Competitive Equilibrium: Existence

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Preliminaries

- Our goal is to prove that there exists a competitive equilibrium in $\mathcal{E}^{pure}$ and $\mathcal{E}^{priv}$.
- Various properties of demand functions and supply functions we have proved play a role here.

Maintained Assumptions

- **Demand Side**
  - $X_i = \Re^L_+$.
  - $\succeq_i$ is rational and continuous ($\rightarrow$ well-defined demand given $p \gg 0$).
  - $\succeq_i$ is locally nonsatiated ($\rightarrow$ Walras' law).
  - $\succeq_i$ is strictly convex ($\rightarrow$ unique demand).
- **Supply Side**
  - $0 \in Y_j$ ($\rightarrow \pi_j(p) \geq 0$)
  - $Y_j \subset \Re^L$ is closed
  - Y_j is bounded above ($\rightarrow$ well-defined supply given $p \gg 0$).
  - $Y_j \subset \Re^L$ is strictly convex ($\rightarrow$ unique supply).

Maintained Assumptions

Remark.

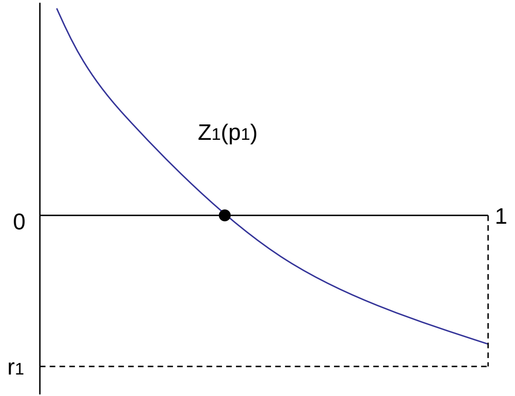
- The strict convexity assumptions on preference and technology can be relaxed.
- The boundedness assumption can be relaxed.
- We may impose conditions on the total production set Y rather than each Y_j.

Example

- Two goods pure exchange economy.
- Only the price ratio matter. Normalize prices: $p_1 + p_2 = 1$.
- By "Walras' law", $z_1(p_1) = 0$ implies $z_2(p_1) = 0$.
- Is there any $p_1 \in [0, 1]$ to satisfy $z_1(p_1) = 0$?

Example

The simplest case.



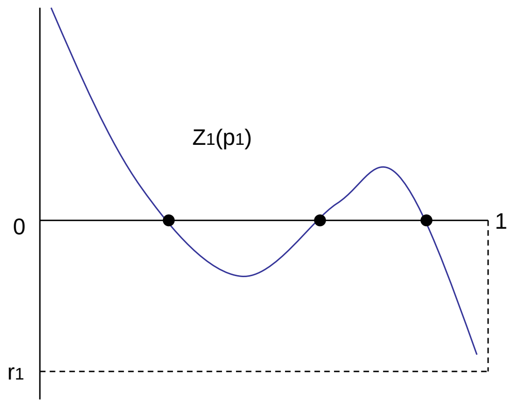
Example

- $z_1(p_1) \to \infty$ as $p_1 \to 0$, and
- $z_2(p_1) \to \infty$ as $p_1 \to 1$ (which implies $\exists p_1 \in (0, 1)$ s.t. $z_1(p_1) < 0$)

is enough for existence.

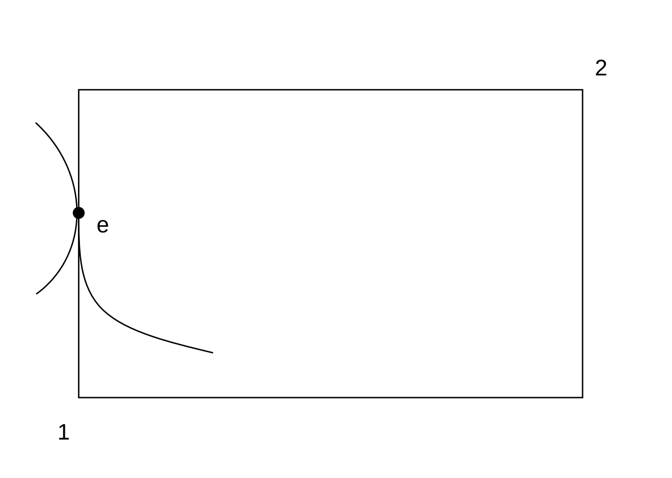
Example

There may be multiple equilibria in general.



Example

What could go wrong? Arrow's example again.



Example

- Note that, in this example,
  - some consumer's endowments are not strictly positive,
  - the demand does not "blow up" near 0 price.
- We prove two existence theorems by assuming that either one of these features is not present.

Existence of Competitive Equilibrium

Excess Demand Function

We start with a simple case: pure exchange economy.

