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A NOTE ON CONSUMER'S SURPLUS, THE DIVISIA INDEX, AND THE MEASUREMENT OF WELFARE CHANGES

BY NEIL BRUCE<sup>1</sup>

1. INTRODUCTION

IN THE CONTROVERSY REGARDING the validity of consumer's surplus analysis, there is an apparent lack of recognition that the issues involved are the same as those in the economic theory of index numbers.<sup>2</sup> This has not only resulted in a duplication of effort but also an unfortunate confusion of the issues themselves. For example, the question of the exactness or consistency of consumer's surplus measures under all circumstances is rather futile in light of the well known impossibility of constructing an index number that is consistent for all forms of the utility function. In addition, recent criticism of consumer's surplus analysis has transcended the question of whether the analysis is partial or general equilibrium in nature and has focused on the path dependency problem of the consumer's surplus expression in its general equilibrium line integral form. The same problem is dealt with in the index number literature with regard to Divisia indexes.

In this note I wish to clarify the equivalence between Divisia quantity indexes and consumer's surplus measures of welfare losses due to market distortions. I will then demonstrate how the consistency of these indexes depends on how the nominal prices used in their construction are normalized or deflated. This is related to two well known results: (i) the validity of consumer's surplus when indifference curves are "parallel" and (ii) the consistency and path independence of Divisia quantity indexes when the utility function is homothetic. Finally, it will be demonstrated that nominal prices can be normalized so as to make the consumer's surplus-Divisia index consistent with the utility function underlying the simple linear expenditure system.

2. CONSUMER'S SURPLUS AND THE INDEX NUMBER PROBLEM

The purpose of constructing a welfare index is to be able to infer from price and quantity data whether the utility of a "typical" consumer has risen or fallen as a result of a discrete change in the commodity and price vectors from some initial values  $X^0, P^0$  to some terminal values  $X^1, P^1$ .<sup>3</sup> Quantity indexes using either  $P^0$  or  $P^1$  as the fixed weights attached to the change in  $X$  are simple to calculate but can be inconsistent (i.e., indicate utility has risen when it actually has fallen and vice versa) with any strictly concave utility function except of the Leontief fixed proportions type. As an improvement, various combinations of the fixed weights indexes have been suggested including Fisher's "ideal" index which has been shown to be consistent with a specific homogeneous utility function.<sup>4</sup> Another alternative is the Divisia quantity index which can be expressed in nominal units as:

$$(1a) \quad D = \int_{X^0}^{X^1} \sum_i^n P_i(X) dX_i$$

<sup>1</sup> I would like to thank A. C. Harberger, F. Flatters, J. MacKinnon, and T. N. Srinivasan for comments and suggestions which substantially improved this note. I am responsible for the remaining errors.

<sup>2</sup> The equivalence was noted by Samuelson [8, p. 196] some time ago, yet the two subjects are usually treated as distinct in modern texts. Silberberg [10, p. 943] noted the similarity between consumer's surplus and Divisia indexes but did not pursue the point.

<sup>3</sup> If the price vector does not change (i.e.,  $P^0 = P^1$ ), then revealed preference criteria can always establish the welfare ranking of  $X^0, X^1$ .

<sup>4</sup> On this subject, see Samuelson and Swamy [9, pp. 575-576] and works cited therein.

where  $P_i(X)$  is the nominal price of  $X_i$  to the consumer. This index avoids the problem of which price weights to choose by expressing the discrete change in  $X$  as a continuum of differential changes weighted by intermediate values of the price vector as it moves from  $P^0$  to  $P^1$ . In the Appendix it is demonstrated that (1a) also represents the change in consumer's surplus (measured in nominal units) when the change in  $X$  is caused by a change in a vector of market distortions (such as commodity taxes) provided all general equilibrium considerations are taken into account. In this context, the intermediate prices used in (1a) are obtained from the general equilibrium supply and demand functions. Thus, in the case of a single distortion, expression (1a) can be represented by the familiar welfare cost triangle bounded by demand and supply curves.

The difficulty with (1a), as index number theoretists know and recent consumer's surplus critics have emphasized, is that it is a line integral and its value is not generally independent of the paths  $X$  and  $P$  follow between their initial and terminal values.<sup>5</sup> Therefore, (1a) can be shown to be inconsistent with a generally specified utility function by calculating its value along a closed path returning to the initial point. In the absence of path independency, the value will be nonzero and, therefore, inconsistent with the utility function. Thus, path independence is a necessary condition for the consumer's surplus-Divisia index to rank utility consistently and this reflects the general index number problem as it applies to Divisia indexes.

