Common Knowledge and Common Prior

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Common Knowledge Assumption

- When we define games, we implicitly introduce lots of common knowledge assumptions.
- Something is common knowledge if everyone knows it, everyone knows that everyone knows it, and so on.
- For example, $N, A_i, u_i$ are all common knowledge for strategic game $G = (N, (A_i), (u_i))$.
- But what does it mean? Is it really a significant assumption?
- To understand the notion of common knowledge better, let’s take a look at so called E-mail game.
Two players, player 1 and player 2, play one of the following games: $G_s$ ("status quo") or $G_o$ ("opportunity").

The game is $G_s$ with probability $1 - p$ and $G_o$ with probability $p \in (0, 1)$.

Only player 1 observes a realization of the game.
If the game is $G_s$, then “stay” (S) is the strictly dominant action.

If the game is $G_o$, then “attack” is optimal if and only if the other player attacks. There are two strict NE for $G_o$: $(A, A)$ and $(S, S)$. The former NE is more efficient.
There is some exchange of information before the game is played:

- If the game is $G_s$, nothing happens.
- If the game is $G_o$, an e-mail message is automatically sent from player 1 to player 2. This message is lost with probability $\epsilon > 0$.
- If player 2 receives a message, then a confirmation e-mail is automatically sent from player 2 to player 1. This message is lost with probability $\epsilon > 0$.
- If player 1 receives a confirmation e-mail, then another confirmation e-mail is automatically sent from player 1 to player 2, which is lost with probability $\epsilon > 0$.
- This process stops when an e-mail is lost (which happens with probability 1).
This game can be regarded as a Bayesian game where \( \Omega = \{ G_s, G_o \} \) and player \( i \)'s type is the number of messages \( i \) sent: 
\[ T_i = \{ 0, 1, 2, 3, \cdots \} \]. Since the true game is \( G_s \) if and only if \( t_1 = 0 \), we drop \( \Omega \).

If player 1's type \( t_1 \) is 0, then player 1 knows that the true state is \( G_s \) (and player 2's type is 0). Hence player 1's optimal choice is \( S \).
For $t_1 > 0$...

- For $t_1 > 0$, there are two possibilities: player 1’s $t_1$th message is lost, which happens with probability $\epsilon$, or player 1’s $t_1$th message reached player 2 but player 2’s $t_1$th message is lost, which happens with probability $(1 - \epsilon)\epsilon$ (conditional on both players have received the $t_1 - 1$ messages).

- Hence 1 believes that 2’s type is $t_1 - 1$ with probability $q = \frac{\epsilon}{\epsilon + (1 - \epsilon)\epsilon} > 1/2$ and $t_1$ with probability $1 - q$.

- This implies that $S$ is the unique best response for player 1 if player 2 plays $S$ when $t_2 = t_1 - 1$.

- Similarly $S$ is the unique best response for player 2 given any $t_2$ if player 1 plays $S$ when $t_1 = t_2$.

- Since $S$ is the unique best response for player 1 when $t_1 = 0$, $S$ must be played by every type by both players.
So we have proved the following result.

**Theorem (Rubinstein, 1989)**

*There exists a unique Bayesian Nash equilibrium for this game and A is never played in equilibrium.*
What do the players know given their types?

- Player 1 of type 1 knows that the true state is $G_0$, but does not know if player 2 knows it.
- Player 1 of type 2 knows that the true state is $G_0$, knows that player 2 knows it, but does not know if player 2 knows that player 1 knows that player 2 knows that the true state is $G_0$.
- If the type profile is $(m, m)$, then the players know that they know that $\cdots \times m \cdots$ that the true state is $G_0$. But they are not sure about the other player’s $m$th order knowledge.

If $m$ is large, then it is “almost common knowledge” that the game is $G_0$. However $(A, A)$, which is a NE when $G_0$ is common knowledge, is not played in any equilibrium.

This may suggest that common knowledge assumption has a strong implication.
State Space Model

- How to model common knowledge formally?
- We formalize the notion of common knowledge in the language of asymmetric information.
We first model one individual’s information.

An information structure for an individual is given by \((\Omega, \mathcal{P})\), where

- \(\Omega\) is a countable set that represents all possible states. For example, one \(\omega\) may be that “it will rain tomorrow”.
- \(\mathcal{P}\) is a partition of \(\Omega\). This individual cannot distinguish any two states in \(\mathcal{P}(\omega)\) for any \(\omega\).
Knowledge Operator

- From this partition, we can derive a knowledge operator

\[ K : 2^\Omega \rightarrow 2^\Omega \] as follows.

\[ K(E) := \{ \omega \in \Omega | \mathcal{P}(\omega) \subset E \} \]

- In words, \( K(E) \) is the set of states where this individual knows that an event \( E \) is true.
Let’s cast the E-mail game into this framework.

