Out-of-Equilibrium Performance of Three Lindahl Mechanisms: Experimental Evidence

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Abstract

We describe an experimental comparison of the out-of-equilibrium performance of three allocation mechanisms designed to achieve Lindahl outcomes as Nash equilibria: the mechanisms due to Walker (1981), Kim (1993), and Chen (2002). We find that Chen's mechanism, which is supermodular, converges closest and most rapidly to its equilibrium. However, we find that the properties that move subjects toward equilibrium in Chen's mechanism typically generate sizeable taxes and subsidies when not in equilibrium, and correspondingly large budget surpluses and deficits, which typically far outweigh the surplus created by providing the public good. The Kim mechanism, on the other hand, converges relatively close to its equilibrium and exhibits much better out-of-equilibrium efficiency properties.

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1 Introduction

In his classic 1919 paper, Erik Lindahl proposed a cost sharing procedure for financing public goods and he maintained that use of his procedure would (in modern terminology) produce Pareto efficient outcomes. Incentive compatible mechanisms that implement Lindahl's outcome as a Nash equilibrium were first proposed by Hurwicz (1979) and Walker (1981). A drawback of these early Lindahl mechanisms was the instability of their equilibria, as shown by Kim (1987).

Several authors have provided solutions to the instability problem by incorporating some form of dynamic stability into the design of the mechanism. Examples include the mechanisms introduced by Vega-Redondo (1989), de Trenqualye (1989), Kim (1993), and Chen (2002). All four mechanisms attain Lindahl outcomes as Nash equilibria, as in the Hurwicz and Walker mechanisms. The first two mechanisms are stable under myopic best reply, and Kim's mechanism is globally stable under a gradient adjustment process. Chen's mechanism is supermodular for some parameter values,[1] and is therefore stable under a wide variety of out-of-equilibrium behavior by participants.

It is not enough, however, to evaluate an allocation mechanism only in terms of the efficiency and stability of its equilibria. Whenever there is a change in the underlying economic conditions (preferences, costs, etc.) and a corresponding change in a mechanism's equilibrium, the mechanism is likely to be out of equilibrium for a period of time — perhaps a considerable period — before it attains an equilibrium or even comes close to an equilibrium. It is therefore important to know how long alternative mechanisms require to reach or approximate an equilibrium, and how far short of efficiency these out-of-equilibrium outcomes will be.

Previous experimental examinations of the stability of public goods allocation mechanisms have found that the Groves-Ledyard mechanism, in which equilibria are Pareto optimal but not Lindahl, converges to near its equilibrium under some assignments of parameter values (Chen & Plott (1996), Chen & Tang (1998)). Many experimental studies have found convergence to Nash equilibrium in mechanisms whose equilibria are not Pareto optimal, such as the mechanism of voluntary contributions and the Vickrey-Clarke-Groves mechanism. We are aware of only two prior studies that examine stability of a Lindahl mechanism: Chen & Tang (1998) and Healy (2004) found that

[1]More precisely, if the mechanism's strategy (message) spaces are compact, then there are some utility functions and some values of the mechanism's parameters for which the resulting game is supermodular.

the Walker mechanism did not converge — as expected, given its theoretical instability under any plausible out-of-equilibrium behavior.

The present paper reports on laboratory experiments designed to compare the out-of-equilibrium performance of the Walker (1981), Kim (1993), and Chen (2002) mechanisms. While each mechanism achieves Lindahl allocations at its Nash equilibria, the mechanisms differ in several important respects. Our primary aim is to compare the effects of these differences on the mechanisms' convergence to equilibrium and on their out-of-equilibrium welfare properties.

We find that the Chen and Kim mechanisms converge toward their respective equilibrium levels of public good provision, but ultimately remain at some distance from equilibrium. As in previous research, the Walker mechanism does not converge. Both the Chen and Kim mechanisms attain levels of the public good that yield a substantial portion of the maximum possible consumer surplus, generally between 70% and 95%. The Walker mechanism produces considerably less than 50% of the possible surplus. Surprisingly, none of the mechanisms converges to the Lindahl taxes.

The most pronounced difference between the mechanisms is in their violations of individual rationality and their failure to balance the budget. On both counts the Kim mechanism performs well, with minimal violations of individual rationality and with budget imbalances that are small relative to the consumer surplus the mechanism produces. In contrast, the parameter values that make the Chen mechanism supermodular produce extreme tax obligations, many violations of individual rationality, and enormous budget deficits and surpluses. On balance, we find the performance of the Kim mechanism to be superior to both the Chen and Walker mechanisms.

The remainder of the paper will proceed as follows. Section 2 summarizes the theoretical concepts that form the basis of our experiment; Section 3 describes the experiment; and Sections 4 and 5 present the experimental results. Section 6 summarizes the results and provides some concluding remarks.

2 Theoretical Preliminaries

2.1 Public Goods and the Lindahl Outcome

We analyze the three mechanisms in the same economic setting: there is a single public good, produced from a second (private) good via a constant-returns-to-scale technology, and there are N individuals, or *participants*, for whom the public good is to be provided. An outcome is denoted by $(x, \tau_1, ..., \tau_N)$: x denotes the level at which the public good is provided, and τ_i denotes the amount of the private good participant i contributes to finance the provision of the public good. It's convenient to think of the private good as money and τ_i as a *tax* or *transfer* paid by i. Each participant evaluates outcomes according to a *utility function* $u_i(x, \tau_i) = v_i(x) - \tau_i$, where $v_i(\cdot)$ is strictly concave and differentiable; v_i is referred to as i's *valuation function.* We assume that no participant's initial holding of the private good is exhausted by his tax τ_i.

Let c denote the per-unit cost of producing the public good — *i.e.*, each unit of the public good requires c units of the private good as input. The Pareto optimal outcomes are the ones that satisfy the Samuelson condition $\sum_{i=1}^N v_i'(x) = c$ and also balance the budget — *i.e.*, ones for which $\sum_{i=1}^N \tau_i = cx$. Note that strict concavity of each v_i ensures that there is at most one value of x that is consistent with Pareto optimality. We refer to this as the *Pareto* value of x, denoted x^*.

Lindahl proposed charging each participant a share s_i of the per-unit cost c for each unit of the public good that is provided — *i.e.*, $\tau_i = s_i x$ for each i, and $\sum_{i=1}^N s_i = c$. If each participant takes his share s_i as given, profit-maximization will lead him to request a public-good provision level x at which his marginal value $v_i'(x)$ is equal to s_i. Lindahl suggested that his procedure would be in equilibrium when the shares $s_1, ..., s_N$ are set so as to induce every participant to request the same amount of the public good. Clearly, such an equilibrium will be Pareto optimal: $\sum_{i=1}^N v_i'(x) = \sum_{i=1}^N s_i = c$.

But how do we determine the shares $s_1, ..., s_N$? At any provision level x, participant i's share s_i should be his marginal value $v_i'(x)$. But if asked to reveal this information, participants typically have an incentive to misrepresent their preferences, hoping to free ride on other participants' payments. In a pioneering paper, Groves & Ledyard (1977) proposed a mechanism whose Nash equilibria are Pareto optimal, but in which participants' shares of the cost are not the Lindahl shares. Hurwicz, Walker, and the subsequent authors mentioned above adapted the Groves-Ledyard idea to create "Lindahl mechanisms" — mechanisms whose Nash equilibria are Lindahl allocations.

2.2 The Mechanisms

A *mechanism* uses messages or actions by the participants to calculate an outcome $(x, \tau_1, ...\tau_N)$, the level at which the public good will be provided and the tax/transfer paid by each of the participants. (Note that this transfer may be negative — a rebate — for some participants.)

The Walker Mechanism Each participant announces a request r_i. The mechanism sums the participants' requests to determine the level at which the public good will be provided:

$$x = \sum_{i=1}^{N} r_i. \tag{1}$$

Denote by $\mathbf{r}$ the profile of all participants' requests: $\mathbf{r} = (r_1, ..., r_N)$. Each participant's tax is given by

$$\tau_i^W(\mathbf{r}) = \left(\frac{c}{N} + r_{i+2} - r_{i+1}\right) x \tag{2}$$

where $N + 2 = 2$ and $N + 1 = 1$. Note that in this mechanism the budget is balanced regardless of the requests, *i.e.*, $\sum_{i=1}^{N} \tau_i^W(\mathbf{r}) = cx$ for all $\mathbf{r}$.

The Chen Mechanism Each participant announces a request r_i; the profile $\mathbf{r}$ determines the public good level x according to (1), just as in the Walker mechanism. The Chen Mechanism requires that each participant also announce a second number, p_i, which the mechanism interprets as a *prediction* about the level at which the public good will be provided. The profiles of requests and predictions are $\mathbf{r} = (r_1, ..., r_N)$ and $\mathbf{p} = (p_1, ..., p_N)$. Each participant's tax in the Chen Mechanism is

$$\tau_i^C(\mathbf{r}, \mathbf{p}; \gamma, \delta) = \left(\frac{c}{N} - \gamma \sum_{j \neq i} r_j + \frac{\gamma}{N} \sum_{j \neq i} p_j\right) x + \frac{1}{2}(p_i - x)^2 + \frac{\delta}{2} \sum_{j \neq i} (p_j - x)^2. \tag{3}$$

where γ and δ are parameters specified by the mechanism's designer. Note that each participant's tax depends on the accuracy of all participants' predictions, through the terms $(p_j - x)^2$. The mechanism is a generalization of Kim's earlier mechanism.

