Optimal Dynamic Auctions for Durable Goods: Posted Prices and Fire-sales *

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Abstract

We consider a seller who wishes to sell a set of durable goods over a finite period of time. Potential buyers enter IID over time and are patient. When there is one good, profit is maximized by awarding the good to the agent with the highest valuation exceeding a cutoff. These cutoffs are determined by a one-period-look-ahead rule and are constant in all periods except the last. In discrete time, this allocation can be implemented with sequential second-price auctions; in continuous time it can be implemented with posted prices and a fire-sale at the end date. When the seller has multiple units, the cutoffs are also determined by a one-period-look-ahead rule. They are deterministic, decrease over time and decrease in the number of units available. We show how the optimal auction can be implemented by posted prices in the continuous-time limit. Unlike the cutoffs, the prices depend not only on the number of remaining units and the remaining time, but also on the history of past sales. These results are extended to allow for a rising/falling number of entrants and partially patient agents.

1 Introduction

We consider a seller who wishes to sell K goods over a finite period of time. Potential buyers arrive to the market over time and, once they arrive, prefer to obtain the good sooner rather than later. At each point in time, the seller thus chooses whether to sell today or incur a costly delay and wait for new buyers - a tradeoff shared by many real-life problems. For example, when an airline sells tickets for a flight, a retailer sells its seasonal inventory, or an owner sells her house, they face the tradeoff between lowering the price today or waiting for new entrants.

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There is a substantial literature on revenue management that analyses how such sellers should price over time (see the book by Talluri and Ryzin (2004)). It is estimated that these techniques have led to a substantial increase in profits for airlines (Davis (1994)), retailers (Friend and Walker (2001)) and car manufacturers (Coy (2000)). Moreover, with the rise of the internet, and the resulting flattening of the supply chain, sophisticated pricing techniques are becoming increasingly feasible. The majority of these models, however, assume that agents are impatient, exiting the market if they do not immediately buy. In this paper, we derive the optimal mechanism when buyers are patient and forward-looking. This seems natural when considering markets such as airline tickets and retailing, where buyers can easily time their purchases. It is also becoming more important as buyers use price prediction tools to aid such inter-temporal arbitrage (e.g. bing.com, where search for flights results contain predictions about changes in prices).

The practical problems that we model in this paper have two key properties. First, the pricing problem is non-stationary. In our examples, the good may expire at a fixed date (e.g. a plane ticket), become much less valuable (e.g. seasonal clothing), or the number of interested buyers may decline over time (e.g. a house). Second, once a buyer arrives, he prefers to purchase the good sooner rather than later. In case of clothing or a house, he has more days to enjoy the good; in case of a flight, he has more time to plan complementary activities.

In our model we capture the non-stationary nature of the problem by assuming that the good expires after time T (or equivalently, that there is no more entry after T). Buyers arrive stochastically over time; motivated by online markets, we assume the number of entrants is not observed by other buyers. Upon entering, each buyer has unit demand, draws a private value from a common distribution, and calculates whether to buy today or wait at the risk of a stock-out. Finally, we model the impatience in the standard way, assuming symmetric proportional discounting.

We start our analysis with the one unit case, where the number of entrants each period is IID. We show the profit-maximizing mechanism allocates the good to the agent with the highest valuation exceeding a cutoff. The cutoff is determined by a one-period-look-ahead policy, whereby the seller is indifferent between serving the cutoff type today and waiting one more period, potentially selling to a new entrant. The optimal cutoffs are independent of the valuations of old agents and take a very stark form: the cutoffs are constant in periods $t < T$, and drop sharply in period T to the static monopoly price. As a result, a buyer who arrives at time t either buys immediately or waits for the "fire-sale" in period T.

The optimal mechanism can be implemented using a sequence of second-price auctions with a deterministic reserve price that declines over time, at a rate faster than the discount rate. Intuitively, today's reserve is set so that the cutoff type is indifferent between buying the good today at the reserve price and waiting for the auction tomorrow. Since he forgoes one period's enjoyment of the good, the price has to drop at least at the speed of the discount rate, but since he is also risking the arrival of new competition, the price has to drop even more.

In the continuous time limit of the game, assuming new buyers enter according to a Poisson process, the optimal allocation can be implemented with deterministic posted prices and a fire-sale at date T. Prices fall faster as the deadline approaches and fall faster when the buyer faces more competition from potential entrants. This implementation, via posted prices and a fire-sale, works even if the seller does not observe when buyers arrive to the market (i.e. even if buyers can conceal their presence).

In Section 4 we consider the effect of inter-temporal changes in the distribution of the number of entrants. For example, a seller of a house may face a stock of pre-existing buyers as well as a flow of entrants. We show that if the number of entrants falls over time, in the sense of first-order stochastic dominance, then the seller's option value of waiting also falls. As a result, the cutoffs decrease over time, so that an agent who enters in period t may end buying in any of the subsequent periods. Conversely, if the number of entrants increases over time, then the cutoffs rise until period $T-1$, so that an agent who enters in period t either buys immediately or waits until the final period. In this case, the one-period-look-ahead policy fails, with the principal being indifferent between serving the marginal agent in period t and waiting until the end of the game. In either case, we show how to implement the optimal allocations with deterministic posted prices and a fire-sale at time T.

We then move to the analysis of the K goods case in Section 5, with the one-unit case being the first step of our inductive solution. When there are k units left, one can think of the units being awarded sequentially within a period, with the k^{th} unit being awarded to the remaining agent with the highest valuation, subject to his valuation exceeding a cutoff x_t^k. In the optimal allocation these cutoffs are deterministic, so are independent of the number of agents who have entered the market and their valuations. The optimal cutoffs fall over time, decrease in k and are therefore determined by a one-period-look-ahead policy. Intuitively, if the seller delays awarding the k^{th} unit by one period then she can allocate it to the highest value entrant rather than the current leader. As the game proceeds, the current leader is increasingly likely to be awarded the good eventually, decreasing the option value of delay and causing the cutoff to fall over time. Similarly, when there are more goods remaining, the current leader is more likely to be awarded the good eventually, again decreasing the option value of delay and causing the cutoff to fall in k.

In the continuous time limit of the game, the optimal mechanism can be implemented by a sequence of posted prices p_t^k and a fire-sale for the remaining units at time T. The resulting price-path thus exhibits a slow decline, with occasional upward jumps when sales occur. Intuitively, when there is no sale, the price will fall because the cutoffs decline and the deadline approaches. The rate of decline is then determined by rate at which the cutoff drops, the interest rate and the probability the cutoff type will lose the good if he delays a little, either to an existing buyer or a new entrant. When a sale does occur, the cutoff to allocate one of the remaining units jumps upwards, as does the price.

In contrast to the impatient buyer model, we assume a buyer's valuation at time t is independent of when he enters the market. In Section 6 we bridge this gap and consider two models of partially patient agents. In the first, agent's values decline deterministically after they enter the market. When there is one unit, we show that cutoffs decline over time but that, because of the declining valuations, an agent who enters at time t either buys immediately or waits until the final period. These cutoffs can be implemented through posted prices and a biased auction in period T. The second model allows buyers to exit randomly. Unlike our other results, the optimal cutoffs depend on the valuations of all entering agents. As a result, they cannot be implemented through simple posted-prices or auctions.

1.1 Literature

There are a number of papers that examine how to sell to patient buyers entering over time. Our results are related to a classic result on "asset selling with recall" (e.g. Bertsekas (1995, p. 177)). Bertsekas derives the welfare-maximising policy with one good, when one agent enters each period and his value is publicly known. We derive the profit-maximising policy for many goods, when several agents enter each period and their values are privately known. We also show how to implement the optimal mechanism in both continuous and discrete time.

Wang (1993) supposes the seller has one object and that buyers arrive at according to a Poisson distribution and experience a fixed per–period delay cost. Wang shows that with an infinite horizon, a profit-maximising mechanism is a constant posted-price. Gallien (2006) characterises the optimal sequence of prices when agents arrive according to a renewal process over an infinite time horizon. Assuming inter-arrival times have an increasing failure rate, Gallien proves that agents will buy when they enter the market (or not at all). In contrast to both Wang (1993) and Gallien (2006), we find that the optimal mechanism may induce delay on the equilibrium path.

Pai and Vohra (2008) consider a model without discounting where agents arrive and leave the market over time. They characterise the profit-maximising mechanism and show that, if the virtual valuation is sufficiently monotone, it is incentive compatible. Mierendorff (2009) considers a two-period version of a similar model and provides a complete characterisation of the optimal mechanism.[1]

[1]There are a number of papers on similar themes. Shen and Su (2007) summarize the operations research literature. For example, Aviv and Pazgal (2008) suppose a seller has many goods to sell to agents who arrive over time and are patient. Aviv and Pazgal restrict the seller to choosing two prices which are independent of the past sales. In economics, Board (2007) assumes a seller sells a single unit to agents whose values vary over time. Hörner and Samuelson (2008) consider a seller with no commitment power who sells a single unit to N

There is also a classic literature studying the sequential allocation of goods to impatient buyers. Karlin (1962) analyses the problem of allocating multiple goods to buyers who arrive sequentially but only remain in the market for one period. In the optimal policy, a buyer is awarded a unit if their valuation exceeds a cutoff. This cutoff is decreasing in the number of units available and increasing in the time remaining. These results have been extended in a number of ways. Derman, Lieberman, and Ross (1972) allow for heterogeneous goods. Albright (1974) allows for random arrivals with positive discount rates. More recently, a number of studies allow buyers' valuations to be private information. Gershkov and Moldovanu (2009a) solve the profit-maximising policy. Gershkov and Moldovanu (2009b) allow the seller to learn about the distribution of valuations over time, introducing correlations in the buyers' valuations. Vulcano, van Ryzin, and Maglaras (2002) suppose N agents to enter each period, and allow the seller to hold an auction. Finally, Said (2009) characterises the optimal dominant strategy mechanism where agents are patient but goods are nonstorable.

Finally, the paper is related to the durable goods literature. Stokey (1979) characterises the optimal strategy for a seller with infinite supply who faces a fixed distribution of buyers. Conlisk, Gerstner, and Sobel (1984) suppose a homogenous set of buyers enters each period, while Board (2008) allows the entering generations to differ.

2 Model

Basics. A seller has K goods to sell. Time is discrete and finite $t \in \{1, \dots, T\}$.[2] There is a common discount rate $\delta \in [0, 1]$.[3]

Entrants. At the start of period t, N_t agents/buyers arrive. N_t is a random variable on $\{0, 1 \dots\}$ and is independent of $\{N_1, \dots, N_{t-1}\}$. N_t is observed by the seller, but not by other agents.[4]

agents by setting a sequence of prices. See Shen and Su (2007) for a survey.

[2]Time T can be interpreted as the last period the seller can sell. For example, a clothing retailer must clear their shelves of summer clothing before Autumn. Alternatively, it is the last time when agents enter the market, since no sales will occur after this point. Our results are straightforward to extend to an infinite horizon by letting $T \to \infty$.

[3]We can interpret the discount rate as the rate of time preference, or as the reduction in valuations over time. For example, consumers' valuations for summer clothes will be lower in July than in June. For the sake of concreteness, we use the time-preference model as in the durable goods literature. To model falling valuations, one should let utility be $v\delta^t - p_t$ rather than $(v - p_t)\delta^t$; under this new specification, the optimal allocations are unaffected, and the prices can be determined using the techniques in this paper.

