Common Knowledge and Common Prior

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Common Knowledge Assumption

- When we define games, we implicitly introduce lots of **common knowledge** assumptions.
- Something is common knowledge if everyone knows it, everyone knows that everyone knows it, and so on.
- For example, N, A_i, u_i are all common knowledge for strategic game $G = (N, (A_i), (u_i))$.
- But what does it mean? Is it really a significant assumption?
- To understand the notion of common knowledge better, let's take a look at so called **E-mail game**.

E-mail Game

- Two players, player 1 and player 2, play one of the following games: G_s ("status quo") or G_o("opportunity").
- The game is G_s with probability $1-p$ and G_o with probability $p \in (0,1)$.
- Only player 1 observes a realization of the game.

Gs	S	A
S	(0,0)	(0,-2)
A	(-2,0)	(-2,-2)

Go	S	A
S	(0,0)	(0,-2)
A	(-2,0)	(1,1)

- If the game is G_s, then "stay" (S) is the strictly dominant action.
- If the game is G_o, then "attack" is optimal if and only if the other player attacks. There are two strict NE for G_o: (A, A) and (S, S). The former NE is more efficient.

- There is some exchange of information before the game is played:
  - If the game is G_s, nothing happens.
  - If the game is G_o, an e-mail message is automatically sent from player 1 to player 2. This message is lost with probability $\epsilon > 0$.
  - If player 2 receives a message, then a confirmation e-mail is automatically sent from player 2 to player 1. This message is lost with probability $\epsilon > 0$.
  - If player 1 receives a confirmation e-mail, then another confirmation e-mail is automatically sent from player 1 to player 2, which is lost with probability $\epsilon > 0$.
  - This process stops when an e-mail is lost (which happens with probability 1).

- This game can be regarded as a Bayesian game where $\Omega = \{G_s, G_o\}$ and player i's type is the number of messages i sent: $T_i = \{0, 1, 2, 3, \cdots\}$. Since the true game is G_s if and only if $t_1 = 0$, we drop Ω.
- If player 1's type t_1 is 0, then player 1 knows that the true state is G_s (and player 2's type is 0). Hence player 1's optimal choice is S.

- For $t_1 > 0$...
  - For $t_1 > 0$, there are two possibilities: player 1's t_1th message is lost, which happens with probability ϵ, or player 1's t_1th message reached player 2 but player 2's t_1th message is lost, which happens with probability $(1-\epsilon)\epsilon$ (conditional on both players have received the $t_1 - 1$ messages).
  - Hence 1 believes that 2's type is $t_1 - 1$ with probability $q = \frac{\epsilon}{\epsilon+(1-\epsilon)\epsilon} > 1/2$ and t_1 with probability $1 - q$.
  - This implies that S is the unique best response for player 1 if player 2 plays S when $t_2 = t_1 - 1$.
  - Similarly S is the unique best response for player 2 given any t_2 if player 1 plays S when $t_1 = t_2$.
- Since S is the unique best response for player 1 when $t_1 = 0$, S must be played by every type by both players.

So we have proved the following result.

Theorem (Rubinstein, 1989)

There exists a unique Bayesian Nash equilibrium for this game and A is never played in equilibrium.

- What do the players know given their types?
    - Player 1 of type 1 knows that the true state is G_o, but does not know if player 2 knows it.
    - Player 1 of type 2 knows that the true state is G_o, knows that player 2 knows it, but does not know if player 2 knows that player 1 knows that player 2 knows that the true state is G_o.
    - If the type profile is (m, m), then the players know that they know that $\cdots \times m \cdots$ that the true state is G_o. But they are not sure about the other player's mth order knowledge.
- If m is large, then it is "almost common knowledge" that the game is G_o. However (A, A), which is a NE when G_o is common knowledge, is not played in any equilibrium.
- This may suggest that common knowledge assumption has a strong implication.

State Space Model

- How to model common knowledge formally?
- We formalize the notion of common knowledge in the language of asymmetric information.

- We first model one individual's information.
- An information structure for an individual is given by $(\Omega, \mathcal{P})$, where
  - Ω is a countable set that represents all possible states. For example, one ω may be that "it will rain tomorrow".
  - $\mathcal{P}$ is a partition of Ω. This individual cannot distinguish any two states in $\mathcal{P}(\omega)$ for any ω.

