

Fundamental Theorems of Welfare Economics

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Pareto Efficiency

Pareto Efficiency

- An allocation is **Pareto efficient** if there is no way to reallocate resources to make everyone weakly better off and someone strictly better off.
- Here is the formal definition.

Pareto Efficiency

An allocation $(x, y) \in X \times Y$ in \mathcal{E} is **Pareto efficient** if it is feasible ($(x, y) \in A$) and there is no other feasible allocation $(x', y') \in A$ such that $x'_i \succeq_i x_i$ for all $i \in I$ and $x'_i \succ_i x_i$ for some $i \in I$.

Pareto Efficiency

Remark.

- There are many Pareto efficient allocations.
- It doesn't say much about the distribution of resources. For example, one individual getting everything can be a Pareto efficient allocation.

Social Welfare Function

- One way to introduce a stronger welfare criterion than Pareto efficiency is to introduce **social welfare function** $W(u_1, \dots, u_I)$
 - ▶ $W(u)$ is typically strictly increasing in each u_i and concave in u .
 - ▶ W is defined with respect to a given representation u .
- **Example.** $W(u) = \sum_{i \in I} a_i u_i$ with $a \gg 0$, $\sum_{i \in I} a_i = 1$.

Pareto Efficiency and Social Welfare

- There is a nice way to characterize Pareto efficient allocations by using social welfare functions.

Pareto Efficiency and Social Welfare

Utility Possibility Set

$$U = \{u \in \mathbb{R}^I \mid \exists (x, y) \in A \text{ s.t. } u \leq u(x)\}$$

Pareto Frontier

$$U^P = \{u \in U \mid \nexists u' \in U \text{ s.t. } u' > u\}$$

Remark.

- If each u_i is continuous and A is compact, then U is closed and bounded above.
- U^P = the set of utility vectors associated with Pareto efficient allocations (Proposition 16.E.1, MWG).

Pareto Efficiency and Social Welfare

Theorem

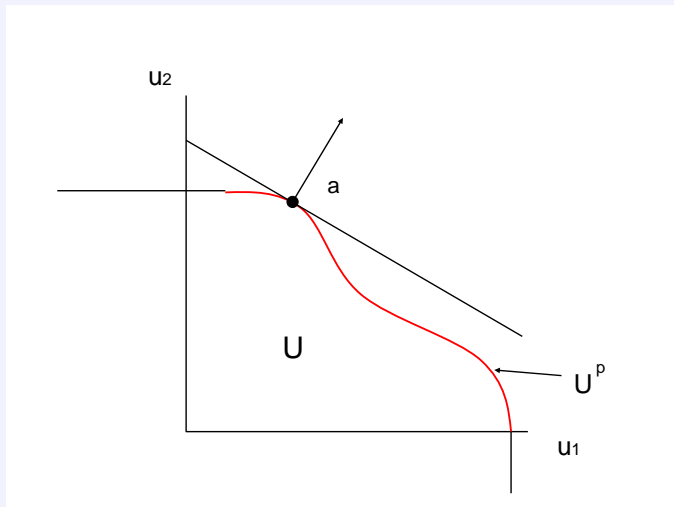
If $(x, y) \in X \times Y$ solves $\max_{(x,y) \in A} \sum_i a_i u_i(x_i)$ for some $a \in \mathbb{R}'_{++}$, then $u(x) \in U^P$.