The Kim Mechanism The Kim Mechanism is the special case of the Chen Mechanism in which $\gamma = 1$ and $\delta = 0$ — *i.e.*, the Kim tax is $\tau_i^K(\mathbf{r}, \mathbf{p}) = \tau_i^C(\mathbf{r}, \mathbf{p}; 1, 0)$.

Properties of the Mechanisms The Walker Mechanism is the simplest of the three, an important consideration when actually implementing a mechanism. Moreover, the mechanism's budget is balanced identically for any profile of requests by the participants, whether the profile is an equilibrium or not. However, as Kim (1987) has shown, the mechanism is unstable under any plausible dynamic behavior by the participants.

The Kim Mechanism is not as simple as the Walker Mechanism: each participant must augment his public-good request with a prediction about the result of the other participants' requests. Further, the mechanism's budget is generally unbalanced when out of equilibrium. However, the mechanism is globally stable under the continuous-time gradient adjustment process, and in some circumstances is stable under discrete-time Cournot best reply, as shown in the Appendix, below.

The Chen Mechanism is clearly the most complicated of the three, and like the Kim Mechanism its budget is typically not balanced except in equilibrium. But because it is supermodular for some combinations of parameter values and individuals' preferences, it is possible, for a range of preferences, to choose corresponding values for γ and δ that will make the mechanism stable under a wide variety of out-of-equilibirium behavior.

3 The Experiment

The experiment consists of applying each of the three Lindahl mechanisms to the same simple public-goods allocation problem, or *environment*. We first describe this common environment, and then the mechanisms.

The environment consists of three participants — *i.e.*, $N = 3$. The participants' valuation functions all have the form $v_i(x) = A_i x - B_i x^2$; the parameter values A_i and B_i are as shown in Table 0. The cost function for providing the public good is $C(x) = 12x$ — *i.e.*, $c = 12$. The unique Pareto public good level x^* is therefore

$$x^* = \frac{\sum_{i=1}^{3} A_i - 12}{2\sum_{i=1}^{3} B_i} = \frac{66 - 12}{(2)(3)} = 9 \ .$$

In order to define the three mechanisms for the experiment, we must specify (a) the message spaces that will be made available to the participants and (b) the values of the Chen Mechanism's parameters γ and δ. The *request space* will be the same for all three mechanisms, consisting of the numbers $r_i \in \{-5, -4.99, -4.98, \ldots, 14.99, 15\}$. The set of possible provision levels for the public good (the sum of the three participants' requests) is therefore the set $\{-15, -14.99, \ldots, 44.99, 45\}$, and we therefore allow the participants to select their predictions p_i from this set. The parameter values for the Chen mechanism are set at $\gamma = 21$ and $\delta = 8$.

Table 0 displays the parameter values and the three mechanisms' equilibrium messages and resulting surplus.

Table 0: Experimental Parameters and Equilibria

	Parameters		Equilibrium Strategies				Lindahl		Surplus at
Player	A_i	B_i	r_W^*	r_K^*	r_C^*	$p_{K,C}^*$	Share	Tax	Equilibrium
1	22	1	-1	3	3	9	4	36	81
2	16	1	5	-3	$\frac{19}{7}$	9	-2	-18	81
3	28	1	5	9	$\frac{23}{7}$	9	10	90	81
Total	66	3	9	9	9		12	108	243

We show in an appendix that with these parameter values the Walker Mechanism is unstable under the discrete-time best reply dynamic, the Kim Mechanism is stable under best reply, and the Chen Mechanism induces a supermodular game and is thus stable under a wide range of dynamic behavior.

Six laboratory sessions were conducted, two sessions with each mechanism. All sessions were conducted in the Economic Science Laboratory at the University of Arizona. All subjects were undergraduate students at the University, recruited via e-mail from the ESL's online subject database.

In each session the subjects were first randomly assigned into groups of three; there were four to six such groups in each session. The subjects' parameter values were revealed to them privately; subjects were not provided with any information about the parameter values of the other two subjects in their group. Each three-person group remained together throughout the session, participating in 40 rounds, or time periods, of that session's mechanism.

In each period the subjects communicated their messages — their requests r_i and, in the Kim and Chen mechanisms, their predictions p_i — from computer terminals to a central server, and they received information in return from the server. Written instructions were provided at each computer terminal. The subjects were given time to read the instructions, after which the experimenter read the instructions aloud and entertained questions. All sessions were conducted by the first author.

The software for the experiment includes two tools to aid subjects in their decision making. Each subject was provided with a "What-if-Scenario" profit calculator[2], which allowed the subject to input hypothetical messages for the other two group members and explore how, against those hypothetical messages, his own decisions would affect his profit. This is a substitute for providing subjects with payoff tables: the complexity of the mechanisms' outcome functions would require multiple extremely complex tables. This calculator, which allows a subject to answer any "what if" question that could have been answered with payoff tables, but to do so more transparently, appears to be a better decision-making aid than payoff tables. Subjects were also able to access a screen that showed, for all prior rounds, all three subjects' messages as well as the resulting public good level and the subject's own profit. Subjects were not required to use these decision aids, but most subjects made use of the profit calculator on almost every round.

As in Healy (2006), we did not use practice rounds, but instead allowed subjects five minutes to practice with the "What-if-Scenario" profit calculator. This provided each subject with some experience using the software, without allowing subjects to learn anything about other subjects' parameters or behavior. After this five-minute practice time with the calculator, each group played 40 periods with one of the three public goods mechanisms.

Each of the 40 decision periods proceeded in the same fashion. Subjects were first asked to submit their requests and, in the Kim and Chen mechanisms, their predictions as well. When all three participants had submitted their messages, the outcome (the public good level and the participants' taxes) was calculated and the following information about the just-completed period was communicated to each participant: all three group members' decisions; the resulting amount of the public good that was provided; and the subject's own revenue, tax, and profit. The subject's cumulative profit was reported only at the end of the experiment, although, as described above, a subject could access a screen displaying all information he'd been provided at prior periods. Subjects were also required to record their information by hand on a record sheet. This task was included in order to ensure that at least some of a subject's attention would be directed to how much he was earning.

[2]See Healy (2006) for a discussion of the "What-if-Scenario" calculator.

At the end of each session one of the 40 periods was selected at random and each subject was paid six cents for every experimental dollar earned in that period.

Subjects remained in the same group for the entire session and were paid privately at the end of the session. No subject participated in more than one session. Sessions typically lasted about 90 minutes.

4 Convergence

We first consider how the subjects' behavior in each mechanism compares with the mechanism's equilibrium: do the subjects' actions tend to become close to their equilibrium actions as they interact repeatedly in the mechanism, and does this convergence occur at different rates in the three mechanisms? Then we will assess how well the mechanisms accomplish the task they were designed for: how well do they converge to the Pareto public good level and to the Lindahl taxes?

4.1 Convergence of Participants' Requests

In order to compare the mechanisms' convergence to their respective equilibrium strategy profiles, we need a measure of the distance between an observed profile of requests, $\boldsymbol{r} = (r_1, r_2, r_3)$, and a mechanism's equilibrium profile $\boldsymbol{r^*} = (r_1^*, r_2^*, r_3^*)$. Here, and throughout our analysis, we use the so-called (average) "city block" metric (or L_1 metric) as our measure of distance. Thus, we define the *request deviation* of group g in period t as

$$R_{gt} := \frac{1}{3}\left(|r_1 - r_1^*| + |r_2 - r_2^*| + |r_3 - r_3^*|\right),$$

i.e., as the average absolute deviation of the participants' requests from their equilibrium values.

Figure 1 shows the time series, for each mechanism, of the period-by-period request deviations averaged across all nine groups who participated in the mechanism, *i.e.*, the *average request deviations*,

$$\overline{R}_t := \frac{1}{9}\sum_{g=1}^{9} R_{gt}.$$

Table 1 presents the information in Figure 1 in an alternative concise format: the table divides the forty periods into four segments of ten periods each, and displays the average deviations $\overline{R}_t$ further averaged across each ten-period block.

There is little evidence that the Walker mechanism is converging. Indeed, in the early periods the requests in the Walker mechanism are moving farther from their equilibrium values. In the last 13 periods the deviations lie within or very close to the interval [4,5]. This failure to converge is qualitatively consistent with the theoretical instability of the mechanism, shown in the appendix.

The deviations in the Kim and Chen mechanisms exhibit a decreasing trend, eventually remaining in the intervals [2,3] and [1,2], respectively, over the final 15 periods. This is qualitatively consistent with these mechanisms' theoretical stability. The Chen mechanism's average deviations from equilibrium are substantially smaller than those of the Kim mechanism in every period with the exception of two early periods.

Result 1: The deviations from equilibrium requests are consistently smallest in the Chen mechanism and largest in the Walker mechanism. There is no evidence of convergence in the Walker mechanism. The requests in the Chen and Kim mechanisms grow closer to equilibrium through most of the experiment, but it is unclear how close to equilibrium they would eventually converge over a longer time horizon.

