Mixed Strategy Nash Equilibrium

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Mixed Strategy

- There is no Nash equilibrium for some games. MP is one such game.
- Let's allow players to randomize over their actions.
  - Distribution of player i's actions: $\alpha_i \in \Delta(A_i)$
  - Player i's **expected payoff** given $\alpha \in \prod_{j\in N} \Delta(A_j)$ is denoted as follows (assuming that A is finite).

$$u_i(\alpha) = \sum_{a\in A} \prod_{j\in N} \alpha_j(a_j) u_i(a).$$

- In this context, we call $\alpha_i \in \Delta(A_i)$ **mixed strategy** and $a_i \in A_i$ **pure strategy**.

Mixed Strategy

- It is easy to verify that, for any $\lambda \in [0,1]$,

$$u_i(\lambda\alpha_i' + (1-\lambda)\alpha'', \alpha_{-i}) = \lambda u_i(\alpha_i', \alpha_{-i}) + (1-\lambda)u_i(\alpha_i'', \alpha_{-i}).$$

  In particular, $u_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) u_i(a_i, \alpha_{-i})$, where a_i means a mixed strategy with probability 1 on a_i.

- When action sets are not finite, the expected payoffs are integrals, i.e.

$$u_i(\alpha) = \int_{a_1 \in A_1} \int_{a_n \in A_n} u_i(a) d\alpha_n \cdots d\alpha_1$$

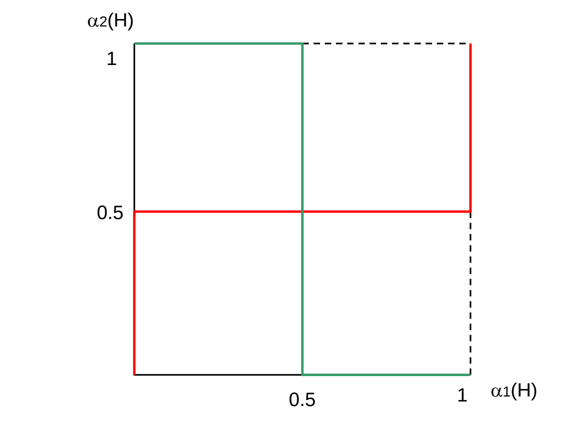
  (**Note:** A_i is a measurable set and u_i is a measurable function on A for $i = 1, ..., n$).

Mixed Strategy Nash Equilibrium

- Call strategic game $(N, (\Delta(A_i)), (u_i))$ the **mixed extension** of $(N, (A_i), (u_i))$, where $u_i : \prod_{j \in N} \Delta(A_j) \to \Re$ is defined as above.
- **Mixed strategy Nash equilibrium** of $(N, (A_i), (u_i))$ is a Nash equilibrium of mixed extension $(N, (\Delta(A_i)), (u_i))$.
- For any finite strategic game, there exists a mixed strategy Nash equilibrium. This is a corollary of the previous existence result.

MSNE for Matching Penny

The red line is player 1's BR. The green line is player 2's BR. $(0.5, 0.5)$ is the mixed strategy Nash equilibrium for MP.



Example: Inspection Game

- Two players: Worker and Manager
- Worker can either work or shirk. Manager can either inspect or do not inspect.
- The cost of work is c. The cost of inspection is g. The worker is paid w if he is not found shirking. The manager gets v if the worker works.

	I	**NI**
W	(w-c, v-w-g)	(w-c, v-w)
S	(0, -g)	(w,-w)

Example: Inspection Game

- What is a Nash equilibrium?
- Find w to maximize the manager's payoff.

Interpretations of Mixed Strategy Nash Equilibrium

- **Deliberate Choice:** Players are literally randomizing or choosing a randomization device (but why do they have to? Maxmin?).
- **Adaptation and Average Behavior:** It is a description of average behavior when players follow some adaptive process.
- **Randomization Device:** Players are playing a pure strategy based on some randomization device (ex. play H if I wake up before 7 and L if after 7).
- **Population Interpretation:** people play the same game with different players over time and a mixed strategy is just a distribution of different pure strategies in population (ex. a half people play H and a half people play T).

Interpretations of Mixed Strategy Nash Equilibrium

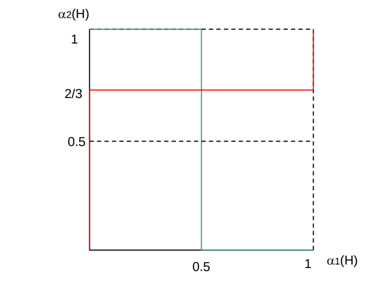
- **Purification (Harsanyi 1973):** Suppose that player i's payoff given $a \in A$ is given by $u_i(a) + \gamma\epsilon_i(a)$, where $\epsilon_i(a)$ is mean zero noise. Then the set of mixed strategy NE in $(N, (\Delta(A_i)), U)$ is a limit of a strict pure strategy Bayesian Nash Equilibrium in $(N, (\Delta(A_i)), (u_i + \gamma\epsilon_i))$ as $\gamma \to 0$.
- **Belief Interpretation (Aumann 1987):** Player 2's mixed strategy represents player 1's belief about player 1's behavior. Both players play a pure strategy (but not much predictive power without imposing consistency between belief and actual behavior).

Matching Penny Experiments

Is MSNE is a good description of reality? Consider the following **asymmetric matching penny**.

	H	T
H	1, -1	-3, 1
T	-1, 1	1, -1

- Modified best response curves:



  - Player 1's equilibrium mixed strategy must the same for MP and AMP.
  - In experiments, people behave differently in the short run. Player 1 plays T more than H in AMP.
  - But typically people's behavior converges to the mixed strategy Nash equilibrium in the long run.

Mixed Strategy in Wimbledon

- Walker and Wooders (2001) examined top tennis players' behavior in Wimbledon games.
- Each game can be regarded as a kind of matching penny game.
  - The server decides whether to serve on the right hand side of the receiver or on the left hand side of the receiver.
  - The receiver decides whether to move to the right or the left.
  - S's optimal choice is to choose the different side from what R chooses. R's optimal choice is to choose the same side as what S chooses. So this is like a matching penny game!

Mixed Strategy in Wimbledon

- If the theory is correct, then both the server and the receiver must be indifferent between their choice of sides. This was confirmed in data. The expected point conditional on serving to the right or serving to the left were found to be roughly the same.
- However, the data is not perfectly consistent with the theory. The theory predicts that players randomize independently over periods. In reality, players switch sides "too much".
- It seems that the behavior of more experienced players fit the theory better. The behavior of amateur players is less consistent with the theory.

O'Neill's four-by-four game

- Consider the following card game (O'Neill 1987).
  - Player 1 and 2 choose one card from $\{A, K, Q, J\}$ simultaneously.
  - Player 1 wins if either (1) both 1 and 2 chose A or (2) no player chose A and different cards were chosen.
  - Otherwise player 2 wins.
  - The loser pays \$1 to the winner.
- The aggregate frequencies of actions is often remarkably close to the theoritical prediction (=MSNE)