

# Preference and Utility

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# Preference Relation

# Preference

- Here we take a classical approach to consumer behavior: we assume that consumers choose the commodity vector they like most within their budget set.
- We need to make this “like most” more precise.

## Preference Relation

**Preference relation** on  $X$  is a subset of  $X \times X$ . When  $(x, y)$  is an element of this set, we say  $x$  is preferred to  $y$  and denote  $x \succeq y$ .

- ▶ We usually use  $\succeq$  to denote a preference relation.
- ▶  $X$  can be any set. For consumer problems,  $X$  is a subset of  $\mathbb{R}_+^L$ .

# Preference

Some basic properties are:

- $\succsim$  on  $X$  is **complete** if either  $x \succsim y$  or  $y \succsim x$  for any  $x, y \in X$
- $\succsim$  on  $X$  is **transitive** if  $x \succsim y$  and  $y \succsim z$  imply  $x \succsim z$  for any  $x, y, z \in X$ .

Are they reasonable? Maybe.

- We can derive two other preference relations from a rational preference.

### Strict Preference

**Strict preference relation**  $\succ$  is defined by  $x \succ y \Leftrightarrow \{x \succeq y \text{ and } y \not\preceq x\}$

### Indifference

**Indifference**  $\sim$  is defined by  $x \sim y \Leftrightarrow \{x \succeq y \text{ and } y \succeq x\}$ .

### Remark

- $\succ$  contains as much information as  $\succeq$  regarding a preference.
- In fact, it is possible to start with a strict preference (with **asymmetry** and **negative transitivity**) and derive a rational preference. So some people prefer to use strict preferences as primitive.

# Preference

## Some critique to transitivity

- 1 How much sugar do you need for a cup of coffee? You are indifferent between no sugar and one grain of sugar, one grain of sugar and two...But are you indifferent between no sugar and 10 spoons of sugar?
- 2 Framing problem.

# Preference

We almost always assume these properties. So let's give them some name.

## Rational Preference

$\succsim$  on  $X$  is **rational** if it is complete and transitive.

From now on, we only consider rational preferences most of the time.

# Preference

There are many other properties we assume from time to time. Let  $X$  be a subset of  $\mathbb{R}_+^L$ .

- $\succeq$  on  $X$  is **locally nonsatiated** if for every  $x \in X$  and  $\epsilon > 0$ , there exists  $y \in X$  such that  $\|y - x\| < \epsilon$  and  $y \succ x$ .
- $\succeq$  on  $X$  is **monotone** (resp. **strongly monotone**) if  $x \gg y$  (resp.  $x > y$ ) implies  $x \succ y$  for any  $x, y \in X$ .
- $\succeq$  on  $X$  is **continuous** if both the **upper contour set**  $\{y \in X : y \succeq x\}$  and the **lower contour set**  $\{y \in X : x \succeq y\}$  are closed for any  $x \in X$  (equivalently, if  $x_n \rightarrow x \in X, y_n \rightarrow y \in X$  and  $x^n \succeq y^n$ , then  $x \succeq y$ ).

# Preference

- $\succsim$  on  $X$  is **convex** if  $\{y \in X : y \succsim x\}$  is convex for any  $x \in X$ .
- $\succsim$  on  $X$  is **strictly convex** if  $y \succsim x$  and  $z \succsim x$  and  $y \neq z$  imply  $\alpha y + (1 - \alpha)z \succ x$  for any  $\alpha \in (0, 1)$ .
- $\succsim$  on  $X = \mathbb{R}_+^L$  is **homothetic** if  $x \sim y \rightarrow \alpha x \sim \alpha y$  for any  $\alpha \geq 0$ .

# Utility Representation

# Utility Representation

It is usually more convenient to work with **utility functions** rather than preferences.

## Definition: Representation of Preference

A utility function  $u : X \rightarrow \mathfrak{R}$  represents  $\succsim$  if  $x \succsim y \Leftrightarrow u(x) \geq u(y)$  for all  $x, y \in X$ .