- **Ω** is a set of all possible \((t_1, t_2)\), where \(t_i\) is the number of messages sent by player \(i\).

- From player 1’s perspective, information partition is
  \((0, 0), \{(1, 0), (1, 1)\} \ldots \) Player 2’s information partition is
  \{(0, 0), (1, 0)\}, \{(1, 1), (2, 1)\} \ldots

![Diagram of the state space model]

Note: here the partition can be interpreted as types.
It is easy to derive the following properties of the knowledge operator.

1. **K1:** $K(\Omega) = \Omega$ ("I know anything that is always true").

2. **K2:** $E \subset F \rightarrow K(E) \subset K(F)$ ("if $F$ is true whenever $E$ is, then I know that $F$ is true whenever I know that $E$ is true").

3. **K3:** $K(E_1 \cap E_2) = K(E_1) \cap K(E_2)$ ("if I know $E_1$ and $E_2$, then I know $E_1$ and I know $E_2$").

4. **K4 (Axiom of Knowledge):** $K(E) \subset E$ ("if I know $E$, then $E$ is true").

5. **K5 (Axiom of Transparency):** $K(E) \subset K(K(E))$ ("if I know $E$, then I know that I know $E$"").

6. **K6 (Axiom of Wisdom):** $\neg K(E) \subset K(\neg K(E))$ ("if I don’t know $E$, then I know that I don’t know $E$").
Common Knowledge

Consider an information structure with $N$ individuals: $\{N, \Omega, (P_i)\}$. Let $K_i$ be $i$’s knowledge operator. Now we can consider interactive knowledge.

- $K^1(E) := \bigcap_{i \in N} K_i(E)$: everyone knows $E$.
- $K^2(E) = \bigcap_{i \in N} K_i(K^1(E))$: everyone knows that everyone knows $E$.
- ...
- $K^\infty(E) := \bigcap_{m=1}^{\infty} K^m(E)$: the set of states in which $E$ is common knowledge.
Event $E \subseteq \Omega$ is **common knowledge** at $\omega \in \Omega$ if $\omega \in K^\infty(E)$.

We say that event $E$ is common knowledge when $E$ is common knowledge at every $\omega \in E$. 
Again it is useful to consider E-mail game as an example.

- When is an event “the realized game is $G_O$” ($= \Omega/\{(0, 0)\}$) is common knowledge?
- When is an event “both players received at least $t$ messages” common knowledge?
Self Evident Events

- We say that $E$ is **self evident** if $\mathcal{P}_i(\omega) \subset E$ for every $\omega \in E$ and every $i \in \mathbb{N}$. For example, $\Omega$ is always self-evident.

- It is easy to show that
  - $E$ is self evident if and only if $K_i(E) = E$ for every $i \in \mathbb{N}$.
  - An event is self evident if and only if it is a union of elements of the meet of the partitions.\(^1\)

- The only self evident event in E-mail game is $\Omega$.

\[^1\text{The meet } \mathcal{P}^* = \prod_i \mathcal{P}_i \text{ is the finest partition such that } \mathcal{P}_i(\omega) \subset \mathcal{P}^*(\omega) \text{ for every } i \in \mathbb{N} \text{ and every } \omega \in \Omega.\]
Theorem

Event $E$ is common knowledge at $\omega \in \Omega$ ($\omega \in K^\infty(E)$) if and only if there exists a self evident event $F$ such that $\omega \in F \subset E$.

Proof.

- For “if”, note that $F = K^n(F) \subset K^n(E)$ by Property 2 and $F$ being self-evident. Hence $F \subset K^\infty(E)$, so $\omega \in K^\infty(E)$.

- For “only if”, we just need to show that $K^\infty(E)$ is self evident.
  
  $\triangleright$ $K_i(K^\infty(E)) \subset K^\infty(E)$ for any $i$ by Property 4.
  
  $\triangleright$ $K^{n+1}(E) \subset K_i(K^n(E))$, hence $K^\infty(E) \subset K_i(K^n(E))$ for any $n$.
  
  $\triangleright$ Since $\lim K_i(A^n) = K_i(\lim A^n)$ for any sequence of decreasing sets, $K^\infty(E) \subset K_i(\lim K^n(E)) = K_i(K^\infty(E))$.
Suppose that player $i$ has a belief $\pi_i \in \Delta(\Omega)$. Hence the information structure is given by $\{N, \Omega, (\mathcal{P}_i), (\pi_i)\}$.

This information structure has a common prior if $\pi_i = \pi$ for all $i \in N$ for some $\pi \in \Delta(\Omega)$.