[4]The assumption that the seller can observe N_t is for definiteness: the optimal allocation and implementation are identical if the seller cannot observe N_t. The assumption that an agent cannot observe when other agents enter is motivated by anonymous markets, such as large retailers and online sellers. If N_t is publicly observed, the optimal allocations are unaffected although, when implementing this allocation, the optimal price at time t is a function of $\{N_1, \dots, N_t\}$. See footnote 7.

Preferences. After he has entered the market, an agent wishes to buy a single unit. The agent is endowed with a privately-known valuation, v_i, drawn IID with density $f(\cdot)$, distribution $F(\cdot)$ and support $[\underline{v}, \overline{v}]$. If the agent buys at time t for price p_t, his utility is $(v - p_t)\delta^t$. Let v_t^k denote the k^{th} highest order statistic of the agents entering at time t. Similarly, let $v_{\leq t}^k$ denote the k^{th} highest order statistic of the agents who have entered by time t.

Mechanisms. Each agent makes report $\tilde{v}_i$ when he enters the market.[5] A mechanism $\langle P_{i,t}, TR_i \rangle$ maps agents' reports into an allocation rule $P_{i,t}$ describing the probability agent i is awarded a good in period t, and a transfer TR_i. Transfers are expressed in time-0 prices, so that a payment p_t contributes $\delta^t p_t$ to TR. A mechanism is *feasible* if (a) $P_{i,t} = 0$ before the agent enters, (b) $\sum_t P_{i,t} \in [0,1]$; (c) $\sum_i \sum_t P_{i,t} \leq K$; and (d) $P_{i,t}$ is adapted to the seller's information, so $P_{i,t}$ can vary only with the reports of agents that have entered by t.

Agent's Problem. Suppose agent i enters the market in period t_i. Upon entering the market, the agent chooses his declaration $\tilde{v}_i$ to maximise his expected utility,

$$u_i(\tilde{v}_i, v_i) = E_0 \left[\sum_{s \geq t_i} v_i \delta^t P_{i,t}(\tilde{v}_i, v_{-i}) - TR_i(\tilde{v}_i, v_{-i}) \middle| v_i \right] \tag{2.1}$$

where E_t denotes the expectation at the start of period t, before agents have entered the market. A mechanism is incentive compatible if the agent wishes to tell the truth, and is individually rational if the agent obtains positive utility.

Seller's Problem. The seller chooses a feasible mechanism to maximise the net present value of profits

$$\Pi_0^K = E_0 \left[\sum_i TR_i \right] \tag{2.2}$$

subject to incentive compatibility and individual rationality.

Incentive Compatibility. As shown in Mas-Colell, Whinston, and Green (1995, Proposition 23.D.2), an allocation rule is incentive compatible if and only if the discounted allocation

[5] For simplicity, we assume the buyer does not know the number of units remaining; when implementing the optimal allocation, we attain the same profits when agents know the number of units available.

probability

$$E_0 \left[\sum_{s \geq t_i} \delta^s P_{i,s}(v) \right] \tag{2.3}$$

is increasing in v_i.

Marginal Revenues. When he enters the market, an agent chooses his declaration $\tilde{v}_i$ to maximise his utility (2.1). Using the envelope theorem and integrating by parts, expected utility is

$$E_0[u_i(v_i, v_i)] = E_0 \left[\sum_{s \geq t_i} \delta^t P_{i,t}(v) \frac{1 - F(v_i)}{f(v_i)} \right]$$

where we use the fact that an agent with value $\underline{v}$ earns zero utility in any profit-maximising mechanism. Profit (2.2) then equals welfare minus agents' utilities,

$$\Pi_0 = E_0 \left[\sum_i \sum_{s \geq t_i} P_{i,s} \delta^s m(v_i) \right]$$

where marginal revenue of agent i is given by $m(v) := v - (1 - F(v))/f(v)$. Throughout we assume the $m(v)$ is increasing in v. Under this assumption, the seller's optimal mechanism is characterised by cutoff rules, allowing us to ignore the monotonicity constraint (2.3).

Continuation Profits. Suppose the seller has k goods at time t. Write "continuation profits", conditional on the information revealed at time t, by[6]

$$\tilde{\Pi}_t^k := E_{t+1} \left[\sum_i \sum_{s \geq t} P_{i,s} \delta^{s-t} m(v_i) \right]. \tag{2.4}$$

Let the expected continuation profit at the start of period t be $\Pi_t^k := E_t[\tilde{\Pi}_t^k]$. When $k = 1$, we omit the superscript.

3 Single Good and Constant Entry

We first derive the optimal solution when the firm has one unit to sell and the number of entrants is IID. By the principle of optimality, the seller maximises continuation profits in every state.

[6] While we call Π_t^k continuation profits, this includes the impact of time t decisions on the willingness to pay of agents who buy in earlier periods.

At time t, profit is

$$\Pi_t = E_t\left[\sum_i P_{i,t} m(v_i) + \left(1 - \sum_i P_{i,t}\right)\delta\Pi_{t+1}\right]$$
$$= E_t\left[\sum_i P_{i,t}(m(v_i) - \delta\Pi_{t+1})\right] + E_t[\delta\Pi_{t+1}] \tag{3.1}$$

Equation (3.1) implies that the good is allocated to maximise the flow profit minus the opportunity cost of allocating the good, $\delta\Pi_{t+1}$. As a result, when the good is awarded, it will be given to the agent with the highest marginal revenue (and the highest valuation).

We can now think of the highest current valuation, v, as a state variable. Let $\Pi_t(v)$ be the profit just before entry in time t, so that

$$\Pi_t(v) = E_t\big[\max\{m(v), m(v_t^1), \delta\Pi_{t+1}(\max\{v, v_t^1\})\}\big] \quad \text{for } t < T \tag{3.2}$$
$$\Pi_T(v) = E_T\big[\max\{m(v), m(v_T^1), 0\}\big]$$

The following result shows that the optimal cutoffs can be characterised by a simple one-period-look-ahead rule.

Proposition 1. *Suppose $K = 1$ and N_t is IID. The optimal mechanism awards the good to the agent with the highest valuation exceeding a cutoff. The cutoffs $\{x_t\}$ are given by:*

$$m(x_t) = \delta E_{t+1}[\max\{m(v_{t+1}^1), m(x_t)\}] \qquad \textit{for } t < T \tag{3.3}$$
$$m(x_T) = 0$$

Consequently, the cutoffs are constant in periods $t < T$.

Proof. The proof is by induction. In period $t = T$, then $m(x_T) = 0$. In period $t = T - 1$, the seller should be indifferent between selling to agent x_{T-1} today and waiting one more period and getting a new set of buyers. Hence

$$m(x_{T-1}) = \delta E_T[\max\{m(x_{T-1}), m(v_T^1)\}]$$

Continuing by induction, fix t and suppose x_s, as defined by (3.3), are optimal for $s > t$. If $v < x_t$ then

$$m(v) < \delta E_{t+1}[\max\{m(v_{t+1}^1), m(v)\}]$$

so the seller strictly prefers to wait one period rather than sell to type v today. Conversely, if

$v > x_t$ then

$$m(v) > \delta E_{t+1}[\max\{m(v_{t+1}^1), m(v)\}]. \tag{3.4}$$

Since N_t is IID, (3.4) implies that $v > x_{t+1}$ so type v will buy tomorrow if he does not buy today. Hence

$$\Pi_{t+1}(v) = E_{t+1}[\max\{m(v_{t+1}^1), m(v)\}]$$

and (3.4) implies that the seller strictly prefers to sell to type v today rather than waiting. Putting this together, x_t is indeed the optimal cutoff. $\square$

Proposition 1 uniquely characterises the optimal cutoffs, and shows they are constant in all periods prior to the last. The intuition is as follows. At the cutoff the seller is indifferent between selling to the agent today and delaying one period and receiving another draw. This indifference rule relies on the assumption that if type x_t does not buy today, then he will buy tomorrow. This is satisfied because the seller faces exactly the same tradeoff tomorrow and therefore is once again indifferent between selling and waiting.

The optimal cutoffs are forward looking, and are therefore independent of the number of agents who have entered in the past, and their valuations. This means that the seller does not have to observe the number of arrivals and, as we show below, can implement the optimal policy through a deterministic sales mechanism.

Proposition 1 is very different to the optimal mechanism when buyers are impatient (e.g. Vulcano, van Ryzin and Marglaras (2002)). In this case, the optimal cutoffs are fully forward-looking, and fall over time as the seller becomes increasingly keen to sell the good. In contrast, when agents are patient, the allocations are determined by a one-period-look-ahead rule.

When $T = \infty$, the cutoffs are determined by (3.3) and are therefore constant in all periods. An agent therefore either buys immediately or never, and we can assume that buyers are impatient without loss of generality (Gallien (2006)). In contrast, when T is finite, it is important to take into account buyers' forward looking behaviour. We further investigate this issue in Section 6.

3.1 Implementation through Sequential Second-Price Auctions

The optimal mechanism allocates the good to the agent with the highest valuation exceeding a cutoff. Corollary 1 shows that this allocation can be implemented through a sequence of second-price auctions with declining reserve prices.

Denote the cutoff in periods $t < T$ by x^* and let $\theta := \delta \Pr(v_t^1 < x^*)$ be an agent's effective discount rate, taking into account the possibility that, if he delays, the good may be sold to a new entrant.

Corollary 1. *Suppose $K = 1$ and N_t is IID. The profit-maximising allocation can be implemented by a sequence of second-price auctions with deterministic reserve prices R_t satisfying $R_T = m^{-1}(0)$ and*

$$R_t = (1 - \theta^{T-t})x^* + \theta^{T-t} E_0 \left[\max\{v^2_{\leq T}, m^{-1}(0)\} \middle| N_1 \geq 1, v^1_{\leq T} = x^* \right] \qquad \text{for } t < T. \tag{3.5}$$

Proof. See Appendix A.1. □

The reserve prices are constructed so that the marginal agent is indifferent between buying today and delaying. When making this calculation, we must condition on the buyer existing; since N_t is IID, we assume the buyer enters in period 1.

The sequential second-price auction has several interesting features. First, while cutoffs are constant in periods $t < T$, the reserve prices decline. In the limit, as $T \to \infty$, then $R_t \to x^*$ for a fixed $t < T$. As a result, the reserve prices are fairly constant when $t << T$ and decrease faster as the deadline nears.

Second, agents below x^* abstain, even though their valuations may exceed the reserve price. Such an agent wishes to delay in order to take advantage of the falling reserve prices.

Third, the reserve prices are deterministic. Intuitively, if an agent has value above x^*, he bids his value, either wins or loses the good, and the game ends. If an agent has value below x^*, he abstains and does not reveal his valuation, so there is no new information arriving to the market to require a contingent change in R_t.[7]

Fourth, the reserve prices depend on the expected number of agents who will enter in the future. This means that as the distribution of N_t grows in the usual stochastic order then (a) the cutoff x^* rises, and (b) the probability of stocking out grows and $x^* - R_t$ shrinks.

