Implementing Efficient Graphs in Connection Networks

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Abstract

We consider the problem of sharing the cost of a network which meets the connection demands of a set of agents. The agents simultaneously choose a path in the network connecting the demand nodes of the agents, and a mechanism splits the total cost of the network formed among the participants.

The recent literature has converged to the *Shapley mechanism* (Sh) which splits the cost of edges equally among its users. Two reasons motivate us to look at alternatives mechanisms. First, Sh is inefficient, asymmetric and discontinuous at equilibrium. Second, Sh requires an amount of information which may not be practical in many settings.

We characterize a class of mechanisms in a setting of minimal information requirement, specifically when the inputs of a mechanism are the total cost of the network formed and the cost of the paths demanded by the agents. The *Average Cost mechanism* (AC) and other asymmetric mechanisms implement the efficient connection. These mechanisms are characterized under three alternative robust properties of efficient implementation.

We also show that efficiency and individual rationality are mutually incompatible. The *Egalitarian mechanism* (EG), a variation of AC that meets individual rationality, is an optimal mechanism (under the price of stability measure) across all individually rational mechanisms. EG outperforms Sh on the grounds of information requirements, stability and symmetry at equilibrium. Moreover, EG is no more inefficient than Sh.

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1 Introduction

We consider the problem of sharing the cost of a congestion-free network which meets the connection demands of a set of agents. The agents simultaneously choose a path in the network connecting the demand nodes of the agents, and a mechanism splits the total cost of the network formed among the participants. This type of problem arises in many contexts ranging from water distribution systems, road networks, telecommunications services and multicast transmission to large computer networks such as Internet.

The Shapley Mechanism ([3]), which divides the cost of every edge equally among its users, has become focal in this setup. Even though Shapley looks a natural mechanism in this setting, there are serious problems associated with it which we discuss as following. First, this method may provide wrong incentives to the players and they may end up choosing an inefficient graph in equilibrium. Indeed, consider the network in figure 1 right. The equilibrium under the Shapley mechanism is (st_1, st_2) which has a total cost equal to 2, whereas the efficient connection network has cost equal to $\frac{3}{2} + \epsilon$. Even the best equilibrium can be as costly as $H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ times the cost of optimal graph, where k is the number of users ([2]). Next issue with Shapley mechanism is its asymmetry at equilibrium. Even though the mechanism is symmetric, at equilibrium it may charge different amounts to agents who are in exactly symmetric situations before the choice of paths by the agents. To see this problem, consider the symmetric network for two agents with common sources and two sinks depicted in the left panel of figure 1. Here, the Nash equilibria of the Shapley mechanism are (st_1, st_1t_2) or (st_2t_1, st_2). Thus agents pay either $(\frac{1}{2}, 1-\epsilon)$ or $(1-\epsilon, \frac{1}{2})$ depending on the equilibrium. Hence, even though the network is symmetric, agents pay different costs at equilibrium under the Shapley mechanism. This example also points to the multiplicity of equilibria and thus the problem of equilibrium selection. Next major concern with this mechanism is that they are not continuous in the network structure. The mechanism is very discontinuous and hence unstable: the two networks in figure 1 can be arbitrarily close under any measure, whereas the equilibriums will be arbitrarily different under the Shapley mechanism. Continuity is also desirable since unavoidable measurement errors in practical life may lead to very unfair outcomes.

Finally, we notice that the amount of information needed for Shapley mechanism may not be practical in many settings. The Shapley mechanism needs as input the paths chosen by each agent. This information can be out of reach in many settings. Consider for instance the network of roads in a state, district or a country to be financed by the users of the roads. The procurement of the information on exact paths used by the drivers needs the compulsory installment of GPS (Global Positioning System) in all the vehicles and the data to be stored and updated by a central taxing authority. Due to privacy issues this may not be possible politically (see for example [11]). However, tax based on the number of miles driven can be implemented without raising that much privacy concerns. Road maintenance taxes, based on the miles driven by every user have been used in pilot programs in Oregon since January 2009, and other

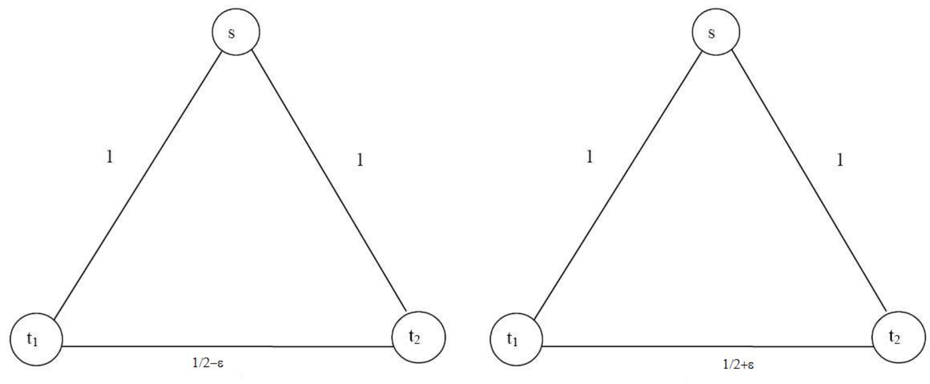

Figure 1: Symmetric networks with a common source and two sinks

states like Ohio, Pennsylvania, Colorado, Florida, Rhode Island, Minnesota and Texas are considering them (see [6, 7, 8, 9, 10, 11]). This kind of environment requires mechanisms where the input is the *total cost of the paths used by the agents* rather than the path itself. Moreover, in spite of the information on the paths being available, it may sometimes be desirable to use just the total costs of the paths rather than the paths itself. Consider, for instance a big or highly dynamic network structure, where agents join and leave the network continuously. It may be impractical to change the formulae of our mechanism every time the network changes. One such example is sharing the cost of a telephone network or Internet where the agreement is generally monthly but there are agents coming in and leaving the network continuously. Notice that, long distance calls being charged the same makes sense irrespective of number of users who share the edges[1]. There are normative concerns too for charging the agents who may not be responsible for their links not being shared by a lot users. Examples are electricity/water supply or postal services to remote villages.

This type of setting demands a new framework which is easy to implement in such settings where the inputs of the mechanism are only the total cost of the agents demand and the total cost of the network formed. This type of problem resembles the classic bankruptcy problem (also referred in the literature as rationing or taxation problem), where a given amount of resources (e.g., money) must be divided among beneficiaries with unequal claims on the resources (see [20] [22] for detailed surveys about the problem).

[1]The choice of path is not a strategy for the telephone user and thus the setting is not exactly the same but the cost-sharing method has a similar motivation, namely its simpler than charging every caller differently based on the path used.

1.1 Overview of the results

We propose mechanisms which implement the efficient graph in a centralized communication network. Our definition of implementation is weaker than that of full implementation. More precisely, we say that an outcome is implemented by a mechanism when that outcome is " a" Nash Equilibrium (NE) in the game induced by the mechanism. We also provide an equilibrium selection rule when multiplicity of equilibria exists. We require the implemented graph to Pareto dominate any other graph which is an equilibrium under that mechanism, whenever possible. The main contribution of the paper is the characterization of mechanisms which implements the efficient graph under such robust equilibria. It turns out that the mechanisms monotonic in total cost, which admits efficient graph as equilibrium and Pareto dominating other equilibrium graphs, also admits efficient graph as a strong equilibrium (Theorem 1). We also give a characterization of the average cost mechanism (AC) ([19] [14]) which divides the total cost of the network equally among its participants (Theorem 2).

The main downplay of AC is that it does not meet individual rationality (IR, also referred in the literature as voluntary participation): agents demanding cheap links may pay more than the cost of their demands, thus they may subsidize agents who demand expensive links. We show that there is no efficient rule that is compatible with IR (Theorem 3*i*). However, we find out that the egalitarian rule (EG), a rule reminiscent to the AC that meets IR, always possesses a pure strategy NE and satisfies IR. EG is optimum across all rules meeting IR under the Price of Stability measure (PoS),[2] the traditional inefficiency measure used in this literature (see [3]). EG is no more wasteful than the Shapley mechanism. It has a price if stability equal to $H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$, where k is the number of agents in the network[3] (Theorem 3*ii*). This is remarkable since, as we have discussed before, EG requires much less information than Sh. Finally, the proportional method, a seemingly natural method in this framework, also admits a pure strategy NE but is far more inefficient than the egalitarian rule (Theorem 3*iii*).

1.2 Related literature

The performance of Sh has been widely studied in recent literature. [3] studies the equilibrium behavior of separable mechanisms, a class of decentralized mechanisms that divides the cost of each edge among its users. The PoS of separable mechanisms with linear cost-sharing function is at least $H(k)$ (which is $O(\log k)$), where k is the number of agents [3]. $H(k)$ is also also the upper bound on PoS(Sh) in general graphs [2], thus Sh is optimal among linear separable mechanisms. PoS(Sh) is achieved in directed graphs. If the graph

[2]PoS is computed by finding the maximum of the ratio of the *best* Nash equilibrium and the efficient graph over all problems.

