Consumer Theory

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October 8, 2012

Utility Maximization

Utility Maximization Problem

We formalize each consumer's decision problem as the following optimization problem.

Utility Maximization

$$\max_{x \in X} u(x) \; s.t. \; p \cdot x \leq w \; (x \in B(p, w))$$

Walrasian Demand

- Let $x(p,w) \subset X$ (**Walrasian demand correspondence**) be the set of the solutions for the utility maximization problem given $p \gg 0$ and $w \geq 0$. Note that $x(p,w)$ is not empty for any such (p,w) if u is continuous.
- We like to understand the property of Walrasian demand. First we prove basic, but very important properties of $x(p,w)$.

Walrasian Demand

Theorem

Suppose that u is continuous, locally nonsatiated, and $X = \Re^L_+$. Then the Walrasian demand correspondence $x : \Re^L_{++} \times \Re_+ \rightrightarrows \Re^L_+$ satisfies

(I) **Homogeneity of degree 0:** $x(\alpha p, \alpha w) = x(p, w)$ for any $\alpha > 0$ and (p, w),

(II) **Walras' Law:** $p \cdot x' = w$ for any $x' \in x(p, w)$ and (p, w),

(III) **Convexity:** $x(p, w)$ is convex for any (p, w) if u is quasi-concave, and

(IV) **Continuity:** x is upper hemicontinuous.

Walrasian Demand

Proof

- (I) follows from the definition of the problem.
- For (II), use local nonsatiation.
- (III) is obvious.
- (IV) follows from the maximum theorem.

Remark.

- $x(p, w)$ is a single point if u is strictly quasi-concave.
- $x(p, w)$ is a continuous function if it is single-valued.
- General remark: it is useful to clarify which assumption is important for which result.

Walrasian Demand

How can we obtain $x(p, w)$?

- If u is differentiable, then we can apply the **(Karush-)Kuhn-Tucker condition** to derive $x(p, w)$ for each $(p, w) \gg 0$.
- They are necessary if the **constraint qualification** is satisfied (which is always the case here), and also sufficient if u is **pseudo-concave** (See "Mathematical Appendix".)

Walrasian Demand

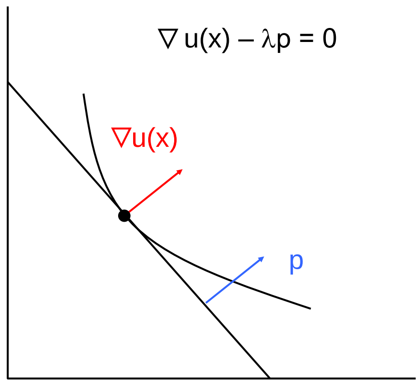
- Suppose that u is locally nonsatiated and the optimal solution is an interior solution. Then the K-T conditions become very simple.

$$\nabla u(x) - \lambda p = 0$$

$$p \cdot x = w$$

- If x_ℓ may be 0 for some ℓ (boundary solution), then $D_\ell u(x) - \lambda p_\ell = 0$ needs to be replaced by $D_\ell u(x) - \lambda p_\ell \leq 0 (= 0 \text{ if } x_\ell > 0)$.

An interior solution looks like:

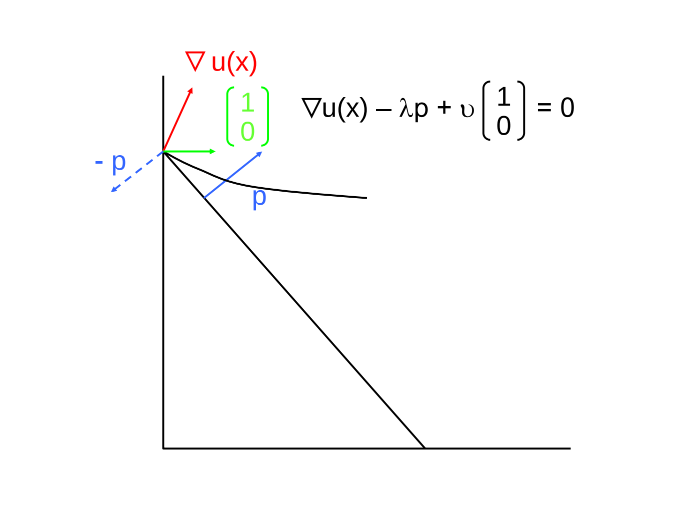


- For a boundary solution, consider the following example with $x_1 = 0$.

$$\begin{aligned} Du_{x_1}(x_1, x_2) - \lambda p_1 &\leq 0 \\ Du_{x_2}(x_1, x_2) - \lambda p_2 &= 0 \end{aligned}$$

- This can be written as $\nabla u(x) - \lambda p + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ for some $\mu \geq 0$.