Proof

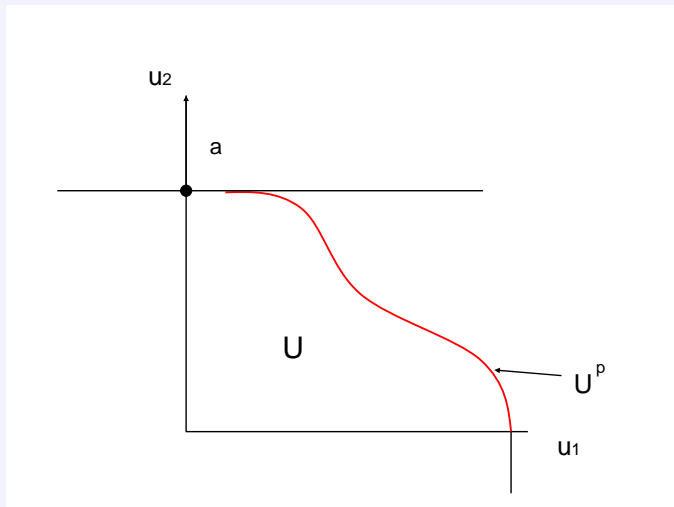
- If $u(x) \notin U^P$, then there exists $(x', y') \in A$ such that $u(x') > u(x)$.
- $\sum_i a_i u_i(x'_i) > \sum_i a_i u_i(x_i)$. A contradiction.

Remark. This implies that U^P is not empty if u_i is continuous.

Example with two consumers.



Why not $a > 0$?



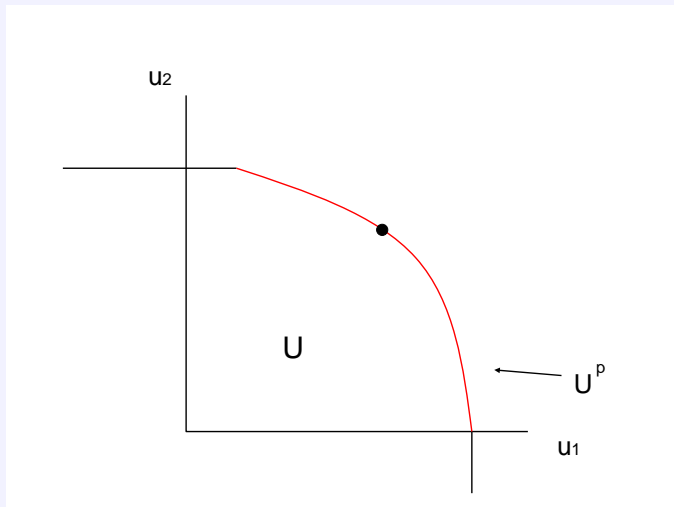
Here is a partial converse.

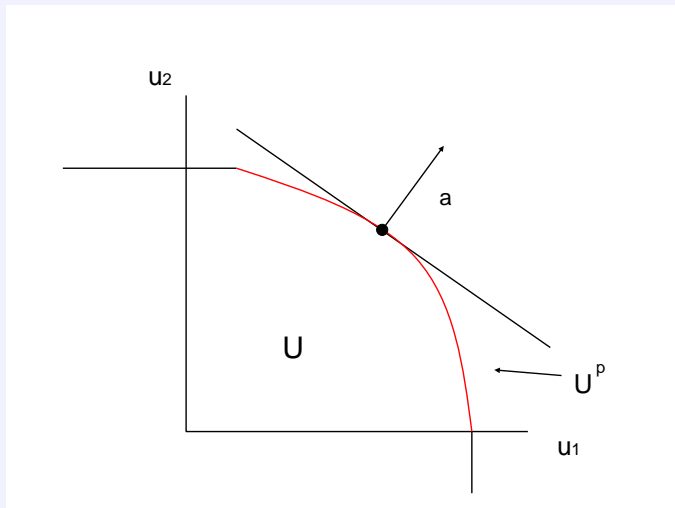
Theorem

Suppose that U is convex. If $u(x) \in U^P$ for some $(x, y) \in A$, then $(x, y) \in A$ solves $\max_{(x, y) \in A} \sum_i a_i u_i(x_i)$ for some $a \in \mathbb{R}'_+ / \{0\}$.

