Consumer Theory

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Utility Maximization

Utility Maximization Problem

We are concerned with the following problem.

Utility Maximization

$$\max_{x \in X} u(x) \ s.t. \ p \cdot x \leq w \ (x \in B(p,w))$$

Walrasian Demand

- Let $x(p, w) \subset X$ (**Walrasian demand correspondence**) be the set of the solutions for the utility maximization problem given $p \gg 0$ and $w \geq 0$. Note that $x(p, w)$ is not empty for any such (p, w) if u is continuous.
- We like to understand the property of Walrasian demand. First we prove basic, but very important properties of $x(p, w)$.

Walrasian Demand

Theorem

Suppose that u is continuous, locally nonsatiated, and $X = \Re^L_+$. Then the Walrasian demand correspondence $x : \Re^L_{++} \times \Re_+ \rightrightarrows \Re^L_+$ satisifes

(I) **Homogeneity of degree 0:** $x(\alpha p, \alpha w) = x(p, w)$ for any $\alpha > 0$ and (p, w),

(II) **Walras' Law:** $p \cdot x' = w$ for any $x' \in x(p, w)$ and (p, w),

(III) **Convexity:** $x(p, w)$ is convex for any (p, w) if u is quasi-concave, and

(IV) **Continuity:** x is upper hemicontinuous.

Walrasian Demand

Proof

- (I) follows from the definition of the problem.
- For (II), use local nonsatiation.
- (III) is obvious.
- (IV) follows from the maximum theorem.

Remark.

- $x(p, w)$ is a single point if u is strictly quasi-concave.
- $x(p, w)$ is a continuous function if it is single-valued.
- General remark: it is useful to clarify which assumption is important for which result.

Walrasian Demand

How can we obtain $x(p, w)$?

- If u is differentiable, then we can apply the **(Karush-)Kuhn-Tucker condition** to derive $x(p, w)$ for each $(p, w) \gg 0$.
- They are necessary if the **constrant qualification** is satisfied (which is always the case here), and also sufficient if u is **pseudo-concave** (See "Mathematical Appendix".)

Walrasian Demand

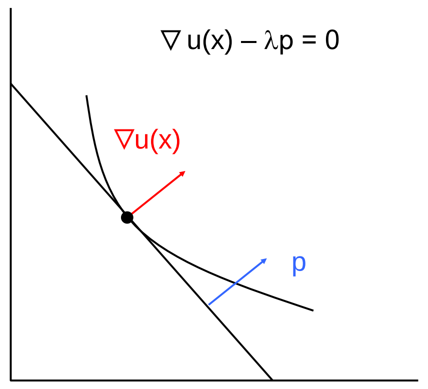
- Suppose that u is locally nonsatiated and the oprimal solution is an interior solution. Then the K-T conditions become very simple.

$$\nabla u(x) - \lambda p = 0$$

$$p \cdot x = w$$

- If x_l can be 0 for some l (boundary solution), then $D_l u(x) - \lambda p_l = 0$ needs to be replaced by $D_l u(x) - \lambda p_l \leq 0 (= 0 \text{ if } x_l > 0)$.

An interior solution looks like:

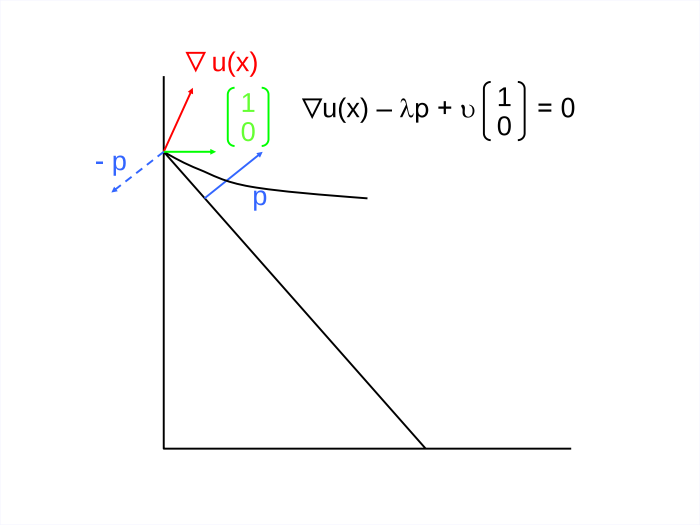


- For a boundary solution, consider the following example with $x_1 = 0$.

$$\begin{aligned} Du_{x_1}(x_1, x_2) - \lambda p_1 &\leq 0 \\ Du_{x_2}(x_1, x_2) - \lambda p_2 &= 0 \end{aligned}$$

- This can be written as $\nabla u(x) - \lambda p + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ for some $\mu > 0$.

