Other Equilibrium Notions

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Trembling Hand Perfect Equilibrium

- We may want to claim that (B, B) is not a reasonable NE in the following game.

	A	B
A	1, 1	0, 0
B	0, 0	0, 0

- One reason would be that people make “mistakes”.

Trembling Hand Perfect Equilibrium

- The following equilibrium notion captures such an idea.

Trembling Hand Perfect Equilibrium

$\alpha^* \in \prod_i \Delta(A_i)$ of a finite strategic game is a **trembling hand perfect equilibrium** if there exists a sequence of completely mixed strategies α^k that converges to α^* such that α_i^* is a best response to α_{-i}^k for every k for every $i \in N$.

- Every trembling hand perfect equilibrium is a MSNE with no weakly dominated strategy. The converse is true for two player games.
- So THPE is a **refinement** of NE.

Theorem

For any finite strategic game, there exists a trembling hand perfect equilibrium.

Proof.

- Let α^ϵ be a mixed strategy profile such that α_i^ϵ is a best response to α_{-i}^ϵ subjective to the constraint that every action is played with at least probability ϵ. Such a mixed strategy profile exists by Nash equilibrium existence theorem.
- Consider a sequence of α^{ϵ_k} such that $\epsilon_k \to 0$. Take a subsequence so that $\alpha^{\epsilon_k} \to \alpha^*$ for some α^*.
- There exists k' such that α_i^* is a best response for every $k \geq k'$ for every $i \in N$ (verify this).

Correlated Equilibrium

- Suppose that players observe a private signal before they play a strategic game (remember some interpretations of mixed strategies).
- Each player plays an optimal action given her private signal.
- What role would such private signals play?

	A	B
A	(3,3)	(1,4)
B	(4, 1)	(0,-0)

- There are two pure strategy NE: (A,B), (B,A) and one mixed strategy symmetric NE: $(0.5, 0.5)$.

- Suppose that a public signal x and y is observed with equal probability.
  - It is possible for players to coordinate and play (A, B) given x and (B, A) given y. Then $(A, B), (B, A)$ is played with probability $1/2$.
- Suppose that there are three states $\{x, y, z\}$, each of which realizes with probability $1/3$. Player 1 can only observe whether x is realized or not. Player 2 can only observe whether z is realized or not.
  - The following strategies are mutually best response: player 1 plays B if and only if x is observed and player 2 plays B if and only if z is observed. Then $(A, A), (A, B), (B, A)$ is played with probability $1/3$.
- These outcome distributions cannot be achieved by any NE.

- Now we introduce the formal definition of **correlated equilibrium**.
- **Information structure** consists of
  - Ω: a finite set of states (ex. $\Omega = \{x, y, z\}$)
  - $\mathcal{P}_i$: player i's information partition (ex. $\mathcal{P}_1 = \{\{x\}, \{y, z\}\}$)
- Player i's strategy is a mapping $s_i : \Omega \to A_i$ that is **adapted to** $\mathcal{P}_i$ (i.e. $s_i(\omega) = s_i(\omega')$ for any ω and ω' in the same element of $\mathcal{P}_i$.). Let S_i be the set of all such strategies adapted to $\mathcal{P}_i$.

Then a correlated equilibrium is simply a Nash equilibrium with respect to these adapted strategies.

Correlated Equilibrium

$(\Omega, \pi, (\mathcal{P}_i), (S_i))$ is a correlated equilibrium for strategic game $(N, (A_i), (u_i))$ if

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(s_i(\omega), s_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\tau_i(\omega), s_{-i}(\omega))$$

for every $\tau_i \in S_i$ for every $i \in N$.

- Upon some reflection, it is easy to see that the set of states can be A without loss of generality. An interpretation is that player i's information is a recommendation of a particular action.
- Then we can show the following (this could be an alternative definition of correlated equilibrium).

Correlated Equilibrium

A distribution of action profile $\sigma \in \Delta(A)$ can be generated by a correlated equilibrium if

$$\sum_{a_{-i} \in A_{-i}} \sigma(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \sigma(a_i, a_{-i}) u_i(a_i', a_{-i})$$

is satisfied for every $a_i', a_i \in A_i$ and for any $i \in N$.