4.2 Public Good Provision Level

The primary raison d'etre of all three mechanisms is to achieve Pareto outcomes. The Pareto outcomes in our experiment are those in which the public good is provided at the level $x^E = x^* = 9$.

Figures 2a and 2b and Table 2 present a simple description of the mechanisms' comparative success at meeting this objective. Averaging across the nine three-person groups, each mechanism tends to yield public good provision levels that exceed the equilibrium (and Pareto) level $x^E = x^* = 9$. Over time the average public good level tends to decrease, moving closer to the Pareto level.

However, the time series of average public good levels depicted in Figures 2a and 2b obscure a substantial amount of variation across the respective nine groups in each mechanism: some groups chose public good levels that were much smaller than x^E and some chose levels much larger than x^E. Using the average therefore obscures the extent to which the public good levels are converging over time, or failing to converge. A clearer picture of the mechanisms' convergence to the equilibrium public good level is provided in Figures 3a and 3b, which display the average absolute deviations from equilibrium,

$$\overline{D}_t := \frac{1}{9}\sum_{g=1}^{9} |x_{gt} - x^E|,$$

where x_{gt} denotes the public good level chosen by group g in period t.

In both the Chen and Kim mechanisms the average deviation $\overline{D}_t$ decreased for approximately the first twenty periods, but thereafter showed no further decrease, remaining for the most part within the interval $[2, 4]$. The average deviations in the Walker mechanism *increased* over about the first twenty periods, displayed extreme period-by-period fluctuations throughout, and never declined to the levels attained by the other two mechanisms. Table 3 presents the average deviations from the Pareto public good level in the 10-period-block format introduced in Table 1, above.

We summarize Figures 3a and 3b and Table 3 in the following statement:

Result 2: The public good levels attained by the Chen and Kim mechanisms become closer to the equilibrium level during the early periods but do not converge more closely to the equilibrium level in subsequent periods. The Walker mechanism's public good level is consistently farther from the equilibrium level than the levels in the other two mechanisms.

The deviations from equilibrium – in effect, the "errors" – produced by the Chen and Kim mechanisms may seem large: even in the later periods, these "errors" are generally between 25% and 40% of the target level $x^* = 9$. However, when we measure a mechanism's performance by the welfare it produces, as in Section 5 below, the Chen and Kim mechanisms appear to be relatively more successful at the task of choosing a public good provision level.

4.3 Lindahl Taxes

We now ask how closely the taxes converge to the Lindahl taxes the mechanisms are designed to achieve. The Lindahl tax profile is $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*, \tau_3^*) = (36, -18, 90)$. At each period t we define group g's deviation from $\boldsymbol{\tau}^*$ as follows:

$$T_{gt} = \frac{1}{3}\left(|\tau_1 - \tau_1^*| + |\tau_2 - \tau_2^*| + |\tau_3 - \tau_3^*|\right),$$

i.e., the average amount that each participant's tax deviates from his Lindahl tax. Figure 4 depicts the time series of average deviations

$$\overline{T}_t = \frac{1}{9} \sum_{g=1}^{9} T_{gt}$$

for the nine three-person groups in each mechanism. Table 4 presents these average deviations in the 10-period-block format introduced in Table 1, above.

Figure 4 and Table 4 reveal striking differences among the three mechanisms: the Kim mechanism's tax profiles are consistently closer to the Lindahl taxes than the tax profiles in the Walker mechanism, which in turn are much closer to the Lindahl taxes than are the tax profiles in the Chen mechanism. Over the last 20 periods the average deviation from the Lindahl tax profile was 30 in the Kim mechanism, 97 in the Walker mechanism, and 408 in the Chen mechanism. In both the Walker and Chen mechanisms these average deviations significantly exceed the target taxes themselves: $(\tau_1^L, \tau_2^L, \tau_3^L) = (36, -18, 90)$ (see Table 0). Thus, neither mechanism is even approximately achieving the Lindahl taxes.

Result 3: All three mechanisms consistently deviate from the Lindahl taxes. The deviations follow a clear ranking: the Kim mechanism's taxes are closest to Lindahl; the Walker mechanism the second closest; and the Chen mechanism's taxes are consistently much farther from Lindahl than are the taxes in the other two mechanisms.

5 Welfare and Budget Comparisons

The attraction of Lindahl mechanisms is that they attain good allocations – *at their equilibria*. The allocations are Pareto efficient at the mechanisms' equilibria; the allocations are individually rational at the mechanisms' equilibria; and the Lindahl taxes the mechanisms impose are proportional to participants' marginal benefits – *at the mechanisms' equilibria*.

But because we can expect the mechanisms to be significantly out of equilibrium for a significant proportion of time, it is important to evaluate the welfare properties of their allocations when out of equilibrium as well as in equilibrium. Indeed, as Figure 3 suggests, the mechanisms may remain significantly out of equilibrium for a very long time.

5.1 Welfare from Provision of the Public Good

We focus our attention first on the welfare associated with the public good levels that were chosen by the nine three-person subject groups in each mechanism, ignoring for the time being any budget imbalances. The induced utility functions in the experiment were quasilinear, *i.e.*, they have the form $u_i(x, \tau_i) = v_i(x) - \tau_i$. Therefore the welfare a mechanism achieves for its participants directly from providing the public good is simply the consumer surplus,

$$S(x) := v_1(x) + v_2(x) + v_3(x) - cx.$$

Assuming a balanced budget (*i.e.*, that the taxes collected, $\tau_1 + \tau_2 + \tau_3$, equal the cost cx of the public good), $S(x)$ is the surplus created by providing x units of the public good instead of not providing any of the public good.

The maximum possible surplus is attained when $x = x^* = 9$. This maximum surplus is $S^* = 243$ (see Table 0). Figures 5a and 5b display the time series of average surplus values,

$$\overline{S}_t := \frac{1}{9} \sum_{g=1}^{9} S_{gt},$$

attained by subjects in each mechanism, where S_{gt} denotes the surplus attained by group g in period t.

As one would expect, the time series in Figures 5a and 5b closely reflect those in Figures 3a and 3b, the average deviations from the Pareto public good level. The surpluses in the Chen and Kim mechanisms show improvement during the early periods and then show little if any improvement over the final twenty or so periods, remaining mostly between 175 and 225 and never exceeding 230. Similarly, the surpluses in the Walker mechanism reflect the Figure 3b deviations from equilibrium provision levels, worsening over the early periods, continuing to fluctuate significantly, and remaining well below the surplus levels attained by the other two mechanisms.

Note that a surplus of 175 is more than 70% of the potential surplus $S^* = 243$, and a surplus of 225 is more than 90% of S^*. Thus, as suggested in the preceding section, while the public good provision levels in the Chen and Kim mechanisms generally deviate by 25% to 40% from the equilibrium level, the provision levels nevertheless yield substantial gains in welfare when compared to non-provision of the public good.

Result 4: The surpluses obtained in the Chen and Kim mechanisms from providing the public good increase during the early periods. In the later periods the surpluses remain approximately between 70% and 95% of the potential surplus, averaging 78% and 77%, respectively, over the final 20 periods. The surpluses in the Walker mechanism display much larger fluctuations and remain well below those attained by the other two mechanisms, averaging only 26% of the potential surplus over the final 20 periods.

5.2 Violations of Individual Rationality

Lindahl allocations, in addition to being Pareto optimal, are also *individually rational*: each participant is at least as well off at the Lindahl allocation as he would have been had the public good not been provided. Formally, Lindahl allocations $(x, \tau_1, \ldots, \tau_n)$ satisfy the inequality $u_i(x, \tau_i) \geq u_i(0, 0)$ for each participant i. Lindahl mechanisms therefore, by their definition, yield individually rational outcomes at their equilibria. However, their disequilibrium outcomes are not Lindahl allocations and therefore need not be individually rational — some participants may be made worse off than if the mechanism were not used and the public good not provided.

Figure 6 and Table 6 describe the violations of individual rationality produced by each of the three mechanisms. In the Kim mechanism IR violations decreased to about five percent of all participant outcomes in the final 20 periods. But in the Chen and Walker mechanisms the number of IR violations did not decrease, remaining at about 30% in the Walker mechanism and at about 40% in the Chen mechanism.

Result 5: The Kim mechanism produces far fewer violations of individual rationality than either the Chen or Walker mechanism.

5.3 Unbalanced Budgets

Until now, we have ignored the fact that both the Chen and Kim mechanisms typically fail to balance the budget when out of equilibrium. An unbalanced budget can reduce or eliminate the welfare gains (the consumer surplus) achieved by providing the public good. For example, if more taxes are collected from the mechanism's participants than required in order to produce the public good, then the participants in the mechanism are sacrificing some, or perhaps all, of their surplus. Conversely, if the taxes fail to cover the cost of the public good, then the amount by which the cost exceeds the taxes must come from outside the mechanism – an additional cost that must be borne by someone.

In either case — tax collections that are too large, or tax collections that are too small — we regard the amount of the budget imbalance as a cost of using the mechanism. For the Walker mechanism this cost is always zero: the mechanism's budget is always balanced, whether in or out of equilibrium. Figure 7 and Table 7 show the average magnitude of the budget imbalance for the Chen and Kim mechanisms. The average budget imbalance produced by the Kim mechanism over the last 20 periods was 48. The Chen mechanism produced imbalances that averaged just under 1000 over the last 20 periods.