Finally, while we assume that the seller uses a second-price auction, we could equally well use a different auction format. One possibility is to use an English auction each period. As in the second-price auction, an agent bids his value, conditional on participation. Given reserve prices (3.5), type x^* is again the lowest type participating. A second possibility is to use a first-price auction in periods $t < T$ and a second-price auction in period T, with the reserve price given by (3.5). In periods $t < T$, agents below x^* abstain, while those above x^* adopt an increasing bidding strategy with $b(x^*) = R_t$. This implements the same allocation as sequential second-price auctions and, by revenue equivalence, raises the same revenue. In period T, agents have different beliefs about the distribution of types for new and old bidders, so a second-price auction can be used to attain an optimal allocation (a first-price auction would not be efficient).

[7] This relies of the fact that one buyer cannot observe the arrival of others (as in online marketplaces). If $\{N_1, \ldots, N_t\}$ is publicly observed then the optimal allocations are identical and the reserve is $R_t = (1-\theta^{T-t})x^* + \theta^{T-t} E_0 \left[\max\{v^2_{\leq T}, m^{-1}(0)\} \middle| \{N_1, \ldots, N_t\}, v^1_{\leq T} = x^* \right]$, which changes with the observed number of agents that arrived.

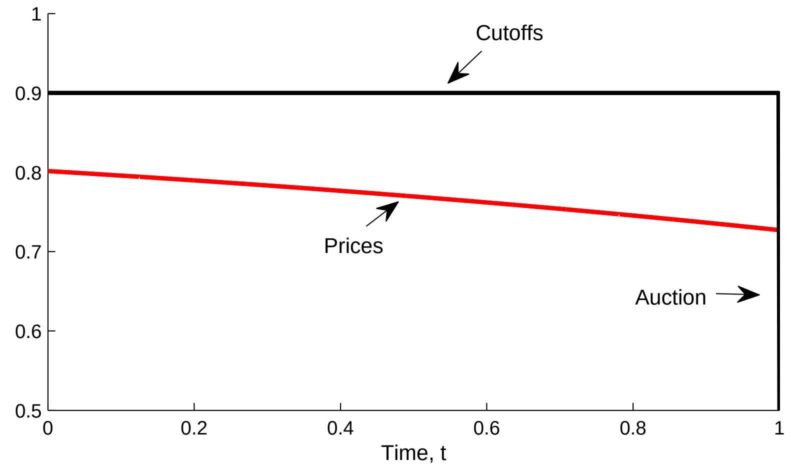

Figure 1: **Optimal Cutoffs and Prices with One Unit in Continuous Time.** When there is one unit, the optimal cutoffs are constant when $t < T$ and drop at time T. The price path is decreasing and concave, with an auction occurring at time T. In this figure, agents enter continuously with Poisson parameter $\lambda = 5$ have values $v \sim U[0,1]$, so the static monopoly price is 0.5. Total time is $T = 1$ and the interest rate is $r = 1/16$.

3.2 Implementation in Continuous Time

The optimal mechanism becomes particularly simple to implement as time periods become very short. Suppose agents enter the market according to a Poisson process with arrival rate λ,[8] and let r be the instantaneous discount rate. Using equation (3.3) the optimal allocation at $t < T$ is given by

$$rm(x^*) = \lambda E\big[\max\{m(v) - m(x^*), 0\}\big] \tag{3.6}$$

where v is distributed according to $F(\cdot)$. Equation (3.6) says the seller equates the flow profit from the cutoff type (the left-hand-side) and the option value of waiting for a new entrant (the right-hand-side). At time T, the optimal cutoff is given by $m(x_T) = 0$. See Figure 1 for an illustration.

The optimal allocation can be implemented by a deterministic sequence of prices with a fire-sale at time T. In the last period, the seller uses a second-price auction with reserve $R_T = m^{-1}(0)$. At time $t < T$ the seller chooses a price p_t, which makes type x^* indifferent between buying and waiting. The optimal allocation can be implemented via a sequence of prices since (a) two agents never arrive at the same time, and (b) agents have the same expectations concerning future prices and bidders' valuations. The final "buy-it-now" price, denoted by $p_T = \lim_{t \to T} p_t$, is chosen so type x^* is indifferent between buying at price p_T and entering the

[8]In discrete time, this means N_t is distributed according to a Poisson distribution with parameter λ.

auction. That is,

$$p_T = E_0\left[\max\{v^2_{\le T}, m^{-1}(0)\}\middle| N_0 = 1, v^1_{\le T} = x^*\right]$$

Note that the buyer conditions on his own existence; since arrivals are independent, we assume that the buyer arrives at time 0 without loss of generality.

When $t < T$, type x^* is indifferent between buying now and waiting dt. This yields

$$(x^* - p_t) = (1 - rdt - \lambda dt)(x^* - p_{t+dt}) + \lambda dt(x^* - p_{t+dt})F(x^*)$$

Rearranging and letting $dt \to 0$,

$$\frac{dp_t}{dt} = -(x^* - p_t)\big(\lambda(1 - F(x^*)) + r\big)$$

so that prices fall faster if (a) prices are lower, (b) the arrival rate is higher, (c) there is a high probability a new arrival will buy the good, or (d) the interest rate is higher. The first property implies that p_t is convex in t, so prices fall faster as the deadline approaches. Empirically, this means that we would expect to see more significant price drops before the fire-sale at time T than far away from the deadline.

3.3 Welfare Analysis

How does the optimal mechanism compare to the welfare-maximising mechanism? Initial welfare is

$$W_0 = E_0\left[\sum_i \sum_{s \ge t_i} P_{i,s}\delta^s v_i\right].$$

Proposition 2. *Suppose $K = 1$ and N_t is IID. The welfare-maximising mechanism awards the good to the agent with the highest valuation exceeding a cutoff. The cutoffs are given by:*

$$\begin{aligned} x_t &= E_{t+1}[\delta \max\{v^1_{t+1}, x_t\}] \qquad \text{for } t < T \\ x_T &= 0 \end{aligned} \tag{3.7}$$

Consequently, the cutoffs are constant in periods $t < T$.

Proof. Analogous to Proposition 1. □

In a one-period problem the standard monopoly distortion implies that the seller awards the good less than is efficient. Corollary 2 shows that, in this dynamic model, the seller additionally awards the good later than is efficient.

Corollary 2. *Suppose N_t is IID and that $(1-F(v))/vf(v)$ is decreasing in v.*[9] *Then the profit-maximising cutoff exceeds the welfare-maximising cutoff for all t.*

Proof. Let x_t^W and x_t^Π be the welfare- and profit-maximising cutoffs. The assumption that $(1-F(v))/vf(v)$ is decreasing in v implies that $m(v)/v$ is increasing in v. If $m(x)>0$ then $m(v)/m(x) \geq v/x$ for $v \geq x$. If $x_t^W > x_t^\Pi$, then (3.3) and (3.7) imply that

$$1 = E_{t+1}\left[\delta \max\left\{\frac{m(v_{t+1}^1)}{m(x_t^\Pi)}, 1\right\}\right] \geq E_{t+1}\left[\delta \max\left\{\frac{v_{t+1}^1}{x_t^\Pi}, 1\right\}\right] > E_{t+1}\left[\delta \max\left\{\frac{v_{t+1}^1}{x_t^W}, 1\right\}\right] = 1$$

yielding the required contradiction. □

4 Varying Entry

This section analyses the optimal mechanism when the expected number of entrants varies over time. In Section 4.1 we suppose fewer agents enter over time, as the stock of potential entrants is used up. In Section 4.2 we suppose more agents enter over time, as word of the market's existence spreads. This analysis forms a bridge between models with no entry (e.g. **?**) and the constant entry model in Section 3. It will also prove useful when deriving the optimal multi-unit mechanism.

4.1 Decreasing Entry

The following result shows that the one-period-look-ahead policy still applies when demand is decreasing. Intuitively, a buyer with value equal to the cutoff at time t will exceed the cutoff at time $t+1$, so the seller will delay allocating the good by only one period.

Proposition 3. *Suppose $K=1$ and N_t is decreasing in the usual stochastic order. Then the optimal cutoffs are characterised by (3.3). These cutoffs are decreasing over time.*

Proof. Since N_t is decreasing in the usual stochastic order, v_t^1 is decreasing in the usual stochastic order and x_t, as defined by (3.3), is decreasing in t. The rest of the proof is the same as Proposition 1. □

In continuous time, these allocations can be implemented by a deterministic price sequence p_t and a fire-sale at date T.[10] Suppose buyers enter with Poisson arrival rate λ_t, which is

[9] This assumption implies that $m(v)$ is increasing, and is implied by the usual monotone hazard assumption.

[10] In discrete time, the optimal cutoffs can be implemented through a sequence of second-price auctions with deterministic reserve prices, as in Section 3.1. In order to do this, however, agents' time of entry must not give them useful information. This is satisfied if N_t comes from a Poisson distribution or if N_t are publicly observed.

decreasing in t. The optimal allocations at time $t < T$ are given by

$$rm(x_t) = \lambda_t E\big[\max\{m(v) - m(x_t), 0\}\big]$$

where v is drawn according to $F(\cdot)$. At time T, the optimal cutoff is given by $m(x_T) = 0$.

At time T, the seller can implement the optimal allocation through a second-price auction with reserve $R_T = m^{-1}(0)$. The final "buy-it-now" price is given by

$$p_T = E_0\left[\max\{v^2_{\le T}, m^{-1}(0)\}\middle| N_0 = 1, v^1_{\le T} = \overline{x}_T\right]$$

where $\overline{x}_T := \lim_{t \to T} x_t$.

Consider $t < T$, and let H_t be the unconditional distribution of $v^1_{\le t}$. The unconditional number of arrivals before time t, denoted $N_{\le t}$, has a Poisson distribution with parameter $\int_0^t \lambda_\tau \, d\tau$. It follows that

$$H_t(x) = \Pr(v^1_{\le t} \le x) = \sum_{k=0}^{\infty} F(x)^k \Pr(N_{\le t} = k) = e^{-(\int_0^t \lambda_\tau \, d\tau)(1-F(x))}. \tag{4.1}$$

Cutoffs are decreasing over time so when a buyer delays, an agent with a lower value has the opportunity to buy. Given that arrivals are independent, this occurs with probability

$$1 - \Pr(v^2_{\le t} \le x_{t+dt} | v^1_{\le t} = x_t, N_0 = 1) = 1 - \frac{H_t(x_{t+dt})}{H_t(x_t)}.$$

Type x_t is indifferent between buying now and waiting dt,

$$(x_t - p_t) = (1 - rdt - \lambda_t dt)(x_t - p_{t+dt})\frac{H_t(x_{t+dt})}{H_t(x_t)} + \lambda_t dt(x_t - p_{t+dt})\frac{H_t(x_{t+dt})}{H_t(x_t)} F(x_t)$$

Using (4.1) and letting $dt \to 0$,

$$\frac{dp_t}{dt} = -(x_t - p_t)\left(-\frac{dx_t}{dt}\left(\int_0^t \lambda_\tau \, d\tau\right) f(x_t) + \lambda_t(1 - F(x_t)) + r\right) \tag{4.2}$$

where we note that $dx_t/dt < 0$. As in Section 3.2, prices fall faster if (a) the arrival rate is higher, (b) there is a high probability a new arrival will buy the good, or (c) the interest rate is higher. In addition, equation (4.2) shows that prices fall faster if (d) the cutoffs fall quickly, or (e) there is a high probability a second agent has a value just below x_t. This second effect means that prices drop faster if buyers think they have more competition from existing buyers.