Proof

- $u(x) \in U^P$ is in the boundary of U .
- Since U is convex, there exists $a (\neq 0) \in \mathbb{R}'$ such that $a \cdot u(x) \geq a \cdot u'$ for any $u' \in U$ by the supporting hyperplane theorem.
- $a > 0$ because $u \in U \rightarrow u' \in U$ for any $u' < u$.
- $\sum_i a_i u_i(x_i) \geq \sum_i a_i u_i(x'_i)$ for any $(x', y') \in A$.





Remark.

- U is convex when each u_i is concave.
- Even if u_i is not concave, there may be a monotone transformation f_i to make $f_i(u_i)$ concave. Remember that concavity is a **cardinal** property.

Welfare Theorems

Main Results

- Welfare properties of competitive (Walrasian) equilibrium?
 - ▶ The first theorem: every competitive equilibrium is Pareto efficient.
 - ▶ The second theorem: every Pareto efficient allocation can be achieved as a competitive equilibrium with appropriate transfer of wealth.

Illustration in Edgeworth Box

First Welfare Theorem

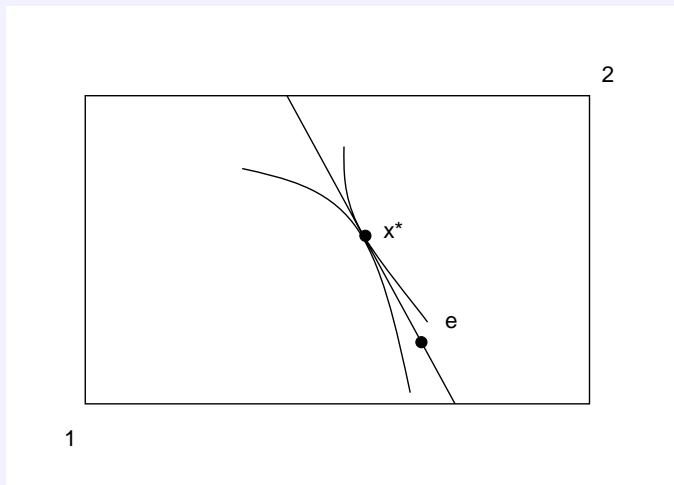
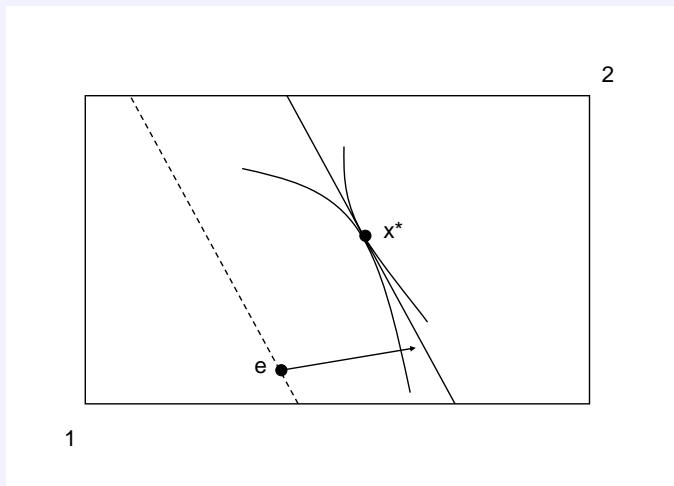


Illustration in Edgeworth Box

Second Welfare Theorem



The First Welfare Theorem

The First Fundamental Theorem of Welfare Economics

Suppose that \succeq_i is locally nonsatiated for every $i \in I$ in \mathcal{E}^{priv} . If

$(x^*, y^*, p^*) \in X \times Y \times \mathbb{R}^L$ is a competitive equilibrium for \mathcal{E}^{priv} , then

(x^*, y^*) is Pareto efficient.