Recall from Table 5 that the Kim mechanism produced, on average, about 187 units of consumer surplus over the last 20 periods. Thus, the average budget imbalance of 48 is a cost worth bearing: the average surplus the Kim mechanism produced was nearly four times as large.

On the other hand, the Chen mechanism over the last 20 periods produced average budget imbalances of 995 and average consumer surplus of only 190. The cost of using the mechanism, on average, was more than five times the consumer surplus it produced. Indeed, the maximum possible consumer surplus is only $S^* = 243$; the average imbalance was more than four times as great as S^*.

Result 6: The Kim mechanism produces budget imbalances that are significantly smaller than the consumer surplus it produces. The Chen mechanism produces budget imbalances that are significantly larger than the consumer surplus it produces.

5.4 The Role of Parameter Values

The large budget imbalances and the numerous violations of individual rationality in the Chen mechanism are related to the mechanism's parameters δ and γ, which were set at $\delta = 8$ and $\gamma = 21$ in our experiment. Smaller values would likely reduce the mechanism's budget imbalances and violations of individual rationality, but supermodularity imposes a lower bound on δ and γ. As shown in the Appendix, supermodularity in the Chen mechanism requires (for quasilinear-quadratic utilities) that $(N-1)\delta + 1 + 2\max\{B_1 \ldots B_N\} \leq \gamma \leq N\delta$. For the economic environment in our experiment ($N = 3$ and $B_1 = B_2 = B_3 = 1$), one easily verifies that the smallest values of δ and γ that satisfy this condition are $\delta = 3$ and $\gamma = 9$. Indeed, related experimental research on efficient mechanisms (*e.g.*, Chen and Gazzale, Chen and Tang, Healy) suggests that large values of "punishment parameters" such as δ are important for inducing convergence. The values $\delta = 8$ and $\gamma = 21$ were chosen to give the Chen mechanism a good chance to converge, and these values also yield simple, integer-valued outcome functions that were likely to be more easily understood by subjects.

6 Conclusion

The Chen and Kim mechanisms perform similarly in some respects: after several early periods of adjustment, each mechanism generally provides the public good at a level that produces 70% to 95% of the possible consumer surplus. However, neither mechanism converges to the Lindahl taxes. The Walker mechanism achieves a much smaller fraction of the possible surplus and fails to converge to either its equilibrium public good level or the Lindahl taxes.

In other important respects, the Kim mechanism significantly outperforms the other two mechanisms. After a few initial periods, the Kim mechanism produces very few violations of individual rationality. These violations occur in the Chen and Walker mechanisms more than 30% of the time, on average. The magnitude of the budget imbalances in the Kim mechanism average about 25% of the consumer surplus the mechanism produces. In the Chen mechanism the budget imbalances average about five times the amount of consumer surplus produced. The Walker mechanism always produces a balanced budget.

Thus, when we take account of the fact that these mechanisms will often be out of equilibrium, the Kim mechanism significantly outperforms the Chen and Walker mechanisms.

7 References

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8 Appendix

8.1 Stability

With the parameters used in the experiment, we show that the Walker mechanism is unstable under the best reply dynamic, the Kim Mechanism is stable under best reply, and the Chen mechanism (with the compact message space used in our experiment) induces a supermodular game and is therefore robustly stable under a wide range of adjustment behavior. Chen (2002) showed that neither the Walker nor the Kim mechanism is supermodular.

The Walker mechanism If there are three players and each one has preferences of the form $u_i(x, y_i) = y_i + A_i x - B_i x^2$, then players' best reply functions in the Walker mechanism can be represented by the following system of linear difference equations,

$$\begin{bmatrix} r_1^{t+1} \\ r_2^{t+1} \\ r_3^{t+1} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{2B_1-1}{2B_1} & -\frac{2B_1+1}{2B_1} \\ -\frac{2B_2+1}{2B_2} & 0 & -\frac{2B_2-1}{2B_2} \\ -\frac{2B_3-1}{2B_3} & -\frac{2B_3+1}{2B_3} & 0 \end{bmatrix} \begin{bmatrix} r_1^t \\ r_2^t \\ r_3^t \end{bmatrix} + \begin{bmatrix} \frac{A_1-c/3}{2B_1} \\ \frac{A_2-c/3}{2B_2} \\ \frac{A_3-c/3}{2B_3} \end{bmatrix}.$$

In our experiment $B_1 = B_2 = B_3 = 1$, so the coefficient matrix reduces to

$$\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} & 0 \end{bmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = 1 + \frac{1}{2}\sqrt{3}\, i$, $\lambda_2 = 1 - \frac{1}{2}\sqrt{3}\, i$, and $\lambda_3 = -2$, all of which lie outside the unit circle. The system is therefore unstable.

The Kim mechanism Kim established that his mechanism is globally stable under the continuous-time gradient adjustment process, but time is discrete in our experiment. We show here that for three players with utility functions of the form $u_i(x, y) = y + A_i x - x^2$, the Kim mechanism is stable under myopic best reply. Express player i's payoff function in the Kim mechanism as follows:

$$\pi_i(r, p) = A_i x - x^2 - (\frac{c}{N} - \sum_{j \neq i} r_j + \frac{1}{N} \sum_{j \neq i} p_j) x - \frac{1}{2} (p_i - x)^2 .$$

The first-order conditions of player i's maximization problem are

$$\begin{aligned} \frac{\partial \pi_i}{\partial r_i} &= A_i - 2x - \frac{c}{N} + \sum_{j \neq i} r_j - \frac{1}{N} \sum_{j \neq i} p_j + p_i - x = 0 \\ \frac{\partial \pi_i}{\partial p_i} &= x - p_i = 0. \end{aligned}$$

If player i is best responding he will choose p_i and r_i in period $t+1$ according to

$$\begin{aligned} r_i^{t+1} &= \frac{1}{2} \left[A_i - \frac{c}{N} + \left(1 - 2 - \frac{1}{N} \right) \sum_{j \neq i} r_j^t - \frac{1}{N} r_i^{t-1} - \frac{(N-2)}{N} x^{t-1} \right] \\ p_i^{t+1} &= r_i^{t+1} + \sum_{j \neq i} r_j^t. \end{aligned}$$

Adding up these conditions for the three individuals, we obtain

$$2x^{t+1} = \sum_i A_i - c - \frac{8}{3} x^t - \frac{4}{3} x^{t-1}.$$

The homogeneous part of this system is $x^{t+1} + \frac{4}{3} x^t + \frac{2}{3} x^{t-1} = 0$, for which the characteristic equation is

$$\lambda^2 + \frac{4}{3} \lambda + \frac{2}{3} = 0.$$

The characteristic roots are $\lambda_1 = -\frac{2}{3} + \frac{1}{3} \sqrt{2}\, i$ and $\lambda_2 = -\frac{2}{3} - \frac{1}{3} \sqrt{2}\, i$, each of which is inside the unit circle:

$$|\lambda_1| = |\lambda_2| = \sqrt{\frac{2}{3}},$$

and the system is therefore globally stable for any initial conditions.

The Chen mechanism The conditions for Chen's mechanism to be supermodular — *i.e.*, for the game defined by the mechanism to be supermodular — depend on the values of the environment parameters (utility and cost functions) and the parameters of the mechanism (γ and δ). For the utility parameters used in our experiment, we derive restrictions on γ and δ that will make Chen's mechanism supermodular.

We first write player i's payoff function as follows:

$$\pi_i(r,p) = A_i x - B_i x^2 - (\frac{c}{N} - \gamma \sum_{j \neq i} r_j + \frac{\gamma}{N} \sum_{j \neq i} p_j) x - \frac{1}{2} (p_i - x)^2 - \frac{\delta}{2} \sum_{j \neq i} (p_j - x)^2 .$$

The function π_i is supermodular in (r_i, p_i), because

$$\frac{\partial^2 \pi_i}{\partial r_i \partial p_i} = 1 \geq 0,$$

and π_i has increasing differences in (r_i, p_j), (r_i, r_j), (p_i, p_j) and (p_i, r_j), $\forall j \neq i$ if and only if the following inequalities hold:

$$\begin{aligned}
\frac{\partial^2 \pi_i}{\partial r_i \partial p_j} &= -\frac{\gamma}{N} + \delta \geq 0 \\
\frac{\partial^2 \pi_i}{\partial r_i \partial r_j} &= -2B_i + \gamma - 1 - \delta (N-1) \geq 0 \\
\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} &= 0 \geq 0 \\
\frac{\partial^2 \pi_i}{\partial p_i \partial r_j} &= 1 \geq 0.
\end{aligned}$$

Therefore, the game is supermodular if and only if $(N-1)\delta + 1 + 2\max\{B_1 \ldots B_N\} \leq \gamma \leq N\delta$. In our experiment, $N = 3$, $\delta = 8$, $\gamma = 21$, and $\max\{B_1 \ldots B_N\} = 1$; hence the game is supermodular. Note that the Kim mechanism is not supermodular: the inequality condition above fails to hold when $\delta = 0$ and $\gamma = 1$.