This boundary solution looks like:



Example

Let's try to solve some example to get the idea.

- Suppose that $u(x) = \sqrt{x_1} + x_2$.
- We can assume an interior solution for x_1. So the K-T conditions become

$$\begin{aligned} \frac{1}{2\sqrt{x_1}} - \lambda p_1 &= 0 \\ 1 - \lambda p_2 \leq 0 \ (= 0 \text{ if } x_2 > 0) \\ p \cdot x &= w \end{aligned}$$

Example

- Normalize p_2 to 1. Then the solution is
  1. $x_1(p,w) = \frac{1}{4p_1^2}, x_2(p,w) = w - \frac{1}{4p_1}, \lambda(p,w) = 1$ when $4p_1w > 1$,
  2. $x_1(p,w) = \frac{W}{p_1}, x_2(p,w) = 0, \lambda(p,w) = \frac{1}{2\sqrt{p_1w}}$ when when $4p_1w \leq 1$.
- Note that there is no wealth effect on x_1 (i.e. x_1 is independent of w) as long as $4p_1w > 1$

Indirect Utility Function

For any $(p, w) \in \Re_{+}^{L+1} \times \Re_{+}$, $v(p, w)$ is defined by $v(p, w) := u(x')$ where $x' \in x(p, w)$. It is not difficult to prove that this **indirect utility function** satisfies the following properties.

Theorem

Suppose that u is continuous, locally nonsatiated, and $X = \Re_{+}^{L}$. Then $v(p, w)$ is

(I) homogeneous of degree 0,

(II) nonincreasing in p_l for any l and strictly increasing in w,

(III) quasi-convex, and

(IV) continuous.

Proof.

- (I) and (II) are obvious.
- (IV) is again an implication of the maximum theorem.
- Proof of (III):
  - Suppose that $\max\{v(p',w'),v(p'',w'')\} \leq \overline{v}$ for any $(p',w'),(p'',w'') \in \Re^L_{++} \times \Re_+$ and $\overline{v} \in \Re$.
  - For any $\alpha \in [0,1]$ and any $x \in B(\alpha p' + (1-\alpha)p'', \alpha w' + (1-\alpha)w'')$, either $x \in B(p',w')$ or $x \in B(p'',w'')$ must hold.
  - Hence

    $$v(\alpha p'+(1-\alpha)p'', \alpha w'+(1-\alpha)w'') \leq \max\{v(p',w'),v(p'',w'')\} \leq \overline{v}.$$

Example: Cobb-Douglas

- Suppose that $u(x) = \sum_{\ell=1}^{L} \alpha_\ell \log x_\ell$, $\alpha_\ell \geq 0$ and $\sum_{\ell=1}^{L} \alpha_\ell = 1$ (Cobb-Douglas utility function).
- The Walrasian demand is

$$x_\ell(p, w) \; = \; \frac{\alpha_\ell w}{p_\ell}$$

  (**Note:** α_ℓ is the fraction of the expense for good ℓ).
- So $v(p, w) = \log w + \sum_{\ell=1}^{L} \alpha_\ell (\log \alpha_\ell - \log p_\ell)$.

Example: Quasi-linear utility

- For the previous quasi-linear utility example,

$$v(p, w) = \sqrt{x_1(p, w)} + x_2(p, w) = \frac{1}{4p_1} + w$$

(assuming an interior solution).

Indirect Utility Function

Exercise.

1. The result of this example generalizes. Suppose that the utility function is in a quasi-linear form: $u(x) = x_1 + h(x_2, ..., x_L)$. Show that the indirect utility function takes the following form: $v(p, w) = a(p) + w$ (assuming interior solutions).
2. Show that the $v(p, w) = b(p)w$ if the utility function is homogeneous of degree 1.

Example: Labor Supply

- Consider the following simple labor/leisure decision promlem:

$$\max_{q,\ell\geq 0}(1-\alpha)\log q+\alpha\log\ell \ s.t. \ pq+w\ell\leq wT+\pi,\ell\leq T$$

where

  - q is the amount of consumed good and p is its price
  - T is the total time available
  - ℓ is the time spent for "leisure" (which determines $h=T-\ell$: hours of work).
  - w is wage (and wh is labor income).
  - π is nonlabor income.