To illustrate, suppose a seller puts her house on the market. There is an initial stock of buyers who have a high probability of seeing the newly listed house, plus a constant inflow of new buyers (where $T = \infty$). In the optimal mechanism, there is an introductory period where cutoffs and price fall quickly, with some buyers strategically waiting. In the limit, where existing buyers see the new house immediately, the seller reduces prices instantly in the form of a Dutch auction. After this introductory period, prices coincide with the cutoffs, and are constant over time, so that no buyer ever delays.

4.2 Increasing Entry

When the number of entrants increases over time, the one-period-look-ahead policy fails. Intuitively, because the number of entrants is rising, the seller wishes to increase the cutoff. Hence an agent who is indifferent at time t will prefer not to buy at time $t+1$, and therefore looks ahead to period T. As a result the optimal allocations depend on the number of entrants in all future periods, not just the adjacent period.

Recursively define the following functions:

$$\begin{aligned} \pi_t(v) &= E_t\big[\max\{m(v_t^1), \delta\pi_{t+1}(\max\{v, v_t^1\})\}\big] \qquad \text{for } t < T \\ \pi_T(v) &= E_T\big[\max\{m(v), m(v_T^1), 0\}\big] \end{aligned} \tag{4.3}$$

This looks similar to equation (3.2), but is simpler because, if the seller delays at time t then she does not return to that buyer until period T.

Proposition 4. *Suppose $K = 1$ and N_t is increasing in the usual stochastic order. Then the optimal cutoffs are given by*

$$\begin{aligned} m(x_t) &= \delta\pi_{t+1}(x_t) \qquad \textit{for } t < T \\ m(x_T) &= 0. \end{aligned} \tag{4.4}$$

These cutoffs are increasing over time, for $t < T$.

Proof. See Appendix A.2. □

When the number of entrants increases over time, the optimal cutoffs (4.4) also increase. As a result, an agent either buys when he enters the market or waits until the final period. This means that, unlike the one-period-look-ahead policies in Propositions 1–3, the optimal cutoffs depend on the future of the game. Consequently, today's cutoff increases if either the game becomes longer, or the future number of entrants rises.

In continuous time, these allocations can be implemented by a deterministic price sequence p_t and a fire-sale at date T.[11] Suppose buyers arrive with Poisson arrival rate λ_t, which is

[11] In discrete time, the optimal cutoffs can be implemented through a sequence of second-price auctions with deterministic reserve prices, as in Section 4.1.

increasing in t. We can define functions corresponding to (4.3) using the end point $\pi_T(v) = v$ and the differential equation

$$r\pi_t(v) = \frac{d\pi_t(v)}{dt} + \lambda_t E\Big[\max\big\{m(v'), \pi_t(\max\{v, v'\})\big\} - \pi_t(v)\Big] \tag{4.5}$$

where v' is the value of the new entrant and is drawn from $F(\cdot)$. Equation (4.5) says that asset value of profits are determined by the increase in their value and the option value from new entrants arriving. We can now define the optimal cutoffs. At time T, the optimal cutoff is given by $m(x_T) = 0$. At time $t < T$, the optimal cutoff is given by $m(x_t) = \pi_t(x_t)$.

At time T, the seller can implement the optimal allocation through a second-price auction with reserve $R_T = m^{-1}(0)$. For $t < T$, the prices are determined so that buyer x_t is indifferent between buying in period t and waiting until the fire-sale. That is,

$$(x_t - p_t) = e^{-r(T-t)} \Pr(v^1_{\geq t} \leq x_t) E\big[x_t - \max\{v^2_{\leq T}, m^{-1}(0)\}\big| N_0 = 1, v^1_{\leq T} = x_t\big] \tag{4.6}$$

where $v^1_{\geq t}$ is the highest order statistic of the buyers who have entered after time t. Using (4.1),

$$\Pr(v^1_{\geq t} \leq x_t) = e^{-(\int_t^T \lambda_\tau \, d\tau)(1-F(x_t))} =: \psi_t.$$

Note that ψ_t increases in t, and that $\psi_T = 1$. Prices are then given by

$$p_t = (1 - \psi_t)x_t + \psi_t E[\max\{v^2_{\leq T}, m^{-1}(0)\}| N_0 = 1, v^1_{\leq T} = x_t], \tag{4.7}$$

where $v^2_{\leq T}$ conditional on $v^1_{\leq T} = x_t$ has distribution function

$$\Pr(v^2_{\leq T} \leq v | N_0 = 1, v^1_{\leq T} = x_t) = \frac{e^{-(\int_0^T \lambda_s \, ds)(1-F(v))}}{e^{-(\int_0^T \lambda_s \, ds)(1-F(x_t))}}.$$

Over time, the optimal posted prices will tend to rise and then fall. Intuitively, as t grows so the cutoff increases, increasing the first term in (4.7). However, as $t \to T$, the fire-sale at T comes closer, decreasing agents willingness to delay and increasing the weight on the second term in (4.7). If we take $T \to \infty$, then the right hand side of (4.6) converges to zero and $p_t \to x_t$ for all t. This follows from the fact that a buyer who delays at time t must wait until period T to have another opportunity to buy.

5 Multiple Units

In this section we suppose the seller has K goods to allocate. Using the principle of optimality, the seller maximises continuation profits at each point in time. Consider period t and suppose the seller has k units.

Lemma 1. *The seller allocates goods to high value agents before low value agents.*

Proof. Suppose in period t the seller sells to agent j but does not sell to agent i, where $v^i \geq v^j$. To be concrete, suppose the seller eventually sells to agent i in period $\tau > t$, where we allow $\tau = \infty$. Now suppose the seller leaves all allocations unchanged but switches i and j. This increases profit by $(1-\delta^{\tau-t})(m(v^i)-m(v^j))$, contradicting the optimality of the original allocation. □

Using Lemma 1, we need only keep track of the k highest valuations. At the start of time t suppose the seller has agents with valuations $\{y^1,\dots,y^k\}$, where $y^i \geq y^{i+1}$. Profit is described by the Bellman equation[12]

$$\tilde{\Pi}_t^k(y^1,\dots,y^k) = \max_{j\in\{0,\dots,k\}} \left[\sum_{i=1}^{j} m(y^i) + \delta\Pi_{t+1}^{k-j}(y^{j+1},\dots,y^k)\right]$$

where $\Pi_{t+1}^k := E_{t+1}[\tilde{\Pi}_{t+1}^k]$. The Bellman equation says the seller receives the marginal revenue from the units she sells today plus the continuation profits from the remaining units. The seller's optimal strategy is thus to sell the first object to the highest value agent, subject to his value exceeding cutoff x_t^k. She then sells the second object to the second highest value agent, subject to his value exceeding cutoff x_t^{k-1}, and so forth. We can thus think of the items being awarded sequentially. When there are k units left, the cutoff is given by the indifference equation

$$m(x_t^k) + \delta\Pi_{t+1}^{k-1}(y^2,\dots,y^k) = \delta\Pi_{t+1}^k(x_t^k, y^2,\dots,y^k).$$

The following Lemma shows that when $\{x_t^k\}$ are decreasing in k we can treat each unit separately, comparing the j^{th} cutoff to the corresponding agent's valuation.

Lemma 2. *Fix t and suppose $\{x_t^k\}$ are decreasing in k. Then unit j is allocated to agent i at time t if and only if*
(a) v_i exceeds the cutoff x_t^j.
(b) v_i has the $(k-j+1)^{th}$ highest valuation of the currently present agents.

Proof. Suppose agent i is allocated good j, then (a) and (b) are satisfied.

Suppose (a) and (b) are satisfied. Then there are $(k-j)$ agents with higher valuations than i. Since the cutoffs are decreasing in k, these valuations exceed their respective cutoffs. Hence object j is allocated to agent i. □

[12] When $j = 0$ the first term in the summation is zero.

Proposition 5 shows the cutoffs are decreasing in k, and explicitly solves for the optimal cutoffs. In the period $t = T$, the seller wishes to allocate the goods to the k highest value buyers, subject to these values exceeding the static monopoly price. Hence,

$$m(x_T^k) = 0 \qquad \text{for all } k. \tag{5.1}$$

Next, consider period $t = T-1$. If she allocates the k^{th} good she gets $m(x_{T-1}^k)$. The opportunity cost is to wait one period and award the good either to agent x_{T-1}^k or the k^{th} highest new entrant. Hence,[13]

$$m(x_{T-1}^k) = \delta E_{T-1}\left[\max\{m(x_{T-1}^k), m(v_T^k)\}\right] \tag{5.2}$$

In periods $t \leq T-1$, the seller is indifferent between selling to the cutoff type today and waiting one more period. This one-period-look-ahead policy can be expressed as

$$\begin{aligned} m(x_t^k) &+ \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1, \dots, v_{t+1}^{k-1})\right] \\ &= \delta E_{t+1}\left[\max\{m(x_t^k), m(v_{t+1}^1)\}\right] + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\{x_t^k, v_{t+1}^1, \dots, v_{t+1}^k\}_k^2)\right]. \end{aligned} \tag{5.3}$$

where the notation $\{x_t^k, v_{t+1}^1, \dots, v_{t+1}^k\}_k^2$ represents the ordered vector of the 2^{nd} to k^{th} highest choices from $\{x_t^k, v_{t+1}^1, \dots, v_{t+1}^k\}$. Notably, equation (5.3) is independent of the state $\{y^2, \dots, y^k\}$ for reasons explained below.

Proposition 5. *Suppose the seller has K units to sell and N_t are IID. The optimal allocation awards unit k at time t to the remaining agent with the highest value exceeding a cutoff x_t^k. The cutoffs are characterised by equations (5.1), (5.2) and (5.3). These cutoffs are deterministic, and decreasing in t and k.*

Proof. See Appendix A.3 and A.4. □

Proposition 5 has a number of important consequences. First, the cutoffs are uniquely determined. Intuitively, the sooner an agent buys a good the more his value affects overall profit. Hence the left hand sides of (5.2) and (5.3) have a steeper slope than the right hand sides.

Second, the cutoffs are independent of the current state $(y^2, \dots, y^k)$. Intuitively, at the cutoff, the seller is indifferent between selling to y^1 and waiting. In either case the allocation to $(y^2, \dots, y^k)$ is unaffected since, in any future state, this decision does not affect their rank in the distribution of agents available to the seller. This fact is used in equation (5.3), where we set $y^j = 0$ for $j \geq 2$.

[13] To be more formal, if $y^1 > v_T^k$, the seller loses $y^1(1-\delta)$ by delaying. If $y^1 < v_T^k$, the seller loses $y^1 - \delta v^k$ by delaying. The seller is this indifferent if y^1 satisfies (5.2).

Third, the cutoffs increase when there are fewer units available (see Figure 2). Intuitively, if the seller delays awarding the k^{th} unit by one period then she can allocate it to the highest value entrant, rather than agent y^1. When there are more goods remaining, agent y^1 is more likely to be awarded the good eventually, reducing the option value of delay and decreasing the cutoff.

