Nash Equilibrium

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January 11, 2012

- In many games, there is no obvious choice (i.e. dominant action).
- In such games, players are in strategic situation: each player's optimal choice depends on the other players' choices.

Coordination Game

	B	C
B	1, 1	0, 0
C	0, 0	1, 1

Chicken Game

	A	B
A	0, 0	4, 1
B	1, 4	3, 3

Matching Penny

One more.

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

- We need a solution concept to make a prediction in such situations. The one that is most common one in Economics is **Nash Equilibrium**
- In Nash equilibrium, every player plays a **best response** against the other players simultaneously.

Best Response

Best Response

$a_i \in A_i$ is a **best response** to $a_{-i} \in A_{-i}$ if

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \text{ for all } a'_i \in A_i$$

For example, B is player 1's best response to A by player 2 in Chicken game.

Nash Equilibrium

- **Nash equilibrium** for a strategic game is a profile of actions such that each action is a best response to the other actions.
- Let $B_i(a_{-i}) \subset A_i$ be the set of player i's best response actions against $a_{-i} \in A_{-i}$. Here is the formal definition of Nash equilibrium.

Nash Equilibrium

$a^* = (a_1^*, \cdots, a_n^*) \in A$ is a Nash Equilibrium if $a_i^* \in B_i(a_{-i}^*)$ for every $i \in N$.

Nash Equilibrium

- How to find a Nash equilibrium? It's easy for games with two players and finite actions.

Chicken Game

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Nash Equilibrium

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  - Player 1's best response against each of player 2's strategy = red circle.

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Nash Equilibrium

- How to find a Nash equilibrium? It's easy for games with two players and finite actions.
  - Player 1's best response against each of player 2's strategy = red circle.
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  - NE = (A,B),(B,A).

Chicken Game

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NE

Nash Equilibrium

- (B, B) and (C, C) are NE for Coordination game. There is no NE for MP.
- A few observations regarding NE and DSE:
  - A dominant strategy equilibrium is a Nash equilibrium.
  - If every player has a strictly dominant action (as in PD), then a profile of strictly dominant actions is the only Nash equilibrium.

Is Nash equilibrium reasonable?

1. **Nash Equilibrium as Self-Enforcing Behavior:** If every player believes that a particular Nash equilibrium is played, then there is no incentive to deviate from it for any player.
2. **Nash Equilibrium as a Steady State of Learning/Evolution:** Suppose that a player plays the same game repeatedly with different players in a large population. If players' behavior converges to a particular action profile, then it is reasonable to think that it must be a Nash equilibrium.[1]

[1] This may be true even if a player plays the same game with the same player repeatedly.

Is Nash equilibrium reasonable?

- There are some issues, which we will deal with in more detail later.
  - **Coordination:** There are often many NE. Why should we expect that the expectations of the players are consistent?
  - **Equilibrium Selection:** Suppose that players play a NE. Even so, as an outsider, we are not sure about which NE is played. Some NE seems to be more reasonable than others. So we need "**refinement**" of NE.
  - **Non-Nash Behavior:** Nash equilibrium is sometimes too strong. In some experiments, people don't play any Nash equilibrium, especially in the short run. We may want to relax the notion of Nash equilibrium or come up with a new notion of equilibrium to capture actual behavior in such games.

Example 1: First Price Auction

We discuss a few examples to illustrate the idea of NE. Let's start with FPA.

- Suppose that n bidders with values $v_1 > v_2 > ... > v_n > 0$ submit bids $b = (b_1, ..., b_n) \in \Re^n$ simultaneously for some good.
- The highest bidder wins and pays his or her own bid. Bidder i's payoff is $v_i - b_i$ when winning the good by bidding $b_i \in \Re_+$. The winner is determined randomly when tied. The bidders maximize their expected payoffs.
- Can you find any Nash equilibrium? Who wins?
- Change the tie-breaking rule: the bidder with the smallest index would win when tied. Can you find any Nash equilibrium? Who wins?

Example 2: Cournot Model of Duopoly

- Firm 1 and firm 2 produce a homogeneous good.
- The firms choose their output $q = (q_1, q_2) \in \Re^2_+$ simultaneously.
- The market price is determined by the inverse demand function: $P(q_1 + q_2)$.
- Firm i's profit is $\pi_i(q) = P(q_1 + q_2)q_i - C_i(q_i)$. $C_i(q_i)$ is firm i's cost function.