[3]An alternative measure is the price of anarchy (PoA). PoA is computed similarly to PoS, but using the *worst* Nash equilibrium instead of the best. EG and Sh are also equally inefficient under PoA. Both rules have a PoA equal to k. However, PoA is not informative since any other symmetric rule has the same PoA.

is undirected, PoS(Sh) is lower than $H(k)$. [1] finds a new upper bound of $O(\log\log k)$ when the graph is single source and there are no steiner nodes. [12] finds a new upper bound of $O(\log k/\log\log k)$ for single source networks when steiner nodes are allowed. [3] shows that the upper bound in two player case with single source is $\frac{4}{3}$. [15] finds out that $\frac{4}{3}$ is also the upper bound in general multicommodity case.

2 The model

We fix the number of agents $\bar{K} = \{1, 2, \ldots, k\}$. A network cost-sharing problem is a tuple $N =< G, K >$, where $G = (V, E)$ is a network which is a directed or undirected such that each edge $e \in E$ has a non negative cost c_e. $K = \{\{s_1, t_1\}, \{s_2, t_2\}, \ldots, \{s_k, t_k\}\}$, where $\{s_i, t_i\} \in 2^V$ for all $i \in \bar{K}$, is the set of sources and sinks that agents want to connect. When there is no confusion, we also denote $K = \bar{K}$ the set of agents. Let the set of all graphs be $\mathbf{G}$, and the set of all network cost-sharing problems be denoted by $\mathbf{N}$.

Given a problem $N \in \mathbf{N}$, a strategy for agent i is a path $P_i \subseteq E$ which connects s_i to t_i. Let the set of paths connecting s_i to t_i be $\Pi_i(N)$. Let $\Pi(N) \equiv \underset{i \in K}{\times} \Pi_i(N)$ is the set of strategy profiles of all agents in network N. $P = \{P_i\}_{i=1}^k \in \Pi(N)$ will be used to denote a strategy profile of the agents. When there is no confusion we denote $\Pi_i(N)$ and $\Pi(N)$ simply as Π_i and Π respectively. Let $G_P = (\underset{i \in \bar{K}}{\cup} P_i)$, the network formed by the choice of paths by different agents. Let $C(P) = \sum_{e \in G_P}(c_e)$ the cost of the graph formed by strategies P.

Let $\mathcal{N} = \cup_{N \in \mathbf{N}} P(N) \times N$ the union of all problems with their respective strategies.

Definition 1 *A cost-sharing mechanism is a mapping $\varphi : \mathcal{N} \rightarrow \mathbb{R}^k_+$ such that*

$$\sum_{i \in K} \varphi_i(P, N) = C(P) \text{ for all } (P, N) \in \mathcal{N}.$$

A cost-sharing mechanism assigns non-negative cost-shares to the users of the network based on their demands such that the total cost of the network formed is exactly collected.

Example 1

- *The Shapley mechanism, Sh, divides the cost of every link equally across it users, that is $Sh_i(P, N) = \sum_{e \in P_i} \frac{c_e}{U(e,P)}$ for all $i \in \bar{K}$, where $U(e, P)$ is the number of users of link e in the strategy profile P.*
- *The proportional to stand-alone mechanism, η^{pr}, divides the cost of the network in proportion to every user's stand-alone cost. That is, $\eta^{pr}_i(P, N) = \frac{SA_i(N)}{SA_1(N)+\cdots+SA_k(N)} C(P)$ for all $i \in \bar{K}$, where $SA_i(N) = \min_{P_i \in \Pi(N)} C(P_i)$ is the stand alone of agent i in network N.*
- *The Average cost mechanism AC divides the cost of the network formed equally across all users. That is $AC_i(P, N) = \frac{C(P)}{k}$ for all $i \in \bar{K}$.*

The Shapley mechanism is a separable mechanism, that is it divides the cost of every link only across its users, and adds those costs for all links in the network formed. Alternative separable mechanisms can be constructed by considering different cost-sharing rules for the links, for instance by giving priority across all users. Nevertheless, Sh is the optimal mechanism (using the price of stability measure, see below) across all separable mechanism ([3]). Sh can be computed in polynomial time.

On the other hand, η^{pr} divides the cost of the network in proportion to the stand alone of the agents. Since the stand-alone of every agent has to be computed for every network, this mechanism uses the full information of the network.

AC divides the cost of the network formed equally across the users of the network. It is the most egalitarian rule, reminiscent to the classic head tax rule where the size of the demands of the agents is not important, only the size of the total cost of the network formed. AC uses less information than Sh or η^{pr}, since only the total cost of the network formed and the number of agents is needed to compute the cost-sharing allocation. There is no need to know the stand-alone of the agents, or the users of certain links. As such, its computation complexity is minimal.

Definition 2 *A cost-sharing mechanism φ is network independent if for any two problems $N =< G, K >$ and $N' =< G', K' >$ and strategies $P \in P(N)$ and $P' \in P(N')$ such that $C(P_i) = C(P'_i)$ for all $i \in \bar{K}$ and $C(P) = C(P')$: $\varphi(P, N) = \varphi(P', N')$.*

Network independence captures those mechanisms that only depend on the cost of the network being formed and the cost of the demands of the agents. Neither Sh nor η^{pr} are network independent. On the other hand, AC only uses the total cost of the network and the number of users, thus it is network independent. More complex network independent mechanism are discussed below.

Let $\mathcal{S}^k = \{(c; y) \in \mathbb{R}_+ \times \mathbb{R}^k_+ | \max_i y_i \leq c \leq \sum_i y_i\}$.

Lemma 1 *A cost-sharing mechanism φ is network independent if and only if there is a unique function $\xi : \mathcal{S}^k \to \mathbb{R}^k_+$ such that $\sum_i \xi_i(c; y) = c$ for all $(c; y) \in \mathcal{S}^k$, and*

$$\varphi(P, N) = \xi(C(P); C(P_1), \dots, C(P_k))$$

for all problems $(P, N) \in \mathcal{N}$.

Proof.

The sufficient part is obvious. We prove the necessity only.

First, for any $(\tilde{c}; \tilde{y}) \in \mathcal{S}^k$ we construct the network $\tilde{N}(c; y)$ as follows. Assume without loss of generality that $\tilde{y}_1 \geq \tilde{y}_2 \geq \dots \geq \tilde{y}_k$. Choose i, $i \in \{1, \dots k\}$ such that:

$$\tilde{y}_1 + \tilde{y}_2 + \dots + \tilde{y}_i \leq c < \tilde{y}_1 + \tilde{y}_2 + \dots + \tilde{y}_{i+1}.$$

Let $\tilde{N}(c;y)$ be a linear network such that every agent has a unique strategy. All agents 1 to i have demand $\tilde{y}_i$ that do not intersect. Agent $i+1$ has demand $\tilde{y}_{i+1}$ such that a segment of length $c-(\tilde{y}_1+\tilde{y}_2+\cdots+\tilde{y}_i)$ does not intersect the other agents, and $\tilde{y}_1+\tilde{y}_2+\cdots+\tilde{y}_{i+1}-c$ intersects the other agents. Agent j, $j>i+1$ has demand $\tilde{y}_j$ contained on the demands of the agents $\{1,\ldots,i+1\}$.

Clearly, the unique strategy of agent k in $\tilde{N}(c;y)$ is y_k, and the network formed by all strategies has cost c. Define $\xi:\mathcal{S}^k\to\mathbb{R}^k_+$ as $\xi(c;y)=\varphi(\tilde{N}(c;y))$.

Second, consider any arbitrary network $N=<G,K>$ and a set of demands P. On one hand, notice that $C(P)\geq C(P_i)$ for every agent i, since $P_i\subseteq P$. On the other hand, notice that $C(P)\leq C(P_1)+\cdots+C(P_k)$, since $P\subseteq P_1\cup P_2\cup\cdots\cup P_k$.

Let $y_i=C(P_i)$ and $c=C(P)$. Then $(c;y)\in\mathcal{S}^k$. By network independence: $\varphi(P,N)=\varphi(N(c;y))=\xi(c;y)$. The uniqueness of ξ follows because it is well defined on $\mathcal{S}^k$.

■

Notice a network independent mechanism is reduced to the function ξ that is similar to a taxation (rationing, bankruptcy) solution ([22, 20]). Since we only work on mechanisms that are network independent, we refer without loss of generality to the function ξ as a mechanism. We describe below some desirable properties on the function ξ.

Definition 3 *A mechanism is continuous if the function $\xi:\mathcal{S}^k\to\mathbb{R}^k_+$ is a continuous function with the Euclidean distance.*

Continuous mechanisms capture the fact that small perturbations on the demand or cost of the network should not change the total allocation of the cost. All the network independent mechanism described in this paper meet continuity. Continuity is used on all the result without referring to it.

