Rationalizability and Iterated Elimination of Dominated Actions

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January 15, 2012

Rationalizability

- What is the implication of "rationality" alone in one-shot game?
- More precisely, assume that a player plays an expected payoff maximizing action given some belief about the other player's action, which must be an optimal action given some belief about the other players' action and so on. If an action can be rationalized by such an infinite sequence of reasoning, we say that the action is **rationalizable**.[1]
- This idea of rationalizability leads to a relaxation of Nash equilibrium.

[1] To be precise, what we assume is a **common knowledge of rationality**.

Rationalizability

Here is one formal definition of rationalizability.

Rationalizability

For a strategic game G, $a_i \in A_i$ is **rationalizable** if there exists $Z_j \subset A_j, j \in N$ such that

- $a_i \in Z_i$
- every $a_j \in Z_j$ is a best response to some belief $\mu_j \in \Delta(Z_{-j})$

It is clear that an action is rationalizable if and only if an action can be rationalized by an infinite sequence of actions and beliefs.

Comments.

- Every Nash equilibrium is rationalizable.
- Some rationalizable action profile is not Nash. For example, every action profile in coordination game is rationalizable. This is because no consistency between a player's belief and the other player's actual behavior is imposed.

Comments.

- Consider a set $Z = \prod_j Z_j \subset A$ that is rationalizable with respect to itself (as in the above definition). Take another such set Z'. Then $Z \bigcup Z'$ is again rationalizable with respect to itself. Hence the maximal such set (the union of all such sets) is the set of all rationalizable actions.
- We assume throughout that a player believes that other players' actions may be correlated. We can instead assume that a player believes that the other players' actions are independent. These assumptions make some difference in the results. (**Note.** This is irrelevant for two-player games).

Traveler's Dilemma

- Consider the following game.
  - Your luggage and your friend's luggage were lost while traveling. Both luggages had exactly the same items in them.
  - The airline company is willing to reimburse any loss up to \$300.
  - The company asks you and your friend separately how much the loss was (must be an integer from \$0 to \$300). The company pays the smaller amount between your report and your friend's report (if the same value is announced, then that value would be paid).
  - In addition, the one who reported a strictly larger value pays \$5 to the one who reported a smaller value.

Traveler's Dilemma

- Is \$100 rationalizable?
- For \$100 to be rationalizable, the player must believe that the other player plays \$101 with some probability (why?). Hence \$101 must be rationalizable. Then \$102 and \$103...
- But \$300 is not rationalizable.
- The only rationalizable action is \$0 (hence $(\$0, \$0)$is the only NE).

Guessing Game

- Here is another tricky game: Guessing Game.
  - There are $n \geq 3$ players
  - Everyone chooses an integer from 0 to 100 simultaneously.
  - The one whose number is closest to the half of the average gets a prize of $100.
- What is the set of rationalizable actions?

Further Applications

- Consider Cournot duopoly model with a linear inverse demand $P(q) = 12 - q_1 - q_2$ and linear cost $3q_i$. Then Cournot-Nash equilibrium is the only rationalizable action profile.
- How about Bertrand duopoly model?

Never-Best Response

- Another way to approach rational behavior is to find nonrationalizable actions and eliminate them.
- We say that an action is a **never-best response** if it is not optimal against any belief about other players' actions. A never-best response action is not rationalizable by definition.

Never-Best Response

$a_i \in A_i$ in G is a **never-best response** if, for any $\mu_i \in \Delta(A_{-i})$, there exists some $a_i' \in A_i$ such that $u_i(a_i', \mu_i) > u_i(a_i, \mu_i)$.

Strictly Dominated Action

- a_i is **strictly dominated** by α_i if α_i is always strictly better than a_i.

Strictly Dominated Action

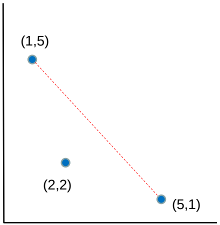
- $a_i \in A_i$ in G is **strictly dominated** by a mixed strategy $\alpha_i \in \Delta(A_i)$ if $u_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.
- $a_i \in A_i$ in G is **strictly dominated** if there is a mixed strategy $\alpha_i \in \Delta(A_i)$ that strictly dominates a_i.

  - Clearly a strictly dominated action is a never-best response, hence not rationalizable.
  - When some action is strictly dominant, then every other action is strictly dominated by it.

Remark.

It is possible that an action is not strictly dominated by any pure strategy, but strictly dominated by a mixed strategy.

	L	R
U	5	1
M	1	5
D	2	2



D is not strictly dominated by any pure strategy, but strictly dominated by $1/2U + 1/2M$.

Strictly Dominated Actions and Never-Best Response

A strictly dominated action is a never-best response. It turns out that the other direction is true as well for finite strategic games.