Fourth, the cutoffs for the last unit are identical to the one unit case and are therefore constant in periods $t \leq T-1$. The other cutoffs are decreasing over time (see Figure 2). The intuition, as above, is based on the fact that if the seller delays awarding the k^{th} unit by one period then she can allocate it to the highest value entrant, rather than agent y^1. As the game progresses, agent y^1 is more likely to be awarded the good eventually, reducing the option value of delay and decreasing the cutoff. Figure 2 shows that the cutoffs decrease rapidly as $t \to T$. Figure 3 shows the corresponding hazard rate of sale. The hazard rate with one unit remaining stays low until $t = T$, at which point it jumps to infinity (because of the fire sale). The hazard rate with two units remaining is qualitatively similar: is low initially and rapidly rises as we approach T. Therefore, even though the multi-unit auction only ever awards one unit at time T, the the pattern of trade close to T still resembles a fire-sale.

Fifth, we can bound the k^{th} unit cutoff from above and below in periods $t < T$. The upper bound is given by

$$m(\overline{x}^k) = \delta E_{t+1}[\max\{m(\overline{x}^k), m(v^1_{t+1})\}] \tag{5.4}$$

If the seller delays by one period she sells the k^{th} unit for the right hand side of (5.4). This is an upper bound because, if the seller delays, then she has a lower state variable for subsequent units, implying that the the right-hand side of (5.4) overestimates the gain from delay. The lower bound is given by

$$m(\underline{x}^k) = \delta E_{t+1}[\max\{m(\underline{x}^k), m(v^k_{t+1})\}] \tag{5.5}$$

which differs from the upper bound expression through the last term. The right-hand side of (5.5) gives the cutoff if the agent is forced to sell all the goods in period $t+1$. This is a lower bound because the seller may choose to delay further, implying that the right-hand side of (5.5) underestimates the gain from delay

5.1 Implementation in Continuous Time

Suppose agents enter according to a Poisson process with parameter λ. In period T, the optimal cutoffs are given by $m(x^k_T) = 0$. In period $t < T$, equation (5.3) becomes

$$rm(x^k_t) = \lambda E\Big[\max\{m(v) - m(x^k_t), 0\} + \Pi^{k-1}_t\big(\min\{v, x^k_t\}\big) - \Pi^{k-1}_t(v)\Big] \tag{5.6}$$

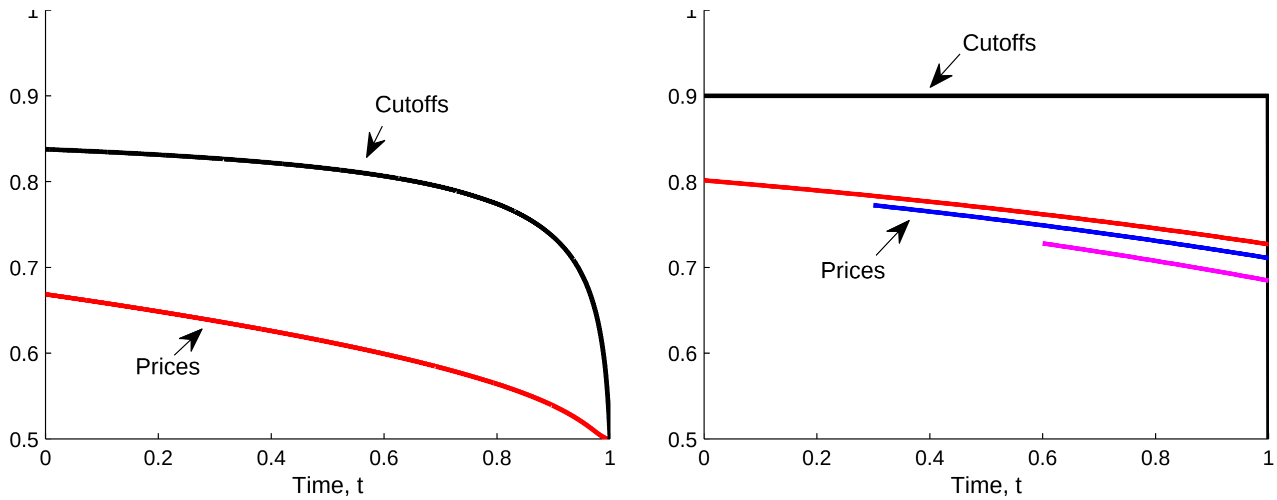

Figure 2: **Optimal Cutoffs and Prices with Two Units.** The **left** panel shows the optimal cutoffs/prices when the seller has two units remaining. The **right** panel shows the optimal cutoffs/prices when the the seller has one unit remaining. The three price lines illustrate the seller's strategy when it sells the first unit at times $t = 0$, $t = 0.3$ and $t = 0.6$. In this figure, agents enter continuously with $\lambda = 5$ and have values $v \sim U[0, 1]$. Total time is $T = 1$ and the interest rate is $r = 1/16$.

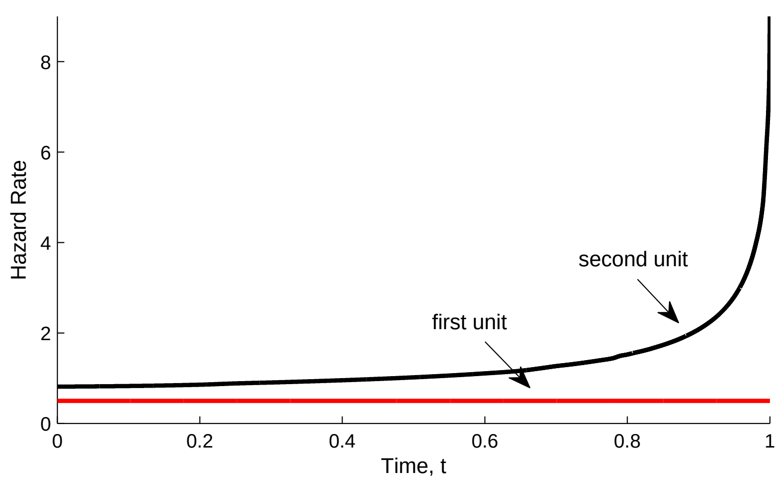

Figure 3: **Hazard Rates with Two Units** This figure shows the probability the first/second unit is sold at time $t + dt$, conditional on there being one/two units remaining at time t. We assume agents enter continuously with Poisson parameter $\lambda = 5$ and have values $v \sim U[0, 1]$. Total time is $T = 1$ and the interest rate is $r = 1/16$.

where v is drawn from $F(\cdot)$. Equation (5.6) states the seller is indifferent between selling today and delaying a little. The cost of delay is the forgone interest (the left-hand side); the benefit is the option value of a new buyer entering the market (the right-hand side). Observe that, since cutoffs are independent of the current state, we can assume there is only one agent and write Π_t^{k-1} a function of one variable. Using Lemmas 5–7, equation (5.6) implies that x_t^k is uniquely determined, and decreasing in k and t. When $k = 1$, x_t^1 is constant for all $t < T$, and jumps down discontinuously at $t = T$. For $k \geq 2$, $\Pi_t^{k-1}(v) \to v$ as $t \to T$, so (5.6) implies that $m(x_t^k) \to 0$, as shown in Figure 2.

We can implement the optimal allocations with prices $\{p_t^k\}$ and a fire-sale at time T for the last unit. We first wish to understand the limit of prices as $t \to T$, giving us a boundary point. For $k \geq 2$, $m(x_t^k) \to 0$ and hence the prices converge to $m^{-1}(0)$. For $k = 1$, seller can use a second-price auction with reserve $m^{-1}(0)$ at time T. When $t < T$, the price converges to

$$p_T = E_0 \left[\max\{v_{\leq T}^2, m^{-1}(0)\} \middle| v_{\leq T}^1 = x^*, s_T(x) \right]$$

where x^* is the constant cutoff with one unit remaining, and $s_T(x)$ denotes the last time the cutoff went below x. Note that p_T depends on when other agents purchased units. In particular, the earlier those units were purchased, the more competition agent with type x^* expects at time T, and the higher is p_T (see Figure 2).

In earlier periods, prices are determined by the cutoff type's indifference condition,

$$\begin{aligned}(x_t^k - p_t^k) = (1 - rdt - \lambda dt) \left[x_t^k - p_{t+dt}^k \right] \frac{H_t(x_{t+dt}^k)}{H_t(x_t^k)} + (1 - rdt - \lambda dt) U_t^{k-1}(x_t^k) \left[1 - \frac{H_t(x_{t+dt}^k)}{H_t(x_t^k)} \right] \\ + (\lambda dt)(x_t^k - p_{t+dt}^k) F(x_t^k) + (\lambda dt) U_t^{k-1}(x_t^k)(1 - F(x_t^k))\end{aligned}$$

where $U_t^{k-1}(x_t^k)$ is the buyer's utility at time t when there are $k-1$ goods left, conditional on $v_{\leq t}^1 = x_t^k$ and the history of the price path.[14] Rearranging and letting $dt \to 0$,

$$\frac{dp_t^k}{dt} = \left[\frac{dx_t^k}{dt} \left(\int_{s_t(x_t^k)}^t \lambda \, d\tau \right) f(x_t) - \lambda(1 - F(x_t^k)) \right] \left[x_t^k - p_t^k - U_t^{k-1}(x_t^k) \right] - r \left(x_t^k - p_t^k \right) \quad (5.7)$$

where $s_t(x_t^k)$ is the last time before t that the lowest cutoff was below x_t^k. This is similar to equation (4.2) except that, when another agent buys the good, then agent x_t^k obtains some outside utility. Since x_t^k is decreasing in t, the price p_t^k falls smoothly over time, but jumps up with every sale.

[14] Using the envelope theorem, utility can be expressed in terms of future allocations, $U_t^{k-1}(x_t^k) = E_t \int_{\underline{v}}^{x_t^k} e^{-r\tau(v)} dv$ where $\tau(v)$ is time buyer v obtains a good.

5.2 Decreasing Entry

We can also extend our results to allow for decreasing entry.

Proposition 6. *Suppose the seller has K units to sell and N_t is decreasing in the usual stochastic order. The optimal cutoffs are characterised by equations (5.1), (5.2) and (5.3). These cutoffs x_t^k are deterministic, and decreasing in t and k.*

Proof. Same as Proposition 5, replacing Lemma 7 with Lemma 7′ in Appendix A.5. □

In continuous time, the optimal allocation can be implemented by posted prices plus an auction for the last unit in period T. The continuous time cutoffs are determined by (5.6), replacing λ with λ_t. Similarly, the price path is determined by (5.7), again replacing λ with λ_t.[15]

6 Partially Patient Agents

One limitation of our analysis is that we do not allow for heterogeneity in the timing of buyers' demands. That is, a type-v agent who enters in period 1 has the same valuation in period t as a type-v agent who enters in period t. This is a problematic assumption for some applications, since buyers may exit the market (for example, a customer may buy another airline ticket), or buyers' valuations may decline relative to the entrants (for example, a customer's value for a seasonal piece of clothing declines after his vacation). In this section we consider these two perturbations of the model: In Section 6.1 we suppose buyers' values decline deterministically relative to those of new entrants; In Section 6.2 we assume that buyers exit stochastically. These results highlight the difficulties these considerations create and bridge our results with the analysis of impatient agents (e.g. Vulcano, van Ryzin, and Maglaras (2002)).