Proof

- Suppose not: $\exists(x', y') \in A$ and $\exists k \in I$ such that $x'_i \succeq_i x_i^*$ for all $i \in I$ and $x'_k \succ_k x_k^*$.
- Since utility maximization \rightarrow cost minimization by local nonsatiation, $p^* \cdot x'_i \geq w_i = p^* \cdot e_i + \sum_j \theta_{i,j} p^* \cdot y_j^*$ for every $i \in I$. And $p^* \cdot x'_k > w_k$ by utility maximization.
- Hence $p^* \cdot \sum_i x'_i > \sum_i w_i$.
- By profit maximization, $p^* \cdot y_j^* \geq p^* \cdot y'_j$ for all $j \in J$.
- Then $\sum_i w_i = p^* \cdot r + \sum_j p^* \cdot y_j^* \geq p^* \cdot r + \sum_j p^* \cdot y'_j$
 \rightarrow a contradiction to feasibility.

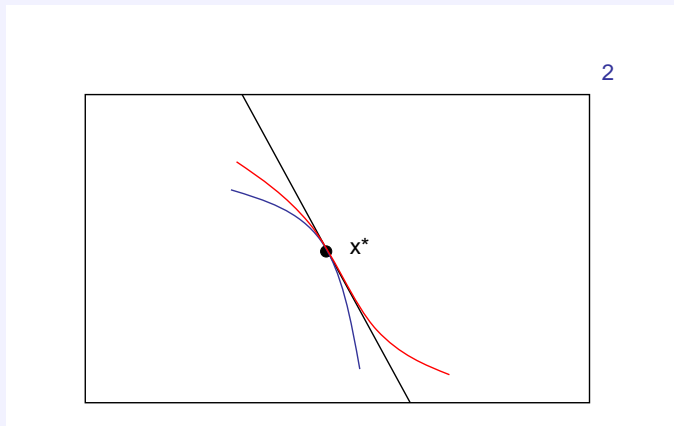
Remark. FFTWE may fail with

- locally satiated preference,
- externality,
- infinite goods (ex. infinite horizon model),
-

Failure of the second theorem: an example

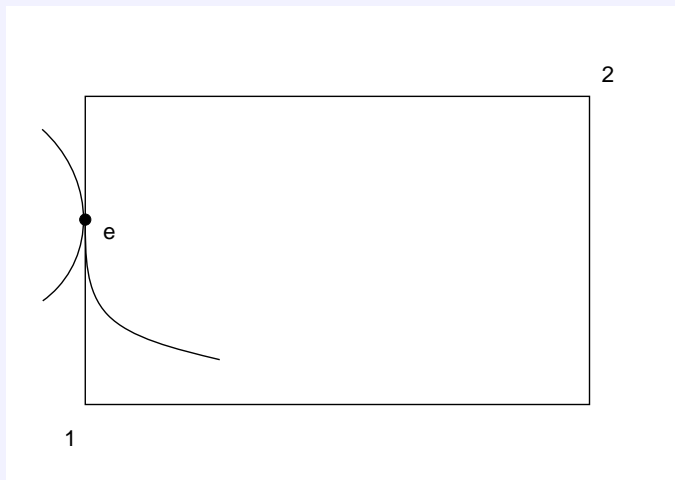
x^* is Pareto efficient, but cannot be achieved by a competitive equilibrium.

Convexity of preference is crucial.



Another Example (Arrow)

We also need cost minimization \rightarrow utility maximization for SFTWE.



Competitive Equilibrium with Transfer

We need the formal definition of competitive equilibrium with transfer.

Here is the definition for \mathcal{E}^{pure} .

Competitive Equilibrium with Transfer

$(x^*, p^*) \in X \times \mathbb{R}_+^L$ is a competitive equilibrium with transfer for \mathcal{E}^{pure} if

there exist $w_i \in \mathbb{R}, i \in I$ such that

- $\sum_i w_i = p^* \cdot r,$
- $x_i^* \in B_i(p^*, w_i)$ and $x_i^* \succeq x_i$ for all $x_i \in B_i(p^*, w_i),$ and
- $\sum_i x_{i,\ell}^* \leq r_\ell$ for all ℓ ($= r$ if $p_\ell^* > 0$).