Example: Labor Supply

- Since the utility function is Cobb-Douglas, it is easy to derive the Walrasian demand: $q(p,w,\pi)=\frac{(1-\alpha)(wT+\pi)}{p}, \ell(p,w,\pi)=\frac{\alpha(wT+\pi)}{w}$ when $\ell < T$.

  (**Note:** if this expression of ℓ is larger than T, $\ell \leq T$ binds. In this case, this consumer does not participate in the labor market ($\ell(p,w,\pi)=T$) and spends all nonlabor income to purchase goods ($q(p,w,\pi)=\frac{\pi}{p}$).)
- It is easy to derive the indirect utility function when $\ell < T$: $v(p,w,wT+\pi)=const.+\log(wT+\pi)-\alpha\log p-(1-\alpha)\log w$.

Cost Minimization

Cost Minization

Next consider the following problem for each $p \gg 0$ and $\underline{u} \in \Re$,

Cost Minimization

$$\min_{x \in X} p \cdot x \ s.t. \ u(x) \geq \underline{u}$$

This problem can be phrased as follows: what is the cheapest way to achieve utility at least as high as $\underline{u}$?

Hicksian Demand

Let $h(p, \underline{u})$ (**Hicksian demand correspondence**) be the set of solutions for the cost minimization problem given $p \gg 0$ and $\underline{u}$.

Remark.

- $h(p, \underline{u})$ is not empty if u is continuous and $\{x \in X : u(x) \geq \underline{u}\}$ is not empty (why?). Let $\underline{U}$ be the set of such $\underline{u}$.
- $h(p, \underline{u})$ is useful for **welfare analysis**, which we do not have time to cover. Read MWG Ch 3-I.

Hicksian Demand

Theorem

Suppose that u is continuous, locally nonsatiated, and $X = \Re^L_+$. Then the Hicksian demand correspondence $h : \Re^L_{++} \times \underline{U} \rightrightarrows \Re^L_+$ is

(I) homogeneous of degree 0 in p,

(II) achieving $\underline{u}$ exactly ($u(x') = \underline{u}$ for any $x' \in h(p, \underline{u})$) if $\underline{u} \geq u(0)$,

(III) convex given any $(p, \underline{u})$ if u is quasi-concave, and

(iv) upper hemicontinuous.

Remark. $h(p, u)$ is a point if u is strictly quasi-concave.

Note on the proof.

- (I), (II), and (III) are straightforward.
- (iv) is slightly more difficult than (iv) for Walrasian demand. We cannot apply the maximum theorem directly because the feasible set is not "locally bounded".

... but we skip the detail.

Expenditure Function

Expenditure function $e(p,\underline{u})$ is defined by $e(p,\underline{u}) := p \cdot x'$ for any $x' \in h(p,\underline{u})$. The proof of the following theorem is left as an exercise.

Theorem

Suppose that u is continuous, locally nonsatiated, and $X = \Re^L_+$. Then the expenditure function $e : \Re^L_{++} \times \underline{U} \to \Re$ is

(I) homogeneous of degree 1 in p,

(II) nondecreasing in p_l for any l and strictly increasing in $\underline{u}$ for $\underline{u} > u(0)$,

(III) concave in p, and

(IV) continuous.

Utility Maximization $\leftrightarrow$ Cost Minimization

Not surprisingly, cost minimization problems are closely related to utility maximization problems. One problem is a flip side of the other in some sense.

Utility Maximization $\leftrightarrow$ Cost Minimization

Utility Maximization $\leftrightarrow$ Cost Minimization

Suppose that u is continuous, locally nonsatiated, and $X = \Re^L_+$.

(I) If $x^* \in x(p, w)$ given $p \gg 0$ and $w \geq 0$, then $x^* \in h(p, v(p, w))$ and $e(p, v(p, w)) = w$.

(II) If $x^* \in h(p, \underline{u})$ given $p \gg 0$ and $\underline{u} \geq u(0)$, then $x^* \in x(p, e(p, \underline{u}))$ and $v(p, e(p, \underline{u})) = \underline{u}$.