# Utility Representation

Once a preference is represented by a utility function, then we can formulate the consumer problem as a constrained optimization problem:

$$\max_{x \in X} u(x) \text{ s.t. } p \cdot x \leq w,$$

or equivalently,

$$\max_{x \in B(p, w)} u(x)$$

, which may be easily solved analytically or numerically.

# Utility Representation

## Examples of Utility Functions

- Cobb-Douglas utility function:  $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$  for  $\alpha \in (0, 1)$ .
- Quasi-linear utility function:  $u(x, m) = v(x) + m$  ( $m$  is *numeraire*).
- Leontief utility function:  $u(x_1, x_2) = \min \{x_1, x_2\}$ .

# Utility Representation

## When can a rational preference be represented by a utility function?

- Consider the easiest case:  $X$  is a finite set. Clearly every rational preference on  $X$  can be represented by some utility function (you will be asked to prove this *formally*).

# Utility Representation

## When can a rational preference be represented by a utility function?

- What if  $X$  is a countable set? For example, this is the case if no good is divisible ( $X = Z_+^L$ ). We can still obtain a representation as follows.
  - ▶ Let  $X_n = \{x_1, \dots, x_n\}$  for  $n = 1, 2, \dots$
  - ▶ For each  $n$ , we can find  $u_n$  to satisfy  $x \succeq y \Leftrightarrow u_n(x) \geq u_n(y)$  for any  $x, y \in X_n$ . In fact, we can keep the same  $u_n$  in each step (i.e.  $u_n(x) = u_{n+1}(x) = \dots$  for any  $x \in X_n$ ).
  - ▶ For each  $x \in X$ , define  $u(x)$  by  $u(x) := u_n(x)$  by taking any large  $n$ . It can be easily verified that (1)  $u$  is well-defined and (2)  $u$  represents  $\succeq$ .

# Utility Representation

You can find a continuous utility function when a preference is (rational and) continuous.

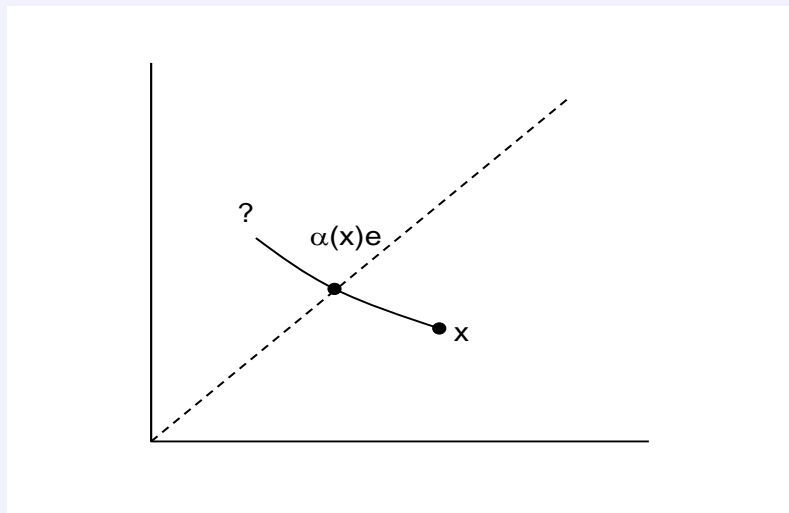
## Theorem (Debreu)

A rational preference  $\succeq$  on  $X \subset \mathbb{R}_+^L$  is continuous if and only if there exists a continuous utility function  $u : X \rightarrow \mathbb{R}$  that represents  $\succeq$ .