This assumption also has a very strong implication. We’ll see two results.
Agree to Disagree

- Common prior assumption has a strong implication on possibles beliefs people can have.
- With common prior, it cannot be common knowledge that different individuals have different beliefs about any event.
- For example, it cannot be common knowledge that one trader believes that there is 60% chance for the price of some stock going up, while another trader believes that there is 60% chance for the price of the same stock going down.
Theorem (Aumann 1976)

Suppose that $\Omega$ is countable and there is a common prior $p$ on $\Omega$. If it is common knowledge at some $\omega \in \Omega$ that the probability of event $E \subset \Omega$ is $q_i, i \in N$, then $q_1 = \ldots = q_n$. 
Proof.

- Let $E^{q_i}$ be the event that player $i$ believes that $E$ is true with probability $q_i$. Let $E' = \bigcap_{i \in N} E^{q_i}$. By assumption, $\omega \in E'$.

- There exists a self evident event $F$ such that $\omega \in F \subset E'$ by the previous theorem.

- $F$ can be partitioned into $P^k_i, k = 1, 2, ... \in \mathcal{P}_i$ for every $i \in N$ (remember that $F$ is an element of the meet).

- By assumption, $\frac{p(E \cap P^k_i)}{p(P^k_i)} = q_i$ for any $k$. Hence $p(E \cap P^k_i) = q_i p(P^k_i)$.

- Summing them up with respect to $k$, we obtain $p(E \cap F) = q_i p(F)$ for every $i$. So $q_i = \frac{E \cap F}{F}$ for all $i \in N$. 

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No Trade Theorem

- When “rational” traders trade, presumably it is common knowledge that both traders are better off by trading.

- Hence the previous result suggests that any kind of purely speculative trade based on differences in beliefs is impossible.

- We show one such result within this framework.
Suppose that there are $n$ traders.

States: $\omega = (\theta, t_1, \ldots, t_n)$.

- $\theta$ determines trader $i$’s preference and endowment $e_i(\theta) \in \mathbb{R}^k$. It can be ex ante observable or not observable.
- $t_i$ is trader $i$’s private signal.
- Assume that there is a common prior $p$ on $\Omega = \Theta \times \prod_{i \in N} T_i$.

Trader $i$’s utility from net trade $x_i \in \mathbb{R}^k$ given $\theta$ is $u_i(e_i(\theta) + x_i, \theta)$. Assume that every trader is strictly risk averse.
Endowment \( e : \Theta \rightarrow \mathbb{R}^{kn} \) is ex ante Pareto-efficient if there is no net trade \( x_i : \Omega \rightarrow \mathbb{R}^k, i = 1...n, \text{ s.t. } \sum_{i \in N} x_i = 0 \) that is Pareto-improving given the common prior \( p \).

Then it cannot be common knowledge that everyone is better off by trading.

No Trade Theorem

Suppose that \( e : \Theta \rightarrow \mathbb{R}^{kn} \) is ex ante Pareto-efficient. If it is common knowledge at some state \( \omega \) that \( e_i + x_i \) is weakly preferred to \( e_i \) for every \( i \in N \) for some feasible net trade \( x \), then it must be common knowledge that the probability of nonzero net trade is 0.
Proof.

- Let $E$ be the event where $e_i + x_i$ is weakly preferred to $e_i$ for every $i \in N$. Then there exists a self evident event $F$ such that $\omega \in F \subset E$.

- Define a new net transfer $x'$ by $x'(\omega) := x(\omega)$ for every $\omega \in F$ and $x'(\omega) := 0$ for every $\omega \in \Omega/F$.

- Then, for any $i$,

\[
E[u_i(e_i(\tilde{\theta}) + x'_i(\tilde{\omega}), \tilde{\theta})] = E[u_i(e_i(\tilde{\theta}) + x_i(\tilde{\omega}), \tilde{\theta})|F] + E[u_i(e_i(\tilde{\theta}), \tilde{\theta})|\Omega/F] \\
\geq E[u_i(e_i(\tilde{\theta}), \tilde{\theta})|F] + E[u_i(e_i(\tilde{\theta}), \tilde{\theta})|\Omega/F] \\
= E[u_i(e_i(\tilde{\theta}), \tilde{\theta})]
\]

- Since $e$ is ex ante Pareto efficient, it must be that

\[
E[u_i(e_i(\tilde{\theta}) + x_i(\tilde{\omega}), \tilde{\theta})|F] = E[u_i(e_i(\tilde{\theta}), \tilde{\theta})|F] \text{ for all } i \in N. \text{ Strict risk averseness implies that net trade must be 0 in } F, \text{ hence no trade is common knowledge.}
\]