Define the **excess demand function** $z : \Re^L_{++} \to \Re^L$ in $\mathcal{E}^{pure}$ as follows.

Excess demand function

$$z(p) := \sum_{i \in I} x_i(p, p \cdot e_i) - \sum_{i \in I} e_i$$

Note that $z(p)$ is not defined when some prices are 0.

Property of Excess Demand Function

What do we know about $z(p)$?

- (I) $z(p)$ is continuous.
- (II) $z(p)$ is homogenous of degree 0.
- (III) $p \cdot z(p) \; = \; 0$ (Walras' Law).
- (IV) $z_\ell(p)$ is bounded below for every $\ell = 1, ..., L$.

Normalization of Prices

Without loss of generality (by (II)), we can restrict our attention to prices in the simplex:

$$\Delta = \{p \in \Re^L_+ | \sum_{\ell \in L} p_\ell = 1\}.$$

Remark. There are many other ways to normalize prices. For example, we may normalize the price of some good to 1 if it is guaranteed to be stritly positive.

Competitive Equilibrium and Excess Demand Function

- Suppose that $z(p^*) \leq 0$. Then $p_\ell^* > 0$ implies $z_\ell(p^*) = 0$ by Walras' law (III).
- Hence $z(p^*) \leq 0 \Leftrightarrow (x(p^*), p^*)$ is a CE.

$(x(p^*) = (x_1(p^*, p^* \cdot e_1), ..., x_I(p^*, p^* \cdot e_I)))$.

Basic Idea of Proof

- Consider a (hypothetical) price adjustment process:

$$p'_\ell \;=\; p_\ell + \max\{0, z_\ell(p)\} \text{ for } \ell = 1, ..., L.$$

- But p' is not in Δ. So normalize it.

$$p'_\ell \;=\; \frac{p_\ell + \max\{0, z_\ell(p)\}}{1 + \sum_\ell \max\{0, z_\ell(p)\}} \text{ for } \ell = 1, ..., L.$$

- It turns out that $z(p) \leq 0$ when $p' \;=\; p$. So the existence of CE boils down to the existence of a fixed point for this process.

Brouwer's Fixed Point Theorem

The proof is based on Brouwer's fixed point theorem.

Brouwer's Fixed Point Theorem

Let $X \subset \Re^N$ be a nonempty, compact and convex set. For any continuous function $f : X \to X$, there exists an $x^* \in X$ such that $f(x^*) = x^*$.

Basic Idea of Proof

- We like to apply Brouwer's fixed point theorem to the price adjustment process to find a fixed point.
  - Since z is continuous, the process is continuous.
  - But z is only defined on $\Re^L_{++}$, not on a compact set Δ.
- We use a pseudo-"excess demand function" $\overline{z}$ to define f, find a fixed point p^* that satisfies $\overline{z}(p^*) \leq 0$, then show that in fact $\overline{z}(p^*) = z(p^*)$.

Detail of Proof

- Pick any $\overline{r} \in \Re^L_+$ such that $\overline{r} \gg r$. Consider the following modified consumer problem.

$$\max_{x_i \in X_i} u_i(x_i) \ s.t. \ p \cdot x_i \leq p \cdot e_i, \ x_i \leq \overline{r}.$$

- Then consumer i's "demand" $\overline{x}_i(p, p \cdot e_i)$ can be defined for every $p \in \Delta$. This defines $\overline{z} : \Delta \to \Re^L$.

More Detail of Proof

- $\overline{z}(p)$ is continuous and satisfies $p \cdot \overline{z}(p) \leq 0$.
  - $\underline{e_i \gg 0}$ guarantees that consumer i's <u>truncated</u> budget set is continuous in Δ. So $\overline{x}_i(p, p \cdot e_i)$ is upper hemicontinuous (hence continuous because $\overline{x}_i$ is a function) in Δ by Maximum theorem.
- Apply Brouwer's fixed point theorem to $\overline{z}$ to find a fixed point p^*.
- Note that $\overline{z}(p^*) = z(p^*)$ because the additional constraint $(x_i \leq \overline{r})$ is not binding at $p^*$and the preferences are strictly convex (this is in the problem set).

Existence in Pure Exchange Economy

There exists a competitive equilibrium in $\mathcal{E}^{pure}$ with all the maintained assumptions and strictly positive initial endowments.

Theorem

There exists a competitive equilibrium $(x^*, p^*) \in X \times \Delta$ for $\mathcal{E}^{pure}$ if $e_i \gg 0$ for every $i \in I$.

