# Bargaining

Ichiro Obara

UCLA

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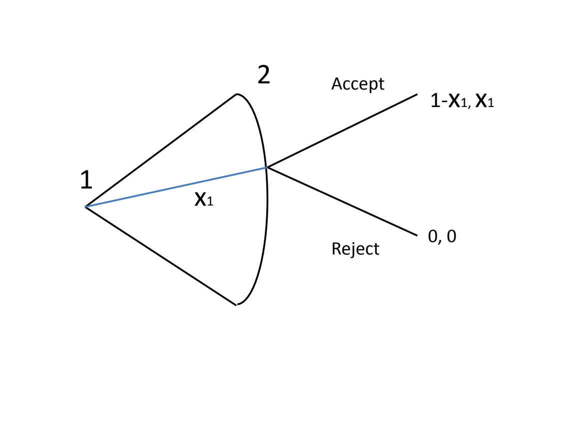
- We often like to model situations where a group of agents whose interests are in conflict make a decision collectively.
- So we need a theory of **bargaining**.
- There are two approaches: cooperative/axiomatic approach and non cooperative approach. Here we take the latter. We first study **alternative offer bargaining**.

# Alternative Offer Bargaining: Finite Horizon Case

- Consider the following finite horizon bargaining game.
  - Two players $i = 1, 2$ are trying to allocate \$1 between them.
  - The game lasts for $K < \infty$ periods. Periods are counted backward, so the game starts in period $K$ and ends in period 1.
  - In any odd period $t$, player 1 makes an offer $x_1 \in [0, 1]$, player 2 accepts or rejects the offer. If the offer is accepted, then the game is over and the players receive $(1 - x_1, x_1)$. Otherwise the game continues to the period $t - 1$.
  - In any even period $t$, player 2 makes an offer $x_2 \in [0, 1]$, player 1 accepts or rejects the offer. If the offer is accepted, then the game is over and the players receive $(x_2, 1 - x_2)$. Otherwise the game continues to period $t - 1$.
  - If the offer is rejected in period 1, then both players receive 0 payoff.
  - Player $i$'s discount factor is $\delta_i \in (0, 1)$.

- For $K = 1$, this game looks like:



- Suppose that $K = 2$.
- Apply backward induction.
  - In period 1, every strictly positive offer is accepted by player 2. Hence the equilibrium offer must be 0, which must be accepted by player 2.
  - In period 2, player 1 accepts any offer strictly larger than $\delta_1$ and rejects any offer strictly smaller than $\delta_1$. Hence the equilibrium offer must be exactly $\delta_1$, which must be accepted by player 1.
- Hence the SPE is unique.
  - Player 1: offer 0 in period 1, accept $x_2$ if and only if $x_2 \geq \delta_1$ in period 2.
  - Player 2: offer $\delta_1$ in period 2, accept any offer in period 1.
- There are many other NE.

- Suppose that $K = 3$.
- Again apply BI.
  - In period 1, every strictly positive offer is accepted by player 2. Hence the equilibrium offer must be 0, which must be accepted by player 2.
  - In period 2, player 1 accepts an offer if and only if the offer is larger than or equal to $\delta_1$. Player 2 always offers $\delta_1$.
  - In period 3, player 2 accepts an offer if and only if the offer is larger than or equal to $\delta_2(1-\delta_1)$. Player 1 offers $\delta_2(1-\delta_1)$.

- In general, we have the following result.
  - In any period $2k-1$, player 1 always offers $x_1(k) = \delta_2(\sum_{m=0}^{k-2}(\delta_1\delta_2)^m) - (\sum_{m=1}^{k-1}(\delta_1\delta_2)^m)$. Player 2 accepts $x_1$ iff $x_1 \geq x_1(k)$.
  - In any period $2k$, player 2 offers $x_2(k) = \delta_1(\sum_{m=0}^{k-1}(\delta_1\delta_2)^m) - (\sum_{m=1}^{k-1}(\delta_1\delta_2)^m)$. Player 1 accepts $x_2$ iff $x_2 \geq x_2(k)$.
- $x_1(k)$ converges to $\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$ and $x_2(k)$ converges to $\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$ as $k \to \infty$.

# Alternative Offer Bargaining: Infinite Horizon

- Now assume that this bargaining game is played indefinitely until some offer is accepted ($K = \infty$). Assume that the game starts with player 1's offer.
- We cannot apply backward induction.
- One-shot deviation principle still holds.
- Remember that SPE is recursive: the continuation play of a SPE at any subgame is itself a SPE in the subgame.