Comments:

- In this formulation, it is easy to see that
  - Every mixed strategy Nash equilibrium is a correlated equilibrium.
  - A convex combination of correlated equilibrium is a correlated equilibrium.
  - In fact, the set of correlated equilibrium distributions is a set of solutions for some system of linear inequalities, which is much easier to solve than finding a fixed point (i.e. NE).

Game Theory in Biology

- Game theory is used in many disciplines. Here we take a look at the most prominent application in biology.
- The influence is not one-way. A biological view on games enriched the game theory. Here we emphasize
  - ESS as a "refinement" of Nash equilibrium for symmetric games.
  - the **population interpretation** of MSNE: mixed strategy as co-existence.

Hawk-Dove Game

Consider the following situation.

- There are many birds of the same type.
- Each bird is randomly paired with another bird and play the following two-by-two game.
  - Each bird can take one of the two strategies: "Hawk" and "Dove".
  - Given (H, D) or (D, H), the bird that chose H takes all the resource $(= V)$ and the bird that chose D gets nothing $(= 0)$.
  - Given (D, D), each gets a half of the pie $(= V/2)$.
  - Given (H, H), each bird wins or loses the battle with equal probability. The winning bird gets the whole pie and the losing bird suffers cost $C > V$.

Hawk-Dove Game

- Each bird's expected payoff is as follows.

	Hawk	Dove
Hawk	(V-C)/2, (V-C)/2	V, 0
Dove	0, V	V/2, V/2

- If this game is played many, many times with different partners, then these payoffs are (almost) the actual average payoff by the **law of large numbers**.

Hawk-Dove Game

- There are three Nash equilibria in this game: two asymmetric pure NE (H, D), (D, H), and a symmetric mixed strategy NE $(V/C, 1 - V/C)$.
- Birds do not know which role they are playing, so the equilibrium must be a symmetric one.
- One interpretation of this mixed NE: V/C and $1 - V/C$ are the **proportion** of birds that play H and D respectively.
- This symmetric NE satisfies some nice stability property. **It is resistant to any mutation in the population**.
  - If more birds take H, then D becomes the optimal choice.
  - If more birds take D, then H becomes the optimal choice.

- **Symmetric Population Game** (A, u) consists of
  - a finite action set A
  - a payoff function $u : A \times A \to \Re$
- Let $u(a, \alpha) = \sum_{a' \in A} \alpha(a')u(a, a')$ where a is own action and $\alpha \in \Delta(A)$ is the distribution of actions.
- Interpretation
  - Each player matches randomly with a different player in the population and play a symmetric 2×2 game $(\{1, 2\}, (A, A), (u_i))$, where $u_1(x, y) = u(x, y)$ and $u_2(x, y) = u(y, x)$.

ESS

- **Evolutionary Stable Strategy (ESS)** is a strategy that is stable. The basic idea of ESS is the following.
  - It must be a symmetric NE.
  - When some "mutants" appear in the population, ESS must perform better than mutants.

Definition of ESS

- **Formal definition of ESS**

Evolutionary Stable Strategy (Maynard Smith and Price (1973))

$\alpha^* \in \Delta(A)$ is **Evolutionary Stable Strategy (ESS)** if for any $\tau \in \Delta(A)$, one of the following conditions hold

1. $u(\alpha^*, \alpha^*) > u(\tau, \alpha^*)$
2. $u(\alpha^*, \alpha^*) = u(\tau, \alpha^*)$ and $u(\alpha^*, \tau) > u(\tau, \tau)$.

Comment.

- The following formulation is equivalent to this definition:

  $\alpha^* \in \Delta(A)$ is ESS if for any $\tau \in \Delta(A)$, there exists $\bar{\epsilon}$ such that

  $$u(\alpha^*, (1-\epsilon)\alpha^* + \epsilon\tau) > u(\tau, (1-\epsilon)\alpha^* + \epsilon\tau)$$

  for any $\epsilon \in (0, \bar{\epsilon})$.

- In words: if α^* is played by the population, then any small number of "mutants" who play $\tau \in \Delta(A)$ does worse than α^*.

