Mathematical Appendix

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1. Miscellaneous Results

This first section lists some mathematical facts that were used or will be used in my lecture, but not included in the appendix of MWG. So it may be updated frequently.

Countable Sets

- A set is **countable** if different integers can be assigned to its different elements. Intuitively, you can literally "count" elements in a countable set.
- The formal definition is that X is countable if and only if there exists a bijection $f : X \to \mathbb{N}$ (natural numbers).
- If X and Y are countable, then $X \times Y$ is countable (to see this, Let $X = Y = \mathbb{N}$. Then $\mathbb{N}^2$ is countable because I can count them as follows: $(0,0),(0,1),(1,0),(2,1),(1,1),(1,2)...)$.

sup and inf

- For any subset Y of real numbers ($Y \subset \Re$), there is the smallest upper bound and the largest lower bound (in $\overline{\Re}$). See any textbook on analysis (or even "supremum" in Wikipedia) to find the definitions.
- Ex. $\sup\{1,2,3\} = 3$, $\sup\{1,2,3,...\} = \infty$, $\inf(0,1) = 0$. Note that $\sup Y$ or $\inf Y$ may not be in Y.
- The following properties follow from the definition almost immediately.
  - For any (nonempty) $Y \subset \Re$, there exists $y_n \in Y, n = 1, 2, ..$ such that $y_n \to \sup Y$ (the same result applies to $\inf Y$).

Sequence

A few useful facts about sequences in $\Re^L$.

- If $x_n, n = 1, 2,$ is bounded, then there exists a convergent subsequence.
- If x_n converges to x^*, then every subsequence of x_n converges to x^*.
- x_n converges to x^* if and only if $x_{\ell,n}$ converges to x_ℓ^* for every $\ell = 1, 2, ..., L$.
- Consider a sequence of numbers ($L = 1$). We use $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup\{x_m, m \geq n\}$ and $\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf\{x_m, m \geq n\}$.
    - Clearly $\limsup_{n\to\infty} x_n \geq \liminf_{n\to\infty} x_n$.
    - If $\liminf x_n \geq \limsup x_n$, then $\liminf x_n = \limsup x_n$ ($= \lim x_n$).

Compactness in $\Re^n$

- $X \subset \Re^n$ is **compact** if and only if it is closed and bounded.
- Another definition of compactness: $X \subset \Re^n$ is compact if and only if every open cover (i.e. a collection of open sets such that their union includes X) of X has a finite cover (i.e. open cover with a finite number of open sets).

2. Correspondence

- A map f which maps each point in $X \subset \Re^M$ to a subset of $Y \subset \Re^N$ is called **correspondence** and denoted by $f : X \rightrightarrows Y$.
- We always assume that $f(x)$ is not an empty set for any $x \in X$.

Closed-Graph Property

$f : X \rightrightarrows Y$ has a closed graph if

1. $y_n \in f(x_n)$
2. $x_n \to x^* \in X$
3. $y_n \to y^*$

for a sequence $(x_n, y_n) \in X \times Y$, then $y^* \in f(x^*)$.

Continuity of Correspondence

upper hemicontinuity

Given $X \subset \Re^M$ $Y \subset \Re^N$, $f : X \rightrightarrows Y$ is **upper hemicontinuous (uhc)** if its image of every compact set in X is bounded and f has a closed graph.

lower hemicontinuity

Given $X \subset \Re^M$ and $Y \subset \Re^N$, $f : X \rightrightarrows Y$ is **lower hemicontinuous (lhc)** if for any $x^* \in X$, any sequence $x_n \in X$ converging to x^* and any $y^* \in f(x^*)$, there exists $y_n \in f(x_n)$ such that $y_n \to y^*$.

$f : X \rightrightarrows Y$ is **continuous** if it is uhc and lhc.

Continuity of Correspondence

Comments.

- If f is a function, then both uhc and lhc imply continuity in the usual sense.
- The first condition for uhc can be replaced by "f is locally bounded at every $x \in X$", i.e. there exists an open neighborhood of x on which f is bounded (why?).
- We say f is **uhc at** $x \in X$ if f is locally bounded at x and satisfies the closed-graph property at x. f is uhc if and only if f is uhc at every $x \in X$.