Proof: Utility Maximization $\rightarrow$ Cost Minimization

- Suppose not, i.e. $\exists x' \in \Re^L_+$ that satisfies $u(x') \geq u(x^*)$ and $p \cdot x' < p \cdot x^*$ $(= w$ by Walras' law).
- By local nonsatiation, $\exists x'' \in \Re^L_+$ that satisfies $u(x'') > u(x^*)$ and $p \cdot x'' < w$. This is a contradiction to utility maximazation.
- Hence $x^* \in h(p, v(p, w))$ and $e(p, v(p, w)) = p \cdot x^* = w$.

Proof: Utility Maximization ← Cost Minimization

- Suppose not, i.e. $\exists x' \in \Re^L_+$ that satisfies $u(x') > u(x^*) \geq \underline{u}$ and $p \cdot x' \leq p \cdot x^*$. Note that $0 < p \cdot x'$ (because $\underline{u} \geq u(0)$).
- Let $x^\alpha := \alpha 0 + (1-\alpha)x' \in X$ for $\alpha \in (0,1)$. Then $u(x^\alpha) > u(x^*)$ and $p \cdot x^\alpha < p \cdot x^*$ ($\underline{\text{because } p \cdot x' > 0}$) for small α. This is a contradiction to cost minimization.
- Hence $x^* \in x(p, e(p, \underline{u}))$ and $v(p, e(p, \underline{u})) = u(x^*) = \underline{u}$ (by $\underline{u} \geq u(0)$).

Some Useful Formulas

- Walrasian demand, Hicksian demand, indirect utility function, and expenditure function are all very closely related. We can expolit these relationships in many ways.
    - Different expressions are useful for different purposes.
    - We can recover one function from another. In particular, we can recover unobserved from observed.
- We already know
    - Utility maximization → Cost minimization
        - $h(p, v(p, w)) = x(p, w)$,
        - $e(p, v(p, w)) = w$.
    - Cost minimization → Utility maximization
        - $x(p, e(p, \underline{u})) = h(p, \underline{u})$,
        - $v(p, e(p, \underline{u})) = \underline{u}$.

Shepard's Lemma

- In the following, we derive a few more important formulas, assuming that $x(p,w)$ and $h(p,\underline{u})$ are $\underline{\mathcal{C}^1\text{(continuously differentiable) functions}}$.
- Let's start with **Shepard's Lemma**.

Theorem

For any $(p,\underline{u}) \in \Re^L_{++} \times \underline{U}$,

$$\nabla_p e(p,\underline{u}) = h(p,\underline{u})$$

Proof

$$\begin{aligned}
\nabla_p e(p,\underline{u}) &= \nabla(p \cdot h(p,\underline{u})) \\
&= h(p,\underline{u}) + D_p h(p,\underline{u})^\top p \\
&= h(p,\underline{u}) + \frac{1}{\lambda} D_p h(p,\underline{u})^\top \nabla u(h(p,\underline{u})) \text{ (by FOC)} \\
&= h(p,\underline{u}) \text{ (by differentiating } u(h(p,\underline{u})) = \underline{u}\text{)}
\end{aligned}$$

Note. I am assuming an interior solution, but this is not necessary (apply Envelope theorem).

Shepard's Lemma

Remark.

- Since $h(p, \underline{u}) = x(p, e(p, \underline{u}))$, this lemma implies

  $$\nabla_p e(p, \underline{u}) = x(p, e(p, \underline{u})).$$

  From this differential equation, we can recover $e(\cdot, \underline{u})$ for each $\underline{u}$ if $D^2 e$ is symmetric.

- If $D^2 e$ is negative semidefinite, then $e(\cdot, \underline{u})$ in fact satisfies all the properties of expenditure functions. Then we can recover the preference that rationalizes $x(p, w)$. See Ch.3-H, MWG.

Slutsky Equation

Slutsky Equation

For all $(p, w) \gg 0$ and $\underline{u} = v(p, w)$,

$$\frac{\partial h_\ell(p, \underline{u})}{\partial p_k} = \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)$$

or more compactly

$$D_p h(p, \underline{u}) = \underbrace{D_p x(p, w)}_{L \times L} + \underbrace{D_w x(p, e)}_{L \times 1} \underbrace{x(p, e)^\top}_{1 \times L}$$

