Why Repeated Games?

- Prisoner’s dilemma game:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1,1</td>
<td>−1,2</td>
</tr>
<tr>
<td>D</td>
<td>2,−1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

- $D$ is the strictly dominant action, hence $(D, D)$ is the unique Nash equilibrium.

- However, people don’t always play $(D, D)$. Why? One reason would be that people (expect to) play this game repeatedly. Then what matters is the total payoff, not just current payoff.

- **Repeated game** is a model about such long-term relationships.
A list of questions we are interested in:

- When can people cooperate in a long-term relationship?
- How do people cooperate?
- What is the most efficient outcome that arise as an equilibrium?
- What is the set of all outcomes that can be supported in equilibrium?
- If there are many equilibria, which equilibrium would be selected?
Repeated Games with Perfect Monitoring

Formal Model

Stage Game

- In repeated game, players play the same strategic game $G$ repeatedly, which is called \textit{stage game}.

- $G = (N, (A_i), (u_i))$ satisfies usual assumptions.
  - \textbf{Player:} $N = \{1, \ldots, n\}$
  - \textbf{Action:} $a_i \in A_i$ (finite or compact&convex in $\mathbb{R}^K$).
  - \textbf{Payoff:} $u_i : A \rightarrow \mathbb{R}$ (continuous).

- The set of \textbf{feasible payoffs} is $\mathcal{F} := \text{co} \{g(a) : a \in A\}$. 
Now we define a repeated game based on $G$.

**History**

- A period $t$ **history** $h_t = (a_1, ..., a_{t-1}) \in H_t = A^{t-1}$ is a sequence of the past action profiles at the beginning of period $t$.
- The initial history is $H_1 = \{\emptyset\}$ by convention.
- $H = \bigcup_{t=1}^{\infty} H_t$ is the set of all such histories.
Strategy and Payoff

- Player $i$’s (pure) strategy $s_i \in S_i$ is a mapping from $H$ to $A_i$.
  - Ex. Tit-for-Tat: “First play $C$, then play what your opponent played in the last period”.

- A strategy profile $s \in S$ generates a sequence of action profiles $(a_1, a_2, ...) \in A^\infty$. Player $i$’s discounted average payoff given $s$ is

$$V_i(s) := (1 - \delta) \sum_{t=1}^\infty \delta^{t-1} g_i(a_t)$$

where $\delta \in [0, 1)$ is a discount factor.
Repeated Game

- This extensive game with simultaneous moves is called **repeated game** (sometimes called **supergame**).
- The repeated game derived from $G$ and with discount factor $\delta$ is denoted by $G^\infty(\delta)$.
- We use subgame perfect equilibrium.
- The set of all pure strategy SPE payoff profiles for $G^\infty(\delta)$ is denoted by $\mathcal{E}[\delta]$. 
Public Randomization Device

- We may allow players to use a publicly observable random variable (say, throwing a die) in the beginning of each period.

- Formally we can incorporate a sequence of outcomes of such **public randomization device** as a part of history in an obvious way. To keep notations simple, we don’t introduce additional notations for public randomization.
Minmax Payoff

- Let $v_i$ be player $i$’s **pure-action minmax payoff** defined as follows.

**Pure-Action Minmax Payoff**

$$v_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} g_i(a).$$

- Intuitively $v_i$ is the payoff that player $i$ can secure when player $i$ knows the other players’ actions.

- Ex. $v_i = 0$ for $i = 1, 2$ in the previous PD.
Minmax Payoff

Minmax payoff serves as a lower bound on equilibrium payoffs in repeated games.

Lemma

Player $i$’s payoff in any NE for $G^\infty(\delta)$ is at least as large as $v_i$.

Proof

Since player $i$ knows the other players’ strategies, player $i$ can deviate and play a “myopic best response” in every period. Then player $i$’s stage game payoff would be at least as large as $v_i$ in every period. Hence player $i$’s discounted average payoff in equilibrium must be at least as large as $v_i$. 
Trigger Strategy

Consider the following PD \((g, \ell > 0)\).

<table>
<thead>
<tr>
<th></th>
<th>(C)</th>
<th>(D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C)</td>
<td>1, 1</td>
<td>(-\ell, 1 + g)</td>
</tr>
<tr>
<td>(D)</td>
<td>(1 + g, -\ell)</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

When can \((C, C)\) be played in every period in equilibrium?
Such an equilibrium exists if and only if the players are enough patient.

**Theorem**

There exists a subgame perfect equilibrium in which \((C, C)\) is played in every period if and only if 

\[
\delta \geq \frac{g}{1+g}.
\]
Proof.

Consider the following **trigger strategy**:

- Play \( C \) in the first period and after any cooperative history \((C, C), \ldots, (C, C)\).
- Otherwise play \( D \).

This is a SPE if the following one-shot deviation constraint is satisfied

\[
1 \geq (1 - \delta)(1 + g)
\]

, which is equivalent to \( \delta \geq \frac{g}{1+g} \).

