Decision Theory under Uncertainty

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Lottery

Lottery

- In many (most?) real life problems, people make a choice when they do not know fully its consequence in advance.

  Example. portfolio choice, career choice, 401K, buying a house etc.

- We need a model for **decision under uncertainty**.

Lottery

- One way to model such decision is to regard it as a problem of choosing an optimal "lottery".

Lottery

Let Z be a finite set of outcomes and $\Delta(Z)$ be the set of all probability distributions on Z. Each $p \in \Delta(Z)$ is called a **lottery**.

- Some notations and remark:
  - For any $p \in \Delta(Z)$, denote the probability of $z \in Z$ by $p(z) \geq 0$. So $\sum_{z \in Z} p(z) = 1$.
  - Let $\delta_z \in \Delta Z$ be the lottery for which z realizes with probability 1.
  - $\Delta(Z)$ can be regarded as a unit simplex in $\Re^{|Z|}$.

Remark.

- An implicit assumption is that probability is objective here. We will discuss **subjective probability** later.
- Z does not need to be a finite set. All the results hold for Z with infinite elements under appropriate assumptions, although some proofs need to be modified and get more technical.

Compound Lottery

- For any $p, q \in \Delta(Z)$ and $a \in [0,1]$, $ap + (1-a)q \in \Delta(Z)$ is defined by $\{ap + (1-a)q\}(z) := ap(z) + (1-a)q(z)$.
- There are at least two ways to interpret $ap + (1-a)q$.
  - This is just a lottery for which the probability of z is $ap(z) + (1-a)q(z)$.
  - This is a **compound lottery**: you first get a lottery p with probability a and q with probability $1-a$, then the outcome of whichever lottery realizes. In this case, uncertainty is gradually resolved.
- We <u>assume</u> that the decision maker does not distinguish these two lotteries that induce the same distribution on Z.

Compound Lottery

Comments.

- it is sometimes reasonable to relax such "reduction axiom" because
  - decision makers may have a preference regarding early/late resolution of uncertainty (would you like to know the results of multiple final exams (say, Micro, Macro, and Econometrics) at the same time or gradually as soon as each result is ready?).
  - decision makers may have a preference on the number of lotteries.

Expected Utility

Expected Utility

- Consider the following lottery: \$100 with 30% and \$50 with 70%.
- The expected value of this lottery is $0.3 \times 100 + 0.7 \times 50$.
- **Expected utility** is a bit more general: $0.3 \times u(100) + 0.7 \times u(50)$. When can we use such function to represent a decision maker's preference over lotteries?

Axioms

- As always, we impose some axioms on the decision maker's preference on $\Delta(Z)$, then derive some functional representation of it.
- First we assume that the preference is rational.

Rationality (A1)

$\succeq$ on $\Delta(Z)$ is a rational preference.

Remark.

- This is the same axiom as before, but it is stronger in some sense. Completeness requires that a decision maker has a preference not only on apples and bananas, but also on objects such as (40% of (2 apples, 3 bananas), 60% of (3 apples, 0 bananas)).
- How do you think about the following example with two lotteries? Lottery A: throw a die and you get $\$100 \times X$ when X comes out. Lottery B: instead you get $\$100 \times X + \$100$ when $X < 6$, but you only get $\$100$ when $X = 6$. B is better than A in 5 cases out of 6, but...

Axioms

The next axiom is continuity.

Continuity (A2)

$\succeq$ on $\Delta(Z)$ is continuous.

Remark.

- (A1) and (A2) already implies that there exists $U : \Delta(Z) \to \Re$ that represents $\succeq$.
- The following axiom, which is weaker than (A2) or 6.B.3 in MWG, is more standard.

Archimedian Axiom (A2')

For any $p, q, r \in \Delta(Z)$ such that $p \succ q \succ r$, there exists $a, b \in (0, 1)$ such that

$$ap + (1-a)r \succ q \succ bp + (1-b)r.$$

Axioms

The last axiom is crucial (and controversial).