Tables

Table 1: Deviations from Equil'm Request Profile

Periods	Chen	Kim	Walker
1 - 10	2.6	3.3	3.9
10 - 20	1.5	3.1	5.4
20 - 30	1.6	2.8	4.8
30 - 40	1.5	2.4	4.4

Table 2: Public Good Level

Periods	Chen	Kim	Walker
1 - 10	13.1	10.5	10.1
10 - 20	10.9	11.4	10.7
20 - 30	10.1	10.0	11.7
30 - 40	8.9	9.6	10.2

Table 3: Deviations from Equil'm Public Good Level

Periods	Chen	Kim	Walker
1 - 10	5.3	4.9	4.6
10 - 20	3.0	4.6	6.4
20 - 30	2.9	3.2	6.4
30 - 40	3.0	3.1	5.1

Table 4: Deviations from Lindahl Tax Profile

Periods	Chen	Kim	Walker
1 - 10	811	40	71
10 - 20	394	39	110
20 - 30	437	29	108
30 - 40	378	30	86

Table 5: Consumer Surplus from the Providing Public Good

Periods	Chen	Kim	Walker
1 - 10	69	134	137
10 - 20	204	150	56
20 - 30	190	188	25
30 - 40	189	186	100

Table 6: Percentage of Subjects with an IR Violation

Periods	Chen	Kim	Walker
1 - 10	44%	20%	23%
10 - 20	31%	15%	37%
20 - 30	37%	4%	29%
30 - 40	43%	6%	28%

Table 7: Mean Absolute Budget Imbalance

Periods	Chen	Kim	Walker
1 - 10	2038	92	0
10 - 20	875	72	0
20 - 30	994	54	0
30 - 40	947	43	0

Figure 1:
Deviation of Requests from Equilibrium

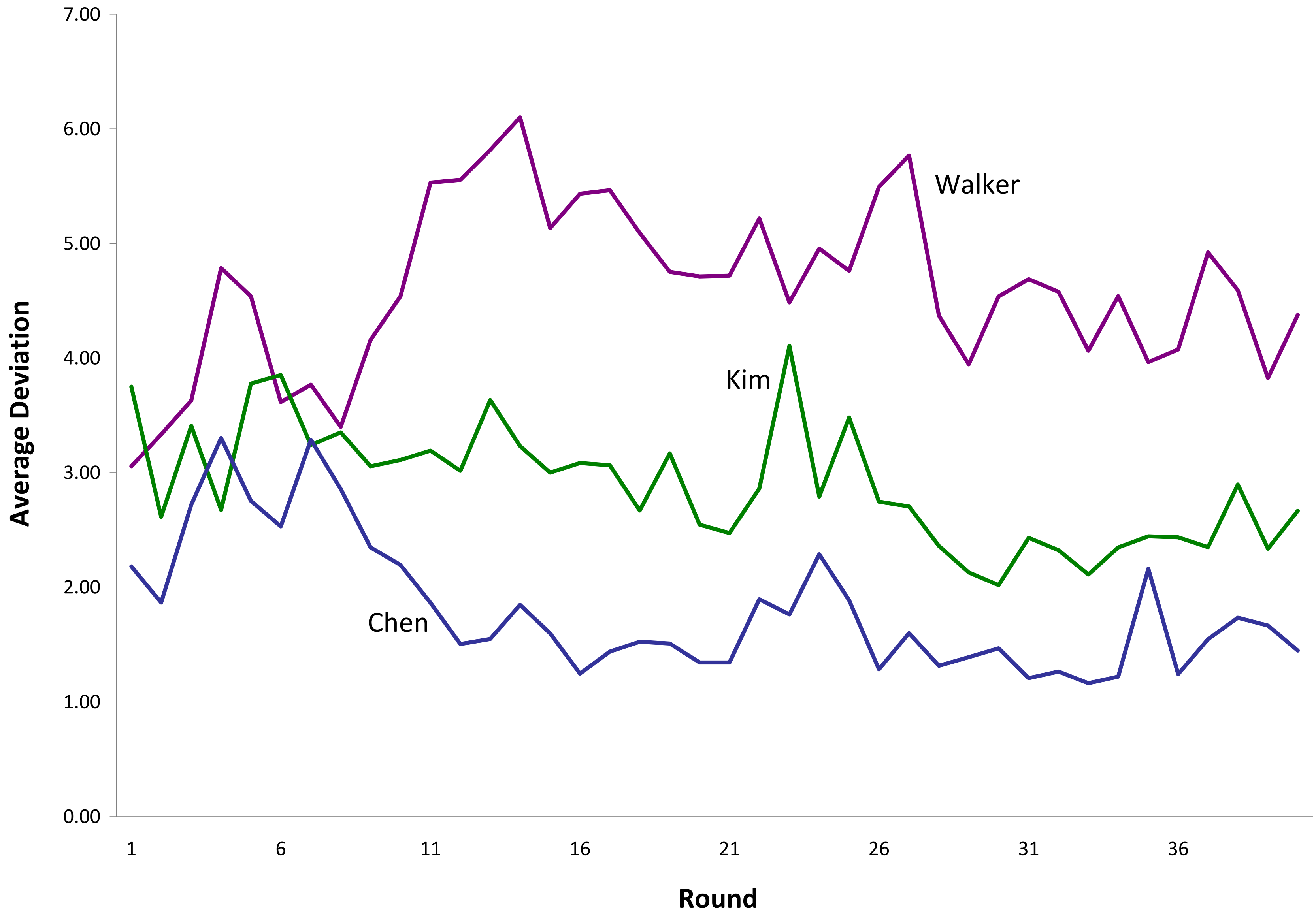

Figure 2a:
Average Public Good Provision (Chen and Kim)

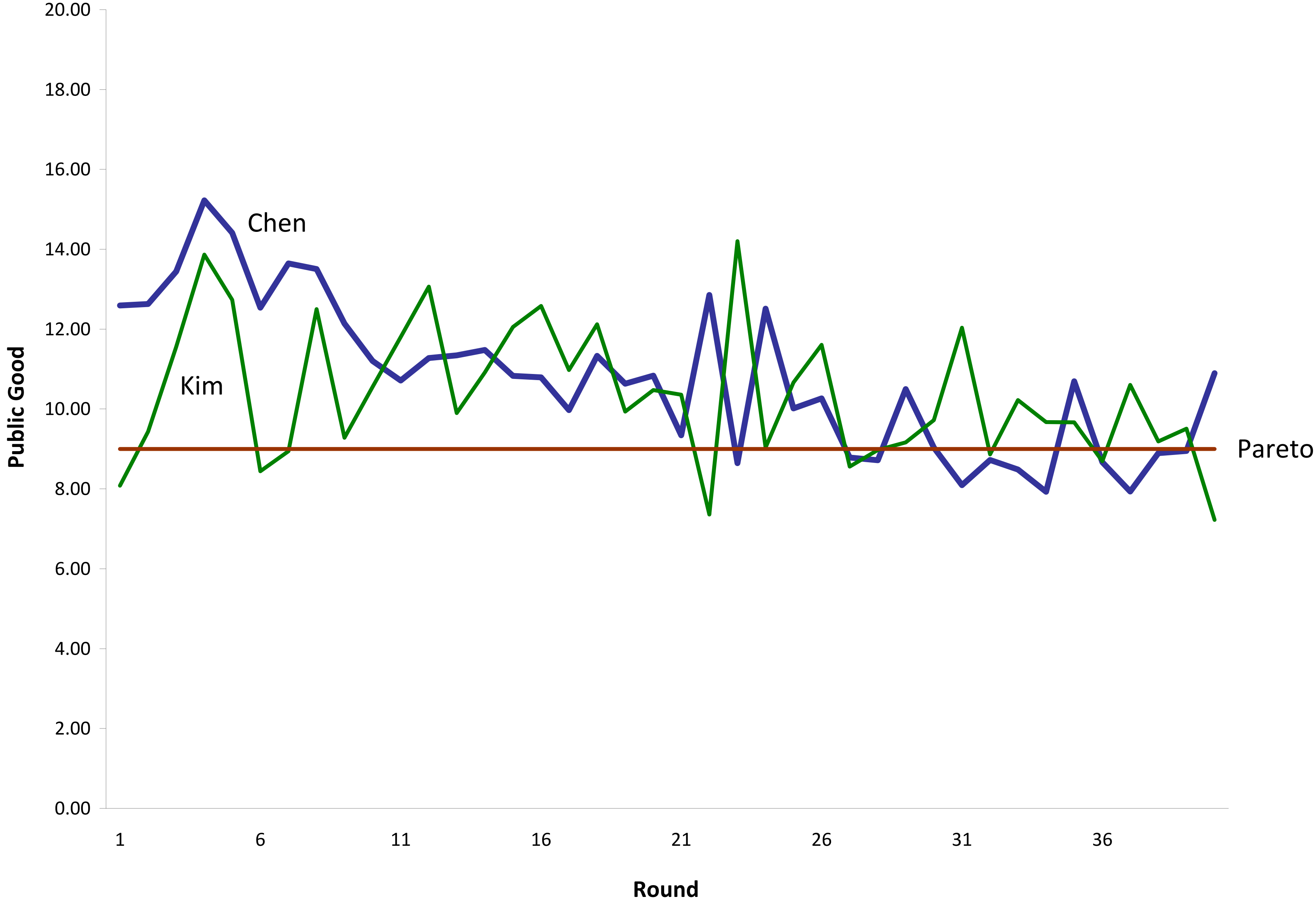

Figure 2b:
Public Good Provsion (Walker)

