Extensive Game with Imperfect Information

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We consider dynamic games where past “moves” (by players or nature) are imperfectly observed.
Example 1: Observational Learning

- $n$ players are lined up.
- Each player has two options: buy a new iPad (B) or do not buy (N).
- The quality of new iPad may be high (H) with probability $p \in (0, 1)$ or low (L) with probability $1 - p$. $p$ is a common prior. The quality is common to all players.
- Player $i$ observes a private signal $s_i \in \{h, l\}$, which is correct with probability $q \in (0, 1)$, and the choice of the preceding players: $(a_1, ..., a_{i-1})$.
- The net payoff from purchasing an iPad is 1 if the quality is good and $-1$ if the quality is bad.
Example 2: Cheap Talk/Information Transmission

- Two players: Manager(M) and Analyst(A). M likes to figure out what is the right size of investment for a new project. A has an access to this information.

- A learns the right size, which is either “small”, “medium”, “large”. He conveys this information to M (but he can lie). Then M chooses the size of the investment.

- A prefers a slightly excessive level of investment: he prefers “medium” when he learns “small”, prefers “large” when he learns “medium” or “large”.

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Extensive Game with Imperfect Information

- **Extensive game** is an extension of extensive game with perfect information.

  - a finite set $N$
  - a set of finite or infinite sequences $H$ such that
    - $\emptyset \in H$
    - $(a^1, \ldots, a^k) \in H \rightarrow (a^1, \ldots, a^\ell) \in H$ for any $\ell < k$
    - $(a^1, \ldots) \in H$ if $(a^1, \ldots, a^k) \in H$ for $k = 1, 2, \ldots$

  with $Z \subset H$ defined by $(a^1, \ldots, a^k) \in Z \iff \exists a^{k+1}, (a^1, \ldots, a^{k+1}) \in H$.

  - a function $P : H/Z \rightarrow N \cup \{c\}$.
  - a function $f_c$ that assigns a probability distribution $f_c(h) \in \Delta(A(h))$ for each $h$ such that $P(h) = c$.
  - a partition $\mathcal{I}_i$ of $H_i = \{h \in H | P(h) = i\}$ for each $i \in N$ such that $A(h) = A(h')$ for any $h, h' \in I_i \in \mathcal{I}_i$ and any $i \in N$.
  - a function $V_i : Z \rightarrow \mathbb{R}$ for $i \in N$. 

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- \(N, H, Z, P, f_c\) and \(V_i\) are the same as before.
- What is new is \(\mathcal{I}_i\): a collection of information sets for each \(i \in N\).
  - Each \(I_i \in \mathcal{I}_i\) is player \(i\)'s information set: the set of histories that player \(i\) cannot distinguish.
- \(A(h) = A(h')\) for any \(h, h' \in I_i\) reflects this informational restriction.
- We denote the set of actions available at \(I_i\) by \(A(I_i)\).
Strategies

- Three kind of strategies:
  - Player $i$’s pure strategy for extensive game $(N, H, P, f_c, (I_i), (V_i))$ is a mapping $s_i$ that assigns an action in $A(I_i)$ for each $I_i \in I_i$. Let $S_i$ be the set of player $i$’s strategies.
  - Player $i$’s mixed strategy is a probability distribution over pure strategies (an element in $\Delta(S_i)$).
  - Player $i$’s behavior strategy is a mapping $\sigma_i$ that assigns a probability measure on $A(I_i)$ for each $I_i \in I_i$. Let $\Sigma_i$ be the set of player $i$’s behavior strategies.

- **Remark.** For mixed and behavior strategies, we assume that the set of actions is finite: $A(h)$ is a finite set for any $H/\mathbb{Z}$. 
Each profile of strategies determines an outcome $O(s) \in Z$ or a distribution of outcomes $\tilde{O}(\sigma) \in \Delta(Z)$.

Player $i$’s payoff given $s$ and expected payoff given $\sigma$ are $V_i(O(s))$ and $E \left[ V_i(\tilde{O}(\sigma)) \right]$, which we denote by $V_i(s)$ and $V_i(\sigma)$ respectively.
Perfect Recall

- For any $h \in H$, let $X_i(h)$ be the sequence of player $i$'s information sets along $h$.
- We almost always assume **perfect recall**: $X_i(h) = X_i(h')$ for any $h, h' \in I_i$ for any $I_i \in \mathcal{I}_i$ for any $i \in \mathcal{N}$.
- Without perfect recall, a player may forget what she knew (what she played etc.)
Example

- Without perfect recall, mixed strategies and behavior strategies are not equivalent unlike in the case of perfect information.

![Diagram showing a game tree with imperfect recall and mixed strategies](image)
Two mixed or behavior strategies are **outcome equivalent** if they generate the same outcome distribution given any pure strategy of the other players.

With perfect recall, we do not need to distinguish behavior strategies and mixed strategies in finite extensive games in the following sense.

**Theorem**

- For any mixed strategy of finite extensive game, there exists an outcome equivalent behavioral strategy.
- For any behavior strategy of finite extensive game, there exists an outcome equivalent mixed strategy.
Nash Equilibrium

For any extensive game \((N, H, P, f_c, (I_i), (V_i))\), there exists a strategic game \((N, (S_i), (V_i))\). So we can define Nash equilibrium for extensive game as usual.

Nash Equilibrium

For extensive game \((N, H, P, f_c, (I_i), (V_i))\), a profile of strategies \(s^*\) is a Nash equilibrium if

\[ V_i(s^*) \geq V_i(s'_i, s^*_{-i}) \]

for any \(s'_i \in S_i\) and any \(i \in N\).

Again our interest is to come up with a reasonable refinement of Nash equilibrium.