By our previous observation, each player’s continuation payoff cannot be lower than 0 after a deviation to \( D \). Hence \( \delta \geq \frac{g}{1+g} \) is also necessary for supporting \((C, C)\) in every period.
Stick and Carrot

Consider the following modified PD.

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1, 1</td>
<td>−1, 2</td>
<td>−4, −4</td>
</tr>
<tr>
<td>D</td>
<td>2, −1</td>
<td>0, 0</td>
<td>−4, −4</td>
</tr>
<tr>
<td>E</td>
<td>−4, −4</td>
<td>−4, −4</td>
<td>−5, −5</td>
</tr>
</tbody>
</table>

The standard trigger strategy supports \((C, C)\) in every period if and only if \(\delta \geq 1/2\).
The following strategy ("Stick and Carrot" strategy) support cooperation for even lower $\delta$.

- **Cooperative Phase**: Play C. Stay in Cooperative Phase if there is no deviation. Otherwise move to Punishment Phase.
- **Punishment Phase**: Play E. Move back to Cooperative Phase if there is no deviation. Otherwise stay at Punishment Phase.

There are two one-shot deviation constraints.

- $1 \geq (1 - \delta)2 + \delta [(1 - \delta)(-5) + \delta]$
- $(1 - \delta)(-5) + \delta \geq (1 - \delta)(-4) + \delta [(1 - \delta)(-5) + \delta]$.

They are satisfied if and only if $\delta \geq \frac{1}{6}$. 
Optimal Collusion

- We study this type of equilibrium in the context of dynamic Cournot duopoly model.
- Consider a repeated game where the stage game is given by the following Cournot duopoly game.
  - \( A_i = \mathbb{R}_+ \)
  - Inverse demand: \( p(q) = \max \{ A - (q_1 + q_2) , 0 \} \)
  - \( \pi_i(q) = p(q)q_i - cq_i \)
- Discount factor \( \delta \in (0, 1) \).
Cournot-Nash equilibrium and Monopoly

- In Cournot-Nash equilibrium, each firm produces $q^C = \frac{A-c}{3}$ and gains $\pi^C = \frac{(A-c)^2}{9}$.

- The total output that would maximize the joint profit is $\frac{A-c}{2}$. Let $q^M = \frac{A-c}{4}$ be the monopoly production level per firm.

- Let $\pi(q) = \pi_i(q, q)$ be each firm’s profit when both firms produce $q$.

- Let $\pi^d(q) = \max_{q_i \in \mathbb{R}^+} \pi_i(q_i, q)$ be the maximum profit each firm can gain by deviating from $q$ when the other firm produces $q$. 
- We look for a SPE to maximize the joint profit.

- The firms like to collude to produce less than the Cournot-Nash equilibrium to keep the price high.

- We focus on **strongly symmetric SPE**. When the stage game is symmetric, an SPE is strongly symmetric if every player plays the same action after any history.
Structure of Optimal Equilibrium

- We show that the best SSSPE and the worst SSSPE has a very simple structure.

- Consider the following strategy:
  - **Phase 1:** Play $q^*$. Stay in Phase 1 if there is no deviation. Otherwise move to Phase 2.
  - **Phase 2:** Play $q_*$. Move to Phase 1 if there is no deviation. Otherwise stay in Phase 2.

- The best SSSPE is achieved by a strategy that starts in Phase 1 (denoted by $s(q^*\infty)$) and the worst SSSPE is achieved by a strategy that starts in Phase 2 (denoted by $s(q_*, q^*\infty)$) for some $q^*, q_*$. 
Let \( \overline{V} \) and \( \underline{V} \) be the best SSSPE payoff and the worst SSSPE payoff respectively (Note: this needs to be proved).

First note that the equilibrium action must be constant for \( \overline{V} \).

- Let \( q^* \) be the infimum of the set of all actions above \( q^M \) that can be supported by some SSSPE. Let \( q^k, k = 1, 2, \ldots \) be a sequence within this set that converges to \( q^* \).

- One-shot deviation constraint implies

\[
(1 - \delta)\pi(q^k) + \delta \overline{V} \geq (1 - \delta)\pi^d(q^k) + \delta \underline{V}
\]

Taking the limit and using \( \pi(q^*) \geq \overline{V} \), we have

\[
\pi(q^*) \geq (1 - \delta)\pi^d(q^*) + \delta \overline{V},
\]

which means that it is possible to support \( q^* \) in every period.
Secondly, we can show that the worst SSSPE can be achieved by 
\( s(q_*, q^{*\infty}) \) ("stick and carrot") for some \( q_* \geq q^C \).