This boundary solution looks like:



Example

Let's try to solve some example.

- Suppose that $u(x) = \sqrt{x_1} + x_2$.
- We can assume an interior solution for x_1. So the K-T conditions become

$$\frac{1}{2\sqrt{x_1}} - \lambda p_1 = 0$$

$$1 - \lambda p_2 \leq 0 \ (= 0 \text{ if } x_2 > 0)$$

$$p \cdot x = w$$

Example

- Then the solution is
  1. $x_1(p,w) = \frac{p_2^2}{4p_1^2}, x_2(p,w) = \frac{w}{p_2} - \frac{p_2}{4p_1}, \lambda(p,w) = 1$ when $4p_1 w > p_2^2$,
  2. $x_1(p,w) = \frac{w}{p_1}, x_2(p,w) = 0, \lambda(p,w) = \frac{1}{2\sqrt{p_1 w}}$ when when $4p_1 w \leq p_2^2$.
- Note that there is no wealth effect on x_1 (i.e. x_1 is independent of w) as long as $4p_1 w > p_2^2$

Indirect Utility Function

For any $(p, w) \in \Re^L_{++} \times \Re_+$, $v(p, w)$ is defined by $v(p, w) := u(x')$ where $x' \in x(p, w)$. It is not difficult to prove that this **indirect utility function** satisfies the following properties.

Theorem

Suppose that u is continuous, locally nonsatiated, and $X = \Re^L_+$. Then $v(p, w)$ is

(I) homogeneous of degree 0,

(II) nonincreasing in p_ℓ for any ℓ and strictly increasing in w,

(III) quasi-convex, and

(IV) continuous.

Proof.

- (I) and (II) are obvious.
- (IV) is again an implication of the maximum theorem.
- Proof of (III):
    - Suppose that $\max\{v(p', w'), v(p'', w'')\} \leq \overline{v}$ for any $(p', w'), (p'', w'') \in \Re^L_{++} \times \Re_+$ and $\overline{v} \in \Re$.
    - For any $\alpha \in [0, 1]$ and any $x \in B(\alpha p' + (1-\alpha)p'', \alpha w' + (1-\alpha)w'')$, either $x \in B(p', w')$ or $x \in B(p'', w'')$ must hold.
    - Hence

      $$v(\alpha p' + (1-\alpha)p'', \alpha w' + (1-\alpha)w'') \leq \max\{v(p', w'), v(p'', w'')\} \leq \overline{v}.$$

Example: Cobb-Douglas

- Suppose that $u(x) = \sum_{\ell=1}^{L} \alpha_\ell \log x_\ell$, $\alpha_\ell \geq 0$ and $\sum_{\ell=1}^{L} \alpha_\ell = 1$ (Cobb-Douglas utility function).
- The Walrasian demand is

$$x_\ell(p, w) = \frac{\alpha_\ell w}{p_\ell}$$

(**Note:** α_ℓ is the fraction of the expense for good ℓ).

- So $v(p, w) = \log w + \sum_{\ell=1}^{L} \alpha_\ell (\log \alpha_\ell - \log p_\ell)$.

Example: Quasi-linear utility

- For the previous quasi-linear utility example,

$$v(p, w) = \sqrt{x_1(p, w)} + x_2(p, w) = \frac{p_2}{4p_1} + \frac{w}{p_2}$$

(assuming an interior solution).

Indirect Utility Function

Exercise.

1. The result of this example generalizes. Suppose that the utility function is in a quasi-linear form: $u(x) = x_1 + h(x_2, ..., x_L)$. Show that the indirect utility function takes the following form: $v(p, w) = a(p) + w$ (assuming interior solutions).
2. Show that the $v(p, w) = b(p)w$ if the utility function is homogeneous of degree 1.