Continuity of Correspondence

- Show that budget constraint correspondence $B(p, w) = \{x \in \Re_+^L : p \cdot x \leq w\}$ is continuous in $\Re_{++}^L \times \Re_+$.
- Constraint for cost minimization problem $\{x \in \Re_+^L : u(x) \geq \underline{u}\}$ is not continuous in $\underline{u}$ (why?). But for any given $p \gg 0$, we can modify this constraint so that the constraint set is locally continuous in $(p, \underline{u})$.
  - Pick any x' such that $u(x') > \underline{u}$.
  - Find $\overline{x} \in \Re_+$ such that $p_\ell \, \overline{x} \geq p' \cdot x'$ for any ℓ for any p' in a neighborhood of p.
  - Then the cost minimizing solution is the same locally even if the constraint set is replaced by $\{x \in \Re_+^L : u(x) \geq \underline{u}, x_\ell \leq \overline{x} \ \forall \ell\}$, which is continuous.

3. Maximum Theorem

Maximum Theorem

Let $F : X \times A (\subset \Re^M \times \Re^N) \to \Re$ be a continuous function and $G : A \rightrightarrows X$ be a continuous correspondence. Let $\gamma(a) \subset X$ be the set of the solutions for $\max_{x \in G(a)} F(x, a)$ given $a \in A$. Then

(i) $\gamma : A \rightrightarrows X$ is upper hemicontinuous, and

(ii) $V(a) := \max_{x \in G(a)} F(x, a)$ is continuous in a.

Proof of uhc

- Since G is uhc, (1) $G(a)$ is compact, hence $\gamma(a)$ is nonempty, and (2) $\gamma(a)$ is locally bounded for every $a \in A$. (2) means that we just need to prove the closed graph property of $\gamma(a)$ for each $a \in A$.
- Take any sequence $(x_n, a_n) \in X \times A$ such that $x_n \in \gamma(a_n)$, $a_n \to a^* \in A$, and $x_n \to x^*$. Note that $x^* \in G(a^*)$ because G is uhc.
- Take any $x' \in \gamma(a^*)$ and $\epsilon > 0$. Since G is lhc, there exists $x'_n \in G(a_n)$ such that $x'_n \to x'$. Since F is continuous, there exists N such that $F(x'_n, a_n) \geq V(a^*) - \epsilon$ for any $n \geq N$. Since $x_n \in \gamma(a_n)$, we have $F(x_n, a_n) \geq F(x'_n, a_n)$. Hence $F(x^*, a^*) \geq V(a^*) - \epsilon$ by continuity. Then $F(x^*, a^*) \geq V(a^*)$ by letting $\epsilon \to 0$. Therefore $x^* \in \gamma(a^*)$.

Proof of continuity

- Take any sequence $a_n \in A$ such that $a_n \to a^* \in A$. The previous proof shows that G is lhc $\Rightarrow \liminf_{n\to\infty} V(a_n) \geq V(a^*)$.
- So we just need to show $\limsup_{n\to\infty} V(a_n) \leq V(a^*)$. Suppose not. Then there exists $\epsilon > 0$ and a subsequence (wlog the original sequence) such that $V(a_n) \geq V(a^*) + \epsilon$.
- Take any sequence $x'_n \in \gamma(a_n)$. Since G is uhc and $\{a_n, n = 1, 2, ..\} \subset A$ is compact, $\{x'_n, n = 1, 2, ..\} \subset X$ is bounded. Hence we can find a subsequence (wlog...) such that $\lim x'_n \to x^*$ for some x^* ($\in G(a^*)$ as G is uhc). Then $F(x^*, a^*) \geq V(a^*) + \epsilon$ holds in the limit by continuity, which is a contradiction. Hence $\limsup_{n\to\infty} V(a_n) \leq V(a^*)$.

Comment.

- The continuity of Walrasian demand correspondences and Hicksian demand correspondences follow from (i).
- The continuity of indirect utility functions and expenditure functions can be proved by using (ii)

4. Kuhn-Tucker Theorem

- Let X be an open convex set in $\Re^L$. Let $f : X \to \Re$ and $g_m : X \to \Re, m = 1, ..., M$ are differentiable functions.
- Consider the following problem.

$$\text{(P): } \max_{x \in X} f(x) \text{ s.t. } g_m(x) \geq 0 \text{ for } m = 1, ..., M.$$

Kuhn-Tucker Theorem

Constraint Qualification: Let $M(x^*)$ be the set of binding constraints at x^* (i.e. $g_m(x^*) = 0$ for $m \in M(x^*)$). We say that the **constraint qualification** (CQ) is satisfied at $x^* \in X$ if either one of the following conditions is satisfied.

Constraint Qualification

- **Linear Independence:** $\{\nabla g_m(x^*) | m \in M(x^*)\} \subset \Re^L$ are linearly independent.
- **Slater Condition:** There exists $x' \in X$ that satisfies $g_m(x') > 0$ for all $m \in M(x^*)$ and g_m are **pseudo-concave** for all $m \in M(x^*)$.