Given a problem $N=<G,K>$, we say P^* is an *efficient* graph if $P^*\in\arg\min\limits_{P\in\Pi(N)}C(P)$. That is, P^* is a graph that connects all the agents at a minimal cost.

Given the problem $N=<G,K>$, the mechanism ξ induces the following non-cooperative game $\Gamma^\xi(N)\equiv<\bar{K},\{\Pi_i(N)\}_{i\in\bar{K}},\{\xi_i\}_{i\in\bar{K}}>$, where the representation of the game is the standard representation of game in normal form. Namely, $\bar{K}=\{1,\ldots,k\}$ is the set of players, $\Pi_i(N)$ is the strategy space of player i, and ξ_i is the (negative of) payoff function of player i which maps a strategy profile to real numbers.

P is a *Nash Equilibrium (NE)* of $\Gamma^\xi(N)$, if $P_i\in\arg\min\limits_{\acute{P}_i\in\Pi_i(N)}\xi_i(\acute{P}_i,P_{-i})$ for all i. Let

$$NE(\Gamma^\xi(N))\equiv\{P\in\Pi(N)|P\text{ is a Nash Equilibrium of }\Gamma^\xi(N)\}$$

be the set of Nash equilibriums of the game $\Gamma^\xi(N)$.

We say that ξ (weakly) *implements* P, if $P\in NE(\Gamma^\xi(N))$.

Definition 4 *The mechanism ξ is efficient (EFF) if it implements an efficient graph for any problem N, that is $P^*\in NE(\Gamma^\xi(N))$ for some efficient graph P^*.*

The definition of efficiency just requires an efficient graph to be selected as a Nash equilibrium. This does not preclude other equilibriums to be selected.

Notice AC is efficient. Indeed, at any strategy profile P^* that implements an efficient graph every agent is paying $\frac{C(P^*)}{k}$. If an agent i deviates from P^* to $\tilde{P}_i$, then he will pay $\frac{C(P_i,P^*_{-i})}{k}$. Clearly, $\frac{C(P_i,P^*_{-i})}{k} \geq \frac{C(P^*)}{k}$ by the optimality of P^*.

Definition 5 *The mechanism ξ Pareto Nash Implements (PNI) an efficient graph if for any problem N, it implements an efficient graph and that graph Pareto dominates any other equilibrium. That is, for any problem N:*

- *There is an efficient graph P^* such that $P^* \in NE(\Gamma^\xi(N))$, and*
- *For any other $P \in NE(\Gamma^\xi(N)) : \xi(P^*) \leq \xi(P)$.*

PNI is a very robust property that guarantees the efficient allocation is selected even when multiplicity of equilibria arise. In the case of multiplicity of equilibria, PNI guarantees that **all** agents would prefer the efficient graph to any other equilibrium. Hence, multiplicity of equilibria is not an issue.

In particular, this guarantees that whenever there are multiplicity of equilibria such that agent i prefers equilibrium P^i to P^j, and agent j prefers equilibrium P^j to P^i, there should exist another equilibrium P^* (the efficient equilibrium) such that agent i prefers equilibrium P^* to P^i and agent j also prefers equilibrium P^* to P^j.

The AC mechanism is also PNI. Indeed, at the efficient graph P^*, this equilibrium would Pareto dominate any other equilibrium $\tilde{P}$ since $\frac{C(P^*)}{k} \leq \frac{C(\tilde{P})}{k}$.

Definition 6 *The mechanism ξ Strongly Nash Implements (SNI) an efficient graph if for any problem N it implements an efficient graph in strong Nash equilibrium. That is for any problem N,*

- *There is an efficient graph P^* such that $P^* \in NE(\Gamma^\xi(N))$, and*
- *for any group of agents $S \subset \{1, \ldots, k\}$, and $P \in \Pi(N)$ such that $P_{-S} = P^*_{-S}$, if $\xi_i(P) > \xi_i(P^*)$ for some $i \in S$, then $\xi_j(P) < \xi_j(P^*)$ for some $j \in S$.*

Under SNI there is no group of agents who can coordinate paths and weakly improve all of them, and at least one agent in the group strictly improve. In particular, this is similar to the Strong Nash equilibrium and to the literature on group strategyproofness ([13, 17]).

On the other hand, SNI is stronger than weakly group strategyproof, where profitable deviations are such that all agents strictly gain. We provide an example below that shows that this property is not enough to derive the main theorem.

The AC mechanism is also SNI. Indeed, at any deviation $\tilde{P}_S$ of the group of agent S from the efficient graph P^*, it should be that $\frac{C(P^*)}{k} \leq \frac{C(\tilde{P}_S,P^*_{N\setminus S})}{k}$ for all $i \in S$. Hence no agent in S would strictly improve by deviating.

Definition 7

- *The mechanism is demand monotonic (DM) if for all feasible problems* $(c; y), (c; \tilde{y}) \in \mathcal{S}^k$ *such that* $y_{-i} = \tilde{y}_{-i}$ *and* $y_i < \tilde{y}_i$: $\xi_i(c; y) \leq \xi_i(c; \tilde{y})$.
- *The mechanism is strongly demand monotonic (SDM) if for all feasible problems* $(c; y), (c; \tilde{y}) \in \mathcal{S}^k$ *such that* $y_{-i} = \tilde{y}_{-i}$ *and* $y_i < \tilde{y}_i$: $\xi_{-i}(c; y) \geq \xi_{-i}(c; \tilde{y})$.

Demand monotonicity is a weak property that requires that whenever the demand of the agent increases, everything else fixed, his payment should not decrease. Notice that does not preclude the payment of other agents would not change. Under SDM, the increase on the demand of one agent does not increase the payment of other agents. In particular, notice that SDM implies DM since all the agent's payments have to add up to a constant.

AC is clearly strongly demand monotonic since $AC(c; y) = AC(c; \tilde{y})$. Thus the increase of the demand of one agent does not change the payments of the other agents.

3 Main result

We now turn to the main result of the paper. We characterize the mechanisms that meet the efficiency properties discussed above.

Theorem 1 *Assume there are three or more agents, then the following statements are equivalent for the mechanism* ξ :

1. ξ *is EFF and SM.*
2. ξ *PNI the efficient graph.*
3. ξ *SNI the efficient graph.*
4. *There is a monotonic function* $f : \mathbb{R}_+ \to \mathbb{R}^k_+$ *such that* $\sum_i f_i(c) = c$ *and for all feasible problems* $(c; y)$, $\xi(c; y) = f(c)$.

The mechanisms characterized by theorem 1 are demand independent, that is the cost-share of every agent do not depend on whether the agents are demanding cheap or expensive links, instead they only depend on the total cost of the network formed. The average cost mechanism, generated by $f(c) = (\frac{c}{k}, \ldots, \frac{c}{k})$, is the only mechanism in this class that treat equal agents equally.

The above statements are independent. Indeed, consider the mechanism

$$\tilde{\xi}(c; y) = (\min\{y_3, \frac{c}{k}\}, \frac{2c}{k} - \min\{y_3, \frac{c}{k}\}, \frac{c}{k}, \ldots, \frac{c}{k}).$$

First notice that $\tilde{\xi}$ implements the efficient graph because at the efficient graph agents $\{3, \ldots, k\}$ do not have the incentive to deviate since by doing so their payment is going to increase. On the other hand, agents $\{1, 2\}$ do not have any incentive to deviate from the efficient equilibrium since the functions $\min\{y_3, \frac{c}{k}\}$ and $\frac{2c}{k} - \min\{y_3, \frac{c}{k}\}$ are weakly monotonic in the total cost of the network and do not depend on their report.

$\tilde{f}$ is also an example of a mechanism that is not SNI, but agents cannot *strictly* improve by coordinating. Hence the mechanisms characterized by Theorem 1 are not weakly group strategyproof.

3.1 Efficient mechanisms for two agents

The example above shows that for three or more agents, EFF is not enough to characterize the demand independent rules. On the other hand, this property is enough when there are two agents. The property is an immediate consequence of a separability lemma described below.

Proposition 1 *Assume there are two agents, $K = \{1, 2\}$. A mechanism is efficient if and only if there is a monotonic function $f : \mathbb{R}_+ \to \mathbb{R}^2_+$ such that $f_1(c) + f_2(c) = c$ and for all feasible problems $(c; y)$, $\xi(c; y) = f(c)$.*

3.2 Equal treatment of equals

Definition 8 *The mechanism satisfies equal treatment of equals (ETE) if for all agents i, j and $(c; y) \in \mathcal{N}^k$ such that $y_i = y_j : \xi_i(c; y) = \xi_j(c; y)$.*

ETE is the standard property of equal responsibility for the cost of the good. Equal agents with the same demand should be allocated the same cost. There is a large class of solutions that meet ETE. We describe in section 4 alternative rules that meet ETE, like the Proportional and Egalitarian solution.

Theorem 2 *A mechanism is EFF and ETE if and only if it is AC.*

Notice this proposition is not directly implied by theorem 1, since we do not need Strong Monotonicity. Instead, it is a separability lemma discussed in section 5.