Slutsky Equation

Proof

- Take any such $(p, w, \underline{u})$. Remember that $h(p, \underline{u}) = x(p, w)$ and $e(p, \underline{u}) = w$.
- Differentiate $h_\ell(p, \underline{u}) = x_\ell(p, e(p, \underline{u}))$ with respect to p_k.
- Then

$$\begin{aligned}
\frac{\partial h_\ell(p, \underline{u})}{\partial p_k} &= \frac{\partial x_\ell(p, e(p, \underline{u}))}{\partial p_k} + \frac{\partial x_\ell(p, e(p, \underline{u}))}{\partial w} \frac{\partial e(p, \underline{u})}{\partial p_k} \\
&= \frac{\partial x_\ell(p, e(p, \underline{u}))}{\partial p_k} + \frac{\partial x_\ell(p, e(p, \underline{u}))}{\partial w} h_k(p, \underline{u}) \\
&= \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)
\end{aligned}$$

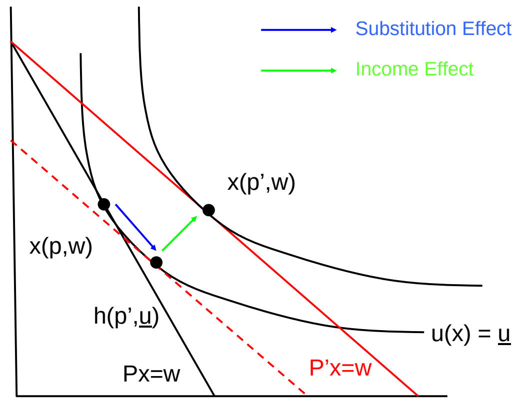
Remark.

- This formula allows us to recover Hicksian demand functions from Walrasian demand functions.
- It is often written as

$$\frac{\partial x_\ell(p,w)}{\partial p_k} = \underbrace{\frac{\partial h_\ell(p,\underline{u})}{\partial p_k}}_{SubstitutionEffect} - \underbrace{\frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w)}_{IncomeEffect}$$

- The (Walrasian) demand curve of good k is downward sloping (i.e. $\frac{\partial x_k(p,w)}{\partial p_k} < 0$) if it is a **normal good** $\left(\frac{\partial x_k(p,w)}{\partial w} \geq 0\right)$. If good k is an **inferior good** $\left(\frac{\partial x_k(p,w)}{\partial w} < 0\right)$, then x_k can be a **Giffen good** $(\frac{\partial x_k(p,w)}{\partial p_k} > 0)$.

Slutsky Equation



Slutsky Matrix

Consider an $L \times L$ matrix $S(p, w)$ whose (ℓ, k)-entry is given by

$$\frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)$$

This matrix is called **Slutsky (substitution) matrix**.

Slutsky Matrix

Properties of Slutsky Matrix

For all $(p, w) \gg 0$ and $\underline{u} = v(p, w)$,

(I) $S(p, w) = D_p^2 e(p, \underline{u})$,

(II) $S(p, w)$ is negative semi-definite,

(III) $S(p, w)$ is a symmetric matrix, and

(IV) $S(p, w)p = 0$.

Proof

- (I) follows from the previous theorem.
- (II) and (III): $e(p, \underline{u})$ is concave and twice continuously differentiable.
- (IV) follows because $h(p, \underline{u})$ is homogeneous of degree 0 in p or $x(p, w)$ is homogeneous of degree 0 in (p, w) + Walras' law.

Remark. This theorem imposes testable restrictions on Walrasian demand functions.

Application: Labor Supply Revisited

- Let's do a Slutsky equation-type exercise for the labor/leisure decision problem. To distinguish wage and income, denote income by I.
- $\frac{d\ell(p,w,I)}{dw}$? Note that w affects ℓ through $I = wT + \pi$. Hence

$$\frac{d\ell(p,w,I)}{dw} = \frac{\partial \ell(p,w,I)}{\partial w} + \frac{\partial \ell(p,w,I)}{\partial I} T$$

- By differentiating $\ell(p,w,\underline{u}) = \ell(p,w,e(p,w,\underline{u}))$ by w ($\ell(p,w,\underline{u})$ is Hicksian demand of leisure), we obtain

$$\frac{\partial \ell(p,w,\underline{u})}{\partial w} = \frac{\partial \ell(p,w,I)}{\partial w} + \frac{\partial \ell(p,w,I)}{\partial I} \ell(p,w,I)$$