Kuhn-Tucker Theorem

Necessity

If $x^* \in X$ solves (P) and the constraint qualification is satisfied at x^*, then there exists $\lambda \in \Re_+^M$ such that:

(1) $\nabla f(x^*) + \sum_{m=1}^{M} \lambda_m \nabla g_m(x^*) = 0$ (**First order conditions**),

(2) $g_m(x^*) \geq 0$ $(= 0$ if $\lambda_m > 0)$ (**Complementary slackness conditions**).

Sufficiency

Suppose that f is pseudo-concave, $g_m, m = 1, ...M$ are all quasi-concave.

If $x^* \in X$ and $\lambda \in \Re_+^M$ satisfies (1) and (2), then x^* solves (P).

Proof

- Let's prove necessity first. Let $x^* \in X$ be an optimal solution of (P). Consider the following linearized programming problem.

$$(P^*): \max_{x \in X} Df(x^*)(x - x^*) \text{ s.t. } Dg_m(x^*)(x - x^*) \geq 0 \text{ for } m \in M(x^*).$$

- We need the following lemma.

Lemma

Suppose that the constraint qualification is satisfied at x^*. If $x^* \in X$ is a solution for (P), then it is also a solution for (P^*).

- **Proof of Lemma:**
    - Suppose not. Then $\exists x' \in X$ such that $Df(x^*)(x' - x^*) > 0$ and $Dg_m(x^*)(x' - x^*) \geq 0$ for all $m \in M(x^*)$.
    - If CQ is satisfied, then there exists $x'' \in X$ such that $Dg_m(x^*)x'' > 0$ for all $m \in M(x^*)$ (prove this for each type of CQ).
    - Then $\widehat{x} = x' + \epsilon x''$ for small $\epsilon > 0$ satisfies $Df(x^*)(\widehat{x} - x^*) > 0$ and $Dg_m(x^*)(\widehat{x} - x^*) > 0$ for all $m \in M(x^*)$. This means that for $x^\alpha = x^* + \alpha(\widehat{x} - x^*)$, $f(x^\alpha) > f(x^*)$ and $g_m(x^\alpha) > 0$ for all $m = 1, ..., M$ if $\alpha > 0$ is small enough. This is a contradiction. So x^* must be a solution for (P^*).

Proof (continued)

- By using **Farkas' Lemma**, it can be shown that x^* is a solution for $(P^*) \Leftrightarrow x^*$ satisfies the K-T condition with some $\lambda \in \Re^M_+$ (this step does not depend on CQ). The proof (only one direction) is in MWG.
- Hence x^* is a solution for $(P) \Rightarrow\ x^*$ is a solution for $(P^*) \Leftrightarrow\ x^*$ satisfies the K-T condition with some $\lambda \in \Re^M_+$.

Proof (continued)

- For sufficiency, we just need to show that x^* is a solution for $(P^*) \;\Rightarrow\; x^*$ is a solution for (P) when u is pseudo-concave and g_m are quasi-concave for all $m \in M(x^*)$. We prove the contrapositive below.
- Suppose that x^* is not a solution for (P). Then $\exists x' \in X$ such that $u(x') > u(x^*)$ and $g_m(x') \geq 0$ for all $m \in M$. Since u is pseudo-concave, g_m is quasi-concave and $g_m(x^*) = 0$ for all $m \in M(x^*)$, $Du(x^*)(x' - x^*) > 0$ and $Dg_m(x^*)(x' - x^*) \geq 0$ for all $m \in M(x^*)$. Hence x^* is not a solution for (P^*).

Farkas' Lemma

What is **Farkas' Lemma**?

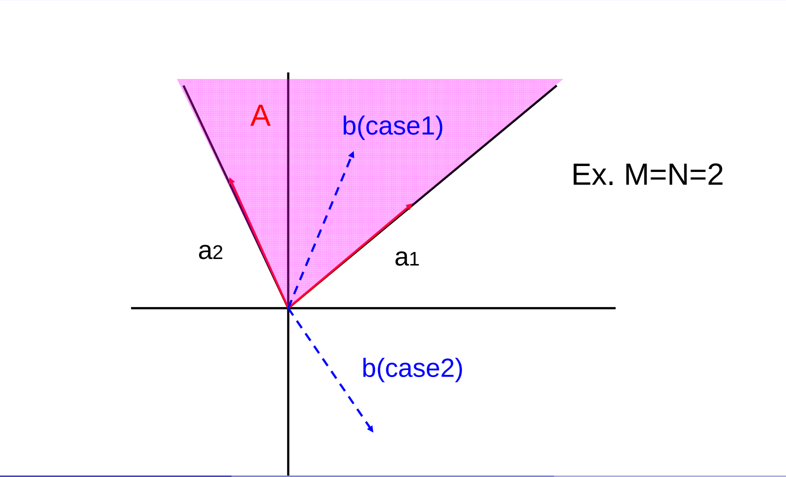
Farkas' Lemma

For any $a_1, ...a_M \in \Re^N$ and $b \in \Re^N_+$, only one of the following conditions hold.