6.1 Declining Values

For the first model, assume that an agent with value v_i who enters in period t_i and buys in period s receives utility

$$\delta^s \beta^{s-t_i} v_i \tag{6.1}$$

where $\beta \in [0,1]$. When $\beta = 1$ this coincides with the model in Section 3; when $\beta = 0$ this coincides with the model of impatient agents. Following the derivation in Section 2, profits are given by

$$\Pi_0 = E_0 \left[\sum_i \sum_{s \geq t_i} P_{i,s} \delta^s \beta^{s-t_i} m(v_i) \right]$$

[15] With increasing entry, the analysis is more complex. The failure of the one-look-ahead-policy means that if the seller delays at time t she may not sell until period $t+7$, between which much may happen.

It will be convenient to think of the state variable at the highest marginal revenue, $\hat{m}$, rather than the highest valuation. Recursively define the following functions:

$$\pi_t(\hat{m}) = E_t\big[\max\{m(v_t^1), \delta\pi_{t+1}(\max\{\beta\hat{m}, m(v_t^1)\})\}\big] \qquad \text{for } t < T \tag{6.2}$$
$$\pi_T(\hat{m}) = E_T\big[\max\{\beta\hat{m}, m(v_T^1), 0\}\big]$$

This looks similar to equation (3.2), but is simpler because, if the seller delays at time t then she does not return to that buyer until period T.

Proposition 7. *Suppose $K = 1$, N_t is IID, and agents have declining values (6.1). At time $t < T$, the optimal mechanism awards the good to the highest value agent who enters in time t, if this value exceeds a cutoff x_t defined by*

$$m(x_t) = \delta\pi_{t+1}^t(m(x_t)). \tag{6.3}$$

These cutoffs have the property that $m(x_t) \geq m(x_{t+1}) \geq \beta m(x_t)$ for $t < T - 1$. At time T, the good is awarded to the agent with the highest discounted marginal revenue, $\beta^{T-t_i}m\left(v_i\right)$.

Proof. See Appendix A.6. □

Proposition 7 tells us that, when agents are only partially patient, the one-period-look-ahead policy fails to hold. In particular, an agent either buys when he enters the market or waits until period T. As in models with impatient agents, the cutoffs fall over time as the seller's options shrink. However, since the seller can always return to an old agent, the rate of decline is bounded below by β.

From equation (6.2) one can see that the seller's profits are increasing in β. That is, the seller prefers agents to be forward looking. While it may seem counterintuitive that allowing inter-temporal arbitrage benefits the seller, delay allows the seller to merge buyers from different cohorts and obtain a more efficient allocation. This suggest that if the seller can run an optimal mechanism, she should embrace price prediction sites such as bing.com, rather than viewing them as a threat to inter-temporal price discrimination.

We now turn to implementation. In period $t < T$, the optimal mechanism awards the good to the entrant with the highest value exceeding the cutoff. If $(1 - F(v))/vf(v)$ is decreasing in v then $m(x_{t+1}) \geq \beta m(x_t)$ implies that $x_{t+1} \geq \beta x_t$ so the time-t cutoff type will not wish to buy at time $t + 1$. As a result, we can implement the optimal mechanism via simple second-price auctions with appropriate reserve prices, despite the new and old entrants being asymmetric. Similarly, in continuous time the optimal mechanism can be implemented via posted prices.

At time T, the optimal mechanism allocates the good to the agent with the highest $\beta^{T-t_i}m\left(v_i\right)$ while a second-price auction would allocate it to the one with the highest $\beta^{T-t_i-T}v_i$. If

$(1-F(v))/vf(v)$ is decreasing in v, $\beta^{T-t_1}v_1 = \beta^{T-t_2}v_2$ implies $\beta^{T-t_1}m(v_1) \geq \beta^{T-t_2}m(v_2)$ for $t_2 > t_1$. As a result, allocation is biased towards agents who enter the market earlier. Intuitively, allocating the object to an older buyer gives away fewer information rents because they have a higher valuation relative to their cohort. We can thus implement the optimal allocation by having agents register with the seller when they arrive in the market. The seller can then give a "discount voucher" to an agent who arrives early. For example, if $v \sim U[0,1]$ then the seller should give an agent who registers in period t a discount of $(1-\beta^{T-t})/2$.[16,17]

6.2 Disappearing Buyers

Another natural way to model the heterogeneity in agents' timing decisions is to allow them to exit probabilistically over time. If entry and exit times are private information of the buyers, the optimal mechanisms are very complicated, as discussed by Pai and Vohra (2008) and Mierendorff (2009). Even if we simplify the model to assume that agents have no private information about their exit times and each agent exits the game with some exogenous probability ρ, the optimal allocations become much more complicated. In particular, the following example illustrates that the striking feature of our model — that the optimal cutoffs depend only on the number of remaining units and time until the deadline — does not hold in a general model with random exits.

Suppose time is discrete, $T=2$ and $K=1$. Suppose there are two entrants in period $t=1$ and one more entering at $T=2$. The discount factor is $\delta=\frac{8}{9}$ and all values are distributed uniformly over $[0,1]$, so that $m(v)=2v-1$. Solving for the optimal cutoffs we have $x_T=\frac{1}{2}$ and $x_1=\frac{3}{4}$, where x_1 solves:

$$m(x_1) = \delta E_{v_3}\left[\max\{m(x_1), m(v_3)\}\right] \tag{6.4}$$

Next, suppose agents independently exit with probability ρ. How does the optimal mechanism change? Without loss suppose that $v_1 \geq v_2$. Then it is optimal to sell the good to agent 1 if and only if

$$m(v_1) \geq \delta E_{v_3}\left[(1-\rho)\max\{m(v_1), m(v_3)\} + \rho(1-\rho)\max\{m(v_2), m(v_3), 0\} + \rho^2 \max\{0, m(v_3)\}\right]$$

This expression is much more complicated than (6.4) because we need to take into account the risk of losing either of the two agents. Importantly, the possibility that agent 1 will exit and agent 2 will stay, makes the decision of whether to award the good to agent 1 today depend on

[16]Proof: The seller wishes award the good to the agent who maximises $\beta^{T-t}(2v-1)$, or equivalently $v\beta^{T-t} + (1-\beta^{T-t})/2$.

[17]This assumes agents cannot register before they are interested in the product. For example, agents only enter when they become aware of the good.

the value of agent 2! For $\delta = 1$ and $\rho = \frac{1}{9}$ the optimal cutoff for agent 1 as a function of agent 2 value is:

$$x_1(v_2) \begin{cases} \approx 0.91 & \text{for } v_2 > 0.91 \\ = \frac{9}{8} - \frac{1}{24}\sqrt{79 - 64v_2^2} & \text{for } v_2 \in [0.5, 0.91] \\ \approx 0.79 & \text{for } v_2 < 0.5 \end{cases}$$

In other words, the optimal cutoff is no longer deterministic - it depends not only on how many goods and periods remain but also on the values of all players that entered so far. This is a general property of the optimal mechanism if buyers exit stochastically. While we can implement such an allocation through a direct revelation mechanism, it seems unlikely that any natural indirect mechanism, such as auctions or posted prices, will work.

7 Conclusion

This paper considers the problem of a seller who wishes to sell a set of durable goods when buyers enter the market over time and are patient. When there is a single good and entrants are IID, the profit-maximising mechanism awards the good to the agent with the highest valuation exceeding a cutoff. These cutoffs can be characterised by a one-period-look-ahead policy and are constant in all periods prior to the last. When the seller owns multiple units, she still uses a one-period-look-ahead policy, yielding cutoffs that are deterministic and fall over time. These optimal allocations can then be implemented by a sequence of prices and an auction for the final good in the period T.

In some cases it may be much easier to solve the firm's problem by assuming buyers are impatient. While, in general, this will lead to a reduction in profits, this is without loss if entry is IID and either there are infinite periods (Gallien (2006)), or markets are large (Segal (2003)), since a constant price is optimal under either scenario. This means that properly modelling patient buyers is most important where the seller's options decline over time, or where the market is small. As an example, this suggests that airline companies should be more concerned with forward-looking customers on their small flights than on their large ones.

The model can be extended in several ways. One could allow the number of entrants to be correlated over time, and for the distribution of valuations to change. Both seller and agents may learn about the distribution of valuations and the number of entering agents as the game progresses. Finally, for many applications, it is important to analyse how competing sellers would interact.

A Omitted Proofs

A.1 Proof of Corollary 1

In period $t = T$, the optimal reserve price is $R_T = m^{-1}(0)$ and it is a weakly dominant strategy for an agent to bid his valuation, $b_T(v) = v$.

Consider period $t < T$. As we verify below, if we set R_t according to (3.5) then an agent with type x^* is indifferent between buying today and delaying, conditional on having the highest valuation. Since $\delta \leq 1$, types $v \geq x^*$ prefer to buy today and bid at least R_t, while types $v < x^*$ prefer to delay and do not bid. For an agent who bids above the reserve price, it is a weakly dominant strategy to bid their valuation.

We now verify the reserve price is defined by (3.5). Consider period $T-1$ and suppose $v^1_{\leq T-1} = x^*$. The reserve is determined by the indifference condition

$$(x^* - R_{T-1}) = \delta E_0 \left[\left(x^* - \max\{v^2_{\leq T-1}, v^1_T, m^{-1}(0)\} \right) \mathbf{1}_{v^1_T < x^*} \middle| N_1 \geq 1, v^1_{\leq T-1} = x^* \right]. \tag{A.1}$$

Rearranging (A.1), the reserve price is

$$\begin{aligned} R_{T-1} &= (1-\theta)x^* + \theta E_0 \left[\max\{v^2_{\leq T-1}, v^1_T, m^{-1}(0)\} \middle| N_1 \geq 1, v^1_{\leq T-1} = x^*, v^1_T < x^* \right] \\ &= (1-\theta)x^* + \theta E_0 \left[\max\{v^2_{\leq T}, m^{-1}(0)\} \middle| N_1 \geq 1, v^1_{\leq T} = x^* \right]. \end{aligned} \tag{A.2}$$

Next, consider period $t \leq T-2$. Type x^* should indifferent between buying and waiting. If he buys in period t he pays the reserve price, R_t; if he waits, we assume he buys in period $t+1$, since x^* is constant.[18] Hence the period-t reserve is determined by the AR(1) equation

$$(x^* - R_t) = \delta E_0[(x^* - R_{t+1})\mathbf{1}_{v^1_{t+1} < x^*} | N_1 \geq 1, v^1_{\leq t} = x^*] \tag{A.3}$$

Rearranging (A.3),

$$R_t = (1-\theta)x^* + \theta E_0[R_{t+1} | N_1 \geq 1, v^1_{\leq t+1} = x^*] \tag{A.4}$$

Using (A.2) and (A.4) the reserve price is given by (3.5).

A.2 Proof of Proposition 4

In period T, the seller awards the good to the agent with the highest value, subject to her marginal revenue exceeding zero, implying that $m(x_T) = 0$. We next claim that x_t are weakly increasing for $t < T$. Suppose, by contradiction, that there exists $t < T-1$ such that $x_t > x_{t+1}$.

[18] Since the cutoffs are constant, we can equally well assume that, if type x^* waits at time t, then he waits until the period $t = T$.