Advocates of consumer's surplus analysis might argue that the indeterminacy of (1a) is a consequence of measuring  $D$  in nominal units and that the problem could be solved by appropriately normalizing the nominal prices used in (1a). In a sense this is correct, but it does not eliminate the index number problem since the "correct" price index with which to deflate nominal prices must be found. This engages us in the equivalent (or dual) problem of constructing a consistent price index. Interestingly, two well known conclusions can be interpreted as specific normalizations of (1a). The first is the case where  $D$  is expressed in units of a numeraire good  $X_n$  by dividing all nominal prices by  $P_n$ . The resulting consumer's surplus (measured in units of  $X_n$ ) can be then shown to consistently measure welfare providing the indifference curves are "parallel" with respect to the  $X_n$  axis. In the second case, all nominal prices are normalized by dividing by nominal income at consumer's prices and the resulting Divisia index is found to be consistent with any homothetic utility function. In fact, both cases are specific cases of a general approach. By way of example, I will demonstrate that by dividing nominal prices in (1a) by a certain geometric price index, the consumer's surplus-Divisia index will be consistent with the Stone-Geary type of utility function.

The following analysis is facilitated by reference to the following theorem.

**THEOREM 1:** *A line integral such as (1a) can be solved as an exact differential equation if there exists an integrating factor  $Y(X)$  such that  $\phi_i(X) = Y(X) \cdot P_i(X)$  satisfies the conditions  $\partial\phi_i/\partial X_j = \partial\phi_j/\partial X_i$  for all  $i$  and  $j$ .<sup>6</sup>*

It is well known that the above conditions will obtain if  $\phi(X) = \{\phi_1(X), \dots, \phi_n(X)\}$  is the gradient vector of a real valued function. If we assume that consumption decisions are made in this economy as if a single "typical" consumer is maximizing a quasi-concave utility function, then the first order conditions for an interior maximum provide that  $U_i(X) = y(X) \cdot P_i(X)$  where  $U_i(X) = \partial U/\partial X_i$  and  $y(X) = \partial U/\partial I$  where  $I$  is the consumer's income.

<sup>5</sup> The path dependency issue has been recognized from Hotelling [5, p. 246] on, but its importance was minimized on the assumption that income effects are "small". Recent critics, notably Silberberg [10] and Mohring [6] take issue with that conclusion.

<sup>6</sup> This is a basic theorem which can be found in most textbooks on vector calculus or differential equations. For example, Boyce and DiPrima [1, p. 35].

Thus it follows that the marginal utility of income is an integrating factor which satisfies the conditions of Theorem 1 since  $U_{ij} = U_{ji}$ . However,  $y(X)$  depends on the underlying utility function and can only be specified when that utility function is specified. This is a manifestation of the general index number problem.

Now divide all  $P_i(X)$  in (1a) by  $P_n(X)$  and use the first order conditions to obtain:

$$(1b) \quad D = \int_{X^0}^{X^1} \sum_i^n R^{in}(X) dX_i$$

where  $D$  is now in units of  $X_n$  and  $R^{in}(X) = U_i(X)/U_n(X)$  is the marginal rate of substitution between  $X_i$  and  $X_n$ . Unfortunately, the  $R^{in}(X)$  terms do not satisfy the required conditions on  $\phi_i$  in Theorem 1 unless the path of integration is confined to a single indifference surface<sup>7</sup> or the utility function is of a specific type. The former is of little use since the level of utility is expected to be changing while the latter requires that  $U_n(X)$  equal a positive constant  $k^{-1}$  which restricts the indifference surfaces to be parallel with respect to the  $X_n$  axis. In this case,  $R^{in}(X) = U_i(X) \cdot k$  so (1b) becomes:

$$(1c) \quad D = k \int_{X^0}^{X^1} \sum_i^n U_i(X) dX_i = k[U(X^1) - U(X^0)]$$

and the consumer's surplus-Divisia index consistently ranks utility in the case of such "parallel" indifference curves.