Knowledge Operator

- From this partition, we can derive a **knowledge operator** $K : 2^{\Omega} \rightarrow 2^{\Omega}$ as follows.

$$K(E) := \{\omega \in \Omega | \mathcal{P}(\omega) \subset E\}$$

- In words, $K(E)$ is the set of states where this individual knows that an event E is true.

- Let's cast the E-mail game into this framework.
  - Ω is a set of all possible (t_1, t_2), where t_i is the number of messages sent by player i.
  - From player 1's perspective, information partion is $(0,0), \{(1,0),(1,1)\} \ldots$ Player 2's information partition is $\{(0,0),(1,0)\}, \{(1,1),(2,1)\} \ldots$

P1	**0**	**1**		**2**		**3**	**......**
Ω	(0,0)	(1,0)	(1,1)	(2,1)	(2,2)	(3,2)	......
P2	0		1		2		......

It is easy to derive the following properties of the knowledge operator.

1. **K1:** $K(\Omega) = \Omega$ ("I know anything that is always true").
2. **K2:** $E \subset F \to K(E) \subset K(F)$ (" if F is true whenever E is, then I know that F is true whenever I know that E is true").
3. **K3:** $K(E_1 \bigcap E_2) = K(E_1) \bigcap K(E_2)$ ("if I know E_1 and E_2, then I know E_1 and I know E_2").
4. **K4(Axiom of Knowledge):** $K(E) \subset E$ ("if I know E, then E is true").
5. **K5(Axiom of Transparency):** $K(E) \subset K(K(E))$ ("if I know E, then I know that I know E")
6. **K6(Axiom of Wisdom):** $\neg K(E) \subset K(\neg K(E))$("if I don't know E, then I know that I don't know E").

Common Knowledge

Consider an information structure with N individuals: $\{N, \Omega, (\mathcal{P}_i)\}$. Let K_i be i's knowledge operator. Now we can consider **interactive knowledge**.

- $K^1(E) := \bigcap_{i \in N} K_i(E)$: everyone knows E.
- $K^2(E) = \bigcap_{i \in N} K_i(K^1(E))$: everyone knows that everyone knows E.
- ...
- $K^\infty(E) := \bigcap_{m=1}^{\infty} K^m(E)$: the set of states in which E is **common knowledge**.

Common Knowledge

Event $E \subset \Omega$ is **common knowledge** at $\omega \in \Omega$ if $\omega \in K^{\infty}(E)$.

We say that event E is common knowledge when E is common knowledge at every $\omega \in E$.

Again it is useful to consider E-mail game as an example.

- When is an event "the realized game is G_O" $(= \Omega / \{(0,0)\})$ is common knowledge?
- When is an event "both players received at least t messages" common knowledge?

Self Evident Events

- We say that E is **self evident** if $\mathcal{P}_i(\omega) \subset E$ for every $\omega \in E$ and every $i \in N$. For example, Ω is always self-evident.
- It is easy to show that
  - E is self evident if and only if $K_i(E) = E$ for every $i \in N$.
  - An event is self evident if and only if it is a union of elements of the meet of the partitions.[1]
- The only self evident event in E-mail game is Ω.

[1] The meet $\mathcal{P}^* = \prod_i \mathcal{P}_i$ is the finest partition such that $\mathcal{P}_i(\omega) \subset \mathcal{P}^*(\omega)$ for every $i \in N$ and every $\omega \in \Omega$.

Theorem

Event E is common knowledge at $\omega \in \Omega$ ($\omega \in K^\infty(E)$) if and only if there exists a self evident event F such that $\omega \in F \subset E$.

Proof.

- For "if", note that $F = K^n(F) \subset K^n(E)$ by Property 2 and F being self-evident. Hence $F \subset K^\infty(E)$, so $\omega \in K^\infty(E)$.
- For "only if", we just need to show that $K^\infty(E)$ is self evident.
  - $K_i(K^\infty(E)) \subset K^\infty(E)$ for any i by Property 4.
  - $K^{n+1}(E) \subset K_i(K^n(E))$, hence $K^\infty(E) \subset K_i(K^n(E))$ for any n.
  - Since $\lim K_i(A^n) = K_i(\lim A^n)$ for any sequence of decreasing sets, $K^\infty(E) \subset K_i(\lim K^n(E)) = K_i(K^\infty(E))$.

□

Common Prior

- Suppose that player i has a belief $p_i \in \Delta(\Omega)$. Hence the information structure is given by $\{N, \Omega, (\mathcal{P}_i), (p_i)\}$.
- This information structure has a common prior if $p_i = p$ for all $i \in N$ for some $p \in \Delta(\Omega)$.
- This assumption also has a very strong implication. We'll see two results.