Then the cutoff x_t is given by

$$m(x_t) = \delta E_{t+1}[\max\{m(x_t), m(v^1_{t+1})\}] \tag{A.5}$$

This follows from the fact that type x_t will buy in period $t+1$ if he does not buy in period t. Now consider period $t+1$ and suppose the seller faces a buyer of value x_{t+1}. If the seller delays he obtains at least $\delta E_{t+2}[\max\{m(x_{t+1}), m(v^1_{t+2})\}]$. Indifference therefore implies that

$$m(x_{t+1}) \geq \delta E_{t+2}[\max\{m(x_{t+1}), m(v^1_{t+2})\}] \tag{A.6}$$

Since N_t is increasing in the usual stochastic order, v^1_{t+2} is larger than v^1_{t+1} in the usual stochastic order, so (A.5) and (A.6) imply $x_{t+1} \geq x_t$, yielding a contradiction.

Fix $t < T$. If the seller sells to type x_t, she obtains $m(x_t)$. If the seller delays, she obtains $\delta\pi_{t+1}(x_t)$, as defined by (4.3), where we use the fact that x_t will not buy in period $t+1$ because the cutoffs are increasing. The seller is indifferent between selling to type x_t and delaying, yielding (4.4), as required.

A.3 Proof of Proposition 5

At time $t = T$ and $t = T-1$ the cutoffs are given by (5.1) and (5.2), as argued in the text. We now claim that $\{x^k_t\}$ are deterministic and decreasing in t and k. This is true for the last two periods. We now continue by induction.

Definitions. At time t, suppose the state is $(y^1, y^2, \ldots, y^k)$. If the seller sells one unit today then continuation profit is

$$\begin{aligned}\Pi^k_t(\text{sell 1 today}) &= m(y^1) + \delta\Pi^{k-1}_{t+1}(y^2, \ldots, y^k) \\ &= m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}^{k-1}_{t+1}(\{y^2, \ldots, y^k, v^1_{t+1}, \ldots, v^k_{t+1}\}^1_{k-1})\right]\end{aligned} \tag{A.7}$$

If the seller sells one or more units tomorrow then she will obtain

$$\Pi^k_t(\text{sell tomorrow}) = \delta E_{t+1}\left[\max\{m(y^1), m(v^1_{t+1})\}\right] + \delta E_{t+1}\left[\tilde{\Pi}^{k-1}_{t+1}(\{y^1, y^2, \ldots, y^k, v^1_{t+1}, \ldots, v^k_{t+1}\}^2_k)\right] \tag{A.8}$$

Denote the difference function by

$$\Delta\Pi^k_t(y^1, \ldots, y^k) = \Pi^k_t(\text{sell 1 today}) - \Pi^k_t(\text{sell tomorrow}).$$

As shown in Lemma 4 in Appendix A.4, $\Delta\Pi_t$ is independent of $\{y^2, \ldots, y^k\}$, so we can write it as a function of y^1 only.

Monotonicity in k and t. Since selling the last unit is identical to selling a single unit the cutoffs are determined by (3.3) and obey $x_t^1 \geq x_{t+1}^1$. By contradiction, let $k \geq 2$ be the smallest number that either (a) $x_t^k > x_t^{k-1}$ or (b) $x_t^k < x_{t+1}^k$.[19]

Case (a). Consider good k in period t. We have

$$x_t^k > x_t^{k-1} \geq x_{t+1}^{k-1} \geq x_{t+1}^k, \tag{A.9}$$

so the k^{th} cutoff is decreasing in t. At the cutoff the seller is indifferent between selling today and waiting. If she sells today she earns $\Pi_t^k(\text{sell today}) \geq \Pi_t^k(\text{sell 1 today})$. If she waits then (A.9) implies that she sells good k tomorrow and $\Pi_t^k(\text{wait}) = \Pi_t^k(\text{sell tomorrow})$. The indifference condition therefore implies that, at the cutoff,

$$\Delta\Pi_t^k(x_t^k) \leq 0. \tag{A.10}$$

Consider good $k-1$ in period t. Since $\{x_t^j\}_{j<k}$ are decreasing in k, $\Pi_t^{k-1}(\text{sell today}) = \Pi_t^{k-1}(\text{sell 1 today})$. Since $x_t^{k-1} \geq x_{t+1}^{k-1}$, $\Pi_t^{k-1}(\text{wait}) = \Pi_t^{k-1}(\text{sell tomorrow})$. As a result,

$$\Delta\Pi_t^{k-1}(x_t^{k-1}) = 0. \tag{A.11}$$

We therefore conclude that

$$0 \geq \Delta\Pi_t^k(x_t^k) > \Delta\Pi_t^k(x_t^{k-1}) \geq \Delta\Pi_t^{k-1}(x_t^{k-1}) = 0. \tag{A.12}$$

yielding the required contradiction. In equation (A.12), the first inequality comes from (A.10). The second comes from $x_t^k > x_t^{k-1}$ and Lemma 5, which says that $\Delta\Pi_t^k(x)$ is strictly increasing in x. The third inequality comes from Lemma 6, which says that $\Delta\Pi_t^k(x)$ is decreasing in k. The final equality comes from (A.11).

Case (b). Consider good k in period t. We have

$$x_t^k < x_{t+1}^k \leq x_{t+1}^{k-1} \leq x_t^{k-1}$$

so $\{x_t^j\}_{j\leq k}$ are decreasing in k. At the cutoff the seller is indifferent between selling today and waiting. If the seller sells today she earns $\Pi_t^k(\text{sell today}) = \Pi_t^k(\text{sell 1 today})$, since $\{x_t^j\}_{j\leq k}$ are increasing in k. If she waits then she obtains $\Pi_t^k(\text{wait}) \geq \Pi_t^k(\text{sell tomorrow})$. The indifference

[19] Since $x_t^{k-1} \geq x_{t+1}^{k-1} \geq x_{t+1}^k$, these two cases are mutually exclusive.

condition implies that, at the cutoff,

$$\Delta\Pi_t^k(x_t^k) \geq 0 \tag{A.13}$$

Consider good k in period $t+1$. Since $\{x_t^j\}_{j\leq k}$ are decreasing in k, $\Pi_{t+1}^k(\text{sell today}) = \Pi_{t+1}^k(\text{sell 1 today})$. Since $x_{t+1}^k \geq x_{t+2}^k$, $\Pi_{t+1}^k(\text{wait}) = \Pi_{t+1}^k(\text{sell tomorrow})$. As a result,

$$\Delta\Pi_{t+1}^k(x_{t+1}^k) = 0. \tag{A.14}$$

We therefore conclude that

$$0 \leq \Delta\Pi_t^k(x_t^k) < \Delta\Pi_t^k(x_{t+1}^k) \leq \Delta\Pi_{t+1}^k(x_{t+1}^k) = 0. \tag{A.15}$$

yielding the required contradiction. In equation (A.15), the first inequality comes from (A.13). The second comes from $x_t^k < x_{t+1}^k$ and Lemma 5, which says that $\Delta\Pi_t^k(x)$ is strictly increasing in x. The third inequality comes from Lemma 7, which says that $\Delta\Pi_t^k(x)$ is increasing in t. The final equality comes from (A.14).

Summary. Given that $\{x_t^k\}$ are decreasing in k and t the optimal cutoffs are given by $\Delta\Pi_t^k(x_t^k) = 0$. Using Lemma 4 we can assume $y^j = 0$ for $j \geq 2$ and write this as (5.3).

A.4 Lemmas for Proof of Proposition 5

Lemma 3. *Fix t and suppose $\{x_s^k\}_{s\geq t+1}$ are decreasing in k. Suppose $y^1 \geq y^2 \geq \ldots \geq y^k$, and let $y^{j-1} \geq \tilde{y}^1 \geq y^j$. Then the difference*

$$\tilde{\Pi}_{t+1}^k(y^1, y^2, \ldots, y^k) - \tilde{\Pi}_{t+1}^k(\tilde{y}^1, y^2, \ldots, y^k)$$

is independent of $\{y^j, \ldots, y^k\}$.

Proof. Suppose the state is $(y^1, y^2, \ldots, y^k)$ and pick $i \geq j$. Since cutoffs are decreasing in k, Lemma 2 says the good is allocated to value y^i if and only if (a) given previous allocations (including those within the period), y^i has the highest value; and (b) y^i exceeds the current cutoff. Since this rule only depends on the rank of y^i, the allocation rule is the same as in state $(\tilde{y}^1, y^2, \ldots, y^k)$. Hence the difference in continuation profits is independent of y^i, as required. □

Lemma 4. *Fix t and suppose $\{x_s^k\}_{s\geq t+1}$ are decreasing in k. Then $\Delta\Pi_t^k(y^1, \ldots, y^k)$ is independent of $\{y^2, \ldots, y^k\}$.*

Proof. *Case 1.* Suppose $y^1 \geq v^1_{t+1}$. Then

$$\Pi^k_t(\text{sell 1 today}) = m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}^{k-1}_{t+1}(\{y^2,\dots,y^k,v^1_{t+1},\dots,v^k_{t+1}\}^1_{k-1})\right]$$
$$\Pi^k_t(\text{sell tomorrow}) = \delta m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}^{k-1}_{t+1}(\{y^2,\dots,y^k,v^1_{t+1},\dots,v^k_{t+1}\}^1_{k-1})\right]$$

Hence $\Delta\Pi^k_t = (1-\delta)m(y^1)$, which is independent of $\{y^2,\dots,y^k\}$.

Case 2. Suppose $y^1 < v^k_{t+1}$. Then

$$\Pi^k_t(\text{sell 1 today}) = m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}^{k-1}_{t+1}(v^1_{t+1},v^2_{t+1}\dots,v^{k-1}_{t+1})\right]$$
$$\Pi^k_t(\text{sell tomorrow}) = \delta E_{t+1}\left[m(v^1_{t+1})\right] + \delta E_{t+1}\left[\tilde{\Pi}^{k-1}_{t+1}(v^2_{t+1}\dots,v^k_{t+1})\right]$$

Hence $\Delta\Pi^k_t$ is independent of $\{y^2,\dots,y^k\}$.