1. $\exists \lambda_m \geq 0, m = 1, ..., M$ such that $b = \sum_{m=1}^{M} \lambda_m a_m$.
2. $\exists y \in \Re^N$ such that $y \cdot b > 0$ and $y \cdot a_m \leq 0$ for $m = 1, ..., M$.

Farkas' Lemma

This just means that b is either in A or not in A in the picture below.



Note on pseudo-concavity

Pseudo-concavity

$f : X \to \Re$ is **pseudo-concave** if $f(x') > f(x) \Rightarrow Df(x)(x' - x) > 0$.

- Easy way to check pseudo-concavity? f is pseudo-concave if
  - it is concave (for example, linear), or
  - it is quasi-concave and $\nabla f(x) \neq 0$ for all $x \in X$.
- K-T may not be sufficient without pseudo-concavity.

  Example. Consider $\max_{x \in [-1,1]} x^3$. Then $x = 0$ satisfies the K-T condition with $(\lambda_1, \lambda_2) = (0, 0)$, but is clearly not optimal. Note that the objective function is strictly quasi-concave.

5. Envelope Theorem

- Suppose that
  - f and g_m depend on some parameter $a \in A$, where A is an open set in $\Re^K$.
  - an optimal solution for (P) given a is given by a differentiable function $x(a)$.
  - constraint qualification is satisfied at $x(a)$ for every $a \in A$.
- Define the optimal value function $v : A \to \Re$ by $v(a) := f(x(a))$.

Envelope Theorem

- We like to compute $\nabla v(a)$.
- Let's first consider the simplest case: assume that there is no constraint (or no constraint is binding). If $x(a)$ is an optimal solution given a, then FOC is $D_x f(x, a) = 0$.
- Then we have

$$\nabla v(a) = \nabla f(x(a), a) = D_x f(x, a) Dx(a) + \nabla_a f(x(a), a) = \nabla_a f(x(a), a)$$

- This can be generalized to the case with (binding) constraints.

Envelope Theorem

Envelope Theorem

Suppose that the binding constraints for (P) do not change in a neighborhood of a. Then

$$\nabla v(a) = \nabla_a f(x(a), a) + \sum_m \lambda_m \nabla_a g_m(x(a), a)$$

, where $\lambda_m, m = 1, ..., M$ are the multipliers in the K-T condition.

6. Implicit Function Theorem

- Consider a circle defined by $f(x, a) = x^2 + a^2 = 1$ and the point $(1/\sqrt{2}, 1/\sqrt{2})$ on the circle.
- x is a function of a in the neighborhood of $(1/\sqrt{2}, 1/\sqrt{2})$. Denote this function by $x(a)$. What is $Dx(a)$ at $a = 1/\sqrt{2}$?
- One way to obtain this value is to derive $x(a)$ explicitly and compute the derivative. But this is not always an easy thing to do.

Implicit Function Theorem

- **Implicit function theorem** allows us to compute this value without deriving $x(a)$ explicitly.
- For this example, just differentiate $f(x(a), a) = 0$ with respect to a. Then you get $Dx(a) = -\frac{D_a f(x,a)}{D_x f(x,a)}$. Hence

$$Dx(a)|_{a=1/\sqrt{2}} = -\frac{\sqrt{2}}{\sqrt{2}} = -1.$$

- This does not work when $D_x f(x, a) = 0$ (at $a = 1$ for example).

Implicit Function Theorem

- This result generalizes. Let $X \in \Re^J$ and $A \in \Re^K$ be open sets and $F : X \times A \rightarrow \Re^J$ be a C^1 (continuously differentiable) function.

Implicit Function Theorem

If *rank* $D_x F(x', a') = J$ at $(x', a') \in X \times A$, then there exist open neighborhoods $V \subset X$ of x', $U \subset A$ of a', and a C^1 function $f : U \rightarrow V$ such that

1. $x = f(a)$ if and only if $F(x, a) = 0$ for $(x, a) \in V \times U$, and
2. $Df(a') = -D_x F(x', a')^{-1} D_a F(x', a')$.

Implicit Function Theorem