Example: Labor Supply

- Consider the following simple labor/leisure decision problem:

$$\max_{q,\ell\geq 0}(1-\alpha)\log q+\alpha\log\ell \;\; s.t. \;\; pq+w\ell\leq wT+\pi, \ell\leq T$$

where

  - q is the amount of consumed good and p is its price
  - T is the total time available
  - ℓ is the time spent for "leisure" (which determines $h=T-\ell$: hours of work).
  - w is wage (and wh is labor income).
  - π is nonlabor income.

Example: Labor Supply

- Since the utility function is Cobb-Douglas, it is easy to derive the Walrasian demand: $q(p,w,\pi) = \frac{(1-\alpha)(wT+\pi)}{p}, \ell(p,w,\pi) = \frac{\alpha(wT+\pi)}{w}$ when $\ell < T$.

  (**Note:** if this expression of ℓ is larger than T, $\ell \leq T$ binds. In this case, this consumer does not participate in the labor market ($\ell(p,w,\pi) = T$) and spends all nonlabor income to purchase goods ($q(p,w,\pi) = \frac{\pi}{p}$).)
- It is easy to derive the indirect utility function when $\ell < T$:

  $v(p,w,wT+\pi) = const. + \log(wT+\pi) - \alpha \log p - (1-\alpha)\log w$.

Cost Minimization

Cost Minization

Next consider the following problem for each $p \gg 0$ and $\underline{u} \in \Re$,

Cost Minimization

$$\min_{x \in X} p \cdot x \ s.t. \ u(x) \geq \underline{u}$$

This problem can be phrased as follows: what is the cheapest way to achieve utility at least as high as $\underline{u}$?

Hicksian Demand

Let $h(p, \underline{u})$ (**Hicksian demand correspondence**) be the set of solutions for the cost minimization problem given $p \gg 0$ and $\underline{u}$.

Remark. $h(p, \underline{u})$ is useful for **welfare analysis**, which we do not have time to cover. Read MWG Ch 3-I.

- Assume local nonsatiation. Then the constraint can be modified locally as follows if u is continuous and $\{x \in X : u(x) \geq \underline{u}\}$ is not empty (denote the set of such $\underline{u}$ by $\underline{U}$.
  - Pick any x' such that $u(x') > \underline{u}$.
  - Pick any $\overline{x} \in \Re_+$ such that $p'_\ell\, \overline{x} \geq p' \cdot x'$ for all ℓ for all p' in a neighborhood of $p \gg 0$.
  - Then the cost minimizing solution is the same locally with respect to $(p, \underline{u})$ when the constraint set is replaced by a compact set $\left\{x \in \Re^L_+ : u(x) \geq \underline{u}, x_\ell \leq \overline{x}\ \forall \ell\right\}$ (because $p' \cdot x \geq p' \cdot x'$ for any x outside of this set).
- This implies that $h(p, \underline{u})$ is not empty around $(p, \underline{u})$ (note that local nonsatiation is not needed for nonemptyness at each $(p, \underline{u}) \in \Re^L_{++} \times \underline{U}$).

Hicksian Demand

Theorem

Suppose that u is continuous, locally nonsatiated, and $X = \Re^L_+$. Then the Hicksian demand correspondence $h : \Re^L_{++} \times \underline{U} \rightrightarrows \Re^L_+$ is

(I) homogeneous of degree 0 in p,

(II) achieving $\underline{u}$ exactly $(u(x') = \underline{u}$ for any $x' \in h(p, \underline{u}))$ if $\underline{u} \geq u(0)$,

(III) convex given any $(p, \underline{u})$ if u is quasi-concave, and

(iv) upper hemicontinuous.

Remark. $h(p, u)$ is a point if u is strictly quasi-concave.

Note on the proof.

- (I), (II), and (III) are straightforward.
- (iv) is slightly more difficult than (iv) for Walrasian demand. We cannot apply the maximum theorem directly because the feasible set is not "locally bounded".

... but we skip the detail.

Expenditure Function

Expenditure function $e(p, \underline{u})$ is defined by $e(p, \underline{u}) := p \cdot x'$ for any $x' \in h(p, \underline{u})$. The proof of the following theorem is left as an exercise.

Theorem

Suppose that u is continuous, locally nonsatiated, and $X = \Re^L_+$. Then the expenditure function $e : \Re^L_{++} \times \underline{U} \to \Re$ is

(I) homogeneous of degree 1 in p,

(II) nondecreasing in p_ℓ for any ℓ and strictly increasing in $\underline{u}$ for $\underline{u} > u(0)$,

(III) concave in p, and

(IV) continuous.