Independence (A3)

For any $p, q, r \in \Delta(Z)$ and $a \in (0, 1)$,

$$p \succ q \Rightarrow ap + (1-a)r \succ aq + (1-a)r$$

Remark. This is slightly weaker than 6.B.4 in MWG.

Exercise: Verify that (A1)-(A3) implies

1. **Betweenness:** For any $p, q \in \Delta(Z)$ and $a \in (0,1)$,

$$p \succ q \Rightarrow p \succ ap + (1-a)q \succ q$$

2. For any $p, q \in \Delta(Z)$ and $a > b \in (0,1)$,

$$p \succ q \Rightarrow ap + (1-a)q \succ bp + (1-b)q$$

3. For any $p, q, r \in \Delta(Z)$ and $a \in (0,1)$,

$$p \sim q \Leftrightarrow ap + (1-a)r \sim aq + (1-a)r$$

Note on Axioms

- It turns out that these axioms lead to the expected utility representation.
- Why is an axiomatic characterization such as this useful?
  - Each axiom can be tested separately.
  - When expected utility theory is rejected by data, we can examine each axiom and replace it with a weaker axiom if it is violated, rather than discarding the whole thing.

Expected Utility Theorem

Expected Utility Theorem

$\succeq$ on $\Delta(Z)$ satisfies (A1) - (A3) if and only if there exists a function $u : Z \to \Re$ such that for any $p, q \in \Delta(Z)$,

$$p \succeq q \Leftrightarrow \sum_{z \in Z} p(z)u(z) \geq \sum_{z \in Z} q(z)u(z)$$

Furthermore, such u is unique up to a positive affine transformation.

Proof

- "if" is trivial. So we prove "only if".
- There exists p^* and p_* in $\Delta(Z)$ such that $p^* \succeq p \succeq p_*$ for any $p \in \Delta(Z)$ (by continuity and compactness). We can assume that $p^* = \delta_{z^*}$ and $p_* = \delta_{z_*}$ for some $z^*, z_* \in Z$(why?). We also assume that $\delta_{z^*} \succ \delta_{z_*}$.
- For every $p \in \Delta(Z)$, define
  $U(p) := \sup \{a \in [0,1] : p \succeq a\delta_{z^*} + (1-a)\delta_{z_*}\}$.
  Then $p \sim U(p)\delta_{z^*} + (1 - U(p))\delta_{z_*}$ (by continuity).
- By (2), such $U(p)$ is unique and $U(p)$ represents $\succeq$ (U is called **vNM expected utility function**).

Proof

- $U(p)$ is linear.
  - For any $a \in [0,1]$,
    $ap + (1-a)q \sim a\left\{U(p)\delta_{z^*} + (1-U(p))\delta_{z_*}\right\} + (1-a)q$ by (3).
  - Using (3) again,
    $$ap + (1-a)q \sim \left[\begin{array}{c} \{aU(p) + (1-a)U(q)\}\,\delta_{z^*} + \\ \{a(1-U(p)) + (1-a)(1-U(q))\}\,\delta_{z_*} \end{array}\right].$$
  - So $U(ap + (1-a)q) = aU(p) + (1-a)U(q)$.
- Define $u(z) := U(\delta_z)$, then $U(p) = \sum_{z\in Z} p(z)u(z)$ for any $p \in \Delta(Z)$ by induction.

Proof

- u is unique up to a positive affine transformation.
  - Suppose that $\succeq$ can be represented in expected utility form with another function $u' : Z \to \Re$.
  - Then there exists $\alpha > 0$ and β such that $u'(z^*) = \alpha u(z^*) + \beta$ and $u'(z_*) = \alpha u(z_*) + \beta$.
  - For any $z \in Z$,

$$\begin{aligned} u'(z) &= a(\delta_z)u'(z^*) + (1 - a(\delta_z))u'(z_*) \\ &= \alpha\left[a(\delta_z)u(z^*) + (1 - a(\delta_z))u(z_*)\right] + \beta \\ &= \alpha u(z) + \beta \end{aligned}$$

Utility for Money

Money Lottery

Here we consider lotteries with monetary outcomes.