Remark. Note that a CE is trivially a CE with transfer with $w_i = p^* \cdot x_i^*.$

The Second Welfare Theorem

We prove the second theorem only for \mathcal{E}^{pure} .

The Second Fundamental Theorem of Welfare Economics

Suppose that $X_i = \mathbb{R}_+^L$ and \succeq_i is locally nonsatiated, **continuous** and convex for all $i \in I$ in \mathcal{E}^{pure} . Consider any Pareto efficient allocation $x^* \in A$. If either (1) $x_i^* \gg 0$ for all $i \in I$ or (2) \succeq_i is strongly monotone for all $i \in I$ and $r \gg 0$, then there exists a price vector $p^* \in \mathbb{R}_+^L$ such that (x^*, p^*) is a competitive equilibrium with transfer for \mathcal{E}^{pure} .

Proof

Let $V_i := \{x_i \in X_i : x_i \succ x_i^*\}$ and $V := \sum_{i \in I} V_i$. V_i is convex. So is V .

Similarly $\bar{V}_i := \{x_i \in X_i : x_i \succeq x_i^*\}$ and $\bar{V} := \sum_{i \in I} \bar{V}_i$ are convex. Let

$$Z = \{z \in \mathbb{R}^L \mid z \leq r\}$$

• **Step 1.** Separate \bar{V} from Z by a price vector.

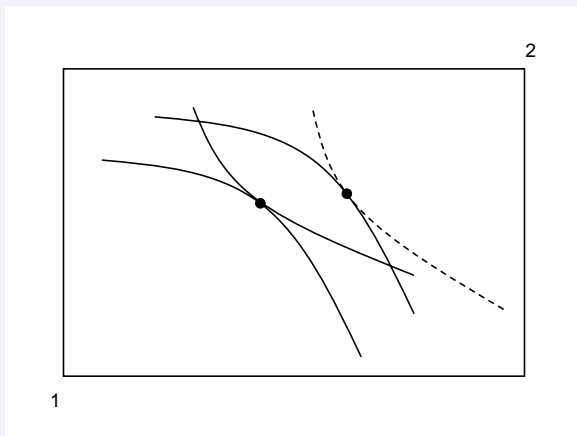
- ▶ $V \cap Z = \emptyset$.
- ▶ By SHT, $\exists p^* \in \mathbb{R}^L / \{0\}$ such that $\forall v \in V, \forall z \in Z, p^* \cdot v \geq p^* \cdot z$.
- ▶ $p^* > 0$.
- ▶ $\forall v \in V, p^* \cdot v \geq p^* \cdot r$.
- ▶ $\forall v \in \bar{V}, p^* \cdot v \geq p^* \cdot r$ (by local nonsatiation).
- ▶ $p^* \cdot \sum_i x_i^* = p^* \cdot r$, hence $p_\ell^* > 0 \Rightarrow \sum_i x_{i,\ell}^* = r_\ell$.

Proof (continued)

- **Step 2.** Set $w_i = p^* \cdot x_i^*$. Then $\sum_i w_i = p^* \cdot r$ is satisfied.
- **Step 3.** Show that consumer i is minimizing the cost within \bar{V}_i .
 $\forall x_i \in \bar{V}_i, p^* \cdot x_i \geq p^* \cdot x_i^*$ (set $x_h = x_h^*$ for all $h \neq i$).
- **Step 4.** Show that cost minimization \rightarrow utility maximization, then we are done. We know that this is true when $w_i = p^* \cdot x_i^* > 0$ (or $p^* \gg 0$ and $x_i^* = 0$). This is indeed the case when either (1) or (2) is satisfied (why?).

Remark.

- How to implement such a reallocation? How can you know the preference of consumers?