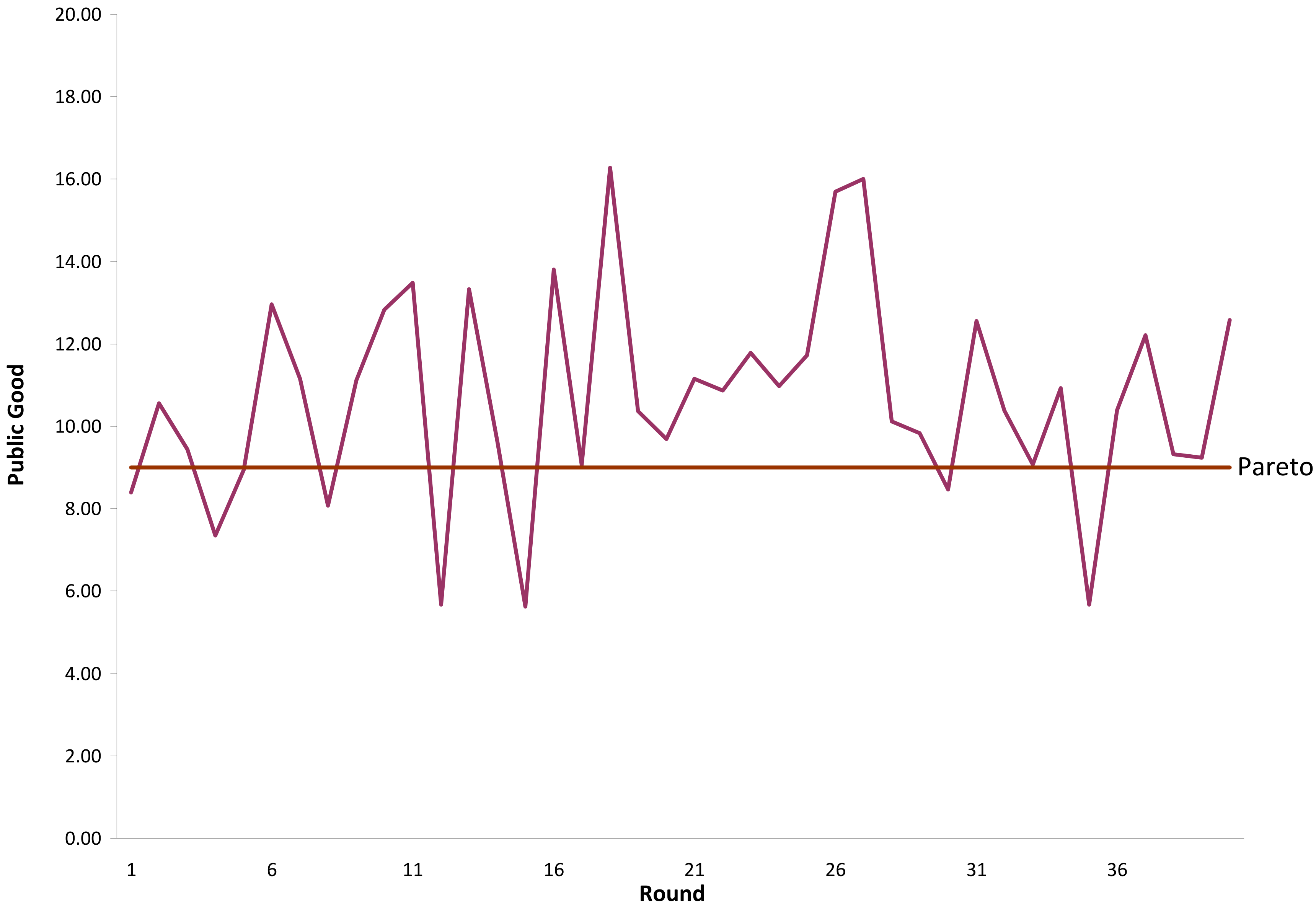

Figure 3a:
Public Good Deviation from Equilibrium (Chen and Kim)

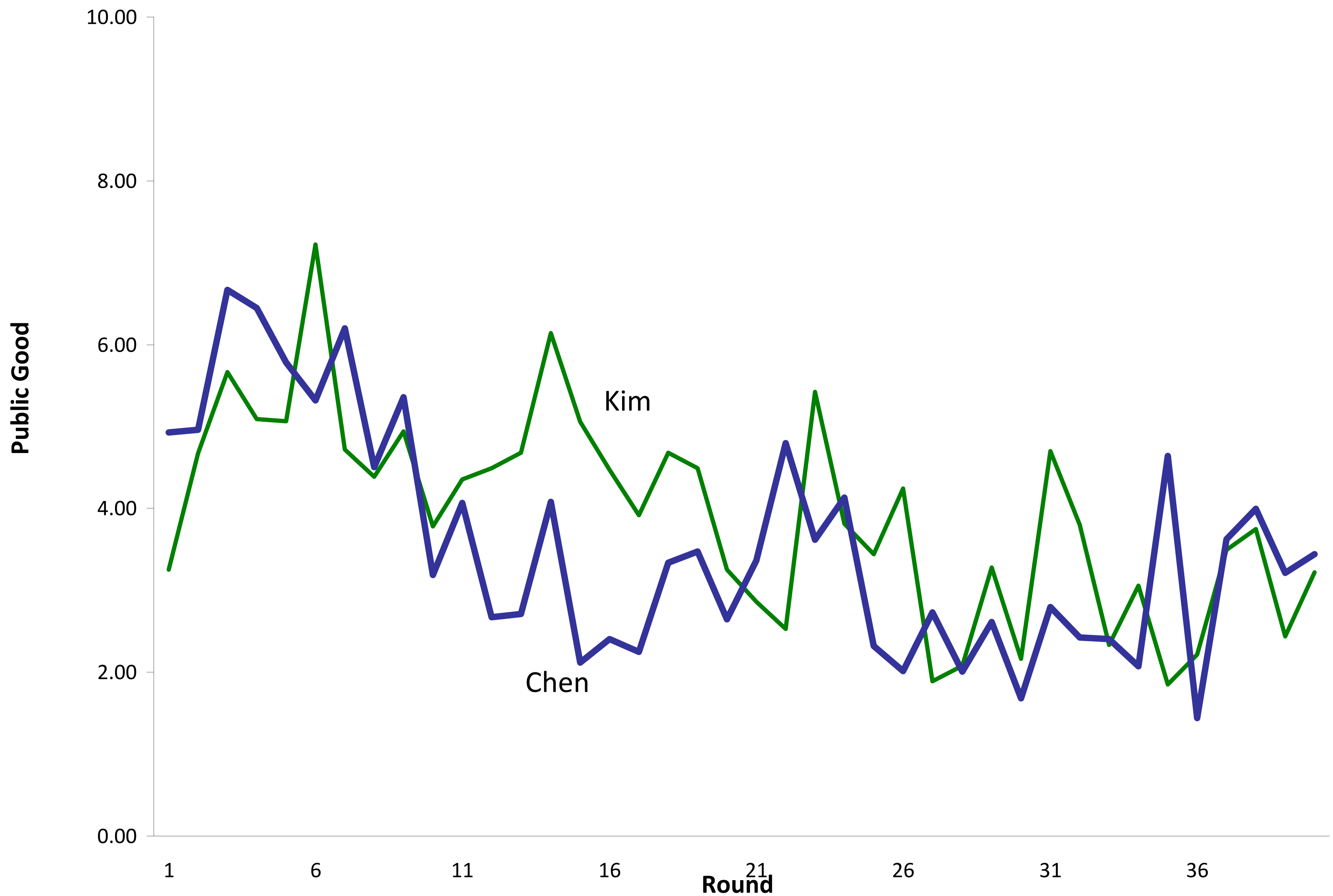

Figure 3b:
Public Good Deviation from Equilibrium (Walker)