- $Z = [\underline{z}, \overline{z}]$.
- A lottery p is associated with a cumulative distribution function F on Z (F is an increasing, right continuous function that satisfies $F(\underline{z}) = 0$ and $F(\overline{z}) = 1$).

Expected Utility

- It is possible to obtain the expected utility representation with similar axioms in this case. The expected utility given F is $\int_{x\in Z} u(x)dF(x)$. $u : Z \to \Re$ is called **Bernoulli utility function**.
- **Remark.**
  - The same expression can be obtained with some bounded Bernoulli utility function even for cases with unbounded Z (such as $X = [0, \infty)$).
  - When F is a simple distribution (only a finite number of outcomes are possible), this expression reduces to the familiar form: $\sum_{z\in Z} p(z)u(z)$.
- We assume that u is a C^1 function and $Du(z) > 0$ for all $z \in Z$. Denote the set of such u by U.

Technical Remark.

- When F has density f (i.e. $F(t) = \int_{\underline{z}}^{t^{\text{`}}} f(x)$),
  $\int_{x \in Z} u(x) dF(x) = \int_{x \in Z} u(x) f(x) dx$.
- This integration (Lebesgue-Stieltjes Integral) can be defined as follows.

$$\int_{x \in Z} u(x) dF(x) = \lim_{\Delta \to 0} \sum_{k=0}^{\infty} \min_{x \in [k\Delta, (k+1)\Delta]} u(x) \left[F((k+1)\Delta) - F(k\Delta) \right]$$

Risk Aversion

Risk Aversion: Definition

A decision maker with Bernoulli utility function $u : Z \to \Re$ is risk averse if for any F

$$\int u(x) dF(x) \leq u\left(\int x dF(x)\right).$$

A decision maker is risk averse if and only if u is concave (Jensen's Inequality).

Certainty Equivalent and Risk Premium

Certainty Equivalent: Definition

Certainty equivalent $C(F,u)$ given F and u is the value to satisfy $u(C(F,u)) = \int u(x)dF(x)$.

- This means that the decision maker is indifferent between receiving $C(F,u)$ certainly and having a lottery F.
- $rp(F,u) = \int xdF(x) - C(F,u)$ is called **risk premium** of lottery F. Clearly $rp(F,u) \geq 0$ for any F if and only if u is concave.

Measure of Risk Aversion

How to measure the risk averseness of a decision maker?

Absolute Risk Aversion

$$r_A(z,u) = -\frac{u''(z)}{u'(z)}$$

Relative Risk Aversion

$$r_R(z,u) = -\frac{zu''(z)}{u'(z)}$$

To understand what these measures mean and why they are useful, solve the optimal portfolio problems (MWG, p.192).

Measure of Risk Aversion

Example

- $u(z) = -e^{-\alpha z}$ for $\alpha > 0$. Then $r_A(z, u) = \alpha$ (CARA).
- $u(z) = \frac{z^{1-\gamma}}{1-\gamma}$ for $\gamma (\neq 1) \geq 0$. Then $r_R(z, u) = \gamma$ (CRRA).

Measure of Risk Aversion

Theorem

For any $u_1, u_2 \in U$, the following statements are equivalent.

- $r_A(z, u_2) \geq r_A(z, u_1)$ for all $z \in Z$.
- There exists a strictly increasing concave function $\phi : u_1(Z) \to \Re$ such that $u_2(z) = \phi(u_1(z))$.
- $C(F, u_2) \leq C(F, u_1)$ for all F.
- $\int u_2(x)dF(x) \geq u_2(z) \Rightarrow \int u_1(x)dF(x) \geq u_1(z)$ for any F and $z \in Z$.

When one (hence all) of these conditions is satisfied, we say u_2 is more risk averse than u_1.

Stochastic Dominance

- How can we compare different lotteries? When is one lottery more attractive or more risky than another? Our definition can be summarized roughly as follows:
  - Lottery F is better than lottery G if every decision maker prefers F to G.
  - Lotter F is less risky than lottery G if every risk averse decision maker prefers F to G.