Now consider a utility function which is homogeneous of degree  $q > 0$ . Then by the first order conditions

$$(2) \quad U_i(X) \cdot X_i = y(X) \cdot P_i(X) \cdot X_i \quad (i = 1, \dots, n),$$

which can be summed over  $n$  and Euler's theorem used to obtain  $q \cdot U(X) = y(X) \cdot I$  where  $I = \sum_i^n P_i(X) \cdot X_i$ . Then normalizing nominal prices in (1a) by dividing by  $I$  yields:

$$(1d) \quad D = \int_{X^0}^{X^1} \sum_i^n \left( \frac{y(X)P_i(X)}{q \cdot U(X)} \right) dX_i = \frac{1}{q} \ln \left( \frac{U(X^1)}{U(X^0)} \right).$$

Since  $\ln U(X)$  is a monotonic transformation of the utility function, the Divisia index will consistently rank welfare with respect to  $U$ . Moreover, since any homothetic utility function is a positive monotonic transformation of some homogeneous function, the Divisia index so normalized will be consistent with any utility function which implies expenditure proportionality.

To the extent that either of these utility functions is acceptable as an approximation to the actual underlying utility function, the consumer's surplus-Divisia index measures of welfare changes (appropriate normalized) are useful in applied welfare economics. Unfortunately, the above utility functions are at odds with a large body of empirical evidence on consumers' behavior, since the first implies zero income elasticities for all non-numeraire goods while the latter implies that all goods have income elasticities of unity. Fortunately, the above cases do not exhaust the possibilities. Consider now the utility function

$$(3) \quad U = \prod_i^n (X_i - S_i)^{m_i}$$

where  $m_i$  is the constant marginal budget share spent on  $X_i$  while the  $S_i$  are constants usually interpreted as habitual or subsistence levels of the  $i$ th good consumed. By the first

<sup>7</sup> In this case,  $R^{in}(X_1, \dots, X_{n-1}, h(X_1, \dots, X_{n-1}))$  where  $X_n = h(X_1, \dots, X_{n-1})$  is obtained by solving  $U_0 = U(X)$  and  $U_0$  is a specified level of utility. Then  $R_j^{in} = (U_n U_{ij} - U_{in} U_j - U_i U_{nj} + U_i U_j U_{nn} U_n^{-1}) U_n^{-2}$  by the implicit function rule. Since  $U_{ij} = U_{ji}$  for all  $i$  and  $j$ , this also equals  $R_i^{jn}$ .

order conditions:

$$(4) \quad U_i(x) = U \cdot m_i(X_i - S_i)^{-1} = y(X) \cdot P_i \quad (i = 1, \dots, n);$$

therefore,  $U^{m_i} m_i^{m_i} (X_i - S_i)^{-m_i} = y(X)^{m_i} \cdot P_i^{m_i}$ . By multiplying over all  $i$  and using  $\sum_i^n m_i = 1$  we obtain  $y = k^{-1} (\prod_i^n P_i^{-m_i})$  where  $k^{-1} = \prod_i^n m_i^{m_i}$  is a positive constant. Then normalizing nominal prices in (1a) by dividing by the geometric average price index  $\bar{P} = \prod_i^n P_i^{m_i}$  is equivalent to multiplying the integral by  $yk$ , so:

$$(1e) \quad D = k \int_{X^0}^{X^1} \sum_i^n U_i(X) dX_i = k[U(X^1) - U(X^0)].$$

Thus the consumer's surplus-Divisia index is normalized so as to consistently rank welfare with respect to the utility function in (3).<sup>8,9</sup>

The importance of this result is that the normalized consumer's surplus measure is now consistent when the underlying utility function approximates (3) in the relevant range, and that this utility function is less restrictive as to the income elasticities of the commodities than the homothetic or parallel indifference curve cases. Of course, there are empirical restrictions implied by (3). For one thing, this utility function implies the linear demand functions:

$$(5) \quad X_i = \sum_j^n a_{ij} \left( \frac{P_j}{P_i} \right) + m_i \left( \frac{I}{P_i} \right) \quad (i = 1, \dots, n)$$

where  $a_{ii} = S_i(1 - m_i)$  and  $a_{ij} = -m_i S_j$  (Stone [11]), but this is a popular empirical assumption, anyway. In addition, it should be noted that this utility function is strongly separable which has several empirical implications.<sup>10</sup>

Two qualifications with regard to the foregoing should be noted. To begin with, the first order equalities were used routinely (as in most of the index number literature) on the basis of an interior maximum. However, the existence of corner maxima anywhere along the path of integration would yield first order inequalities and, upon substitution, inequalities in expressions (1c) to (1e). Thus the completeness of the ranking obtained using (1a) would be in question. Secondly, the gain in generality suggested by the use of a nonhomothetic utility function of the Stone-Geary form may be of limited value since the existence of a community utility function (an assumption on which the entire analysis is predicated) usually depends on identical and homothetic preferences across all individuals.