Application: Labor Supply Revisited

- Hence

$$\begin{aligned}\frac{d\ell(p,w,I)}{dw} &= \underbrace{\frac{\partial\ell(p,w,\underline{u})}{\partial w}}_{Substitution\ \ Effect} - \underbrace{\frac{\partial\ell(p,w,I)}{\partial I}\ell(p,w,I)}_{Income\ \ Effect\ \ I} + \underbrace{\frac{\partial\ell(p,w,I)}{\partial I}T}_{Income\ \ Effect\ \ II} \\ &= \frac{\partial\ell(p,w,\underline{u})}{\partial w} + \frac{\partial\ell(p,w,I)}{\partial I}\left(T - \ell(p,w,I)\right)\end{aligned}$$

- In terms of labor supply $h = T - \ell$, this becomes

$$\frac{dh(p,w,I)}{dw} = \frac{\partial h(p,w,\underline{u})}{\partial w} - \frac{\partial h(p,w,I)}{\partial I}h(p,w,I)$$

Roy's Identity

The last formula is so called **Roy's identity**.

Roy's Identity

For all $(p, w) \gg 0$,

$$x(p, w) = -\frac{1}{D_w v(p, w)} \nabla_p v(p, w)$$

Proof

- For any $(p, w) \gg 0$ and $\underline{u} = v(p, w)$, we have $v(p, e(p, \underline{u})) = \underline{u}$.
- Differentiating this, we have

$$\begin{aligned} \nabla_p v(p, e(p, \underline{u})) + D_w v(p, e(p, \underline{u})) \nabla_p e(p, \underline{u}) &= 0 \\ \nabla_p v(p, e(p, \underline{u})) + D_w v(p, e(p, \underline{u})) h(p, \underline{u}) &= 0 \\ \nabla_p v(p, w) + D_w v(p, w) x(p, w) &= 0. \end{aligned}$$

- Rearrange this to get the result.

Note on Differentiability

- When are Walrasian and Hicksian demand functions (continuously) differentiable?
- Assume that
  - u is differentiable, locally nonsatiated, and $X = \Re^L_+$ (then all the previous theorems can be applied).
  - $\underline{u} > u(0), w > 0$.
  - u is pseudo-concave.
  - prices and demands are strictly positive.
- Then Walrasian demand and Hicksian demand are characterized by the following K-T conditions respectively

Note on Differentiability

For Walrasian demand,

$$\nabla u(x) - \lambda p = 0$$

$$w - p \cdot x = 0$$

For Hicksian demand,

$$p - \lambda \nabla u(x) = 0$$

$$u(x) - \underline{u} = 0$$

Note on Differentiability

- We focus on the utility maximization problem (The same conclusion applies to the cost minimization problem).
- The **implicit function theorem** implies that $x(p, w)$ is a C^1 (continuously differentiable) function if the derivative of the left hand side with respect to (x, λ)

$$\begin{pmatrix} D^2u(x) & -p \\ -p^\top & 0 \end{pmatrix}$$

is a full rank matrix.

- By FOC, what we need to show is that

$$\begin{pmatrix} D^2u(x) & -\frac{1}{\lambda}Du(x)^\top \\ -\frac{1}{\lambda}Du(x) & 0 \end{pmatrix}$$

  is full rank.

- This is satisfied when u is **differentiably strictly quasi-concave** (check it).

Definition

$u : X(\subset \Re^L) \to \Re$ is differentiably strictly quasi-concave if $\Delta x^\top D^2u(x)\Delta x < 0$ for any $\Delta x(\neq 0) \in \Re^L$ such that $Du(x)\Delta x = 0$.

Axiomatic Approach to Demand Functions

Axiomatic Approach to Demand Functions

- Here again we take a “choice-based approach” instead of “preference-based approach”.
- Consider the following three axioms on demand function $x : \Re^{L+1}_{++} \to \Re$, which we assume to be C^1 (continuously differentiable) throughout.

Homogeneity of degree 0 (A1)

$x(p, w)$ is homogenious of degree 0 if $x(p, w) = x(\alpha p, \alpha w)$ for any (p, w) and $\alpha > 0$.

Walras' Law (A2)

$x(p, w)$ satisfies **Walras' law** if $p \cdot x(p, w) = w$ for any (p, w).

Weak Axiom of Revealed Preference (A3)

$x(p, w)$ satisfies **Weak Axiom of Revealed Preference (WARP)** if for any (p, w) and (p', w')

$$p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w)$$
$$\Rightarrow p' \cdot x(p, w) > w'$$

Axiomatic Approach to Demand Functions

- It can be easily verified that utility maximization implies
  - (A1)
  - (A2) with local nonsatitation
  - (A3) with strict quasi-concavity.
- So these axioms may provide a somewhat weaker theory than utility maximization.