First-Order Stochastic Dominance

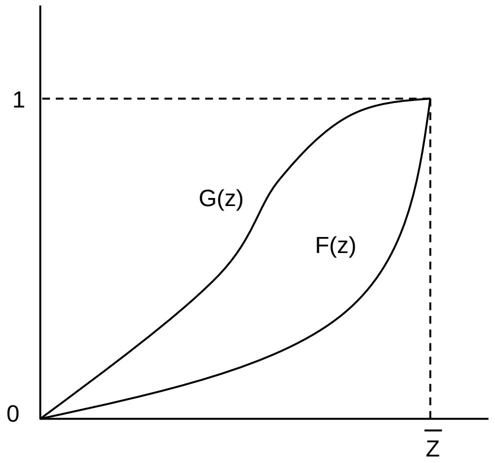
Definition of FOSD

F **first-order stochastically dominates** G if

$$F(z) \leq G(z)$$

for all $z \in Z$.

First-Order Stochastic Dominance



First-Order Stochastic Dominance

Theorem

F first-order stochastically dominates G if and only if

$\int u(x)dF(x) \geq \int u(x)dG(x)$ for any $u \in U$.

First-Order Stochastic Dominance

Proof.

- By integration by parts,

$$\begin{aligned}\int u(x)dF(x) &= [-u(x)(1-F(x))]_{\underline{z}}^{\overline{z}} + \int (1-F(x))Du(x)dx \\ &= u(\underline{z}) + \int (1-F(x))Du(x)dx\end{aligned}$$

- Since $1-F(x) \geq 1-G(x)$, $\int u(x)dF(x) \geq \int u(x)dG(x)$.

Note: If F FOSD G "strictly" for some $z > \underline{z}$, then $\int u(x)dF(x) > \int u(x)dG(x)$ for every u.

Second-Order Stochastic Dominance

- Suppose that the mean of F and G is the same. So FOSD does not rank F and G strictly.
- It is reasonable to say that G is more risky than F if every risk averse individual prefers F to G.

Second-Order Stochastic Dominance

Definition of SOSD

Suppose that $\int x dF(x) = \int x dG(x)$. F **second-order stochastically dominates** G if

$$\int_{\underline{z}}^{t} F(x)dx \leq \int_{\underline{z}}^{t} G(x)dx$$

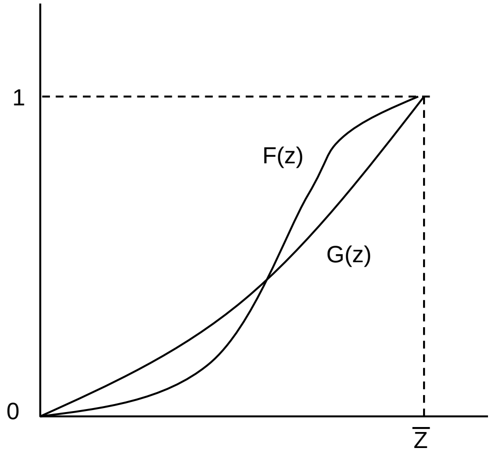
for all $t \in Z$.

Second-Order Stochastic Dominance

There are a variety of ways to capture SOSD.

- Let p_F and p_G be the lotteries associated with F and G respectively. Then F SOSD G if and only if $p_G = p_F + \epsilon$, where ϵ is some "noise" (G is a **mean preserving spread** of F).

If F crosses G once from below, then F SOSD G (assuming the same mean).



Second-Order Stochastic Dominance

Theorem

Suppose that $\int xdF(x) = \int xdG(x)$. Then F second-order stochastically dominates G if and only if $\int u(x)dF(x) \geq \int u(x)dG(x)$ for every concave $u \in U$.

Subjective Probability

State and Act

- You are betting on who will be the next president. You and your friend bet \$10 on candidate A and B. You will collect \$20 if you win, 0 if not.
- This is a kind of lottery. The set of possible outcomes (Z) is $(\$10, -\$10)$ (in net amount). If you bet on A, then you get \$10 with probability $Pr(\text{"A win"})$ and lose \$10 with $Pr(\text{"B win"})$.