### 3. CONCLUSIONS

This note provides additional support to the general use of consumer's surplus-Divisia index type welfare measurement in applied works, although qualifications can undoubtedly be made with respect to some particular applications. The generality of this index has been extended in that its consistency can be established for a nonhomothetic utility function popular in earlier demand studies. In any case, it should now be clear that the usefulness of

<sup>8</sup> Note that (3) reduces to the usual Cobb-Douglas function when all  $S_i = 0$ . Thus the Cobb-Douglas can be normalized by either nominal income or the geometric price index. Also note that the numeraire good case can be considered a special case of (3) where  $m_i = 0$ ,  $i \neq n$ , and  $m_n = 1$ . Thus,  $\bar{P} = P_n$ .

<sup>9</sup> T. N. Srinivasan has suggested a generalization to me. Letting  $U(X) = F(h(Y))$  where  $Y_i = \alpha_i X_i + \beta_i$  and  $\alpha_i > 0$  for all  $i$  and where  $h$  is homogeneous, it is easy to establish that (1a) can be normalized by dividing nominal prices by  $I - \sum_i^n (\beta_i / \alpha_i) P_i(X)$  so as to yield a consistent index of  $U$ . In the case of the  $S$ -branch generalized linear expenditure system (see Brown and Heien [2]), the  $\beta_i / \alpha_i$  terms represent committed quantities of  $X_i$  so the normalizing factor is supernumerary income.

<sup>10</sup> See Samuelson [8] and Goldberger [3]. Goldberger shows that expression (3) is the only directly additive utility function with constant marginal budget shares.

consumer's surplus analysis should be judged in the context of the index number problem and against other (necessarily imperfect) welfare standards. As a case in point, consider the suggestion that consumer's surplus measures be abandoned in favor of the path independent compensating or equivalent variations (Mohring [6, p. 362] and Silberberg [10, p. 950-951]). If such alternatives were operational, that would end the issue, but they are not. For example, evaluating the compensating variation requires knowledge of the consumption bundle the consumer would choose at terminal prices but with his initial utility level. In fact, such measures could be interpreted as consumer's surplus or Divisia indexes calculated using constant utility demand functions. Unfortunately, the market data do not reveal such relations as matter of course, so additional a priori information about the consumer's indifference map must be assumed. Against such criteria, no operational index number including consumer's surplus measures, can compete.

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APPENDIX

Let the consumer maximize  $U(X)$  subject to  $P \cdot X = I + R$  where  $R$  are the tax proceeds redistributed as a lump sum payment and  $I$  is factor income (inclusive of economic profits if nonzero). Let competitive producers maximize  $I = p \cdot X$  subject to  $\pi(X) = 0$  where  $p$  is a vector of producers' prices and  $\pi(X)$  is a convex production set. By the first order conditions:

$$U_i = yP_i = yp_i(1 + t_i) = \lambda\pi_i(1 + t_i)$$

where  $t$  is a vector of *ad valorem* tax rates. Assuming  $t \rightarrow X$  is a univalent mapping, then by the usual change of variables technique, equation (1a) can be written

$$(1a') \quad D = - \int_{t^0}^{t^1} \sum_i^n \sum_j^n t_j p_j \frac{\partial X_i}{\partial t_j} dt_j$$

since  $\sum p_i dX_i = 0$  by the production constraint. In (1a') the minus sign results from adopting the convention of measuring welfare costs and  $\partial X_i / \partial t_j$  is a general equilibrium substitution term. In the special case where  $\pi(X)$  is linear, units of the commodities can be set such that all  $p_i = 1$ . Then by considering all  $t_i = 0$  and unchanging except  $t_k$  we get

$$(1a'') \quad D = -\frac{1}{2}(t_k^0)^2 \frac{\partial^2 X_k}{\partial t_k^2}$$

where  $\overline{\partial X_k / \partial t_k}$  is an intermediate value and  $t_k^0 = 0$ . Expression (1a'') is the familiar "triangle" formula.

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