- Take any path \( Q' = (q'_1, q'_2, ..., ) \) to archive the worst SSSPE.
- Since \( \pi(q'_t) \leq \pi(q^*) \) for all \( t \) and \( \pi(q) \) is not bounded below, we can find \( q_* \geq q'_1 \) such that \( Q_* = (q_*, q^*, ..., ) \) generates the same discounted average payoff as \( Q' \).
- Then \( c(q_*, q^{*\infty}) \) is a SPE that archives the worst SSSPE payoff because

\[
V(Q') \geq (1 - \delta) \pi^d(q'_1) + \delta V(Q') \\
\downarrow
\]

\[
V(Q_*) \geq (1 - \delta) \pi^d(q_*) + \delta V(Q_*) .
\]
To summarize, we have the following theorem.

**Theorem (Abreu 1986)**

There exists $q^* \in [q^M, q^C]$ and $q^* \geq q^C$ such that $s(q^*\infty)$ achieves the best SSSPE and $s(q^*, q^*\infty)$ achieves the worst SSSPE.

- **Note:** This can be generalized to the case with nonlinear demand function and many firms.
Q: How many SPE? Which payoff can be supported by SPE?

A: Almost all “reasonable” payoffs if $\delta$ is large.
What does “almost all” mean?

We know that player $i$’s (pure strategy) SPE payoff is never strictly below $v_i$. We show that every feasible $v$ strictly above $v$ can be supported by SPE. This is so called **Folk Theorem** in the theory of repeated games.
Definitions

- $v \in \mathcal{F}$ is **strictly individually rational** if $v_i$ is strictly larger than $v_{-i}$ for all $i \in I$. Let $\mathcal{F}^* \subset \mathcal{F}$ be the set of feasible and strictly individually rational payoff profiles.

- Normalize $v_i$ to 0 for every $i$ without loss of generality.

- Let $\bar{g} := \max_i \max_{a,a' \in A} |g_i(a) - g_i(a')|$. 
For the repeated PD, the yellow area is the set of strictly individually rational and feasible payoffs.
There are many folk theorems. This is one of the most famous ones.

**Theorem (Fudenberg and Maskin 1986)**

Suppose that $F^*$ is full-dimensional (has an interior point in $\mathbb{R}^N$). For any $\nu^* \in F^*$, there exists a strategy profile $s^* \in S$ and $\delta \in (0, 1)$ such that $s^*$ is a SPE and achieves $\nu$ for any $\delta \in (\underline{\delta}, 1)$. 
Folk Theorem

Idea of Proof

- Players play $a^* \in A$ such that $v^* = g(a^*)$ every period in equilibrium.

- Any player who deviates unilaterally is punished by being minmaxed for a finite number of periods.

- The only complication is that minmaxing someone may be very costly, even worse than being minmaxed.

- In order to keep incentive of the players to punish the deviator, every player other than the deviator is “rewarded” after minmaxing the deviator.
Proof

- **Step 1.** Pick $v^j \in F^*$ for each $j \in N$ so that $v^*_j > v^j_j$ for all $j$ and $v^i_j > v^j_i$ for all $i \neq j$. We assume that there exists $a^* \in A, j = 1, ..., N$ such that $v^* = g(a^*)$ and $v^j = g(a^j)$ for simplicity (use a public randomization device otherwise).

- **Step 2.** Pick an integer $T$ to satisfy $\bar{g} < T \min_{i,j} v^i_j$. 
Proof

- **Step 3.** Define the following strategy.

  - **Phase I.** Play $a^* \in A$. Stay in Phase I if there is no unilateral deviation from $a^*$. Go to Phase II(i) if player $i$ unilaterally deviates from $a^*$.
  
  - **Phase II(i).** Play $a(i) \in A$ (the action minmaxing player $i$) for $T$ periods and go to Phase III(i) if there is no unilateral deviation. Go to Phase II(j) if player $j$ unilaterally deviates from $a(i)$.
  
  - **Phase III(i).** Play $a^i \in A$. Stay in Phase III(i) if there is no unilateral deviation from $a^i$. Go to Phase II(j) if player $j$ unilaterally deviates from $a^i$.  

**Proof**

- **Step 4.** Check all one shot deviation constraints.

  - **Phase I**
    \[(1 - \delta) \bar{g} \leq (1 - \delta) (\delta^+, ..., +\delta^T) v_j^* + \delta^{T+1} \left( v_j^* - v_j^i \right) \text{ for all } j \in N\]

  - **Phase II(i)(the first period):** IC is clearly satisfied for \(i\). For \(j \neq i\),
    \[(1 - \delta^{T+1}) \bar{g} \leq \delta^{T+1} \left( v_j^i - v_j^i \right)\]

  - **Phase III(i)**
    \[(1 - \delta) \bar{g} \leq (1 - \delta) (\delta^+, ..., +\delta^T) v_j^i + \delta^{T+1} \left( v_j^i - v_j^i \right) \text{ for all } j \in N\]

These constrains are satisfied for some large enough \(\delta\) and any \(\delta \in (\delta, 1)\).
References