### Theorem

There exists the unique subgame perfect equilibrium in the infinite-horizon alternative offer bargaining game. In equilibrium,

- player 1 always offers $\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$ and accepts player 2's offer $x_2$ if and only if $x_2 \geq \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$.
- player 2 always offers $\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$ and accepts player 1's offer $x_1$ if and only if $x_1 \geq \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$.

## Proof.

- First we show that the strategy profile is in fact a SPE. We first check one-shot deviation constraints for player 1.
  - When player 1 makes an offer, player 1's one-shot deviation constraint is $1 - \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \geq \delta_1 \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}\right)$.
  - When player 2 makes an offer, player 1's one-shot deviation constraint boils down to $\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} = \delta_1 \left(1 - \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$.
- One-shot deviation constraints for player 2 are the same.

□

## Proof.

- Let $M_i$ be the supremum of player $i$'s SPE payoffs when it is player $i$'s turn to make an offer. Let $m_i$ be the infimum of player $i$'s SPE payoffs when it is player $i$'s turn to make an offer.
- We show that $M_i = m_i$ for $i = 1, 2$, hence the SPE payoff is unique.
  - Any offer $x_1 > \delta_2 M_2$ would be accepted. Hence $m_1 \geq 1 - \delta_2 M_2$. Similarly $m_2 \geq 1 - \delta_1 M_1$.
  - Any offer $x_1 < \delta_2 m_2$ would be rejected. So $M_1 \leq \max\left\{1 - \delta_2 m_2, \delta_1^2 M_1\right\}$. Since $M_1 > 0$, it must be the case that $M_1 \leq 1 - \delta_2 m_2$. Similarly $M_2 \leq 1 - \delta_1 m_1$.
  - Then
    - $M_1 \leq 1 - \delta_2 m_2 \leq 1 - \delta_2(1 - \delta_1 M_1)$, so $M_1 \leq \frac{1-\delta_2}{1-\delta_1\delta_2}$.
    - $m_1 \geq 1 - \delta_2 M_2 \geq 1 - \delta_2(1 - \delta_1 m_1)$, so $m_1 \geq \frac{1-\delta_2}{1-\delta_1\delta_2}$.
  - Hence $M_1 = m_1 = \frac{1-\delta_2}{1-\delta_1\delta_2}$. Similarly $M_2 = m_2 = \frac{1-\delta_1}{1-\delta_1\delta_2}$.

**Proof.**

- Since player 2 is guaranteed to get $\frac{1-\delta_1}{1-\delta_1\delta_2}$ as a proposer, player 2 accepts any offer if and only if it is more than or equal to $\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$. Hence player 1's offer must be $\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$ in any period.
- Similarly, since player 1 is guaranteed to get $\frac{1-\delta_2}{1-\delta_1\delta_2}$ as a proposer, player 1 accepts any offer if and only if it is more than or equal to $\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$. Hence player 2's offer must be $\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$ in any period.

□

## Comments

- Player $i$ with higher $\delta_i$ would get more at any point of the game for any given $\delta_{-i}$.
- Player $i$'s payoff converges to 1 as $\delta_i \to 1$.
- Player $i$'s payoff converges to 0 as $\delta_i \to 0$ if she is not the first mover. If she is the first mover, she would still get $1 - \delta_{-i}$ even when $\delta_i = 0$.
- Suppose that $\delta_1 = \delta_2 = \delta$. Then the SPE payoff profile is $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$, which converges to $(0.5, 0.5)$ as $\delta \to 1$.
- Let $\exp(-r_i\Delta)$ be player $i$ 's discounting factor, where $r_i$ is player $i$'s discounting rate and $\Delta$ is the duration of each period. Then the SPE payoff profile converges to $\left(\frac{r_2}{r_1+r_2}, \frac{r_1}{r_1+r_2}\right)$ as $\Delta \to 0$.

# Bargaining in Legislature

- Next we consider a different bargaining model with many players.
  - $n$ (odd) players trying to allocate \$1 among them. Let $X = \left\{ x \in \Re^n_+ | \sum_i x_i \leq 1 \right\}$ be the set of feasible allocations.
  - The game is played for $K$ periods.
  - In any period $t$, a player is chosen as a proposer with probability $1/n$.
  - The proposer suggests how to divide \$1, i.e. chooses some $x = (x_1, ..., x_n)$ from $X$.
  - The players vote publicly and sequentially (in some order). If the proposal is approved by the majority, then it is implemented and the game is over. Otherwise the game moves on to the next period.
  - If no proposal is approved by the end of the game, then everyone receives 0 payoff.
  - Each player's discount factor is $\delta \in (0, 1)$.

