Supermodular Games

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- We study a class of strategic games called **supermodular game**, which is useful in many applications and has a variety of nice theoretical properties.
- A game is supermodular if the marginal value of one player's action is increasing in the other players' actions.
- We are also interested in supermodularity between actions and exogenous parameters.

Lattice

We need to introduce a few mathematical concepts first. Let's start with **lattice**.

- For each $x, y \in \Re^k$, define $x \wedge y, x \vee y \in \Re^k$ as follows.
  - $(x \wedge y)_i := \min\{x_i, y_i\}$ (**"meet"**)
  - $(x \vee y)_i := \max\{x_i, y_i\}$ (**"join"**)
- A set is lattice if it includes the join and the meet of any pair in the set.

Lattice

$X \subset \Re^k$ is a **lattice** if $x \wedge y, x \vee y \in X$ for every $x, y \in X$.

Lattice

Remark.

- We are restricting our attention to a special class of lattices (sublattices of $\Re^k$). The theory can be much more general.
- $X \subset \Re^k$ and $Y \subset \Re^m$ is a lattice if and only if $X \times Y \subset \Re^{k+m}$ is a lattice.

Lattice

Examples

- Interval $[0, 1]$.
- A set of $x \in \Re^k$ such that $x_i \geq x_{i+1}$ for $i = 1, ..., k - 1$.

Greatest and Least Element of Lattice

- Notion of greatest and least:
  - $x^* \in X$ is a **greatest** element in X if $x^* \geq x$ for any $x \in X$.
  - $x_* \in X$ is a **least** element in X if $x^* \leq x$ for any $x \in X$.
- Nonempty compact lattice A has the greatest element and the least element (why?). We denote them by $\overline{A}$ and $\underline{A}$ respectively.

Increasing Differences

A function $f(x, y)$ satisfies **increasing differences** if the marginal gain from increasing x is larger when y is larger.

Increasing Differences

Let $X, Y \subset \Re^k$ be lattices. A function $f : X \times Y \to \Re$ satisfies **increasing differences** in (x, y) if

$$f(x', y') - f(x, y') \geq f(x', y) - f(x, y)$$

for any $x' \geq x$ and $y' \geq y$.

- f satisfies **strictly increasing differences** in (x, y) if the inequality is strict for any $x' > x$ and $y' > y$.
- This formalizes the notion of **complementarity**.

Supermodularity

A closely related concept is **supermodularity**.

Supermodularity

Let $X \subset \Re^k$ be a lattice. A function $f : X \to \Re$ is **supermodular** if

$$f(x \vee x') + f(x \wedge x') \geq f(x) + f(x')$$

for every $x, x' \in X$.

Supermodularity

- When is f supermodular?
  - It is easy to see that a function f on lattice $X \times Y$ satisfies increasing differences in (x, y) if and only if f satisfies increasing differences for any pair of (x_i, y_j) given any x_{-i}, y_{-j}.
  - Similarly a function f on lattice X is supermodular if and only if f satisfies increasing differences with respect to any pair of variables (x_i, x_j) given any $x_{-i,j}$ (show it).
  - When f is twice continuously differentiable on $X = \Re^k$, f is supermodular if and only if $\frac{\partial^2 f}{\partial x_i, \partial x_j} \geq 0$ for any x_i, x_j.

Note. It is sometimes useful to work with $\log f$ instead of f (**log supermodularity**).

Supermodularity

Example

- In the simplest Bertrand game with n firms, each firm's profit is $\pi_i(p) = (p_i - c_i)(a - p_i + b\sum_{j \neq i} p_j)$. Hence $\pi_i(p)$ is supermodular in p.
- For Cournot game with n firms, $\pi_i(q)$ is not supermodular. However, when $n = 2$, it is supermodular with respect to firm 1's production and the negative of firm 2's production: each firm's profit function $\pi_i(q_1, -q_2)$ satisfies increasing differences in $(q_1, -q_2)$.

Monotone Comparative Statics

- We prove an important preliminary result: **Monotone Comparative Statics**.
- When there is a **complementarity** between choice variable x and parameter t, we often show that the optimal solution increases in t by using the implicit function theorem as follows.
  - FOC: $f_x(x,t) = 0$. Then $x'(t) = -\frac{f_{x,t}(x,t)}{f_{x,x}(x,t)}$.
  - SOC: $f_{x,x} < 0$. Then $x'(t) \geq 0$ if and only if $f_{x,t} \geq 0$.
- We prove the same thing without any differentiability.

Monotone Comparative Statics

Monotone Comparative Statics

Let $X \subset \Re^k$ be a compact lattice and $T \subset \Re^m$ be a lattice. Suppose that $f : X \times T \to \Re$ is supermodular and continuous on X for each $t \in T$, and satisfies increasing differences in (x, t). Let

$$x^*(t) = \operatorname*{argmax}_{x \in X} f(x, t)$$

be the set of the optimal solutions given t. Then

- $x^*(t) \subset X$ is a nonempty compact lattice
- $x^*(t)$ is increasing in **strong set order**, i.e. $x \in x^*(t) \& x' \in x^*(t') \Rightarrow x \vee x' \in x^*(t')$ and $x \wedge x' \in x^*(t)$ when $t' > t$.
- In particular, $\overline{x}^*(t') \geq \overline{x}^*(t)$ and $\underline{x}^*(t') \geq \underline{x}^*(t)$ when $t' > t$.

Proof.