Analytical Approach to Welfare Theorems

Analytical Approach

We consider a pure exchange economy. Assume that

- $X_i = \mathbb{R}_+^L$.
- u_i is differentiable.
- u_i is locally nonsatiated and pseudo-concave (\rightarrow **quasi-concave**).
- There is at least one consumer, say consumer 1, whose utility function is differentially strictly increasing ($Du_1(x_1) \gg 0$).

Analytical Approach

Consider the following programming problem.

$$\begin{aligned}
 (P) \quad & \max_{x \in \mathbb{R}_+^{I \times L}} u_1(x_1) \text{ s.t.} \\
 & u_i(x_i) \geq \underline{u}_i, i = 2, \dots, I, \\
 & \sum_i x_i \leq r.
 \end{aligned}$$

Analytical Approach

Lemma

$x^* \in \mathbb{R}_+^{I \times L}$ is Pareto efficient if and only if x^* solves (P) with $\underline{u}_i = u_i(x_i^*)$ for $i = 2, \dots, I$.

Proof

- “Only if” follows from the definition of Pareto efficiency.
- If x^* solves (P) but is not Pareto efficient, then you can find another feasible allocation $x' \in A$ to achieve a higher value for (P) than x^* . This is a contradiction.

Pareto Efficiency and K-T conditions

- Thanks to the assumptions imposed on u_i , the K-T conditions are necessary and sufficient for the optimal solution $x^* \gg 0$ of (P) . So $x^* \in \mathbb{R}_{++}^{I \times L}$ is Pareto efficient if and only if the following conditions are satisfied:

K-T conditions for PE

$\exists a_2, \dots, a_I \geq 0$ and $q \in \mathbb{R}_+^L$ such that

- 1 $a_i Du_i(x_i^*) - q = 0$ for all $i \in I$ where $a_1 = 1$.
- 2 $\sum_i x_i^* \leq r$ and $\sum_i x_{i,\ell}^* = r_\ell$ if $q_\ell > 0$.

Pareto Efficiency and K-T conditions

These K-T conditions can be simplified as follows.

- 1 $\exists a \in \mathbb{R}'_{++}$ and $q \in \mathbb{R}^L_{++}$ such that $a_i Du_i(x_i^*) - q = 0$ for all $i \in I$.
- 2 $\sum_i x_i^* = r$.

CE with transfer

$(x^*, p^*) \in \mathbb{R}_{++}^{I \times L} \times \mathbb{R}_+^L$ is a CE with transfer $w_i = p^* \cdot x_i^*$ for \mathcal{E}^{pure} if and only if the following conditions are satisfied:

- 1 $\exists \lambda \in \mathbb{R}_+^I$ such that $Du_i(x_i^*) - \lambda_i p^* = 0$ for all $i \in I$.
- 2 $\sum_i x_i^* \leq r$ and $\sum_i x_{i,\ell}^* = r$ if $p_\ell^* > 0$.

They can be simplified (note that $p^* \gg 0$) as follows.

- 1 $\exists \lambda \in \mathbb{R}_{++}^I$ such that $Du_i(x_i^*) - \lambda_i p^* = 0$ for all $i \in I$.
- 2 $\sum_i x_i^* = r$.

Fundamental Welfare Theorems via K-T Conditions

Then we can prove the first theorem and the second theorem at the same time by one line.

Theorem

$x^* \in \mathfrak{R}_{++}^{I \times L}$ is Pareto efficient in \mathcal{E}^{pure} if and only if there exists $p^* \in \mathfrak{R}_{++}^L$ such that $(x^*, p^*) \in \mathfrak{R}_{++}^{I \times L} \times \mathfrak{R}_{++}^L$ is a CE with transfer for \mathcal{E}^{pure} .

Proof

Set $p^* = q$ and $\lambda_i = \frac{1}{a_i}$ for all $i \in I$.