Example 2: Cournot Model of Duopoly

- Suppose that $P(q) = 12 - q_1 - q_2$(or 0 if negative) and $C_i(q_i) = 3q_i$. Find a Nash equilibrium.
- Consider a simple adaptive process $(q_1(0), q_2(1), q_1(2), ...)$ where $q_i(t)$ is a best response to $q_{-i}(t-1)$. Verify that it converges to a Nash equilibrium given any starting point $(q_1(0))$.
- Consider n firms with the same identical linear cost function and the same inverse demand function.
  - Can you find a Nash equilibrium?
  - Is there any relationship between the market price and the number of firms in equilibrium?

Example 3: Bertrand Model of Duopoly

- There are two firms 1 and 2 that produce <u>similar, but different</u> products (**differentiated products**).
- The firms choose their prices $p = (p_1, p_2) \in \Re_+$ simultaneously.
- Each firm's demand q_i depends on both prices (their products are imperfect substitutes, ex. coffee and tea).
- Firm i's profit is $\pi_i(p) = p_i q_i(p) - C_i(q_i(p))$.

Example 3: Bertrand Model of Duopoly

- Suppose that $q_i(p) = a - p_i + bp_j$ and each firm's marginal cost is c.

  Assume that $b < 2$.
- Find a Nash equilibrium.

Example 4: Voluntary Contribution of Public Good

- There is a village with n residents.
- Each resident's wealth is W. Assume that W is large enough.
- The residents decide simultaneously how much they donate their money, which will be used to provide public goods in the village.
- Resident i's payoff is $W - x_i + \sqrt{y}$ where x_i is player i's private donation and $y = \sum_{j=1}^{n} x_j$ is the amount of public goods.
- Find a Nash equilibrium.

Example 5: Rent Seeking Game

- There are n lobbying groups.
- Only one of them "wins" and gains $V > 0$.
- Group i chooses its "effort level" $x_i \in \Re_+$ at the cost of $cx_i \geq 0$.
- The probability that group i wins is $\frac{x_i}{\sum_{j=1}^{n} x_j}$.
- Group i's expected payoff is $\frac{x_i}{\sum_{j=1}^{n} x_j} V - cx_i$.
- Find a Nash equilibrium.

Example 6: AdAuction

- M links on a webpage are auctioned among $n > M$ bidders
- The top link gets the largest number of clicks, then the second top link and so on.
- Suppose that bidders need only one link and their values per click are $v_1 > ... > v_n$.
- Bidders bid how much they would pay per click. The highest bidder wins the top link and pays the second highest bid. The second highest bidder wins the second link and pays the third highest bid...
- This **ad auction** or **Pay-per-Click auction** is a generalized second price auction (it reduces to SPA when $M = 1$).

Example 6: AdAuction

- Bidder i's payoff given $b = (b_1, ..., b_n)$ is

$$u_i(b) = \sum_{k=1}^{M} \mathbf{1}(b_i = b^{(k)}) \left(v_i - b^{(k+1)} \right) x_k$$

where $\mathbf{1}(b_i = b^{(k)}) = 1$ if and only if bidder i's bid is the kth highest bid (otherwise 0) and x_k is the number of clicks at the kth link (ignoring any tie).

Example 6: AdAuction

- Suppose that
  - $M = 2$ and $n = 3$.
  - The first link generates 100 click/day. The second link generates 60/day.
  - The value of one click is $v_1 = \$20$, $v_2 = \$10$, and $v_3 = \$5$ for bidder 1,2, and 3 respectively.
- Is it a dominant action to bid v_i for bidder i?
- Find a Nash equilibrium.

Existence of NE

- There is sometimes no Nash equilibrium (ex. MP, complete information FPA with random tie-breaking).
- When does a Nash equilibrium exist?

Existence of NE

Nash Existence Theorem

Suppose that A_i is a nonempty compact convex subset of $\Re^k$ and u_i is continuous in A and quasi-concave in A_i given any $a_{-i} \in A_{-i}$ for every $i \in N$ for strategic game $(N, (A_i), (u_i))$. Then there exists a Nash equilibrium .

Kakutani's Fixed Point Theorem

We use Kakutani's fixed point theorem

Kakutani's Fixed Point Theorem

Let X be a nonempty compact and convex subset of $\Re^k$ and $f : X \rightrightarrows X$ be a correspondence. Suppose that f is convex-valued and upper hemicontinuous (has a closed graph). Then there exists x^* such that $x^* \in f(x^*)$.

