Extensive Game with Imperfect Information

Ichiro Obara

UCLA

March 7, 2012

Extensive Game with Imperfect Information

- We consider dynamic games where past "moves" (by players or nature) are imperfectly observed.

Example 1: Observational Learning

- n players are lined up.
- Each player has two options: buy a new iPad (B) or do not buy (N).
- The quality of new iPad may be high (H) with probability $p \in (0, 1)$ or low (L) with probability $1 - p$. p is a common prior. The quality is common to all players.
- Player i observes a private signal $s_i \in \{h, l\}$, which is correct with probability $q \in (0, 1)$, and the choice of the preceding players: $(a_1, ..., a_{i-1})$.
- The net payoff from purchasing an iPad is 1 if the quality is good and -1 if the quality is bad.

Example 2: Cheap Talk/Information Transmission

- Two players: Manager(M) and Analyst(A). M likes to figure out what is the right size of investment for a new project. A has an access to this information.
- A learns the right size, which is either "small", "medium", "large". He conveys this information to M (but he can lie). Then M chooses the size of the investment.
- A prefers a slightly excessive level of investment: he prefers "medium" when he learns "small", prefers "large" when he learns "medium" or "large".

Extensive Game with Imperfect Information

- **Extensive game** is an extension of extensive game with perfect information.
  - a finite set N
  - a set of finite or infinite sequences H such that
    - $\emptyset \in H$
    - $(a^1, ..., a^k) \in H \rightarrow (a^1, ..., a^\ell) \in H$ for any $\ell < k$
    - $(a^1, ...) \in H$ if $(a^1, ..., a^k) \in H$ for $k = 1, 2, ...$

    with $Z \subset H$ defined by $(a^1, ..., a^k) \in Z \Leftrightarrow \nexists a^{k+1}, (a^1, ..., a^{k+1}) \in H$.
  - a function $P : H/Z \rightarrow N \bigcup \{c\}$.
  - a function f_c that assigns a probability distribution $f_c(h) \in \Delta(A(h))$ for each h such that $P(h) = c$.
  - a partition $\mathcal{I}_i$ of $H_i = \{h \in H | P(h) = i\}$ for each $i \in N$ such that $A(h) = A(h')$ for any $h, h' \in I_i \in \mathcal{I}_i$ and any $i \in N$.
  - a function $V_i : Z \rightarrow \Re$ for $i \in N$.

- N, H, Z, P, f_c and V_i are the same as before.
- What is new is $\mathcal{I}_i$: a collection of **information sets** for each $i \in N$. Each $I_i \in \mathcal{I}_i$ is player i's information set: the set of histories that player i cannot distinguish.
- "$A(h) = A(h')$ for any $h, h' \in I_i \cdots$" reflects this informational restriction.
- We denote the set of actions available at I_i by $A(I_i)$.

Strategies

- Three kind of strategies:
  - Player i's pure strategy for extensive game $(N, H, P, f_c, (\mathcal{I}_i), (V_i))$ is a mapping s_i that assigns an action in $A(I_i)$ for each $I_i \in \mathcal{I}_i$. Let S_i be the set of player i's strategies.
  - Player i's mixed stratey is a probability distribution over pure strategies (an element in $\Delta(S_i)$).
  - Player i's behavior strategy is a mapping σ_i that assigns a probability measure on $A(I_i)$ for each $I_i \in \mathcal{I}_i$. Let Σ_i be the set of player i's behavior strategies.
- **Remark.** For mixed and behavior strategies, we assume that the set of actions is finite: $A(h)$ is a finite set for any H/Z.

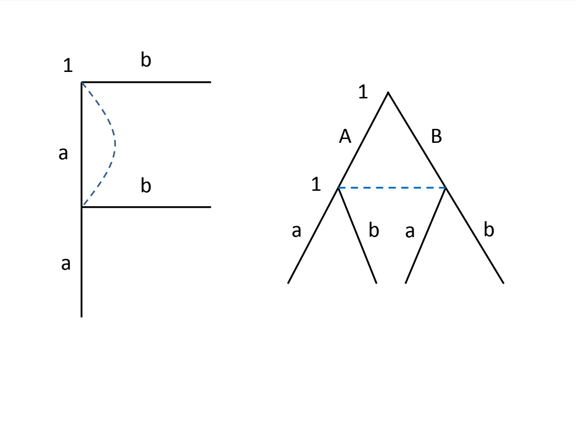
- Each profile of strateges determines an outcome $O(s) \in Z$ or a distribution of outcomes $\widetilde{O}(\sigma) \in \Delta(Z)$.
- Player i's payoff given s and expected payoff given σ are $V_i(O(s))$ and $E\left[V_i(\widetilde{O}(\sigma))\right]$, which we denote by $V_i(s)$ and $V_i(\sigma)$ respectively.

Perfect Recall

- For any $h \in H$, let $X_i(h)$ be the sequence of player i's information sets along h.
- We almost always assume **perfect recall**: $X_i(h) = X_i(h')$ for any $h, h' \in I_i$ for any $I_i \in \mathcal{I}_i$ for any $i \in N$.
- Without perfect recall, a player may forget what she knew (what she played etc.)

Example

- Without perfect recall, mixed strategies and behavior strategies are not equivalent unlike in the case of perfect information.



- Two mixed or behavior strategies are **outcome equivalent** if they generate the same outcome distribution given any pure strategy of the other players.
- With perfect recall, we do not need to distinguish behavior strategies and mixed strategies in finite extensive games in the following sense.

Theorem

- For any mixed strategy of finite extensive game, there exists an outcome equivalent behavioral strategy.
- For any behavior strategy of finite extensive game, there exists an outcome equivalent mixed strategy.

Nash Equilibrium

- For any extensive game $(N, H, P, f_c, (\mathcal{I}_i), (V_i))$, there exists a strategic game $(N, (S_i), (V_i))$. So we can define Nash equilibrium for extensive game as usual.

Nash Equilibrium

For extensive game $(N, H, P, f_c, (\mathcal{I}_i), (V_i))$, a profile of strategies s^* is a Nash equilibrium if

$$V_i(s^*) \geq V_i(s_i', s_{-i}^*)$$

for any $s_i' \in S_i$ and any $i \in N$.

- Again our interest is to come up with a reasonable refinement of Nash equilibrium.