Proof

- Define $f : \Delta \to \Delta$ by $f_\ell(p) := \frac{p_\ell + max\{0, \overline{z}_\ell(p)\}}{1+\sum_\ell \max\{0, \overline{z}_\ell(p)\}}, \ell = 1, ..., L$.
- By the Maximum theorem, $\overline{z}$ is continuous. Hence f is continuous. So there exists a fixed point $p^* \in \Delta$ of f by Brouwer's fixed point theorem.
- Multiply both sides by $\overline{z}_\ell(p^*)$ and sum them up across all goods.

$$\frac{p^* \cdot \overline{z}(p^*) + \sum_{\ell \in L} \overline{z}_\ell(p^*) \max\{0, \overline{z}_\ell(p^*)\}}{1 + \sum_{\ell \in L} \max\{0, \overline{z}_\ell(p^*)\}} = p^* \cdot \overline{z}(p^*).$$

Then we can show that $\overline{z}(p^*) \leq 0$ by using $p^* \cdot \overline{z}(p^*) \leq 0$.

- Since $\succeq_i$ is strictly convex and $\overline{x}_i^*(p^*, p^* \cdot e_i) < \overline{r}$ for all $i \in I$, we have $\overline{x}_i^*(p^*, p^* \cdot e_i) = x_i^*(p^*, p^* \cdot e_i)$, hence $\overline{z}(p^*) = z(p^*) \leq 0$.

Existence in Private Ownership Economy

- We provide another proof of existence, which follows MWG more closely.
- We incorporate the production sector this time.

Excess Demand Function

- We define **excess demand function** $z : \Re^L_{++} \to \Re^L$ by

$$z(p) \; := \; \sum_{i \in I} x_i(p) - \sum_{i \in I} e_i - \sum_{j \in J} y_j(p)$$

where $x_i(p)$ is defined by

$$x_i(p) := x_i(p, w_i(p))$$

$\left(w_i(p) = p \cdot e_i + \sum_{j \in J} \theta_{i,j} p \cdot y_j(p) \right)$.

- Clearly CE $\Leftrightarrow z(p) = 0$.

Property of Excess Demand Functions

It is not difficult to verify that $z(p)$ satisfy the following four conditions as before.

- (I) $z(p)$ is continuous.
- (II) $z(p)$ is homogenous of degree 0.
- (III) $p \cdot z(p) = 0$.
- (IV) $z_\ell(p)$ is bounded below for every $\ell = 1, ..., L$.

Property of Excess Demand Functions

- We do not assume $e_i \gg 0$ this time. Instead we introduce the 5th property of $z(p)$.
  - (V) $\|z(p_n)\| \to \infty$ for any $\{p_n\}_n \subset int\Delta$ that converges to a boundary point of Δ.
- For example, this property is satisfied for $\overline{z}$ with strongly monotonic preferences and $r \gg 0$.

Proof.

- Suppose not. Take any sequence $p_n \to p' \in \partial\Delta$ such that $\{p_n\}_n \subset int\Delta$ and $\|z(p_n)\|$ is bounded. Then we can find a convergent subsequence such that, for every $i \in I$,
  - $w_i(p_n) \to w_i(p')$ (because $\pi_j(p) \in [0, p \cdot \overline{y}]$)
  - $x_i(p_n, w_i(p_n)) \to x_i' \in B_i(p', w_i(p'))$ (because $\overline{y}_j(p_n) \leq \overline{y}$).
- $\exists i \in I$ such that $w_i(p') > 0$ because $r \gg 0$ and $\overline{\pi}_j(p') \geq 0$.
- For any α in $(0,1)$ and any $x_i \in B_i(p', w_i(p'))$, $\alpha p' \cdot x_i < p' \cdot e_i$. Hence $p_n \cdot (\alpha x_i) < w_i(p_n)$ for large n.
- $x_i(p_n, w_i(p_n)) \succeq_i \alpha x_i, \to x_i' \succeq_i \alpha x_i$.
- $x_i' \succeq_i x_i$ by $\alpha \to 1$ (x_i' maximizes i's utility given p').
- This is a contradiction because $x_i(p', w_i(p'))$ must be empty.

Existence in Private Ownership Economy

The following theorem says that, for <u>any</u> function z that satisfies (I)-(V), there exists a strictly positive solution for $z(p) = 0$.

Theorem

Suppose that $z : \Re^L_{++} \to \Re^L$ satisfies (I), (II), (III), (IV), and (V). Then there exists $p^* \in \Re^L_{++}$ such that $z(p^*) = 0$.

Kakutani's Fixed Point Theorem

We use Kakutani's fixed point theorem for this proof.

Kakutani's Fixed Point Theorem

Let $X \subset \Re^N$ be a nonempty, compact and convex set. Suppose that a correspondence $f : X \rightrightarrows X$ satisfies the following:

- $f(x)$ is nonempty for every $x \in X$.
- $f(x)$ is convex-valued for every $x \in X$.
- f has a closed graph.