Utility Maximization $\leftrightarrow$ Cost Minimization

Not surprisingly, cost minimization problems are closely related to utility maximization problems. One problem is a flip side of the other in some sense.

Utility Maximization $\leftrightarrow$ Cost Minimization

Utility Maximization $\leftrightarrow$ Cost Minimization

Suppose that u is continuous, locally nonsatiated, and $X = \Re^L_+$.

(I) If $x^* \in x(p, w)$ given $p \gg 0$ and $w \geq 0$, then $x^* \in h(p, v(p, w))$ and $e(p, v(p, w)) = w$.

(II) If $x^* \in h(p, \underline{u})$ given $p \gg 0$ and $\underline{u} \geq u(0)$, then $x^* \in x(p, e(p, \underline{u}))$ and $v(p, e(p, \underline{u})) = \underline{u}$.

Proof: Utility Maximization → Cost Minimization

- Suppose not, i.e. $\exists x' \in \Re^L_+$ that satisfies $u(x') \geq u(x^*)$ and $p \cdot x' < p \cdot x^*$ ($= w$ by Walras' law).
- By local nonsatiation, $\exists x'' \in \Re^L_+$ that satisfies $u(x'') > u(x^*)$ and $p \cdot x'' < w$. This is a contradiction to utility maximization.
- Hence $x^* \in h(p, v(p, w))$ and $e(p, v(p, w)) = p \cdot x^* = w$.

Proof: Utility Maximization $\leftarrow$ Cost Minimization

- Suppose not, i.e. $\exists x' \in \Re^L_+$ that satisfies $u(x') > u(x^*) \geq \underline{u}$ and $p \cdot x' \leq p \cdot x^*$. Note that $0 < p \cdot x'$ (because $\underline{u} \geq u(0)$).
- Let $x^\alpha := \alpha 0 + (1-\alpha)x' \in X$ for $\alpha \in (0,1)$. Then $u(x^\alpha) > u(x^*)$ and $p \cdot x^\alpha < p \cdot x^*$ (because $p \cdot x' > 0$) for small α. This is a contradiction to cost minimization.
- Hence $x^* \in x(p, e(p, \underline{u}))$ and $v(p, e(p, \underline{u})) = u(x^*) = \underline{u}$ (by $\underline{u} \geq u(0)$).

Some Useful Formulas

- Walrasian demand, Hicksian demand, indirect utility function, and expenditure function are all very closely related. We can exploit these relationships in many ways.
  - Different expressions are useful for different purposes.
  - We can recover one function from another. In particular, we can recover unobserved from observed.
- We already know
  - Utility maximization $\rightarrow$ Cost minimization
    - $h(p, v(p, w)) = x(p, w)$,
    - $e(p, v(p, w)) = w$.
  - Cost minimization $\rightarrow$ Utility maximization
    - $x(p, e(p, \underline{u})) = h(p, \underline{u})$,
    - $v(p, e(p, \underline{u})) = \underline{u}$.

Shepard's Lemma

- In the following, we derive a few more important formulas, assuming that $x(p, w)$ and $h(p, \underline{u})$ are $\mathcal{C}^1$(continuously differentiable) functions.
- Let's start with **Shepard's Lemma**.

Theorem

For any $(p, \underline{u}) \in \Re^L_{++} \times \underline{U}$,

$$\nabla_p e(p, \underline{u}) = h(p, \underline{u})$$

Proof

$$
\begin{aligned}
\nabla_p e(p, \underline{u}) &= \nabla(p \cdot h(p, \underline{u})) \\
&= h(p, \underline{u}) + D_p h(p, \underline{u})^\top p \\
&= h(p, \underline{u}) + \frac{1}{\lambda} D_p h(p, \underline{u})^\top \nabla u(h(p, \underline{u})) \text{ (by FOC)} \\
&= h(p, \underline{u}) \text{ (by differentiating } u(h(p, \underline{u})) = \underline{u})
\end{aligned}
$$

Note. I am assuming an interior solution, but this is not necessary (apply Envelope theorem).

Shepard's Lemma

Remark.

- Since $h(p, \underline{u}) = x(p, e(p, \underline{u}))$, this lemma implies

$$\nabla_p e(p, \underline{u}) = x(p, e(p, \underline{u})).$$

  From this differential equation, we can recover $e(\cdot, \underline{u})$ for each $\underline{u}$ if $D^2 e$ is symmetric.

