha \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

or $p = A\mu$. The function $A: \mu \mapsto A\mu$ maps the unit simplex Δ into itself. If we take $\alpha = 1$, then the matrix A is the identity matrix, and so A maps the simplex **onto** itself. Thus, any $p \in \Delta$ can be attained as a solution to this model for some appropriate choice of parameters.

Exercise 6.10

See Figure 19. Each M_j contains its boundary points and hence is closed. It is also evident that

$$M_1 \cup M_2 \cup M_3 = \Delta$$

so $\{ M_j \mid j = 1, 2, 3 \}$ cover the simplex. Letting e^1, e^2, and e^3 denote the vertices of the unit simplex, we have

$$e^1 \in M_1, \quad e^2 \in M_2, \quad e^3 \in M_3$$

Figure 18: Free disposal equilibrium.

$$\{e^1, e^2\} \subset M_1 \cup M_2, \quad \{e^1, e^3\} \subset M_1 \cup M_3, \quad \{e^2, e^3\} \subset M_2 \cup M_3$$

and

$$\{e^1, e^2, e^3\} \subset M_1 \cup M_2 \cup M_3.$$

Thus, all the requirements of the KKM Theorem are satisfied.

Exercise 6.11

By the KKM Theorem, there exists a $p^* \in \cap_{j \in J} M_j$; i.e., a price vector p^* such that

$$\Delta_w \Phi_j(p^*) \leq 0 \quad \text{for all} \quad j \in J. \tag{$*$}$$

The strong form of Walras' Law imposes the restriction

$$p^* \cdot \Delta_w \Phi(p^*) = \sum_{j \in J} p_j^* \Delta_w \Phi_j(p^*) = 0. \tag{$**$}$$

Using $(*)$ and the fact that $p^*|j \geq 0$, we conclude that each term in the sum $(**)$ is equal to zero. Thus, $\Delta_w \Phi_j(p^*) = 0$ whenever $p_j^* > 0$.

Exercise 6.12

Using Euclidean coordinates, let $x = (x_1, x_2)$ denote an arbitrary point in S. Let

$$v^1 = (0,0), \quad v^2 = (1,0), \quad \text{and} \quad v^3 = (0,1).$$

(a) $M_1 = \{x \in S \mid x_1 + x_2 < 1/2\}$ and $M_2 = M_3 = \{x \in S \mid x_1 + x_2 \geq 1/2\}$ satisfy all the requirements of the KKM Theorem except that M_1 is not closed. The intersection of these sets is empty.

(b) $M_1 = \{x \in S \mid x_1 + x_2 \leq 1/4\}$ and $M_2 = M_3 = \{x \in S \mid x_1 + x_2 \geq 1/2\}$ satisfy all the requirements of the KKM Theorem except that they do not cover the simplex S. Their intersection is empty.

(c) $M_1 = \{x \in S \mid x_1 + x_2 \leq 1/2\}$, $M_2 = \{x \in S \mid x_1 \geq 3/4\}$, and $M_3 = \{x \in S \mid x_1 + x_2 \geq 1/2\}$ are closed sets whose union covers S, but their intersection is empty. The conditions which fail are

$$\text{co}\{v^1, v^2\} \subset M_1 \cup M_2 \quad \text{and} \quad \text{co}\{v^2, v^3\} \subset M_2 \cup M_3.$$

Exercise 6.13

There are $(n-1)!$ distinct permutations.

Exercise 6.14

Figure 19: Sets M_j in example economy.

$$\tilde{v}^1 = b = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\tilde{v}^2 = \tilde{v}^1 + \tilde{e}^2 = \begin{bmatrix} 5 \\ 5 \\ 2 \\ 4 \end{bmatrix}$$

$$\tilde{v}^3 = \tilde{v}^2 + \tilde{e}^1 = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 4 \end{bmatrix}$$

$$\tilde{v}^4 = \tilde{v}^3 + \tilde{3}^3 = \begin{bmatrix} 6 \\ 4 \\ 3 \\ 3 \end{bmatrix}$$

Exercise 6.15

Left to reader.

Exercise 6.16