4 Individually rational mechanisms

Definition 9 *The mechanism is individually rational if for all $(c; y) \in \mathcal{S}^k$: $\xi_i(c; y) \leq y_i$ for all i.*

Individually rational mechanisms rule out cross-subsidies, that is no agent should pay more that the cost of their demand.

Notice neither AC nor any mechanism discussed in theorem 1 meet individual rationality. Therefore, the traditional incompatibility of efficiency, budget balance and individual rationality also holds in this problem.[4]

[4]Nevertheless, this incompatibility only holds since we consider Network Independent mechanisms. If we remove network independence then there is large class of mechanisms

On the other hand, there is a large class of individually rational mechanisms that are network independent: most of the mechanisms discussed in the rationing/bankruptcy literature meet IR, see for instance [22, 20].

Definition 10 *

- *The proportional mechanism (PR): $PR_i(c;y) = \frac{y_i}{y_1+\cdots+y_k}c$*
- *The egalitarian mechanism (EG): $EG_i(c;y) = \min\{y_i, \lambda\}$ where λ solves $\sum_i \min\{y_i, \lambda\} = c$.*

PR and EG are the traditional and most compelling mechanisms in the cost-sharing literature. PR divides the cost in proportion to their demands. On the other hand, EG divides the cost equally across the agents subject to no agent paying more than their demand.

Contrary to the traditional analysis of this problem. The games induced by PR and EG are not potential games, therefore the previous potential techniques used in the analysis of this problems do not work anymore. We do not know if any mechanism (induced by a rationing method) always has a pure strategy Nash equilibrium. Nevertheless, we show below that PR and EG always have a pure strategy Nash equilibrium and provide algorithms to compute them.

Lemma 2 *PR and EG always admit a pure strategy Nash equilibrium.*

Even if the existence of equilibrium of other individually rational mechanisms is unknown, no mechanism would be more efficient than EG.

Theorem 3

i. There is no mechanism that is individually rational and EFF. Any individually rational mechanism has a PoS at least $H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

ii. The PoS of EG is $H(k)$.

iii. The PoS of PR is of order k.

Since the Shapley mechanism has a price of stability equal to $H(k)$, then EG is as inefficient Shapley. No other individually rational mechanism can be more efficient than EG and Shapley. On the other hand, the traditional proportional mechanism is extremely inefficient, since its price of stability is bounded by k, its maximal loss approaches that in the limit.

that always implement the efficient network and at the same time meet individual rationality. For instance, consider the proportional to stand-alone mechanism η^{pr} discussed above. η^{pr} is individually rational because no agent pays more than his stand alone, which in turns is less than his demand. On the other hand, η^{pr} implements the efficient allocation because the cost-share of every agent is in proportion to the cost of the network, therefore any deviation of the efficient graph that increases the total cost of the network formed would increase the cost share of all the agents.

5 Conclusions

This paper provides a new perspective to the problem of cost-sharing in networks. In particular, we provide new concepts of implementation and characterize the class of mechanisms that meet three robust definitions of efficiency. The average cost mechanism is the benchmark mechanism characterized by this paper.

It is also shown that efficiency and individual rationality are not compatible. The egalitarian mechanism is optimal across the mechanisms that are are individually rational. EG is not efficient, but is an optimal mechanism across all individually rational mechanisms using the price of stability measure. We also show that EG outperforms the Shapley mechanism on the grounds of efficiency, stability and fairness.

We do not know if EG is the unique optimum mechanism within the individually rational mechanisms, but conjecture this is true. We know that other mechanisms, like the proportional mechanism, are much more inefficient than EG. The difficulty we encounter in tackling this question is that even for simple rationing mechanisms, there is no general technique to evaluate whether or not these mechanisms even have a pure strategy Nash equilibrium.

Finally, we conjecture that our main characterization theorem discussed above on Pareto-Efficient implementation can be extended to a more general class of mechanisms that only depend on the path chosen by the agents and the network formed (notice this class contains the Shapley mechanism and all traditional separable mechanisms).

6 Proofs

6.1 Proof of Lemma 2

6.1.1 Existence of equilibrium for PR

Proof. We prove a stronger property which is that the best response (br) dynamics (one agent at a time) of any arbitrary fixed ordering of agents converges to a NE, no matter from where we start the br dynamics. Suppose on contrary, that for some fixed ordering of agents the br dynamics from some point "s" does not converge. This means that there is a cycle of a finite length l – $s(1) \rightarrow s(2) \rightarrow s(3) \rightarrow .... \rightarrow s(l) \rightarrow s(1)$. Say, without loss of generality, this cycle includes deviations by the set of agents $M = \{1, 2, ..., m\} \subseteq K$. The strategy of agents in K/M is fixed at s^{-M}. Notice that l is at least as big as $2m$. This is so because after the l best responses we arrive at the original strategy profile i.e., $s(1)$. Since, every agent in M is a part of the cycle which in turn means that they change their strategy at least once. Therefore, it must be the case that every agent in M takes its turn at least twice so that they reach the original profile i.e., $s(1)$. Let's assume that agent $i \in M$ takes its turn in the br dynamics $n_i > 1$ number of times so that $\sum_{i \in M} n_i = l$. Let the strategies played by the agent i in the cycle be $s^{i;1}, s^{i;2}, ..., s^{i;n_i}, s^{i;1}$ and so on. Let's call the agent who takes his turn of br in the movement from s_t to s_{t+1} as agent a_t. Therefore, $s(1) = (s^{1;1}, s^{2;1}, ..., s^{m;1}, s^{-M})$, $s(2) = (s^{a_1;2}, s_{-a_1}(1))$, $s(3) = (s^{a_2;2}, s_{-a_2}(2))$, ...,$s(l-1) = (s^{a_{l-1};n_{a_{l-1}}}, s_{-a_{l-1}}(l-2))$, $s(l) = (s^{a_l;n_{a_l}}, s_{-a_l}(l-1))$. Here, we use the standard notation where $s_{-i}(t)$ represents the strategy profile of $K\backslash\{i\}$ fixed at that in $s(t)$. We abuse the notation and say that the cost of $s^{p;i}$ is equal to $s^{p;i}$. Here the cost of the network formed by the strategy profile $s(i) = C(G_{s(i)})$. Now, $\xi_j^{pr}(C(G_{s(i)}); s(i)) = s^{j;p}A_i$ where A_i is fixed for any particular $s(i)$ and $s^{j;p}$ represents the strategy of agent j in $s(i)$. The fixed A_i for an $s(i)$ is the ratio of $C(G_{s(i)})$ to the sum of the costs of individual paths in $s(i)$.

Now every step of the cycle corresponds to an inequality which we will present as following:

Step 1: $s(1) \rightarrow s(2) \implies$

$$s^{a_1;2} \times A_2 < s^{a_1;1} \times A_1 \tag{1}$$

Step 2: $s(2) \rightarrow s(3) \implies$

$$s^{a_2;2} \times A_3 < s^{a_2;1} \times A_2 \tag{2}$$

Step 3: $s(3) \rightarrow s(4) \implies$

$$s^{a_3;t} \times A_4 < s^{a_3;t-1} \times A_3; t = \left\{ \begin{array}{c} 3\ if\ a_3 = a_1 \\ 2 \text{ otherwise} \end{array} \right\} \tag{3}$$

|
|

Step p: $s(p) \rightarrow s(p+1) \implies$

$$s^{a_p;t} \times A_{p+1} < s^{a_p;t-1} \times A_p; t \in \{1, 2, ..., n_{a_p}\} \tag{p}$$

|
|
Step l: $s_l \rightarrow s_1 \implies$

$$s^{a_l;n_{a_l}} \times A_1 < s^{a_l;n_{a_l}-1} \times A_l \tag{l}$$

If we multiply the systems (2), (3), ..., (l) together[5], then everything else cancels out and we are left with $s^{a_1;2} \times A_2 > s^{a_1;1} \times A_1$ which contradicts the inequality (1). Therefore, we conclude that there can not be any cycle no matter what ordering of agents and what initial point we follow for the best response dynamics. ■

[5]Notice, we can do that since everything here is positive

6.1.2 Existence of equilibrium for EG and $POS(EG) = H(k)$

Proof. We prove by induction on the number of players that EG has an equilibrium and the $POS(EG) = H(k)$.

The base of induction is one player. This case is trivially true for one player since the game is just an optimization exercise and any optimal graph, which is a cheapest path of connecting her demand nodes, is a NE .

