Government Subsidies for Research Programs Facing "If" and "When" Uncertainty

David Besanko*
Northwestern University

Jianjun Wu†
University of Arizona

Abstract

Although Governments have long played a significant role in supporting private and public R&D investments, they often end up subsidizing research programs with little or no knowlege of the projects'viability. However, most fundamental research such as the stem cell project involve both "if" and "when" uncertainty while previous literature on R&D subsidy has largely ignore the "if" part. In this paper, we introduce a two-armed bandit framework to model the impact of government subsidy on private R&D investment on projects with both types of uncertainty. We focus on two cases: monopoly and competitive R&D. We derive an individual firm's optimal R&D investment decision and noncooperative firms' symmetric Markov Perfect equilibrium investment strategies. If there is no shadow cost of public funding, we show that the first best welfare can be attained through a pure matching policy for the one firm case and a combination of matching and unrestricted subsidy policy for the multiple-firm case. An unrestricted component is necessary when subsidizing multiple firms because of the free riding problem that happens in the latter case. In contrast, if there is shadow cost of public funding, we show that earmark and unrestricted subsidies are never optimal for the monopoly case for a large set of parameters. In fact, numerical examples demonstrate that a pure matching policy is optimal for all cases in the monopoly and for cases in the R&D competition with not very large spillover.

1 Introduction

Governments have long played a role in supporting universities, non-profit organizations, and even private firms in their investments in R&D. The National Science Foundation reports that in 2006, $347.9 billion was spent on R&D in the United States, with the federal government responsible for directly funding about 28 percent ($97.7 billion) of this total.[1] The U.S. government's support for R&D includes money

*Department of Management and Strategy, Kellogg School of Management, Northwestern University, Evanston, IL 60208. Email: d-besanko@kellogg.northwestern.edu

†Department of Economics, University of Arizona, Tucson, AZ 85721-0108. Email: jwu@eller.arizona.edu.

[1] Table 1 in National Science Foundation, Division of Science Resources Statistics. 2008. National Patterns of R&D Resources: 2007 Data Update. NSF 08-318. Arlington, VA (hereafter NSF 2008). Available at http://www.nsf.gov/statistics/nsf08318/

spent on R&D directly by the U.S. government (27 percent) and by federally funded national laboratories (13 percent); the remaining 60 percent is divided between industries and universities in similar portions (Scotchmer,2004). Government involvement in basic research is even more substantial, with the U.S. federal government funding responsible for about 60 percent of the total amount spent on basic research in 2006.[2] Since this data only pertains to direct federal expenditures on R&D and not to other forms of support the government provides for R&D, they understate the substantial role that the U.S. government plays in stimulating and supporting R&D activity.

Government intervention in the provision of R&D arises for a variety of reasons. It may, first and foremost, be a response to a set of market imperfections that result in a less than socially optimal provision of R&D by private organizations (Arrow, 1962). These imperfections include: (1) difficulties in defining property rights to the full set of benefits created by an innovation, which means that innovators cannot fully appropriate the benefits their discoveries create (even with comprehensive patent protection); (2) challenges stemming from the unique nature of innovative knowledge (e.g., increasing returns and difficulties in judging whether it is valuable until after it is already acquired) that limit private demand for that knowledge and, in turn, make it difficult for markets for that knowledge to operate efficiently, further limiting the appropriability; (3) interdependencies and spillovers across research projects which can lead to innovators free riding on the efforts of other innovators; and (4) the inherent riskiness of R&D, coupled with capital market and insurance market imperfections, which can lead to privately profitable R&D projects not being funded. Government intervention may also be a response to macroeconomic benefits created by R&D activity. These include setting the stage for the emergence of strategically important industries in a developing nation or increasing the rate of long-term economic growth in a developed country (Grossman and Helpman, 1994). Finally, government intervention can also arise for political economy reasons, as when a firm or industry is able to influence the political decision makers to secure support for its R&D activities.

Government intervention itself can take a variety of forms. For example, governments may directly engage in R&D activity through government agencies, such as the National Institute of Health in the United States. The government also directly funds FFRDCs that are administered by universities and non-profit organizations. Examples include the Argonne National Laboratory (funded by the U.S. Department of Energy and administered by the University of Chicago) or the National Defense Research Institute (funded by the U.S. Department of Defense and administered by the RAND Corporation). The government might

[2] See Table 2 in NSF (2008).

also sponsor R&D activities in universities or not-for-profit organizations through programs of grants to researchers in those organizations. Governments may also subsidize private R&D investments through guaranteed loans (e.g., the $25 billion in loans provided to the Big Three automobile manufacturers in the United States to support the development of "green car" technologies), tax deductions or tax credits (e.g., the U.K. allows small and medium-size companies to deduct up to 175 percent of expenditures on R&D activities when determining their net income for tax purposes), direct payments to producers that develop new products or processes (e.g., the Chinese government provides direct subsidies for firms that manufacture wind power generators), or direct funding of R&D projects undertaken by firms (e.g., in Israel firms may apply to the government to receive R&D funding for specific projects). Governments can also indirectly enable private R&D by relaxing antitrust restrictions, as when the U.S. government gave firms partial antitrust immunity to form research consortia under the National Cooperation Act of 1984. Finally, governments can provide "infrastructure support" for private R&D through the use of tax dollars to support research parks or incubators, or through financial support for institutions that can fund the fruits of new product or technology development (e.g., in the U.S., the Small Business Administration has programs that provide funding for venture capital firms).