Then there exists an $x^* \in X$ such that $x^* \in f(x^*)$.

Proof

- Define a correspondence $f : \Delta \rightrightarrows \Delta$ as follows:
  - For z for $p \in int\Delta$, use the same f as before.
  - For $p \in \partial\Delta$ (boundary), assign a ***subset*** of Δ to f as follows:

$$f(p) := \{q \in \Delta | q_\ell = 0 \text{ if } p_\ell > 0\}$$

- f is nonempty, convex-valued. We show that f has a closed graph.
  - This is trivial in $int\Delta$.
  - Consider a sequence $(p_n, q_n) \to (\widetilde{p}, \widetilde{q})$ such that $\widetilde{p} \in \partial\Delta$ and $q_n \in f(p_n)$. We just need to show $\widetilde{q} \in f(\widetilde{p})$. Suppose $\widetilde{p}_\ell > 0$ for some ℓ. We assume that $p_n \gg 0$ (otherwise...). Then $q_{n,\ell} \to 0$ because $z_\ell(p_n)$ is bounded above by (III) and (IV) but the denominator of $f_\ell(p_n) \to \infty$ by (V).
  - Since there is no restriction on $f_\ell(\widetilde{p})$ if $\widetilde{p}_\ell = 0$, we have $\widetilde{q} \in f(\widetilde{p})$.

Proof (continued)

- By Kakutani's fixed point theorem, there exists a fixed point $p^* \in \Delta$ of f.
- It must be the case that $p^* \gg 0$ by the definition of f at the boundary. Since $p^* \cdot z(p^*) = 0$ by Walras' law and $z(p^*) \leq 0$ can be shown as before, $z(p^*) = 0$ is proved.

Existence in Private Ownership Economy

So we obtain the following theorem for private ownership economies with all the maintained assumptions.

Theorem

There exists a competitive equilibrium $(x^*, y^*, p^*) \in X \times Y \times int\Delta$ for $\mathcal{E}^{priv}$ if the excess demand function z satisfies (I)-(V).

Large Economy and Nonconvexity

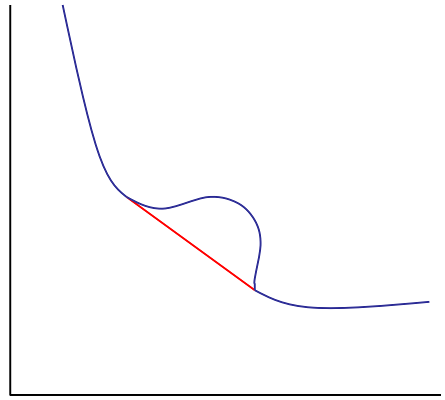
Nonconvexity

- We can prove the existence of CE with (not necessarily strictly) convex preference and technology.
- However, we cannot drop convexity altoghether. Convexity is crucial for the existence of CE. For example, demand functions may not be continuous with nonconvex preferences. Also remember that convexity was important for the second theorem.
- Is there any way to dispense with convexity?

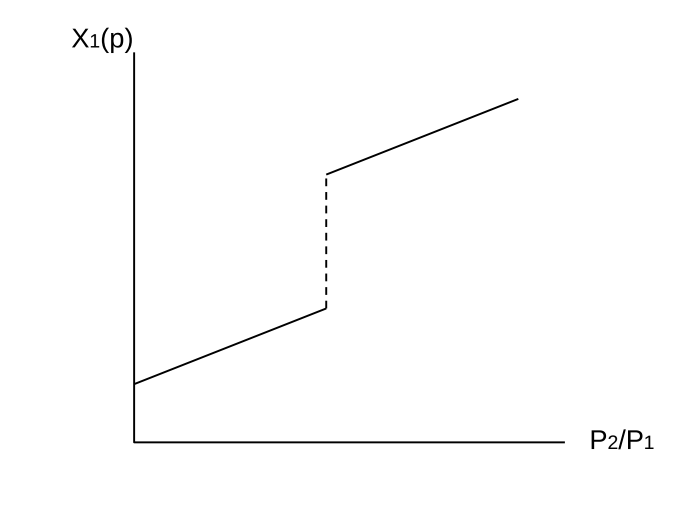
Large Economy

- The problems associated with nonconvexity can be mitigated in a "large economy", where preferences and technologies are heterogeneous, but there are many consumers with the same preference and many producers with the same technology.
- The idea is that, even if an individual preference is nonconvex, the aggregate preference is convex. Then even if an individual demand correspondence is discontinuous, the aggregate (average) demand correspondence is upper hemicontinuous.

Large Economy and Nonconvexity



Large Economy and Nonconvexity



Large Economy and Nonconvexity