We now assume that for all networks with number of agents $m < n$, with an efficient efficient graph G^*_m there exists a NE which costs no more than $H(m) * C(G^*_m)$. We claim that for a network with n agents, with an efficient graph G^*_n there exists a NE of cost no more than $H(n) * C(G^*_n)$. Let's call the set $N \equiv \{1, 2, 3, ..., n\}$. Let's start from the efficient graph G^*_n . Now, under $\xi^{uni}(G^*_n)$[6] there can be two cases. Either there exist an agent t s.t. $\xi^{uni}_t(G^*_n)$ $< \frac{G^*_n}{n}$ or $\xi^{uni}_j(G^*_n) = \frac{G^*_n}{n}$,$\forall j \in N$. If it is the the first case then pick the agent with the lowest cost share[7] and call this agent, "agent i". If it is the second case then there can be two cases. Either there exist an agent who has a profitable deviation or there doesn't exist such an agent. If there doesn't exist such an agent then our claim is trivially true since G^*_n is a NE. If such agents exist then pick one of them and call her "agent i". Now, ask the agent i to take her best response. There can be two cases– either $\xi^{uni}_i(G^*_n) = \lambda(y) = \frac{C(G^*_n)}{n}$ or $\xi^{uni}_i(G^*_n) = y_i < \frac{C(G^*_n)}{n}$. In both the cases, the only way agent i has a profitable best deviation is when she moves to a cheapest path P^*_i (which is also called the stand alone of agent i) connecting her demand nodes s.t.,

$$C(P^*_i) < \xi^{uni}_i(G^*_n) \leq \frac{C(G^*_n)}{n} \tag{6.1}$$

Now, there can be two cases.

Case 1: There exists such a cheapest path P^*_i and agent i moves to P^*_i.

In this case, the new network has a cost $\grave{C}$ s.t., $C(G^*_n) \leq \grave{C} \leq C(G^*_n) + C(P^*_i)$. Let's consider an efficient graph G^*_{-i} for connecting all the agents in $N/\{i\}$. Notice that since there are less nodes to be connected and all edges are still available, we have the following inequality

$$C(G^*_{-i}) \leq C(G^*_n) \tag{6.2}$$

Now, ignoring agent i there will be a network game with the the player set $N/\{i\}$. From the induction hypothesis it follows that there exists a graph configuration G^{NE}_{-i} which is a NE of this game and

$$C(G^{NE}_{-i}) \leq H(n-1) * C(G^*_{-i}). \tag{6.3}$$

We claim that if we add player i to the set $N/\{i\}$ then the configuration $\tilde{G}_n$, where i is playing P^*_i and $N/\{i\}$ are fixed at the configuration G^{NE}_{-i}, is a NE of

[6] $\xi^{uni}(G)$ is defined in the obvious way, where $d(G) = C(G)$ and $y_i = C(P_i)$ where P_i is the path chosen by agent i.

[7] In fact we can pick any agent with the cost share less than $\frac{C(G^*_n)}{n}$. It doesn't matter for the proof.

the game amongst the player set N. Let's denote the demand profile in $\tilde{G}_n$ and G^{NE}_{-i} by y and y_{-i} respectively. First notice that the optimality of G^*_n implies that $C(\tilde{G}_n) \geq C(G^*_n)$. But, this means that agent i who is paying $C(P^*_i) < \frac{C(G^*_n)}{n} \leq \frac{C(\tilde{G}_n)}{n}$ does not have any profitable deviation. This is so because P^*_i is the cheapest path to connect her demand nodes and it is impossible to bring the $\lambda(y)$ below $\frac{C(G^*_n)}{n}$ given the optimality of G^*_n. Let's think about the players in $N/\{i\}$ under $\tilde{G}_n$. By the addition of player i in the network, the strategy space of players in $N/\{i\}$ remain unchanged. Only thing which may change is the cost shares of the agents. For all $j \neq i$, $\xi^{uni}_j(G^{NE}_{-i}) \geq \xi^{uni}_j(\tilde{G}_n)$.This happens so because $\lambda(y) \leq \lambda(y_{-i})$. Thus, the agents whose cost shares were below $\lambda(y)$ remain unaffected. Also, they will not have any profitable deviation since they are already paying their stand alone costs and by introduction of new player i their strategy set remains unchanged and thus their stand alone remains unchanged. For the agents whose cost shares were above $\lambda(y)$, their stand alone must be above $\lambda(y)$. Therefore, the only deviation $\grave{y}_j$ which is profitable to such an agent j is the one which brings the $\lambda(y_{-j}, \grave{y}_j)$ below $\lambda(y)$. But, such a deviation would have been profitable in the game with the player set $N/\{i\}$ under the configuration G^{NE}_{-i} contradicting G^{NE}_{-i} being a NE. Thus we have shown that $\tilde{G}_n$ is a NE. Only thing which remains to be shown is that $C(\tilde{G}_n) \leq H(n) * C(G^*_n)$. Since $\tilde{G}_n$ is the union of the edges of G^{NE}_{-i} and P^*_i we must have

$$
\begin{aligned}
C(\tilde{G}_n) &\leq C(G^{NE}_{-i}) + C(P^*_i) && (1)\\
&\leq H(n-1) * C(G^*_{-i}) + C(P^*_i) && (2)\\
&\leq H(n-1) * C(G^*_n) + C(P^*_i) && (3)\\
&\leq H(n-1) * C(G^*_n) + \frac{C(G^*_n)}{n} && (4)\\
&= H(n) * C(G^*_n) && (5)
\end{aligned}
$$

Here the second, third and fourth inequalities comes from (6.3), (6.2) and (6.1) respectively.

Case 2: There doesn't exist a br deviation for agent i. But, from the way we have chosen our agent i, this means that y_i under G^*_n is less than $\frac{C(G^*_n)}{n}$. Then we call the existing choice of the path by agent i as P^*_i and everything else follows exactly as in the case 1 above.

■

6.2 Preliminary Lemmas

Definition 11 *The mechanism is monotonic in cost if for all feasible problems* $(c; y), (c'; y) \in \mathcal{N}^K$ *such that* $c < c'$: $\xi(c; y) \leq \xi(c'; y)$.

Lemma 3 *If the mechanism* ξ *is efficient then it is monotonic in total cost.*

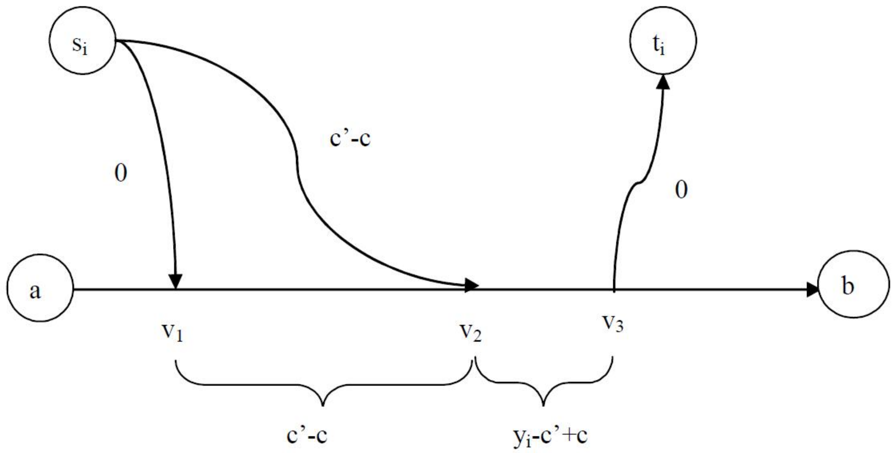

Figure 2:

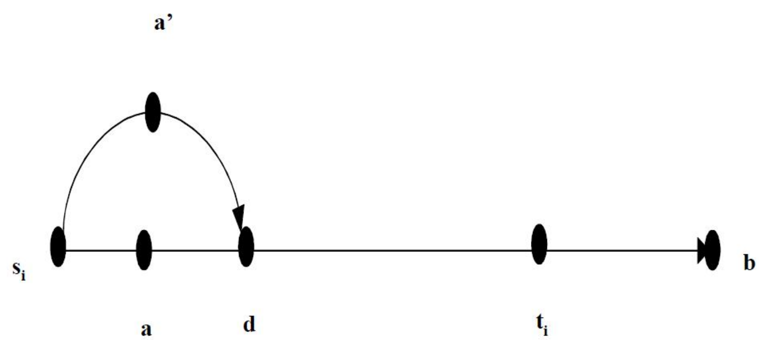

Figure 3:

Proof.