State and Act

- But what is $Pr(\text{"A win"})$? This is not an objective probability. Your belief may not be shared by your friend (probably this is why you and your friend agreed to bet in the first place) or outside observers (like us economists). So this is **subjective probability**.
- Furthermore, there is no guarantee that such well-defined number exists even subjectively.
- How should we model this type of uncertainty?

State and Act

- A fundamentally new idea: derive subjective probability (belief) from decision.
- Decision makers choose from many **acts**. An act f is a mapping from the finite set of **states/states of nature** $S = \{1, ..., S\}$ to the finite set of outcomes Z.
- We impose axioms on the preference/decisions on the set of all acts. This time our goal is to derive **both utility and subjective probability**.
- If the axioms are satisfied, we can treat the decision maker as an expected utility maximizer with some subjective belief on Z.

Anscombe-Aumann Approach

- Here is one simple approach to the problem. Enrich the choice set by considering all acts from S to $\Delta(Z)$. A crux of the matter is that the distribution on Z at each state is objective (This type of act f is sometimes called a **horse-race lottery**).
- Can we derive some representation like

$$\sum_{s \in S} \mu(s) \sum_{z \in Z} f_s(z) u(z)?$$

- Let $\mathcal{F}$ be the set of all such acts. We consider a preference $\succeq$ on $\mathcal{F}$. Note that $\mathcal{F}$ can be identified with $(\Delta(Z))^S$

Anscombe-Aumann Approach

- We impose usual axioms on $\succeq$.
  - (A1) $\succeq$ on $\mathcal{F}$ is rational,
  - (A2) $\succeq$ on $\mathcal{F}$ is continuous (in $(\Delta(Z))^S$), and
  - (A3) $\succeq$ on $\mathcal{F}$ satisfies independence, i.e. for any $f, g, h \in \mathcal{F}$ and $a \in (0, 1)$,

$$f \succ g \Rightarrow af + (1-a)h \succ ag + (1-a)h$$

- By using (A1)-(A3) and exactly the same proof, we can obtain an affine function U on $\mathcal{F}$ (and a bit more) that represents $\succeq$.

Warning! These three axioms are exactly the same as before. But remember that we are imposing the same axioms on a larger domain (so our assumptions are stronger and less justified).

Anscombe-Aumann Approach

We need two more axioms on $\succeq$ to get what we want. Let $\succeq^*$ be the preference on $\underline{\Delta(Z)}$ defined by $p \succeq^* q \Leftrightarrow (p, ..., p) \succeq (q, ..., q)$.

- (A4) $\succeq$ on $\mathcal{F}$ satisfies **monotonicity**, i.e. for any $f, g \in \mathcal{F}$ if $f(s) \succeq^* g(s)$ for all $s \in S$ (and $f(s) \succ^* g(s)$ for some s), then $f \succeq g$ ($f \succ g$ respectively).
- (A5) $\succeq$ on $\mathcal{F}$ is not **trivial**, i.e. $f \succ g$ for some $f, g \in \mathcal{F}$.

Anscombe-Aumann Approach

Then we have the following representation theorem.

Theorem

$\succeq$ on $\mathcal{F}$ satisfies (A1) - (A4) if and only if there exists $\mu(\gg 0) \in \Delta(S)$ and $u : Z \to \Re$ such that for any $f, g \in \mathcal{F}$,

$$f \succeq g \Leftrightarrow \sum_{s \in S} \mu(s) \sum_{z \in Z} f_s(z) u(z) \geq \sum_{s \in S} \mu(s) \sum_{z \in Z} g_s(z) u(z)$$

Furthermore, u is unique up to a positive affine transformation. If in addition (A5) is satisfied, then μ is also unique.

Savage Approach

- It is possible to dispense with objective probability altogether. Given that a preference on the set of all acts ($f : S \to Z$) satisfies certain axioms, we can obtain the following representation of the preference:

$$\sum_{s \in S} \mu(s) u(f(s))$$

for some probability distribution μ on S and some function u on Z. This remarkable result is due to Savage (1954).