- If $D^2 e$ is negative semidefinite, then $e(\cdot, \underline{u})$ in fact satisfies all the properties of expenditure functions. Then we can recover the preference that rationalizes $x(p, w)$. See Ch.3-H, MWG.

Slutsky Equation

Slutsky Equation

For all $(p, w) \gg 0$ and $\underline{u} = v(p, w)$,

$$\frac{\partial h_\ell(p, \underline{u})}{\partial p_k} = \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)$$

or more compactly

$$D_p h(p, \underline{u}) = \underbrace{D_p x(p, w)}_{L \times L} + \underbrace{D_w x(p, e)}_{L \times 1} \underbrace{x(p, e)^\top}_{1 \times L}$$

Slutsky Equation

Proof

- Take any such $(p, w, \underline{u})$. Remember that $h(p, \underline{u}) = x(p, w)$ and $e(p, \underline{u}) = w$.
- Differentiate $h_\ell(p, \underline{u}) = x_\ell(p, e(p, \underline{u}))$ with respect to p_k.
- Then

$$\begin{aligned}
\frac{\partial h_\ell(p, \underline{u})}{\partial p_k} &= \frac{\partial x_\ell(p, e(p, \underline{u}))}{\partial p_k} + \frac{\partial x_\ell(p, e(p, \underline{u}))}{\partial w} \frac{\partial e(p, \underline{u})}{\partial p_k} \\
&= \frac{\partial x_\ell(p, e(p, \underline{u}))}{\partial p_k} + \frac{\partial x_\ell(p, e(p, \underline{u}))}{\partial w} h_k(p, \underline{u}) \\
&= \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)
\end{aligned}$$

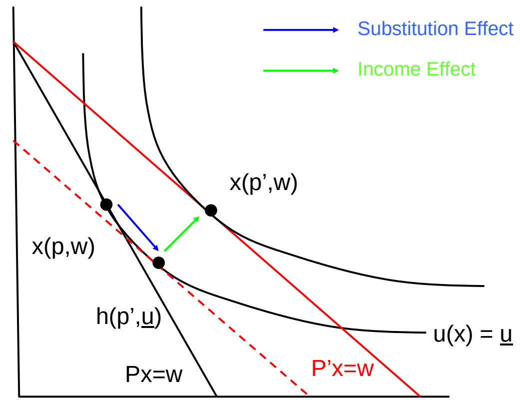
Remark.

- This formula allows us to recover Hicksian demand functions from Walrasian demand functions.
- It is often written as

$$\frac{\partial x_\ell(p,w)}{\partial p_k} = \underbrace{\frac{\partial h_\ell(p,\underline{u})}{\partial p_k}}_{SubstitutionEffect} - \underbrace{\frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w)}_{IncomeEffect}$$

- The (Walrasian) demand curve of good k is downward sloping (i.e. $\frac{\partial x_k(p,w)}{\partial p_k} < 0$) if it is a **normal good** $\left(\frac{\partial x_k(p,w)}{\partial w} \geq 0\right)$. If good k is an **inferior good** $\left(\frac{\partial x_k(p,w)}{\partial w} < 0\right)$, then x_k can be a **Giffen good** $(\frac{\partial x_k(p,w)}{\partial p_k} > 0)$.

Slutsky Equation



Slutsky Matrix

Consider an $L \times L$ matrix $S(p, w)$ whose (ℓ, k)-entry is given by

$$\frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)$$

This matrix is called **Slutsky (substitution) matrix**.

Slutsky Matrix

Properties of Slutsky Matrix

For all $(p, w) \gg 0$ and $\underline{u} = v(p, w)$,

(I) $S(p, w) = D_p^2 e(p, \underline{u})$,

(II) $S(p, w)$ is negative semi-definite,

(III) $S(p, w)$ is a symmetric matrix, and

(IV) $S(p, w)p = 0$.

Proof

- (I) follows from the previous theorem.
- (II) and (III): $e(p, \underline{u})$ is concave and twice continuously differentiable.
- (IV) follows because $h(p, \underline{u})$ is homogeneous of degree 0 in p (or $x(p, w)$ is homogeneous of degree 0 in (p, w) + Walras' law.)

Remark. This theorem imposes testable restrictions on Walrasian demand functions.