- $x^*(t)$ is nonempty and compact for each t by Weierstrass theorem.
- For any $x, x' \in x^*(t)$, $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$. Since X is a lattice, it must be the case that $f(x \wedge x') = f(x \vee x') = f(x) = f(x')$. Hence $x^*(t)$ is a lattice
- For any $x \in x^*(t), x' \in x^*(t')$, we have $f(x,t) - f(x \wedge x', t) \geq 0$. By ID and SM, $f(x \vee x', t') - f(x', t') \geq 0$. This means $x \vee x' \in x^*(t')$. Thus $x \leq x \vee x' \leq \overline{x}^*(t')$ for any $x \in x^*(t)$, hence $\overline{x}^*(t') \geq \overline{x}^*(t)$.
- By the same proof, $\underline{x}^*(t') \geq \underline{x}^*(t)$.

□

Monotone Comparative Statics

- If f satisfies strictly increasing differences, then $x^*(t)$ is increasing in the sense that $x' \geq x$ for any $x' \in x^*(t')$ and $x \in x^*(t)$ when $t' > t$.
    - This means that any selection from $x^*(t)$ such as $\overline{x}^*(t)$ is nondecreasing.
- The above proof works even when the choice set $X(t)$ is increasing in strong set order.

Supermodular Game

What is the implication of all these to strategic games?

Supermodular Game

- A strategic game $G = (N, (A_i), (u_i))$ is **supermodular** if
  - Each $A_i \subset \Re^k$ is a compact lattice
  - u_i is continous and supermodular on A_i for every $a_{-i} \in A_{-i}$, and satisfies increasing differences in (A_i, A_{-i}) for each $i \in N$.

Theorem

There exists a pure strategy Nash equilibrium for any supermodular game.

- **Remark.**
  - Note that no concavity assumption is imposed, no continuity is assumed with respect to a_{-i} and no mixed strategy is needed.
  - For a finite strategic game, u_i is automatically continuous and A_i is automatically compact. So we just need A_i to be a lattice and u_i to satisfy supermodularity/increasing differences.

Proof.

- For this proof, we assume that u is continuous.
- For any $a_{-i} \in A_{-i}$, $B_i(a_{-i})$ is a nonempty compact lattice by MCS. Hence $B(a) = (B_1(a_{-1}), ..., B_n(a_{-n}))$ is nonempty compact lattice.
- Let $a^*(0) \in A$ be the greatest action profile in A. Let $a^*(t), t = 0, 1, 2, ...$ be a sequence such that $a^*(t+1) = \overline{B}(a^*(t))$. Then $a^*(t)$ is a decreasing sequence by MCS. Since a decreasing sequence in a compact set in $\Re^k$ converges within the set. There exists $a^* \in A$ such that $a^* = \lim_{t \to \infty} a^*(t)$.
- We show that a^* is a NE. For any i and $a_i \in A_i$, $u_i(a_i^*(t+1), a_{-i}^*(t)) \geq u_i(a_i, a_{-i}^*(t))$. Then $u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*)$ by continuity.

Comments

- We can find the least NE if we start from the least action profile.
- Consider a parametrized strategic game, where $u_i(a, t)$ depends on some exogenous parameter t. If each u_i satisfies increasing difference in (a_i, t) (in addition to all the other assumptions), then it follows from the above proof that the greatest NE and the least NE is increasing in t.
- To drop continuity, use **Tarski's fixed point theorem**.

Tarski's Fixed Point Theorem

Let $X \in \Re^k$ be a compact lattice and $f : X \to X$ be a nondecreasing function. Then there exists $x^* \in X$ such that $f(x^*) = x^*$.

Example

- Consider a Bertrand competition model with n firms, where firm i's demand function $q_i(p, \theta)$ depends on every firm's price and market condition θ. Firm i's cost function is $c_i q_i(p)$.
  - The logarithm of firm i's profit function satisfies increasing differences in (p_i, c_i) for $p_i \geq c_i$.
  - Also suppose that firm i's profit function satisfies increasing differences in $(p_i, (p_{-i}, \theta))$ (when is this the case?).
- Assuming that there is a natural upper bound on p_i, the following results follow without assuming any explicit functional form for q_i:
  - There exists the highest equilibrium price vector and the lowest equilibrium price vector.
  - The highest equilibrium price vector and the lowest equilibrium price vector increase when c_i increases for any i or when θ increases.

Supermodularity and Rationalizability

- Let $A_i(t) = \{a_i \in A_i : a_i \leq a^*(t)\}$.
- Every $a_i \not\leq a_i^*(t)$ is strictly dominated by $a_i \wedge a^*(t+1)$ in strategic game $(N, (A_i(t)), (u_i))$, because for any $a_{-i} \in A_{-i}(t)$,

$$\begin{aligned} 0 &< u_i(a^*(t+1), a^*_{-i}(t)) - u_i(a_i \vee a^*(t+1), a^*_{-i}(t)) \\ &\leq u_i(a^*(t+1), a_{-i}) - u_i(a_i \vee a^*(t+1), a_{-i}) \text{ (by ID)} \\ &\leq u_i(a_i \wedge a^*(t+1), a_{-i}) - u_i(a_i, a_{-i}) \text{ (by SM)} \end{aligned}$$

- This means that no action above a^* survives IESDA, hence no action above a^* is rationalizable.
- The largest rationalizable action profile and the largest NE (hence the largest MSNE) coincide for supermodular games with continuous u.

References

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