Compensated Price Change

- We can derive some interesting properties of demand functions directly from these axioms.
- Fix some (p, w). Consider the change of price from p to p'. Adjust w to $w' = p' \cdot x(p, w)$ so that the consumer can still afford $x(p, w)$.
- This change from (p, w) to (p', w') is called **compensented price change**.

Compensated Price Change

Theorem

Suppose that (A2) and (A3) are satisfied. For any compensated price change from (p, w) to $(p', w') = (p', p' \cdot x(p, w))$, we have

$$(p' - p) \cdot \left[x(p', w') - x(p, w)\right] \leq 0$$

, with strict inequality if $x(p', w') \neq x(p, w)$.

Compensated Price Change

Proof

$$\begin{aligned} & (p' - p) \cdot \left[x(p', w') - x(p, w)\right] \\ = \;& \underbrace{p' \cdot \left[x(p', w') - x(p, w)\right]}_{0 \text{ by (A2)}} - \underbrace{p \cdot \left[x(p', w') - x(p, w)\right]}_{\geq 0 \text{ by (A3)}} \\ \leq \;& 0 \end{aligned}$$

Compensated Price Change

Remark.

- If the price of good ℓ goes up, keeping the other prices fixed, the consumption of good ℓ must go down after compensation.
- This is different from $\frac{\partial x_\ell(p,w)}{\partial p_\ell} < 0$!

Slutsky Matrix

- So we have $\Delta p \cdot \Delta x \leq 0$ for any compensated price change, where $\Delta p = p' - p$ and $\Delta x = x(p', w') - x(p, w)$.
- For any compensated price change,

$$\begin{aligned} \Delta x &\approx D_p x(p, w)\Delta p + D_w x(p, w)\Delta w \\ &= \left\{ D_p x(p, w) + D_w x(p, w) x(p, w)^\top \right\} \Delta p \\ &= S(p, w)\Delta p. \end{aligned}$$

Slutsky Matrix

- Fix any $\Delta p \in \Re^L$ and apply this formula to the price change of $\lambda \Delta p$.

  Then $\Delta x = S(p,w)\,(\lambda \Delta p) + o(\lambda)$.

  ($o(\lambda)$ is an approximation term s.t. $\frac{1}{\lambda} o(\lambda) \to 0 \left(\in \Re^L\right)$ as $\lambda \to 0$).

- By the last theorem

$$(\lambda \Delta p)^\top S(p,w)\,(\lambda \Delta p) + (\lambda \Delta p)^\top o(\lambda) \leq 0.$$

- Divide both sides by λ^2 and let $\lambda \to 0$. Then we have the following result.

Slutsky Matrix

Theorem

Suppose that $x(p, w)$ satisfies (A1)-(A3). Then for any $\Delta p \in \Re^L$,

$$\Delta p^\top S(p, w) \Delta p \leq 0.$$

i.e. $S(p, w)$ is negative semi-definite.

Slutsky Matrix

Remark.

- It can be shown that $S(p, w)$ is symmetric when $L = 2$. But it is not symmetric in general (when $L > 2$).
- So (A1)-(A3) provides a weaker theory than the preference-based theory, which implies the symmetry of $S(p, w)$ in addition to (A1)-(A3).

Rationalizability

- When can $x(p, w)$ be rationalizable? The previous result is related to this question.

Rationalizable Demand

$x : \Re^L_{++} \times \Re_+ \to \Re^L_+$ is rationalized by a preference $\succeq$ if $x(p, w) \succ x$ for every $x\,(\neq x(p, w)) \in B(p, w)$ (i.e. $x(p, w)$ is the unique optimum).

Remark. This is slightly different from Afriat's formulation, which requires the actual demand to be one of the optimum.

Rationalizability

- It turns out that if $S(p, w)$ is negative semi-definite and symmetric and satisfies (A1) - (A3), then there exists a rational preference that rationalizes $x(p, w)$ (MWG 3-H).
- The previous result shows that (A1) - (A3) implies negative semi-definiteness. So (A1) - (A3) + (A4): $S(p, w)$ is symmetric is necessary and sufficient for $x(p, w)$ to be rationalizable.

Rationalizability

This result is important for at least two reasons.

1. Theoretically, it implies that there is no more property that follows from the assumption of utility maximization.
2. For applied works, it allows one to start with any parametric demand function that satisfies (A1)-(A4), rather than specifying preferences and deriving demand functions.