Application: Labor Supply Revisited

- Let's do a Slutsky equation-type exercise for the labor/leisure decision problem. To distinguish wage and income, denote income by I.
- $\frac{d\ell(p,w,I)}{dw}$? Note that w affects ℓ through $I = wT + \pi$. Hence

$$\frac{d\ell(p,w,I)}{dw} = \frac{\partial \ell(p,w,I)}{\partial w} + \frac{\partial \ell(p,w,I)}{\partial I} T$$

- By differentiating $\ell(p,w,\underline{u}) = \ell(p,w,e(p,w,\underline{u}))$ by w ($\ell(p,w,\underline{u})$ is Hicksian demand of leisure), we obtain

$$\frac{\partial \ell(p,w,\underline{u})}{\partial w} = \frac{\partial \ell(p,w,I)}{\partial w} + \frac{\partial \ell(p,w,I)}{\partial I} \ell(p,w,I)$$

Application: Labor Supply Revisited

- Hence

$$\frac{d\ell(p,w,I)}{dw} = \underbrace{\frac{\partial\ell(p,w,\underline{u})}{\partial w}}_{Substitution\ \ Effect} - \underbrace{\frac{\partial\ell(p,w,I)}{\partial I}\ell(p,w,I)}_{Income\ \ Effect\ \ I} + \underbrace{\frac{\partial\ell(p,w,I)}{\partial I}T}_{Income\ \ Effect\ \ II}$$
$$= \frac{\partial\ell(p,w,\underline{u})}{\partial w} + \frac{\partial\ell(p,w,I)}{\partial I}\left(T - \ell(p,w,I)\right)$$

- In terms of labor supply $h = T - \ell$, this becomes

$$\frac{dh(p,w,I)}{dw} = \frac{\partial h(p,w,\underline{u})}{\partial w} - \frac{\partial h(p,w,I)}{\partial I}h(p,w,I)$$

Roy's Identity

The last formula is so called **Roy's identity**.

Roy's Identity

For all $(p, w) \gg 0$,

$$x(p, w) = -\frac{1}{D_w v(p, w)} \nabla_p v(p, w)$$

Proof

- For any $(p, w) \gg 0$ and $\underline{u} = v(p, w)$, we have $v(p, e(p, \underline{u})) = \underline{u}$.
- Differentiating this, we have

$$\begin{aligned}
\nabla_p v(p, e(p, \underline{u})) + D_w v(p, e(p, \underline{u})) \nabla_p e(p, \underline{u}) &= 0 \\
\nabla_p v(p, e(p, \underline{u})) + D_w v(p, e(p, \underline{u})) h(p, \underline{u}) &= 0 \\
\nabla_p v(p, w) + D_w v(p, w) x(p, w) &= 0.
\end{aligned}$$

- Rearrange this to get the result.

Note on Differentiability

- When are Walrasian and Hicksian demand functions (continuously) differentiable?
- Assume that
  - u is differentiable, locally nonsatiated, and $X = \Re_+^L$ (then all the previous theorems can be applied).
  - $\underline{u} > u(0), w > 0$.
  - u is pseudo-concave.
  - prices and demands are strictly positive.
- Then Walrasian demand and Hicksian demand are characterized by the following K-T conditions respectively

Note on Differentiability

For Walrasian demand,

$$\nabla u(x) - \lambda p = 0$$
$$w - p \cdot x = 0$$

For Hicksian demand,

$$p - \lambda \nabla u(x) = 0$$
$$u(x) - \underline{u} = 0$$

Note on Differentiability

- We focus on the utility maximization problem (The same conclusion applies to the cost minimization problem).
- The **implicit function theorem** implies that $x(p, w)$ is a C^1 (continuously differentiable) function if the derivative of the left hand side with respect to (x, λ)

$$\begin{pmatrix} D^2u(x) & -p \\ -p^\top & 0 \end{pmatrix}$$

is a full rank matrix.

- By FOC, what we need to show is that

$$\begin{pmatrix} D^2u(x) & -\frac{1}{\lambda}Du(x)^\top \\ -\frac{1}{\lambda}Du(x) & 0 \end{pmatrix}$$

is full rank.

- This is satisfied when u is **differentiably strictly quasi-concave** (check it).

Definition

$u : X(\subset \Re^L) \to \Re$ is differentiably strictly quasi-concave if $\Delta x^\top D^2u(x)\Delta x < 0$ for any $\Delta x(\neq 0) \in \Re^L$ such that $Du(x)\Delta x = 0$.