Interpretation of ESS

- Here is one biological interpretation for ESS.
  - Each strategy such as α^* is genetically programmed and inherited from a generation to the next generation (Different individuals with the same mixed strategy α^* may behave differently ex post by choosing different pure strategies).
  - A higher payoff $=$ higher fitness $\Rightarrow$ more offsprings.
  - If α^* is an ESS, any small mutation would die out after a few generations.

ESS as a refinement of NE

- It is clear from the definition that $u(\alpha^*, \alpha^*) \geq u(\tau, \alpha^*)$ for any τ. So *ESS* is a symmetric NE.
- Consider the following game:

	A	B
A	2, 2	0, 0
B	0, 0	1, 1

  - There are three symmetric NE.
  - (A, A) and (B, B) are **strict NE**, hence ESS. But the MSNE $(1/3, 2/3)$ is not.

- It can be shown that there always exists an ESS for any two-by-two symmetric game.
- ESS may not exist with more than two actions.

	A	B	C
A	γ,γ	1,-1	-1,1
B	-1,1	γ,γ	1,-1
C	1,-1	-1,1	γ,γ

$0<\gamma<1$

- $(1/3, 1/3, 1/3)$ is the unique symmetric MSNE, but it can be invaded by any of A,B,C.

Asymmetric Matching Penny Experiments

- Consider the following asymmetric MP game.

	H	T
H	1, -1	-3, 1
T	-1, 1	1, -1

- We know that player 1's equilibrium strategy must be $(0.5, 0.5)$, but player 1 plays H more than T in experiments. Maybe people choose "nonoptimal" strategy with positive probability for whatever reason?

Quantal Response Equilibrium

- Consider a finite strategic game. Let $u_i(a_i, \alpha_{-i})$ be player i's expected payoff when player i chooses a_i and player j plays a mixed strategy α_j.
- Let's assume that player i plays the following mixed strategy α_i instead of the best response strategy against α_{-i}.

$$\alpha_i(a_i) \; = \; \frac{\exp(\lambda u_i(a_i, \alpha_{-i}))}{\sum_{a_i' \in A_i} \exp(\lambda u_i(a_i', \alpha_{-i}))}$$

Quantal Response Equilibrium

- This defines a parametric family of noisy best response strategy $BR_i^\lambda(\alpha_{-i}) \in \Delta(A_i)$ such that
  - Every strategy is played with positive probability.
  - A strategy with a larger expected payoff is played more often.
  - This strategy is totally random when $\lambda = 0$. It converges to a best response strategy as $\lambda \to \infty$.
- This can be interpreted as (1) short-run nonoptimal behavior (incomplete learning) or (2) an optimal strategy with private payoff shocks.

Quantal Response Equilibrium

Quantal Response Equilibrium is defined as a fixed point of these noisy best response mappings.

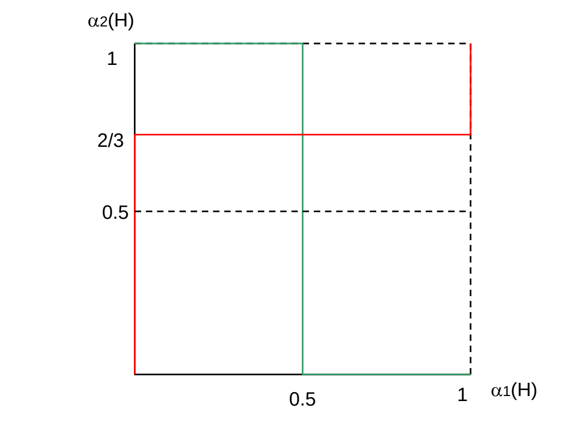
Quantal Response Equilibrium

$\alpha_i^* \in \Delta(A_i), i \in N$ is a **Quantal Response Equilibrium** with parameter $\lambda \in \Re_+$ if it satisfies $\alpha_i^* = BR_i^\lambda(\alpha_{-i}^*)$ for all $i \in N$.

There exists a QRE by Brouwer's fixed point theorem.

Quantal Response Equilibrium

These are the original best responses for Asymmetric Matching Penny.



Quantal Response Equilibrium

These are the noisy best responses given some $\lambda > 0$.