Consider two feasible problems $(c; y)$ and $(\grave{c}; y)$, where $\grave{c} > c$ and $(\grave{c} - c) < \min_{i \in K}\{y_i\}$. Suppose, there exists an agent i and an efficient ξ such that $\xi_i(\grave{c}; y) < \xi_i(c; y)$. Then we can have a network configuration which will contradict the efficiency of ξ. Consider a network where, agents $j \neq i$ have just one strategy each P_j which costs y_j. Agent i has two strategies P_i and P_i' both of which cost y_i but P_i makes the total cost of the network c and P_i' makes the total cost go up to $\grave{c}$. To see what kind network will generate these problems, consider the following two cases. Case 1: $c \leq \sum_{j \neq i} y_j$. In this case we can have a configuration as shown in figure 2. Here, the demands of agents in $K\backslash\{i\}$ is contained in the interval $a \to b$ which costs c. This is possible since when $c = \sum_{j\neq i} y_j$, we can have $a \to b$ as the concatenation of the demand links of the agents $j \neq i$ .When $c < \sum_{j\neq i} y_j$ we can have the demand links overlapping e.g., when $\max\limits_{j\neq i}\{y_j\} = c$, then $a \to b$ is the demand link of the biggest demander and all other demands overlap with his. $P_i = s_i \to v_1 \to v_2 \to v_3 \to t_i$ and $P_i' = s_i \to v_2 \to v_3 \to t_i$. All the costly links of P_i is contained in $\{\bigcup\limits_{j\neq i} P_j\}$ whereas there are links of cost $c' - c$ which are not contained in $\{\bigcup\limits_{j\neq i} P_j\}$ under P_i'. Again, this is possible since c' and c are close enough to guarantee that for all i we can have such paths. Case 2: $\sum_{j\in K} y_j > c > \sum_{j\neq i} y_j$. In this case we can have a configuration as shown in figure 3. Here, the interval $a \to b$ is the concatenation of the demand links of agents in $K\backslash\{i\}$. Thus $|a \to b| = \sum_{j\neq i} y_j$, $|s_i \to a| = c - \sum_{j\neq i} y_j$, $|a \to d| = c' - c$. $|s_i \to a \to d| = |s_i \to a' \to d| = c' - \sum_{j\neq i} y_j$. $P_i = s_i \to a \to d \to t_i$ and $P_i' = s_i \to a' \to d \to t_i$. Notice that it may be the case that $t_i = b$. Now clearly in both the cases, i will have a profitable deviation from the efficient graph of cost c thus contradicting the efficiency of ξ. Thus we have shown that efficient ξ must be monotonic in total cost in some open neighborhood of c for all c. Therefore, we can extend the argument to conclude that ξ must be monotonic in total cost in general.

■

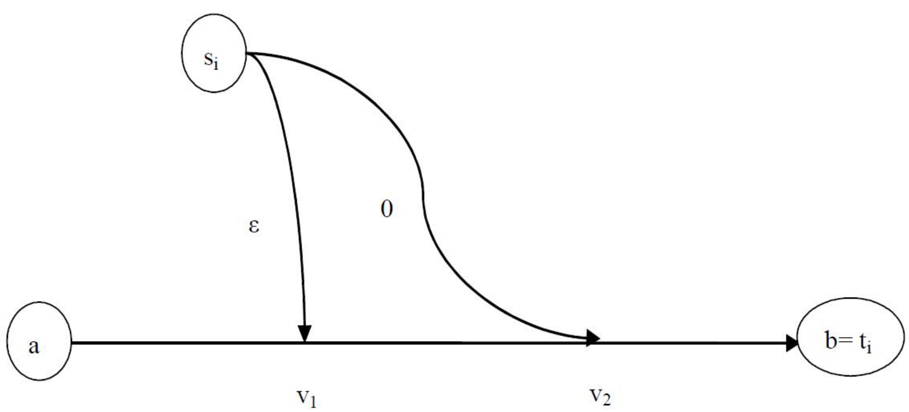

Figure 4:

Lemma 4 (Separability Lemma) *If the mechanism ξ is efficient then $\implies$ $\xi(C; y) = (\xi_1(C; y_{-1}), \xi_2(C; y_{-2}), \ldots.., \xi_k(C; y_{-k}))$. That is, any efficient mechanism is separable and assigns the costs shares to the agents independently of their demand.*

Proof. If we prove that for any feasible problems $(c; y)$ and $(c; \tilde{y}_i, y_{-i})$, any continuous and efficient ξ must have $\xi_i(c; y) = \xi_i(c; \tilde{y}_i, y_{-i})$ then we are done. Consider a feasible problem $(c; y)$.Consider a graph as shown in Figure 4 which generates this problem. The sources and sinks of agents $j \neq i$ lie on the the ray $a \to b$ according the demand profile, i.e., the agent with the highest demand covers most of the span on $a \to b$ and so on. Thus, an agent $j \neq i$ has one strategy which generates the demand y_j. Agent i has two strategies– either connect $s_i - t_i$ through v_1 or through v_2. The demands of agent i when connecting through v_1 and v_2 are $\tilde{y}_i$ and y_i respectively. Now, the total cost when i uses v_1 and v_2 are respectively "$c + \epsilon$" and "c". Notice, by moving the position of v_2 and arranging the demand links of the agents $j \neq i$, we can generate all the feasible problems $(c; y_i, y_{-i})$. Also, by moving the position of v_1 and arranging the demand links of the agents $j \neq i$, we can generate all the feasible problems $(c + \epsilon; y_i, y_{-i})$. Consider an efficient ξ which is continuous. Efficiency of ξ requires the following inequality

$$\xi_i(c; y_i, y_{-i}) \leq \xi_i(c + \epsilon; \tilde{y}_i, y_{-i}) \tag{6}$$

Using continuity we get

$$\xi_i(c; y_i, y_{-i}) \leq \xi_i(c; \tilde{y}_i, y_{-i}) \tag{7}$$

Similarly, switching the position of v_1 and v_2 and using continuity again we get

$$\xi_i(c; y_i, y_{-i}) \geq \xi_i(c; \tilde{y}_i, y_{-i}) \tag{8}$$

Thus, we conclude that $\xi_i(c; y_i, y_{-i}) = \xi_i(c; \tilde{y}_i, y_{-i})$ for all feasible problems $(c; y_i, y_{-i})$ and $(c; \tilde{y}_i, y_{-i})$.

∎

6.3 Proof of Proposition 1

Consider a problem $(c; y_1, y_2) \in S^2$.

By separability lemma: $\xi_1(c; y_1, y_2) = \xi_1(c; c, y_2)$.

By budget balance: $\xi_2(c; y_1, y_2) = \xi_2(c; c, y_2)$. Thus, $\xi(c; y_1, y_2) = \xi(c; c, y_2)$.

By separability lemma: $\xi_2(c; c, y_2) = \xi_2(c; c, c)$.

By budget balance: $\xi_1(c; c, y_2) = \xi_1(c; c, c)$. Thus, $\xi(c; c, y_2) = \xi(c; c, c)$.

Hence $\xi(c; y_1, y_2) = \xi(c; c, c)$.

6.4 Proof of Theorem 1

6.4.1 1. $\implies$ 4.

Proof.

Consider a continuous ξ which is efficient and strongly monotonic. Consider two arbitrary feasible problems $(c; y)$ and $(c; \tilde{y})$. We will prove that $\xi(c; y) = \xi(c; \tilde{y}) = f(c)$. The monotonicity of f comes from lemma 1. Let $a = \frac{1}{k} \sum_{i \in K} y_i$

and $\tilde{a} = \frac{1}{k}\sum_{i\in K}\tilde{y}_i$. Assume without loss of generality that $y_1 \leq y_2 \leq y_3 \leq ...... \leq y_k$ and $\tilde{y}_1 \leq \tilde{y}_2 \leq \tilde{y}_3 \leq ...... \leq \tilde{y}_k$.

Step 1: $\xi(c;y) = \xi(c;a,a,...,a)$ and $\xi(c;\tilde{y}) = \xi(c;\tilde{a},\tilde{a},....,\tilde{a})$

Proof:

Consider the following problems: $P_0 = (c;y)$, $P_1 = (c;a,y_2,y_3,...y_k)$, $P_2 = (c;a,a,y_3,y_4,...,y_k)$, ...,$P_k = (c;a,a,......,a)$. Notice first that feasibility of P_0 implies the feasibility of $P_1, P_2, ..., P_k$. This is true because maximum of the demand profile doesn't go above y_k in all these problems and sum of the individual demands is always at least $k*a = \sum_{i\in K} y_i$. Similarly, if we define the counterpart problems $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, ..., \tilde{P}_k$ where $\tilde{P}_i = (c;\tilde{a},\tilde{a},...,\tilde{a},\tilde{y}_{i+1},\tilde{y}_{i+2},...,\tilde{y}_{k-1},\tilde{y}_k)$, then again all of them will be feasible.

Now, due to the separability lemma (lemma 2) we must have $\xi_1(P_0) = \xi_1(P_1)$. But then, strong monotonicity and budget balancedness implies $\xi_{-1}(P_0) = \xi_{-1}(P_1)$. Thus, we have $\xi(P_0) = \xi(P_1)$. Using the same argument we have $\xi(P_i) = \xi(P_{i+1})$ and $\xi(\tilde{P}_i) = \xi(\tilde{P}_{i+1})$ for all $0 \leq i \leq k-1$. Thus, we have $\xi(P_0) = \xi(P_k)$ and $\xi(\tilde{P}_0) = \xi(\tilde{P}_k)$ as desired.