## Proof.

- “if” is trivial. We prove “only if” in the following.
- Assume (1)  $X = \mathfrak{R}_+^L$  and (2)  $\succeq$  is monotone for simplicity (neither is needed for the theorem).
- For every  $x \in X$ , there exists unique  $\alpha(x) \in \mathfrak{R}_+$  that satisfies  $\alpha(x)e \sim x$  by continuity (prove this formally).
- Define  $u$  by  $u(x) := \alpha(x)$ .
- Then  $x \succeq x' \Leftrightarrow \alpha(x)e \succeq \alpha(x')e \Leftrightarrow u(x) \geq u(x')$ .
- Verify that  $u$  is continuous.

# Utility Representation



## Remark.

- What if continuity is dropped? Can a plain rational preference  $\succeq$  be always represented by some  $u$ ? NO.
- The following rational preference is not continuous and cannot be represented by any utility function.

- ▶ **Lexicographic Preference on  $\mathbb{R}_+^2$**

For any  $x, y \in \mathbb{R}_+^2$ ,  $x \succeq y$  if and only if either (1)  $x_1 > y_1$  or (2)  $x_1 = y_1$  and  $x_2 \geq y_2$ .

- ▶ **Proof.** Define a function  $f$  from  $\mathbb{R}_+$  to  $Q$  (rational number) by associating each  $x$  with  $f(x) \in Q$  such that  $u(x, 1) < f(x) < u(x, 2)$ . Then a different rational number is assigned to different  $x$ , a contradiction.

## Properties of Preferences in terms of Utilities

- $\succeq$  on  $X$  is **locally nonsatiated**  $\Leftrightarrow$  for every  $x \in X$  and  $\epsilon > 0$ , there exists  $y \in X$  such that  $\|y - x\| < \epsilon$  and  $u(y) > u(x)$ .
- $\succeq$  on  $X$  is **monotone (strongly monotone)**  $\Leftrightarrow x \gg y$  (resp.  $x > y$ ) implies  $u(x) > u(y)$  for any  $x, y \in X$ .
- $\succeq$  on  $X$  is **convex**  $\Leftrightarrow u$  is quasi-concave, i.e.  $u(y) \geq u(x)$  and  $u(z) \geq u(x)$  imply  $u(\alpha y + (1 - \alpha)z) \geq u(x)$  for any  $\alpha \in [0, 1]$ .
- $\succeq$  on  $X$  is **strictly convex**  $\Leftrightarrow u$  is strictly quasi-concave, i.e.  $u(y) \geq u(x)$  and  $u(z) \geq u(x)$  with  $y \neq z$  imply  $u(\alpha y + (1 - \alpha)z) > u(x)$  for any  $\alpha \in (0, 1)$ .
- $\succeq$  is **homothetic**  $\Leftrightarrow u(\alpha x) = u(\alpha y)$  for any  $\alpha \geq 0$  and  $x, y \in X$  such that  $u(x) = u(y)$ .

## Ordinal Property and Cardinal Property

- Let  $f : \mathfrak{R} \rightarrow \mathfrak{R}$  be any *strictly increasing* function. Then  $u(x)$  and  $f(u(x))$  represents the same preference because  $u(x) \geq u(y) \Leftrightarrow f(u(x)) \geq f(u(y))$ .
  - ▶ Ex.  $u(x) = x^{0.5}$  and  $\log u(x) = 0.5 \log x$ .
- The properties (of utilities) that are preserved under any such strictly monotone transformation are **ordinal properties**. The properties that are not are **cardinal properties**.
- Example.
  - ▶ Monotonicity and quasi-concavity are ordinal properties.
  - ▶ Concavity and decreasing marginal utility are cardinal properties.

# Ordinal Property and Cardinal Property

## Comment.

- If we regard a utility function as merely one convenient representation of the underlying preference, then we should be careful about the interpretation of any result that depends on its cardinal properties.
- On the other hand, we may like to use a more convenient representation with nice cardinal properties if that does not affect our results (Example. Some quasi-concave utility function can be transformed into a concave utility function. This doesn't change the preference, whereas concave functions are easier to use than quasi-concave functions).