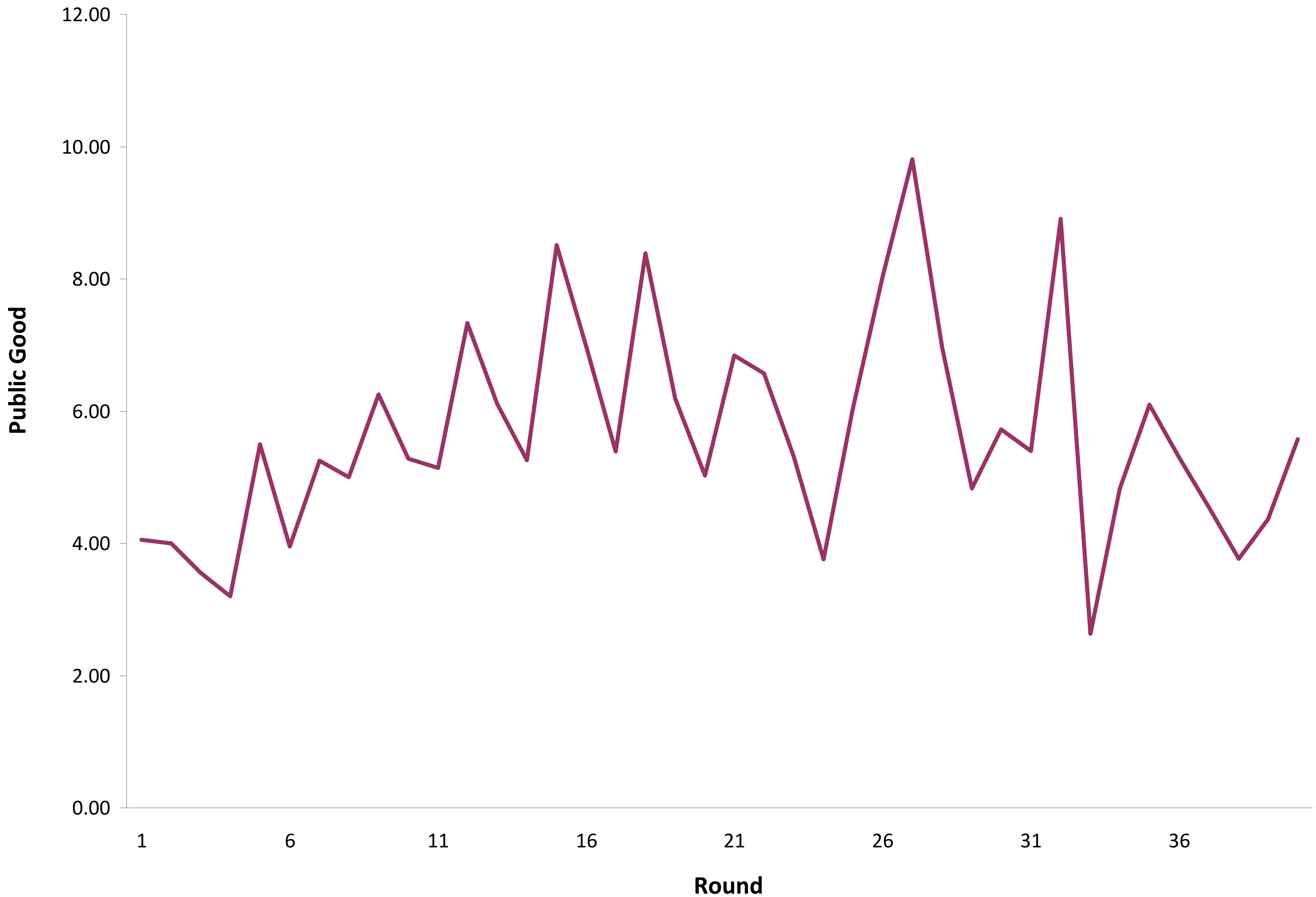

Figure 4:
Deviations from the Lindahl Tax Profile

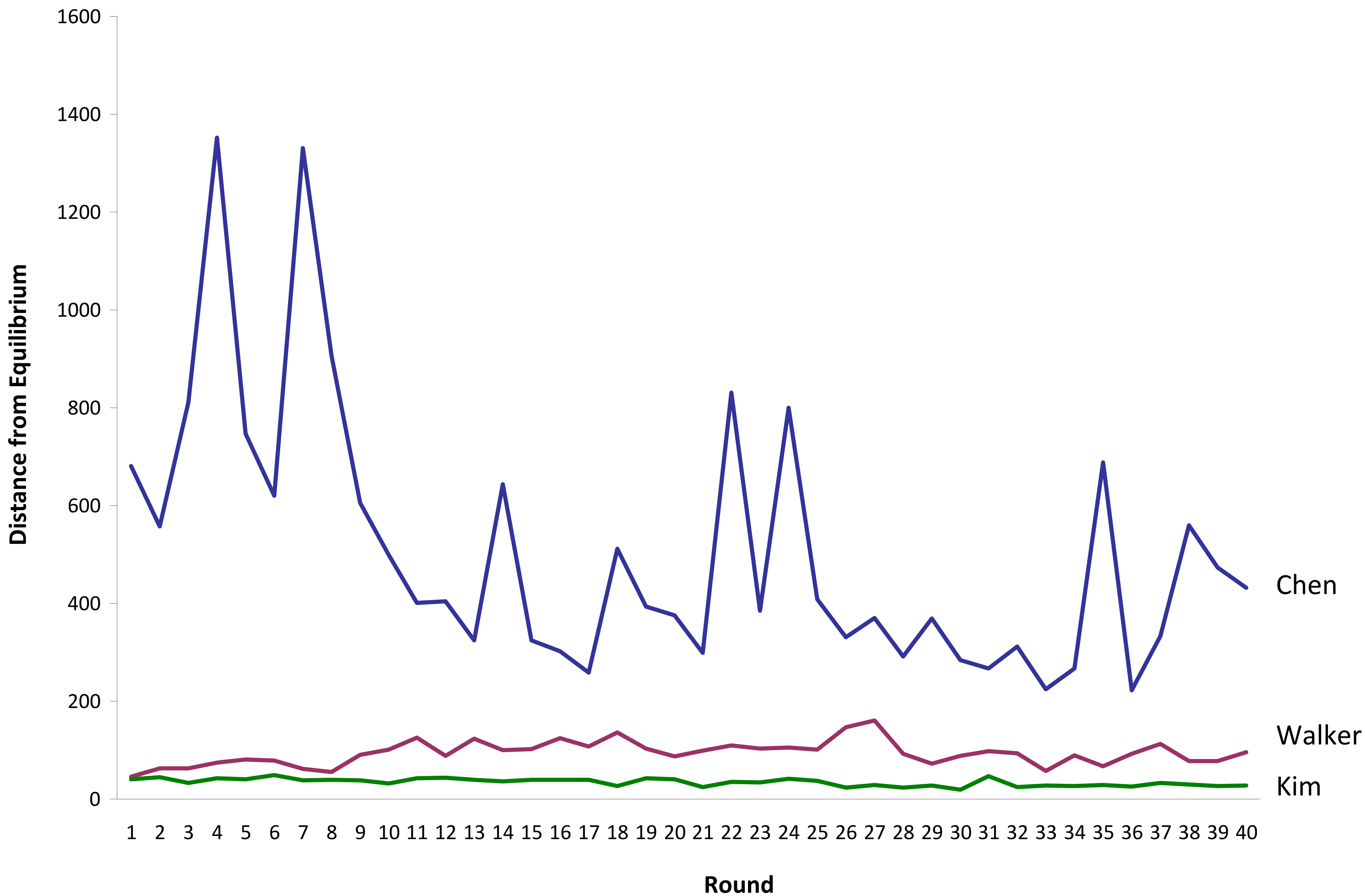

Figure 5a:
Average Surplus (Chen and Kim)

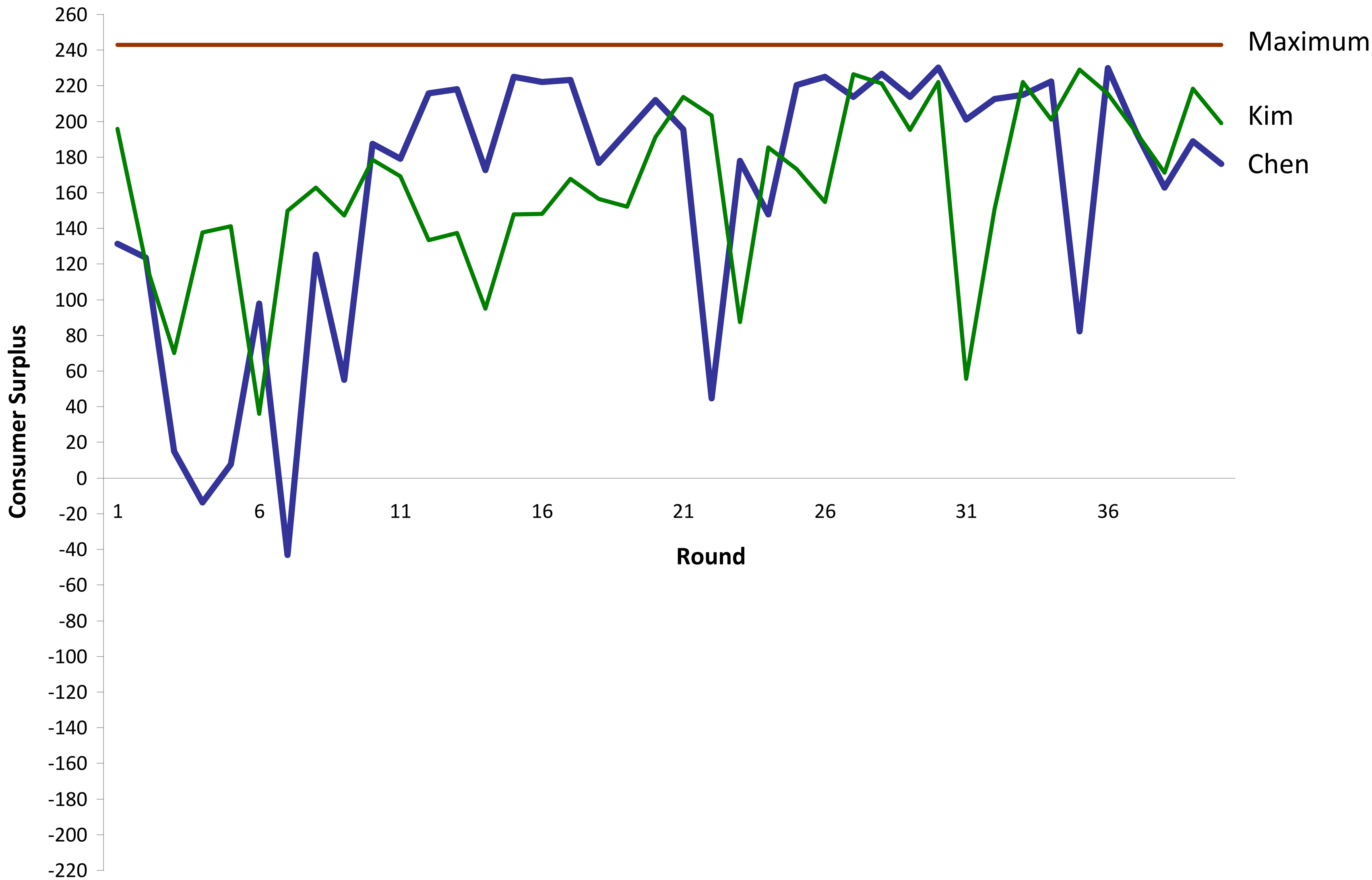

Figure 5b:
Average Surplus (Walker)