- But we skip this. The axioms and the proof are more complicated than the ones for AA.

Paradoxes and Nonexpected Utilities

The Allais Paradox

Consider the following two gambles.

Gamble A

$500,000$ with probability 1

Gamble B

$2,500,000$ with probability 0.1, $500,000$ with probability 0.89, 0 with probability 0.01.

Which gamble do you prefer?

The Allais Paradox

Here is another pair of gambles.

Gamble C

$500,000$ with probability 0.11, 0 with probability 0.89.

Gamble D

$2,500,000$ with probability 0.1, 0 with probability 0.9.

Which gamble do you prefer?

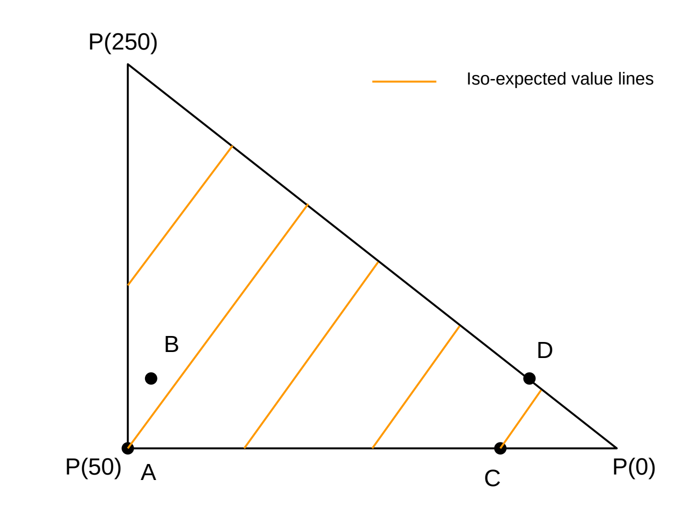
The Allais Paradox

- If you are an expected utility maximizer, then your choice should be either (A, C) or (B, D).
- Why is this happening?
  - Maybe (A3) is bad.
  - There may be something special about 0 probability event.

Machina's Triangle

- The set of probability distributions for the Allais paradox example can be represented by a triangle.

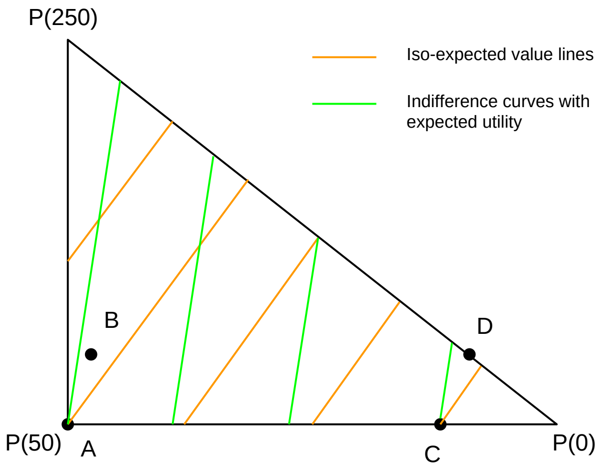
Each orange line corresponds to a particular expected value.



Machina's Triangle

- How about indifference curves of the expected utility?
- (A1) and (A2) implies the existence of a utility function, hence well-defined indifference curves on the triangle.
- Independence (A3) implies that indifference curves are straight and parallel.

With risk aversion, the indifference curves with expected utility are steeper than the iso-expected value line. In this case, $A \succ B$ and $C \succ D$.



The Ellsberg Paradox

There are many versions of this story. Here is one.

- There are two urns A and B with 100 balls in each urn. Each ball is either red or blue.
- Suppose that there are 49 red balls and 51 blue balls in urn A, but the composition of color is not known for urn B.