Case 3. Suppose $v^{j-1}_{t+1} > y^1 \geq v^j_{t+1}$ for $j \in \{2,\dots,k\}$. Then

$$\Pi^k_t(\text{sell 1 today}) = m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}^{k-1}_{t+1}(v^1_{t+1},v^2_{t+1}\dots,v^{j-1}_{t+1},\{y^2,\dots,y^k,v^j_{t+1},\dots,v^k_{t+1}\}^1_{k-j})\right]$$
$$\Pi^k_t(\text{sell tomorrow}) = \delta E_{t+1}[m(v^1_{t+1})] + \delta E_{t+1}\left[\tilde{\Pi}^{k-1}_{t+1}(y^1,v^2_{t+1}\dots,v^j_{t+1},\{y^2,\dots,y^k,v^j_{t+1},\dots,v^k_{t+1}\}^1_{k-j})\right]$$

Since $\{x^k_s\}_{s\geq t+1}$ are increasing in k, we can apply Lemma 3, implying that $\Delta\Pi^k_t$ is independent of $(y^2,\dots,y^k)$. □

Lemma 5. *$\Delta\Pi^k_t(y^1)$ is strictly increasing in y^1.*

Proof. Using equation (A.7),

$$\frac{d}{dy^1}\Pi^k_t(\text{sell 1 today}) = m'(y^1)$$

Using equation (A.8) and the envelope theorem,

$$\frac{d}{dy^1}\Pi^k_t(\text{sell tomorrow}) = m'(y_1)\delta^{\tau^k_1(y^1)-t}$$

where $\tau^k_1(y^1)$ is the time y^1 buys when he's in first position at time t and there are k goods to sell. The result follows from the fact that $\tau^k_1(y^1) > t$ and $\delta < 1$. □

Lemma 6. *Fix t and suppose $\{x^k_s\}_{s\geq t+1}$ are decreasing in k. Then $\Delta\Pi^k_t(y^1)$ is increasing in k.*

Proof. Let $\{y^1,\dots,y^k\}$ and $\{\tilde{y}^1,\dots,\tilde{y}^k\}$ be arbitrary vectors, where $y^j \geq \tilde{y}^j$ for each j. Using equation (2.4),

$$\begin{aligned}\tilde{\Pi}^k_{t+1}(y^1,\dots,y^k) - \tilde{\Pi}^k_{t+1}(\tilde{y}^1,\dots,\tilde{y}^k) &\geq \tilde{\Pi}^k_{t+1}(y^1,\dots,y^{k-1},\tilde{y}^k) - \tilde{\Pi}^k_{t+1}(\tilde{y}^1,\dots,\tilde{y}^{k-1},\tilde{y}^k) \\ &= \delta^{-(t+1)}\int_{\{\tilde{y}^1,\dots,\tilde{y}^{k-1}\}}^{\{y^1,\dots,y^{k-1}\}} (m'(z^1)\delta^{\tau^k_1(z^1)},\dots,m'(z^{k-1})\delta^{\tau^k_{k-1}(z^{k-1})})d(z^1,\dots,z^{k-1}) \\ &\geq \delta^{-(t+1)}\int_{\{\tilde{y}^1,\dots,\tilde{y}^{k-1}\}}^{\{y^1,\dots,y^{k-1}\}} (m'(z^1)\delta^{\tau^{k-1}_1(z^1)},\dots,m'(z^{k-1})\delta^{\tau^{k-1}_{k-1}(z^{k-1})})d(z^1,\dots,z^{k-1}) \\ &= \tilde{\Pi}^{k-1}_{t+1}(y^1,\dots,y^{k-1}) - \tilde{\Pi}^{k-1}_{t+1}(\tilde{y}^1,\dots,\tilde{y}^{k-1}) \qquad \text{(A.16)}\end{aligned}$$

The first line comes from the fact that $y^k \geq \tilde{y}^k$. The second line use the envelope theorem, where τ^k_1 is the stopping time of the agent in the 1st position when there are k object for sale. The third line follows from the fact that stopping times increase when the seller has one less object since $\{x^k_s\}_{s\geq t+1}$ are decreasing in k. The final line again uses the envelope theorem.

Looking at equations (A.7) and (A.8), observe that the vector

$$\{y^2,\dots,y^k,v^1_{t+1},\dots,v^{k-1}_{t+1}\}^1_{k-1}$$

is pointwise larger than the vector

$$\{y^1,y^2,\dots,y^k,v^1_{t+1},\dots,v^{k-1}_{t+1}\}^2_k.$$

The result follows from equation (A.16). □

Lemma 7. *Fix t and suppose $\{x^k_s\}_{s\geq t+1}$ are decreasing in s and k. Then $\Delta\Pi^k_{t+1}(y^1) \geq \Delta\Pi^k_t(y^1)$.*

Proof. If $y^1 \geq v^1_{t+1}$ then $\Delta\Pi^k_t(y^1) = (1-\delta)m(y^1)$ is independent of t, as shown in Lemma 4. We thus assume $y^1 < v^1_{t+1}$ and let $\tilde{y}^1 = \max\{y^1, v^k_{t+1}\}$. Lemma 4 implies that the values below $\tilde{y}^1$ do not affect $\Delta W^k_t(y^1,\dots,y^k)$, so we can set $y^j = 0$ for $j \geq 2$. For shorthand, write

$$\tilde{\Pi}^{k-1}_{t+1}(z) := \tilde{\Pi}^{k-1}_{t+1}(z, v^2_{t+1}\dots,v^{k-1}_{t+1}).$$

Then using Lemma 4,

$$\Delta\Pi^k_t(y^1) = m(y^1) - \delta E_{t+1}[m(v^1_{t+1})] + \delta E_{t+1}[\tilde{\Pi}^{k-1}_{t+1}(v^1_{t+1}) - \tilde{\Pi}^{k-1}_{t+1}(\tilde{y}^1)]$$

Using the envelope theorem,

$$\tilde{\Pi}^{k-1}_{t+1}(v^1) - \tilde{\Pi}^{k-1}_{t+1}(y^1) = \delta^{-1}\int_{\tilde{y}^1}^{v^1_{t+1}} m'(z)\delta^{\overline{\tau}^{k-1}(z)-t}\,dz$$

where $\overline{\tau}^k(z)$ is the time the object is allocated to type z, holding $\{v_{t+1}^2, \dots, v_{t+1}^{k-1}\}$ constant. As t increases the cutoff x_t^k decreases and $\overline{\tau}^k(z) - t$ falls. Hence $\delta^{\overline{\tau}^k(z)-t}$ and $\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1) - \tilde{\Pi}_{t+1}^{k-1}(\tilde{y}^1)$ increases. Since N_t is IID, $\Delta\Pi_t^k$ increases, as required. □

A.5 Lemma 7′ for Proof of Proposition 6

Lemma 7′. *Fix t and suppose $\{x_s^k\}_{s \geq t+1}$ are decreasing in s and k. Then $\Delta\Pi_{t+1}^k(y^1) \geq \Delta\Pi_t^k(y^1)$.*

Proof. Let $\hat{v}_{t+2}^j$ be the order statistics at time $t+2$ if the number of bidders N_{t+2} were drawn from $g_{t+1}(\cdot)$. Define

$$\Delta\hat{\Pi}_{t+1}^k(y^1) = m(y^1) + \delta E_{t+2}\left[\tilde{\Pi}_{t+2}^{k-1}(\{y^2, \dots, y^k, \hat{v}_{t+2}^1, \dots, \hat{v}_{t+2}^k\}_{k-1}^1)\right]$$
$$- \delta E_{t+2}\left[\max\{m(y^1), m(\hat{v}_{t+2}^1)\}\right] + \delta E_{t+2}\left[\tilde{\Pi}_{t+2}^{k-1}(\{y^1, y^2, \dots, y^k, \hat{v}_{t+2}^1, \dots, \hat{v}_{t+2}^k\}_k^2)\right]$$

where we have replaced v_{t+2}^j with $\hat{v}_{t+2}^j$, for $j \in \{1, \dots, k\}$. Since N_t is decreasing in the usual stochastic order, $\hat{v}_{t+2}^j$ exceeds v_{t+2}^j in the usual stochastic order. Since each entrant buys earlier under "sell tomorrow", this change increases Π_t^k(sell tomorrow) more than Π_t^k(sell 1 today). Hence $\Delta\Pi_{t+1}^k(y^1) \geq \Delta\hat{\Pi}_{t+1}^k(y^1)$.

We now prove that $\Delta\hat{\Pi}_{t+1}^k(y^1) \geq \Delta\Pi_t^k(y^1)$. Since $\hat{v}_{t+2}^j$ and v_{t+1}^j have the same distribution, we can assume that $\hat{v}_{t+2}^j = v_{t+1}^j$ for each j.

Case 1. When $y^1 \geq v_{t+1}^1$ then $\Delta\hat{\Pi}_{t+1}^k(y^1) = \Delta\Pi_t^k(y^1) = (1-\delta)m(y^1)$, as required.

Case 2. Suppose $y^1 < v_{t+1}^1$ and let $\tilde{y}^1 = \max\{y^1, v_{t+1}^k\}$. Lemma 4 implies that the values below $\tilde{y}^1$ do not affect $\Delta\Pi_t^k(y^1, \dots, y^k)$, so we can set $y^j = 0$ for $j \geq 2$. For shorthand, write

$$\tilde{\Pi}_{t+1}^{k-1}(z) := \tilde{\Pi}_{t+1}^{k-1}(z, v_{t+1}^2 \dots, v_{t+1}^{k-1}).$$

Then using Lemma 4,

$$\Delta\Pi_t^k(y^1) = y^1 - \delta E_{t+1}[m(v_{t+1}^1)] + \delta E_{t+1}[\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1) - \tilde{\Pi}_{t+1}^{k-1}(\tilde{y}^1)]$$

Using the envelope theorem,

$$\tilde{\Pi}_{t+1}^{k-1}(v^1) - \tilde{\Pi}_{t+1}^{k-1}(y^1) = \delta^{-1}\int_{\tilde{y}^1}^{v_{t+1}^1} m'(z)\delta^{\overline{\tau}^{k-1}(z)-t}\,dz$$

where $\overline{\tau}^k(z)$ is the time the object is allocated to type z, holding $\{v_{t+1}^2, \dots, v_{t+1}^{k-1}\}$ constant. Since the cutoff types are decreasing in k, agent z buys the first time (a) he has the highest valuation, and (b) his type exceeds the cutoff. Since (a) future order statistics are decreasing in t, and (b) future cutoffs decrease in t, $\overline{\tau}^k(z) - t$ decreases in t. Hence $\delta^{\overline{\tau}^k(z)-t}$ and $\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1) - \tilde{\Pi}_{t+1}^{k-1}(\tilde{y}^1)$

increases in t. Since $\hat{v}^1_{t+2}$ and v^1_{t+1} have the same distribution, $\Delta\hat{\Pi}^k_{t+1}(y^1) \geq \Delta\Pi^k_t(y^1)$, as required. □

A.6 Proof of Proposition 7

We first show that for $t < T-1$, $m(x_{t+1}) \geq \beta m(x_t)$. By contradiction, let t be the last time this inequality is not satisfied, so $m(x_{t+1}) < \beta m(x_t)$. It follows that, if the seller chooses not to sell to type x_t at time t then she will sell at time $t+1$. Hence the time-t cutoff is determined by

$$m(x_t) = \delta E_{t+1}[\max\{\beta m(x_t), m(v^1_{t+1})\}] \tag{A.17}$$

where the left-hand side is the payoff today, and the right-hand side is the payoff from delaying using the fact that $m(x_{t+1}) < \beta m(x_t)$. At time $t+1$ the cutoff satisfies

$$m(x_{t+1}) \geq \delta E_{t+2}[\max\{\beta m(x_{t+1}), m(v^1_{t+2})\}] \tag{A.18}$$

where the right-hand side is lower bound on the value from delaying. From (A.17) and (A.18), $m(x_{t+1}) \geq m(x_t)$, which contradicts the assumption that $m(x_{t+1}) < \beta m(x_t)$.

Since $m(x_{t+1}) \geq \beta m(x_t)$, we know that if type x_t does not obtain a good in period t then he will not obtain one until period T. The resulting indifference equation yields equation (6.3).

Finally, since the seller can always choose allocate the good by period $T-1$, the profit function obeys $\pi_t(v) \geq \pi_{t+1}(v)$. Equation (6.2) thus implies that x_t decreases over time.

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