Step 2: $\xi(c;a,a,...,a) = \xi(c;\tilde{a},\tilde{a},....,\tilde{a})$

Proof:

Notice first that feasibility of $(c;a,a,...,a)$ & $\xi(c;\tilde{a},\tilde{a},....,\tilde{a})$ implies that any problem $(c;\hat{a})$ where some of the $\hat{a}_i = a$ and other $\hat{a}_i = \tilde{a}$ is also feasible. Now, lemma 2 implies $\xi_1(c;a,\tilde{a},...,\tilde{a}) = \xi_1(c;\tilde{a},\tilde{a},....,\tilde{a})$. Now, there can be three cases–$a < \tilde{a}$, $a > \tilde{a}$ or $a = \tilde{a}$. In the first two cases strong monotonicity and budget balancedness implies $\xi_{-1}(c;a,\tilde{a},...,\tilde{a}) = \xi_{-1}(c;\tilde{a},\tilde{a},....,\tilde{a})$ and we get $\xi(c;a,\tilde{a},...,\tilde{a}) = \xi(c;\tilde{a},\tilde{a},....,\tilde{a})$. The third case trivially implies $\xi(c;a,\tilde{a},...,\tilde{a}) = \xi(c;\tilde{a},\tilde{a},....,\tilde{a})$ since its the same problem so the solution must be the same. Similarly, we get $\xi(c;\tilde{a},\tilde{a},....,\tilde{a}) = \xi(c;a,\tilde{a},...,\tilde{a}) = \xi(c;a,a,\tilde{a},...,\tilde{a}) = ... = \xi(c;a,a,...,a)$.

■

6.4.2 2. $\Longrightarrow$ 1.

Proof.

We know that ξ PNI efficient graph implies ξ is efficient. We will prove that ξ PNI efficient graph implies ξ is strongly monotonic. Consider a ξ which PNI efficient graph and a feasible problem $(c;y)$ and assume without loss of generality that $y_1 < y_2 < ..... < y_k$[8]. Now, consider a graph as shown in figure 5 below.

Here every agent has two strategies– either use the path in the solid graph or use that in the dotted graph. Let's call the solid graph as "**" and the dotted graph as "*". Let "*" be a small perturbation of "**" as following. The cost of path of an agent $j \neq i$ in both the graphs is y_j. The cost of path of agent i in "**" and "*" are y_i and $\tilde{y}_i$ where $\tilde{y}_i$ is in a neighborhood of y_i and $\tilde{y}_i > y_i$ and $|\tilde{y}_i - y_i| < \min_{j,k\in K}|y_j - y_k|$. This restriction guarantees the ranking to be preserved in the perturbed problem. Let the total cost of "**" and "*" be "$c-\epsilon$" and "c"

[8]The case of weak inequality will follow from the assumption of continuity on our method

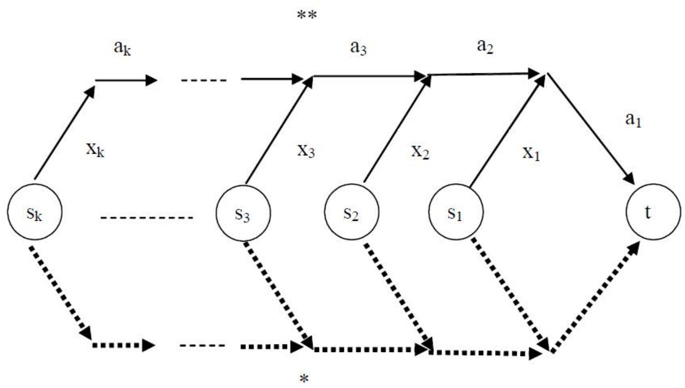

Figure 5:

respectively. First we will show that this graph generates all feasible problems $(c; y)$. This happens if and only if the following system has a solution:

$$\begin{aligned}
x_1 + a_1 &= y_1 \\
x_2 + a_2 + a_1 &= y_2 \\
x_3 + a_3 + a_2 + a_1 &= y_3 \\
&: \\
&: \\
x_k + a_k + a_{k-1} + ... + a_1 &= y_k \\
\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} a_i &= c \\
\forall i &\in K;\ x_i, a_i \geq 0
\end{aligned}$$

We use Farka's Lemma to prove that this system has a solution:

From the Farka's lemma we know that $Ax = b$; $x \geq 0$ has a solution if and only if $A^T z \geq 0$; $b^T z < 0$ doesn't have a solution.

Here, the $(k+1) \times (2k)$ matrix A, vector x and vector b are defined as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 & .... & 1 & 0 & 0 & .... \\ 0 & 1 & 0 & .... & 1 & 1 & 0 & .... \\ .... & .... & .... & .... & .... & .... & .... & .... \\ 1 & 1 & 1 & .... & 1 & 1 & 1 & .... \end{bmatrix}$$

$x = \begin{bmatrix} x_1 & x_2 & .... & x_k & a_1 & a_2 & .... & a_k \end{bmatrix}^T$

$b = \begin{bmatrix} y_1 & y_2 & .... & y_k & c \end{bmatrix}^T$

$A^T z \geq 0$; $b^T z < 0$ gives the following $(2k + 1)$ inequalities;

$$z_1 + z_2 + ... + z_{k+1} \geq 0 \tag{1}$$

$$z_2 + z_3 + ... + z_{k+1} \geq 0 \tag{2}$$

$$: \tag{:}$$

$$z_k + z_{k+1} \geq 0 \tag{k}$$

$$z_1 + z_{k+1} \geq 0 \tag{k+1}$$

$$z_2 + z_{k+1} \geq 0 \tag{k+2}$$

$$: \tag{:}$$

$$z_k + z_{k+1} \geq 0 \tag{2k}$$

$$y_1 z_1 + y_2 z_2 + ... + y_k z_k + c z_{k+1} < 0 \tag{2k+1}$$

Now, do the following operation on the first k inequalities: $y_1 \times (1) + (y_2 - y_1) \times (2) + ... + (y_k - y_{k-1}) \times (k)$, to get,

$$y_1 z_1 + y_2 z_2 + ... + y_k z_k + y_k z_{k+1} \geq 0 \tag{2k+2}$$

Now, for the inequalities (2k+1) and (2k+2) to be compatible, it must be the case that $z_{k+1} < 0$. Let, this be the case and let (2k+2) and (2k+1) hold. Then, (2k+1) implies:

$$y_1 z_1 + y_2 z_2 + ... + y_k z_k + (\sum_{i \in K} y_i) z_{k+1} < 0 \tag{2k+3}$$

This is true because feasibility requires $\sum_{i \in K} y_i \geq c$. Now, if we do the following operation on inequalities $(k+1)$ through $(2k)$: $y_1 \times (k+1) + y_2 \times (k+2) + ... + y_n \times (2k)$, then we get,

$$y_1 z_1 + y_2 z_2 + ... + y_k z_k + (\sum_{i \in K} y_i) z_{k+1} \geq 0 \tag{2k+4}$$

which contradicts (2k+3) to give us the desired result.

We now prove the strong monotonicity of ξ. Clearly, the efficiency of ξ implies that "**" is a NE but since "*" is a perturbation of "**", we will have "*" as a NE for the perturbation small enough. The fact that ξ PNI the efficient graph implies the following inequality

$$\xi(c-\epsilon; y_i, y_{-i}) \leq \xi(c; \tilde{y}_i, y_{-i})$$

Using continuity we get,

$$\xi(c; y_i, y_{-i}) \leq \xi(c; \tilde{y}_i, y_{-i})$$

Now consider a perturbation where every thing is exactly the same but "**" costs "$c+\epsilon$". Using the same argument of Pareto Nash implementability and continuity we get

$$\xi(c; y_i, y_{-i}) \geq \xi(c; \tilde{y}_i, y_{-i})$$

Thus we conclude that $\xi(c; y_i, y_{-i}) = \xi(c; \tilde{y}_i, y_{-i})$ for $\tilde{y}_i$ in an open neighborhood of y_i. But we can, repeatedly using the open neighborhood argument, show that this is true for any arbitrary y_i and $\tilde{y}_i$ as long as $(c; y_i, y_{-i})$ and $(c; \tilde{y}_i, y_{-i})$ are both feasible.

■

6.4.3 3. $\implies$ 4.

Consider a continuous ξ which implements the efficient graph in Strong NE. Consider two arbitrary feasible problems $(c; y)$ and $(c; \tilde{y})$. We will prove that $\xi(c; y) = \xi(c; \tilde{y}) = f(c)$. The monotonicity of f comes from lemma 1. Let $a = \frac{1}{k}\sum_{i \in K} y_i$ and $\tilde{a} = \frac{1}{k}\sum_{i \in K} \tilde{y}_i$. Assume without loss of generality that $y_1 \leq y_2 \leq y_3 \leq ...... \leq y_k$ and $\tilde{y}_1 \leq \tilde{y}_2 \leq \tilde{y}_3 \leq ...... \leq \tilde{y}_k$.