# Two Period Case

- First consider the case with $K = 2$.
- In the second period, whoever is chosen as a proposer can request everything.
- In the first period, the proposer can buy one vote by paying $\delta/n$. Hence the proposer can pay $\delta/n$ to $\frac{n-1}{2}$ players so that his proposal just gets the majority votes.

# Infinite Horizon Case

- Suppose that $K = \infty$.
- We first focus on **symmetric stationary SPE**, where (1) the distribution of proposals is the same independent of histories, (2) every player except for the proposer is treated symmetrically by the equilibrium proposal and (3) the equilibrium voting behavior is the same across all the players.

**Theorem**

For any $\delta \in (0,1)$, there exists the unique symmetric stationary subgame perfect equilibrium . In equilibrium, the proposer always proposes to distribute $\delta/n$ to randomly selected $\frac{n-1}{2}$ players. Player $i$ votes for the proposal if and only if the proposal assigns player $i$ at least $\delta/n$.

## Proof.

- It is easy to prove that this is a SPE by checking one-shot deviation constraints. In the following, we show that this is the only symmetric stationary SPE.
- Take any SPE. Every player's equilibrium payoff in the beginning of each period must be the same. Denote this by $v$.
- Each proposer is guaranteed to receive at least $1 - \delta v \frac{n-1}{2}$ by paying $\delta v$ to $\frac{n-1}{2}$ players. Since this is larger than $\delta v$, the proposal must be approved with probability 1.

□

## Proof.

- To minimize expense, it must be the case that the proposer pays exactly $\delta v$ to $\frac{n-1}{2}$ players. Note that each other player is in the coalition with probability 0.5. So $v$ must satisfy

$$v = \frac{1}{n}\left(1 - \delta v \frac{n-1}{2}\right) + \frac{n-1}{n}\frac{1}{2}\delta v$$

  Hence $v = 1/n$.

- Then clearly the equilibrium strategy must be unique and as stated in the theorem.

□

# Multiple Equilibria

- Once stationarity is dropped, then many allocations can be supported by SPE.
- In fact, any allocation can be supported if there are many players and the players are patient.

### Theorem

Suppose that $n \geq 5$ and $\delta \in \left(\frac{n+1}{2(n-1)}, 1\right)$. Then any $x \in X$ can be achieved by a subgame perfect equilibrium where

- every proposer proposes $x$ if there has been no deviation by any proposer. This proposal is accepted by every player immediately.
- if player $j$ deviates and proposes $y \neq x$, then (1) it is rejected by some majority $M(y)$ that does not include $j$, (2) the next proposer proposes $z(y) \in X$ such that $z_j(y) = 0$ and everyone in $M(y)$ votes for $z(y)$.
- if the next proposer $k$ proposes $w \neq y$ instead of $y$ in the previous step, then repeat the previous step with $(z(w), k)$ instead of $(z(y), j)$.

**Proof.**

- No proposer has an incentive to deviate from $x$ because then the continuation payoff is 0.
- Consider $(j, y)$-phase. We define $M(y)$ and $z(y)$ as follows:
  - $M(y)$ is a group of $\frac{n+1}{2}$ players such that $j \notin M(y)$ and $\sum_{i \in M(y)} y_i$ is minimized.
  - $z_i(y) = 0$ for $i \notin M(y)$ and $z_i(y) = \frac{y_i}{\sum_{k \in M(y)} y_k}$ for $i \in M(y)$.

□

## Proof.

- No proposer in $(j, y)$-phase (even player $j$) does not have an incentive to deviate and propose something different from $z(y)$ because then the continuation payoff is 0.
- Everyone votes for $x$ and every player in $M(y)$ votes for $z(y)$ (a deviation just causes a delay).
- Finally we need to make sure that $M(y)$ rejects $y$ in favor of $z(y)$ in the next period.
  - This is trivially satisfied for $i \in M(y)$ such that $y_i = 0$.
  - For $i \in M(y)$ with $y_i > 0$, we need $\delta z_i(y) \geq y_i$, which is $\delta \geq \sum_{k \in M(y)} y_k$. The least upper bound of the RHS is $\frac{n+1}{2(n-1)}$, which is less than 1 if $n \geq 5$.

**Remark.**

- In this construction, if $i$ is the pivotal voter (voting $\frac{n+1}{2}$th "yes") in $M(y)$ and $z_i(y) = 0$, then $i$ is playing a weakly dominated strategy by voting for $z_i(y)$. This can be fixed easily by considering a slightly more complicated transfer: it is possible to perturb $z_i(y)$ slightly so that $\delta z_i(y) > y_i$ holds for every $i \in M(y)$.

# References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