Theorem

For a finite strategic game G, $a_i \in A_i$ is never-best response if and only if it is strictly dominated.

Note: For this result, we need to allow correlation in beliefs.

Proof.

- Suppose that $a_i^* \in A_i$ in G is never-best response.
- Define an auxiliary two-player zero-sum game where
  - Player 1's action set is $A_i / \{a_i^*\}$, player 2's action set is A_{-i}.
  - Player 1's payoff is $v_1(a) = u_i(a_i, a_{-i}) - u_i(a_i^*, a_{-i})$ (and $v_2 = -v_1$).
- Since a_i^* is never-best response, $\min_{\alpha_2} \max_{\alpha_1} v_1(\alpha_1, \alpha_2) > 0$.
- Since there exists a Nash equilibrium for a finite strategic game, $\max_{\alpha_1}, \min_{\alpha_2} v_1(\alpha_1, \alpha_2) > 0$ by the minimax theorem.
- This means that there exists $\alpha_i^* \in \Delta(A_i / \{a_i^*\})$ such that $u_i(\alpha_i^*, a_{-i}) > u_i(a_i^*, a_{-i})$ for any $a_{-i} \in A_{-i}$.

□

Iterated Elimination of Strictly Dominated Actions

- Let's eliminate strictly dominated actions from the game because no rational player plays such actions.
- Note that, after strictly dominated actions are eliminated, even more actions can be strictly dominated within the remaining game. Such actions would not be played if a player is rational and believes that the other players are rational. So eliminate them.
- Then further actions can be eliminated if a player is rational, believes that the other players are rational, and believes that the other players believe that the other players are rational....
- For a finite game, this process of successive eliminations stop at some point.

Iterated Elimination of Strictly Dominated Actions

Iterated Elimination of Dominated Actions

$X = \prod_{j \in N} X_j \subset A$ in G **survives iterated elimination of strictly dominated actions** if there exists a finite sequence of sets $\prod_{j \in N} X_j^t, t = 0, 1, ..., T \subset A$ that satisfies the following conditions for every $j \in N$.

- $X_j^0 = A_j$ and $X_j^T = X_j$
- every $a_j \in X_j^t / X_j^{t+1}$ is strictly dominated in a finite strategic form game $(N, (X_i^t), (u_i))$ for $t = 0, 1, ..., T - 1$.
- No action in X_j^T is strictly dominated in $(N, (X_i^T), (u_i))$

As you might guess, this process leads to the set of all rationalizable actions for finite strategic games

Theorem

For a finite strategic game G, $X = \prod_{j \in N} X_j \subset A$ survives iterated elimination of strictly dominated actions if and only if X_j is the set of all rationalizable actions for every $j \in N$

Note: Observe that this result implies that the way in which strictly dominated actions are deleted does not matter.

Proof.

- If $a_i \in A_i$ is rationalizable, then there exist $Z_j \subset A_j, j \in N$ that sustains a_i. Clearly no action profile in $Z = \prod_j Z_j$ is eliminated in the first step of IESDA. Then no action profile in Z is eliminated in the second step and so on. Hence $Z \subset X$ and $a_i \in Z_i \subset X_i$.
- Suppose that X survives IESDA. Since no action in X_j is strictly dominated in $(N, (X_i), (u_i))$ and an action is a never-best response if and only if it is strictly dominated, every action in X_j is rationalizable.

□

Weakly Dominated Action

- There is a weaker notion of domination. We say that a_i is **weakly dominated** by α_i if α_i is always at least as good as a_i and sometimes strictly better than a_i.

Weakly Dominated Action

- $a_i \in A_i$ in G is **weakly dominated** by a mixed strategy $\alpha_i \in \Delta(A_i)$ if $u_i(\alpha_i, a_{-i}) \geq u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$ with strict inequality for some $a_{-i} \in A_{-i}$
- $a_i \in A_i$ in G is **weakly dominated** if there is a mixed strategy $\alpha_i \in \Delta(A_i)$ that weakly dominates a_i.

- For example, bidding any value different from the true value in second price auction is weakly dominated by bidding the true value.
- More generally, if there is a weak dominant action, then every other action is weakly dominated by it.

Iterated Elimination of Weakly Dominated Actions

- We can define iterated elimination of weakly dominated strategies.
- However the outcome of a successive eliminations may depend on the way in which weakly dominated actions are eliminated.

- Consider the following example.

	C	D
A	1,0	0,1
B	0,0	0,1

  - (A, D) would survive if B and C are eliminated simultaneously or B is eliminated first and C is eliminated second.[2]
  - (A, D) and (B, D) would survive if only C is deleted first.

[2] A finite strategic game is **dominance solvable** if the unique outcome survives when every weakly dominated action is eliminated in each step.