Note: We always assume that a correspondence's value ($f(x)$ in this case) is not empty.

Proof

- For any a_{-i}, $B_i(a_{-i})$ is nonempty (by continuity and compactness), convex (by quasi-concavity) and upper hemicontinuous (by Maximum theorem).
- The product $B = (B_1, ..., B_n)$ is a correspondence from A to A and is nonempty, convex-valued and upper hemicontinuous.
- By Kakutani's fixed point theorem there is a fixed point a^* where $a_i^* \in B_{-i}(a_{-i}^*)$ for every $i \in N$. Clearly a^* is a NE.

Zero-Sum game

- A two-player strategic game G is a **zero-sum game** (**a strictly competitive game**) if $u_1(a) + u_2(a) = 0$ for every $a \in A$.
  - **Note:** If $u_1(a) + u_2(a) = K$ for every $a \in A$ for some constant K (constant-sum game), then we can subtract K from one player's payoff without loss of generality to turn it into a zero-sum game.
- For this class of games, the set of NE has a simple structure.

Maxmin

Maxmin

$a_i \in A_i$ is a **maxminimizer** for player i if

$$\min_{a_{-i} \in A_{-i}} u_i(a_i^*, a_{-i}) \geq \min_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \; \forall a_i \in A_i.$$

- That is, a maximizer is a (part of) solution for $\max_{a_i} \min_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})$.
- A maximizer maximizes the payoff in the worst case scenario. So it may be a reasonable choice when someone makes a very conservative choice.

Minimax Theorem

This is a version of **Minimax theorem**.

Minimax Theorem

Let $G = (\{1,2\}, (A_i), (u_i))$ be a zero-sum game where A_i is compact and u_i is continuous for every $i \in N$. Then $a^* \in A$ is a Nash equilibrium if and only if, for $i = 1, 2$,

1. a_i^* is a maxminimizer.
2. $u_i(a^*) = \max_{a_i \in A_i} \min_{a_{-i} \in A_{-i}} u_i(a) = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a)$.

Proof.

- Suppose that $a^* \in A$ is a NE.
  - It's easy to show that $u_1(a^*) \geq \min_{a_2} \max_{a_1} u_1(a) \geq \max_{a_1} \min_{a_2} u_1(a)$.
  - Since $u_2(a_1^*, a_2^*) \geq u_2(a_1^*, a_2)$ for any a_2, $u_1(a_1^*, a_2) \geq u_1(a^*)$ for any a_2. Then $\max_{a_1} \min_{a_2} u_1(a) \geq u_1(a^*) \leq \min_{a_2}$. Hence $u_1(a^*) = \max_{a_1} \min_{a_2} u_1(a) = \min_{a_2} \max_{a_1} u_1(a)$, which also implies that a_1^* is a maxminimizer for player 1 (the same for $i = 2$).
- Suppose that a_i^* solves $\max_{a_i} \min_{a_{-i}} u_i(a)$ and $u_i(a^*) = \max_{a_i} \min_{a_{-i}} u_i(a) = \min_{a_{-i}} \max_{a_i} u_i(a)$ for $i = 1, 2$.
  - The assumption implies $u_2(a_1, a_2^*) \geq \max_{a_2} \min_{a_1} u_2(a)$ for any a_1.
  - Since $u_2(a_1, a_2^*) = -u_1(a_1, a_2^*)$ and $\max_{a_2} \min_{a_1} u_2(a) = \min_{a_1} \max_{a_2} u_2(a) = -\max_{a_1} \min_{a_2} u_1(a) \geq -u_1(a_1^*, a_2^*)$, we have $u_1(a_1^*, a_2^*) \geq u_1(a_1, a_2^*)$ for any a_1. The same proof applies to player 2. Hence a^* is a NE.

Comments.

- Every NE generates exactly the same payoff for both players in zero-sum games.
- When there exists a NE for a zero-sum game, we can find it by solving for a maxminimizer for both players.
- If $\widehat{a}$ and $\widetilde{a}$ are a NE for a zero-sum game, then $(\widehat{a}_1, \widetilde{a}_2)$ and $(\widetilde{a}_1, \widehat{a}_2)$ are also NE (NE are **interchangeable**).
- For a zero-sum game, we cannot distinguish standard utility maximizing players playing a NE and maxminimizing players (if there exists a NE).