The Ellsberg Paradox

- First a subject picks a ball from either urn A or urn B, and win $1000 if it is red. Suppose that the subject picked from urn A in this first experiment.
- After the ball is returned to urn A, the subject is asked to pick another ball. This time the prize is $1000 when the color is blue. Suppose that the subject picked from urn A in this second experiment.
- Can this guy have a well-defined subjective probability (regarding urn B)?

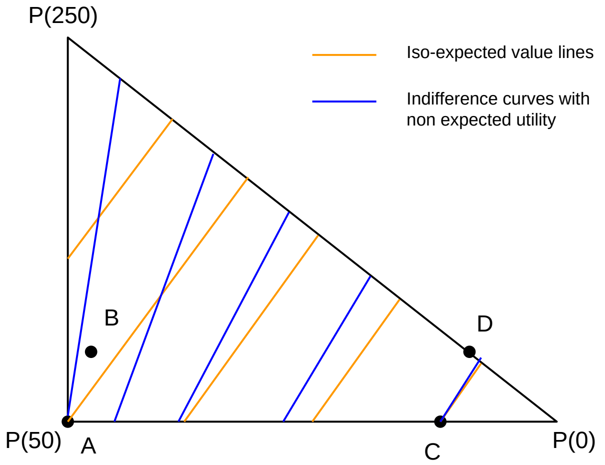
Nonexpected Utility

- There have been many attempts to relax EU and SEU to explain a variety of systematic violations of the theory. Different axioms were relaxed by different people. Some people introduced a particular type of utility function directly without any axiom.
- Let's take a look at a few examples.

1. Relaxing Independence

- To explain the pattern such as $A \succ B$ and $D \succ C$, we need to relax the independence axiom (Example: replace independence with betweenness).
- Then we can generate the indifference curves which are "fanning out" and rationalize $A \succ B$ and $D \succ C$.

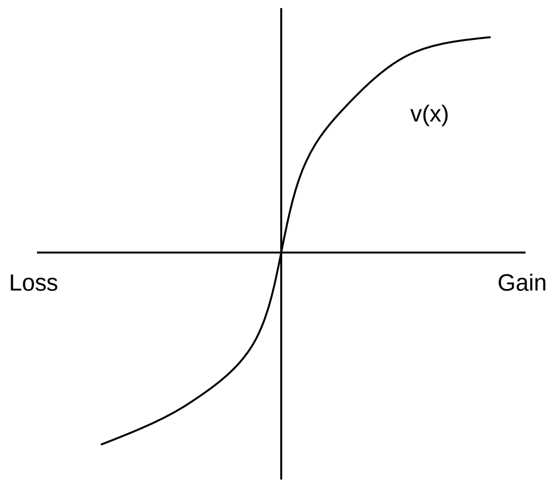
1. Relaxing Independence



2. Prospect Theory

- **Prospect theory** tweaks both probability and utility at the same time. It uses the following expression $\sum_{z \in Z} \pi(p(z))v(z-r)$.
- Value v is defined with respect to the change from the **reference point/status quo** r. It is concave with respect to **gain** and convex with respect to **loss** (**Diminishing sensitivity**). It exhibits **loss aversion** when $|v(x)| < |v(-x)|$ for $x \geq 0$.
- π is non-linear weighting function, overweighting low probability events and underweighting high probability events.

2. Prospect Theory



3. Maxmin Expected Utility

- Consider a decision maker who is uncertain about the underlying probability.
- Assume that DM does not have a unique subjective probability, but rather has a set of possible subjective probabilities.
- This theory assumes that DM chooses an act that maximizes the following:

$$\min_{\mu \in C} \sum_{s \in S} \mu(s) u(f(s))$$

- This explains the Ellsberg Paradox.

4. Regret Theory

- A decision maker may care not only about consequences, but also about the difference between what one received and what one could have received. Regret theory takes into account this psychological effect.
- Here is one example. Consider a set of acts F. When choosing $f \in F$, **regret** at state s is given by $R_{f,F}(s) = \max_{g \in F} \{g(s) - f(s)\}$. Assume that a decision maker minimizes the maximized regret. Then DM's preference can be represented by

$$\max_{f \in F} \min_{s \in S} R_{f,F}(s).$$