If government subsidies for private R&D help correct market failures, then government subsidies to firms that engage in R&D may stimulate more R&D investment and increase social welfare. However, some scholars, policy makers, and politicians have raised concern over government subsidies for R&D. Government subsidies may encourage firms to undertake R&D investments that are socially inefficient. Alternatively, it is suggested that subsidies for R&D may crowd out private investment in R&D. "Crowding out" would occur if government subsidies ended up displacing some of the money that firms would have spent on R&D in the absence of the subsidies. With crowding out, private R&D expenditures (excluding the portion paid for by government subsidies) are less than private R&D expenditures in the absence of government subsidies. In extreme cases of crowding out, total R&D expenditures (including the portion paid for by government subsidies) would be no greater, or possibly even less) than private R&D expenditures in the absence of government subsidies.

The theoretical literature on subsidization of R&D has generally focused on deterministic models of process innovations. For example, Hinloopen (1997, 2000) shows that regardless of whether firms engage in cooperative or non-cooperative R&D, subsidizing R&D optimally will lead to more R&D investment and higher social welfare. Romano (1989) shows that R&D subsidy also leads to higher social welfare for new product R&D bears only "when" uncertainty. The empirical evidence on R&D subsidization, however, is mixed. Irwin and Klenow (1996) find that government subsidies for R&D led to lower private investments in semiconductor industries. In contrast, Lach (2002) finds that R&D subsidies greatly stimulate the private investment in R&D for small business in Israel. The empirical work almost certainly deals in a broader context than the available theoretical contributions. It includes not only process R&D but also new product R&D in which there is uncertainty about both the timing of the new product discovery and even the ultimate technological or commercial viability of the product or technology.

Our goal in this paper is to fill the gap between the extant theory and empirical work that explore the impact of government subsidies on R&D. We focus on "programmatic" R&D activity aimed at achieving a breakthrough on a fundamental research(e.g., development of a fuel efficient electric vehicle or development of next-generation technologies for manufacturing semiconductors). In this context it is natural to imagine that there is considerable uncertainty, both about how long it might take until someone discovers the new product or technology ("when" uncertainty), and about whether the breakthrough can even be achieved at all ("if" uncertainty). The presence of both "if" and "when" uncertainty is characteristic of R&D that we tend to think of as more "basic" than "applied." This is precisely the domain in which government involvement in subsidizing private R&D is most substantial. For example, according to the National Science Foundation, of the \$250.3 million spent by private industry (firms plus industry administered FFRDCs) on R&D in the U.S. in 2006, \$29.3 million (or 10.7 percent) came from the federal government. By contrast, more than 18 percent of private industry spending on basic research came from funding by the federal government.[3]

The framework we employ is similar to the two-arm bandit model of R&D in Besanko and Wu (2008). In this framework, firms make R&D investments aimed at achieving a significant R&D breakthrough. As time passes and the breakthrough is not achieved, firms become more pessimistic about the likelihood that this path of inquiry will ever pay off, and if they become sufficiently pessimistic, they will eventually terminate the project. Conditional on the project being viable, the likelihood and timing of a breakthrough depends on how persistent the firms are, i.e., how willing they are to continue to fund the project over time.

Using this framework, we examine the implications of government subsidies for R&D. Under the general subsidy mechanism considered, the firm receives government funding for R&D, which may be tied to its

[3] These calculations are based on data in Tables 1 and 2 in NSF (2008).

actual R&D effort. The general mechanism subsumes three specific funding schemes commonly used in practice: a *pure matching subsidy*, an *earmarked subsidy*, and a *pure unrestricted subsidy*. Under a matching subsidy, the government reimburses *ex post* a fraction $\phi \in (0, 1]$ of the firm's actual R&D expenses. Under an earmarked subsidy, the government commits to fund a project until a breakthrough occurs, subject to the requirement that the firm provides a minimum mandated level of R&D effort on the focal project. Under an unrestricted subsidy, the government makes an open-ended commitment to fund the project until a breakthrough is made. Unlike an earmarked subsidy there is no formal requirement that the firm actually spend the money on the focal R&D project.