Agree to Disagree

- Common prior assumption has a strong implication on possibles beliefs people can have.
- With common prior, it cannot be common knowledge that different individuals have different beliefs about any event.
- For example, it cannot be common knowledge that one trader believes that there is 60% chance for the price of some stock going up, while another trader believes that there is 60% chance for the price of the same stock going down.

Theorem (Aumann 1976)

Suppose that Ω is countable and there is a common prior p on Ω. If it is common knowledge at some $\omega \in \Omega$ that the probability of event $E \subset \Omega$ is $q_i, i \in N$, then $q_1 =, ..., = q_n$.

Proof.

- Let E^{q_i} be the event that player i believes that E is true with probability q_i. Let $E' = \bigcap_{i \in N} E^{q_i}$. By assumption, $\omega \in E'$.
- There exists a self evident event F such that $\omega \in F \subset E'$ by the previous theorem.
- F can be partitioned into $P_i^k, k = 1, 2, ... \in \mathcal{P}_i$ for every $i \in N$ (remember that F is an element of the meet).
- By assumption, $\frac{p(E \bigcap P_i^k)}{p(P_i^k)} = q_i$ for any k. Hence $p(E \bigcap P_i^k) = q_i p\left(P_i^k\right)$.
- Summing them up with respect to k, we obtain $p(E \bigcap F) = q_i p\left(F\right)$ for every i. So $q_i = \frac{E \bigcap F}{F}$ for all $i \in N$.

□

No Trade Theorem

- When "rational" traders trade, presumably it is common knowledge that both traders are better off by trading.
- Hence the previous result suggests that any kind of purely speculative trade based on differences in beliefs is impossible.
- We show one such result within this framework.

- Suppose that there are n traders.
- States: $\omega = (\theta, t_1, ..., t_n)$.
  - θ determines trader i's preference and endowment $e_i(\theta) \in \Re^k$. It can be ex ante observable or not observable.
  - t_i is trader i's private signal.
  - Assume that there is a common prior p on $\Omega = \Theta \times \prod_{i \in N} T_i$.
- Trader i's utility from net trade $x_i \in \Re^k$ given θ is $u_i(e_i(\theta) + x_i, \theta)$.

  Assume that every trader is strictly risk averse.

- Endowment $e : \Theta \to \Re^{kn}$ is ex ante Pareto-efficient if there is no net trade $x_i : \Omega \to \Re^k, i = 1...n, \ s.t. \sum_{i \in N} x_i = 0$ that is Pareto-improving given the common prior p.
- Then it cannot be common knowledge that everyone is better off by trading.

No Trade Theorem

Suppose that $e : \Theta \to \Re^{kn}$ is ex ante Pareto-efficient. If it is common knowledge at some state ω that $e_i + x_i$ is weakly preferred to e_i for every $i \in N$ for some feasible net trade x, then it must be common knowledge that the probability of nonzero net trade is 0.

Proof.

- Let E be the event where $e_i + x_i$ is weakly preferred to e_i for every $i \in N$. Then there exists a self evident event F such that $\omega \in F \subset E$.
- Define a new net transfer x' by $x'(\omega) := x(\omega)$ for every $\omega \in F$ and $x'(\omega) := 0$ for every $\omega \in \Omega/F$.
- Then, for any i,

$$\begin{aligned} E[u_i(e_i(\widetilde{\theta}) + x_i'(\widetilde{\omega}), \widetilde{\theta})] &= E[u_i(e_i(\widetilde{\theta}) + x_i(\widetilde{\omega}), \widetilde{\theta})|F] + E[u_i(e_i(\widetilde{\theta}), \widetilde{\theta})|\Omega/F] \\ &\geq E[u_i(e_i(\widetilde{\theta}), \widetilde{\theta})|F] + E[u_i(e_i(\widetilde{\theta}), \widetilde{\theta})|\Omega/F] \\ &= E[u_i(e_i(\widetilde{\theta}), \widetilde{\theta})] \end{aligned}$$

- Since e is ex ante Pareto efficient, it must be that $E[u_i(e_i(\widetilde{\theta}) + x_i(\widetilde{\omega}), \widetilde{\theta})|F] = E[u_i(e_i(\widetilde{\theta}), \widetilde{\theta})|F]$ for all $i \in N$. Strict risk averseness implies that net trade must be 0 in F, hence no trade is common knowledge.

□