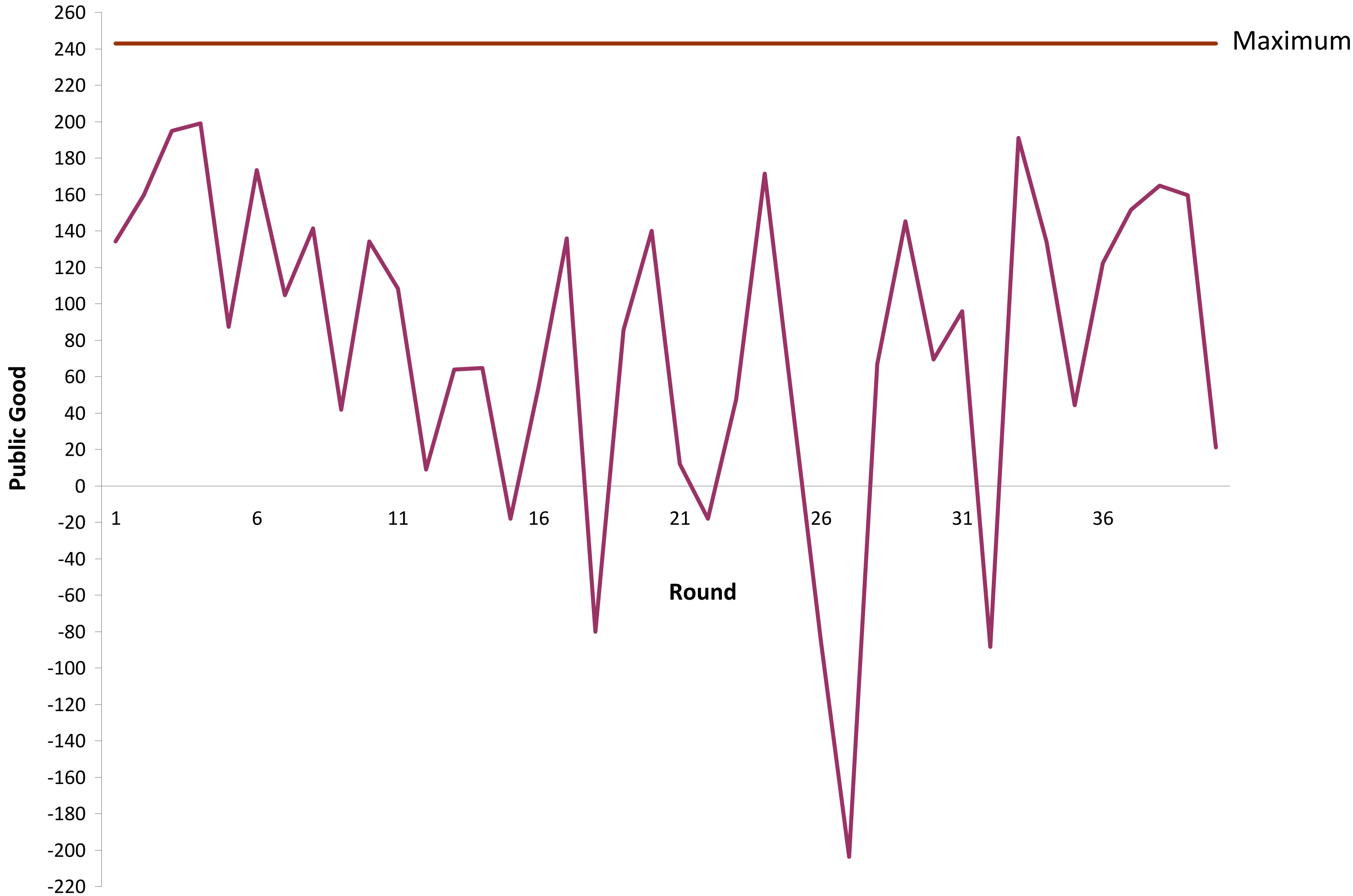

Figure 6:
Percentage of Participants with Violations of Individual Rationality

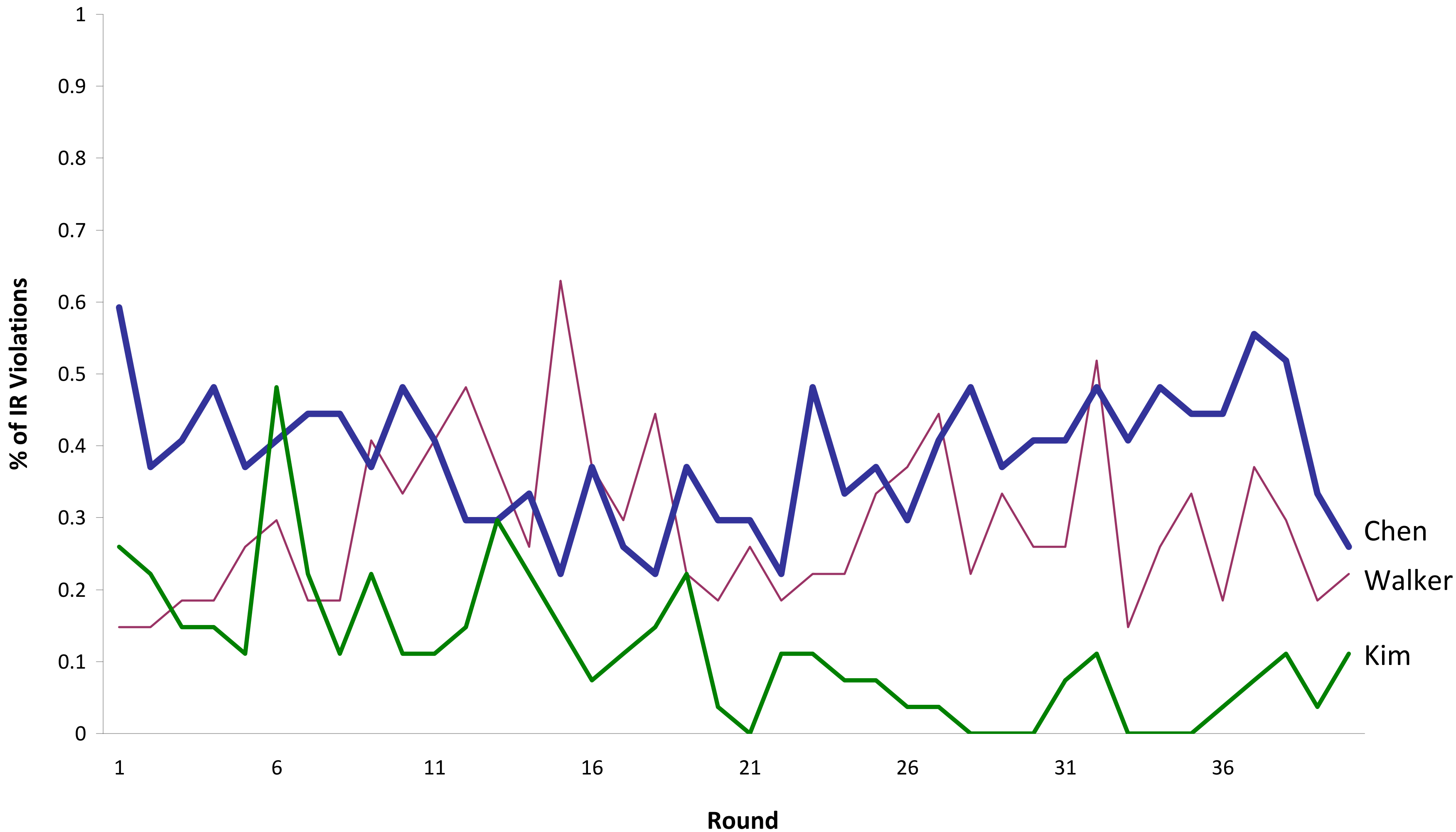

Figure 7:
Average Budget Imbalance