Step 1: $\xi(c; y) = \xi(c; a, a, ..., a)$ and $\xi(c; \tilde{y}) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$

Proof:

Consider the following problems: $P_0 = (c; y)$, $P_1 = (c; a, y_2, y_3, ...y_k)$, $P_2 = (c; a, a, y_3, y_4, ..., y_k)$, ...,$P_k = (c; a, a, ..., a)$. Notice first that feasibility of P_0 implies the feasibility of $P_1, P_2, ..., P_k$. This is true because maximum of the demand profile doesn't go above y_k in all these problems and sum of the individual demands is always at least $k * a = \sum_{i \in K} y_i$. Similarly, if we define the counterpart problems $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, ..., \tilde{P}_k$ where $\tilde{P}_i = (c; \tilde{a}, \tilde{a}, ..., \tilde{a}, \tilde{y}_{i+1}, \tilde{y}_{i+2}, ..., \tilde{y}_{k-1}, \tilde{y}_k)$, then again all of them will be feasible.

Now, due to the separability lemma (lemma 2) we must have $\xi_1(P_0) = \xi_1(P_1)$. Also, strong Nash implementability implies that $\xi_{-1}(P_0) = \xi_{-1}(P_1)$. To see this, suppose that it is not the case and for some agent $j \neq 1$, we have $\xi_j(P_0) \neq \xi_j(P_1)$. Assume without loss of generality that $\xi_j(P_0) < \xi_j(P_1)$. This means $\exists \grave{j} \in K \backslash \{1, j\}$ s.t., $\xi_{\grave{j}}(P_0) > \xi_{\grave{j}}(P_1)$, because of budget balancedness. Consider a network where all the agents $2, 3, ..., k$ have just one strategy which costs $y_2, y_3, ..., y_k$ and agent 1 has two strategies, where one of them costs y_1 and the other costs a. In both the cases, the total cost of the network is c. Thus one of the configurations generates the problem P_0 and the other P_1. Now both the configurations of the network is efficient and therefore at least one of them must be a strong NE under ξ. But clearly none of them is a strong NE. From P_1

the group $\{1, j\}$ has a profitable deviation and from P_0 the group $\{1, \grave{j}\}$. Thus, we have $\xi(P_0) = \xi(P_1)$. Using the same argument we have $\xi(P_i) = \xi(P_{i+1})$ and $\xi(\tilde{P}_i) = \xi(\tilde{P}_{i+1})$ for all $0 \leq i \leq k-1$. Thus, we have $\xi(P_0) = \xi(P_k)$ and $\xi(\tilde{P}_0) = \xi(\tilde{P}_k)$ as desired.

Step 2: $\xi(c; a, a, ..., a) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$

Proof:

Notice first that feasibility of $(c; a, a, ..., a)$ & $\xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$ implies that any problem $(c; \hat{a})$ where some of the $\hat{a}_i = a$ and other $\hat{a}_i = \tilde{a}$ is also feasible. Now, lemma 2 implies $\xi_1(c; a, \tilde{a}, ..., \tilde{a}) = \xi_1(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$. And again, the strong Nash implementability implies $\xi_{-1}(c; a, \tilde{a}, ..., \tilde{a}) = \xi_{-1}(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$. The proof of this statement is analogous to the one in step 1. Thus we have $\xi(c; a, \tilde{a}, ..., \tilde{a}) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$. Similarly, we get $\xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a}) = \xi(c; a, \tilde{a}, ..., \tilde{a}) = \xi(c; a, a, \tilde{a}, ..., \tilde{a}) = ... = \xi(c; a, a, ..., a)$.

■

The results "4. $\Longrightarrow$ 1.," "4. $\Longrightarrow$ 2" and "4. $\Longrightarrow$ 3" are straightforward and the proof is omitted.

6.5 Proof of Theorem 2

Proof. The "if" part is clear. For, "only if" consider an arbitrary feasible problem $(c; y)$. Assume without loss of generality that $y_1 \geq y_2 \geq y_3 \geq .... \geq y_k$. Let,$a = \frac{1}{k}\sum_{i=1}^{k} y_i$. Consider a problem $(c; a, a, ..., a)$ and suppose that ξ is continuous, efficient and satisfies ETE. Notice, the feasibility of $(c; y)$ implies the feasibility of $(c; a, a, ..., a)$ and any other problem $(c; \hat{y})$ where $\hat{y}_i = y_i$ for all $i \in \{1, 2, ..., l\}$ and $\hat{y}_i = a$ for all $i \in \{l, l+1, ..., k-1, k\}$. Now, the ETE property of ξ implies

$$\xi(c; a, a, ..., a) = (c/k, c/k, ..., c/k) \tag{9}$$

Using lemma 2 and applying ETE again we get,

$$\xi(c; y_1, a, ..., a) = (c/k, c/k, ..., c/k) \tag{10}$$

Now again applying lemma 2, and ETE we have,

$$\xi(c; y_1, y_2, a, a, ..., a) = (x_1, c/k, x, x, ..., x) \tag{11}$$

But if we change the ordering of 1 & 2 while arriving the above profile then we should have,

$$\xi(c; y_1, y_2, a, a, ..., a) = (c/k, x_2, x, x, ..., x) \tag{12}$$

But since the ordering is immaterial so we must have $x_1, x_2, x = c/k$. And thus we have,

$$\xi(c; y_1, y_2, a, a, ..., a) = (c/k, c/k, ..., c/k) \tag{13}$$

Repeating the same argument, we conclude that $\xi(c; y) = (c/k, c/k, ..., c/k)$
■

6.6 Proof of Theorem 3

6.6.1 Incompatibility of Efficiency and IR

Proof.

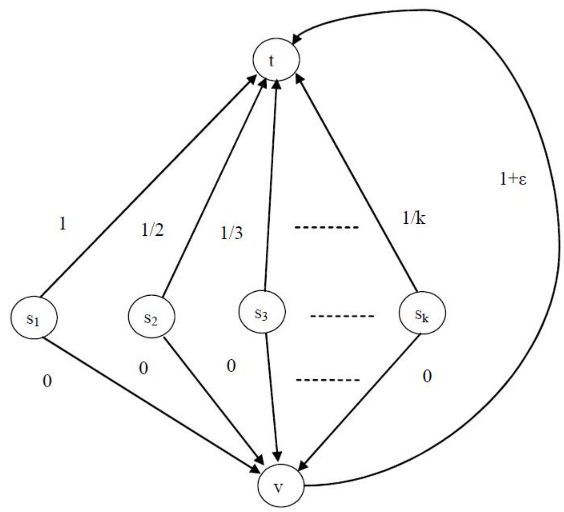

Figure 6:

We show by an example that any individually rational cost sharing rule must have a PoS of at least $H(k)$. Consider a situation as shown in figure 6. Here, every agent i has two strategies– either connect its demand nodes directly where the cost of the path is $1/i$ or connect through the the path where link costs are 0 and $1+\epsilon$. Consider any arbitrary cost sharing method ξ which satisfies individual rationality. We will show that the only equilibrium under such method will be where every agent is using their direct path to t. Suppose, this is not the case. This means there can be two cases. First case is where all the agents use a free link to v and then the common link of cost $1 + \epsilon$ to t. But then at least one of the agents must be paying more than $1/k$. Lets assume that this agent is the kth agent in some configuration[9] of the graph. Then he will have a profitable deviation to go to the direct link of cost $1/k$ under any individually rational rule. The other case to consider is when s agents are using their direct link

[9] It is important to note that just one such configuration is enough since PoS is measure of performance of the best NE in the worst case example.

and $k - s$ agents are sharing the common link to "v". Then it follows from individual rationality of the s agents that at least one of the agents in $k - s$ must be paying more than $1/(k - s)$. Notice, that in this case there exists an unused direct link, say $s_j \rightarrow t$, of cost $1/s_j$ which is at most $1/(k - s)$. Now in some configuration of the graph agent j will be the agent who is paying the above said amount of more than $1/(k - s)$ and thus he would like to deviate. We have just shown that the only NE in some configuration of this example has a cost equal to $H(k)$ whereas the efficient graph has a cost equal to $1 + \epsilon$ where everyone uses a costless link to node v and then the common link to t. ■

6.6.2 Lower bound for PoS(PR)

Proof. Consider a situation as shown in figure 6. We show that the unique equilibrium of the proportional method is of order k. Let, the costs of links $s_i \rightarrow t$ be x_i and the other things be exactly the same as in figure 6. Straightforward computations show that the $k - th$ agent will deviate from the efficient graph of cost $1 + \epsilon$ if $x_k \leq \frac{1-k+\sqrt{(k-1)^2+4k(k-1)}}{2k}$. As k grows, x_k converges to the golden number $\frac{\sqrt{5}-1}{2}$ in contrast to $1/k$ for the uniform method which goes to zero. Also $x_{t-1} > x_t$ for all $t = 2, 3, .., k$ and $x_1 = 1$. Thus the lower bound on the PoS of proportional method is $\sum_{i=1}^{k} x_i$ which is of order k. ■

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