In the paper, we focus on two cases: monopoly (by which we mean a firm that faces no R&D competition) and competition. A monopolist's optimal R&D investment decision is "bang-bang" rule: depending on its posterior beliefs about the project's viability, it either invests "flat out" in R&D at each instant in time (i.e., it sets its R&D effort at its maximum technologically feasible level) or nothing at all. Relative to the case of no subsidies, the matching component of the subsidy expands the range of posterior beliefs over which the monopolist invests "flat out," while the earmarked and unrestricted components of the subsidy shrinks that range. Thus, increases in the matching rate stimulate private spending on R&D, while increases in unrestricted funding or the minimum mandated R&D effort crowd out private spending. However, with minimum mandated R&D effort, under an earmarked subsidy, R&D effort on behalf of the project continues below the point at which the firm continues to invest flat out, and thus in principle, an earmarked subsidy could stimulate overall investment in R&D. However, as we show, when there is no shadow cost of public funds (i.e., subsidies are pure transfers between taxpayers and firms), the optimal subsidy mechanism is a pure matching subsidy that induces the firm to follow the first-best R&D policy irrespective of prior beliefs about the viability of the project. With a positive shadow cost of public funds, the first-best investment policy cannot, in general, be achieved. However, through a combination of analytical sufficient conditions and calculations based on empirically plausible parameter values, it appears that the optimal subsidy policy (i.e., that maximizes *ex ante* welfare given the subsidy policy) continues to be a pure matching policy. However, the matching rate is generally less than the matching rate that implements the first-best solution in the absence of a shadow costs. In some cases, depending on prior beliefs and the appropriability of the social returns from R&D, the optimal matching rate may be 0 (i.e., it may not be optimal to subsidize the firm).

Under competition, an additional complication arises that does not exist under monopoly: the possibility that firms may free ride on the R&D efforts of other firms. As we show below, the free-rider problem implies that in a symmetric equilibrium, investment by firms is no longer "bang-bang." Instead, it may involve a range of posterior beliefs over which each firm invests a positive amount in R&D which is less than the technological maximum. When the free-rider problem can arise, an unrestricted component to the subsidy and a minimum mandated level of R&D — which had unambiguously adverse incentive effects under monopoly — can actually eliminate the free-rider problem. Indeed, unless there is an unrestricted component of the subsidy or a minimum mandated R&D effort, the free-rider problem will always arise. When there is a zero shadow costs of public funds, as in the case of monopoly, there exists a subsidy policy that induces firms in equilibrium to follow the first-best investment policy. However, unlike the case of monopoly, that policy is not a pure matching policy. Instead, it involves a combination of a matching component and an unrestricted component. Numerical examples show that for small positive shadow costs of public funds, the optimal subsidy policy continues to involve only mathcing and unrestricted components; for large shadow cost of public funds, the optimal subsidy policy reverts to a pure matching policy when the spillover is not very large.

Most theoretical papers on R&D subsidy policy employ the model developed by d'Aspremont and Jacquemin (1998) and Kamien et. al. (1992) in which firms' investments in R&D deterministically reduce production costs. Hinloopen (1997) shows that in this context an R&D subsidy leads to more investment and social welfare improvement. By contrast, by employing a two-arm bandit model, our paper is the first one that address the subsidy to the R&D projects with both "if" and "when" uncertainty. Unlike the cost-reducing R&D policy discussed in previous papers, we show that the structure of R&D policy matters in the stimulating R&D investment and affecting social welfare.

The paper is divided into 5 sections. Section 2 lays out the model. Section 3 considers the case of monopoly, while Section 4 consider the N-firm case. Section 5 explores the possibility that firms can form a research joint venture. Section 6 summarizes and concludes.

2 The Model

We present a model of R&D investment based on the exponential bandit framework of Keller, Rady, and Cripps (2005). We state the model with N firms, with the analysis of monopoly corresponding to the special case of $N = 1$.

Each of the N firms faces an opportunity to invest in an R&D program aimed at achieving a significant scientific, engineering, or commercialization breakthrough. The breakthrough might be development of a new vaccine for an infectious disease, the development of a commercially viable design of an electric vehicle, or a new encryption protocol to strengthen cybersecurity. A key point is that the R&D program is deemed sufficiently important that the government is, in principle, committed to supporting firms' efforts to achieve a breakthrough until a breakthrough occurs.

Ex ante the firms do not know if a breakthrough is possible. Let $p_0 = \Pr(\widetilde{\eta} = 1) \in (0, 1)$ denote the firms' common prior that the project is viable, i.e., that the breakthrough will eventually be achieved by some firm. Conditional on the project being viable, the time the breakthrough occurs is random. Higher R&D investment is assumed to increase the likelihood that the breakthrough occurs sooner rather than later. Specifically, let k_t^i denote firm i's R&D effort at time t. Conditional on the project being viable, the hazard rate of success on the R&D project is $\lambda k_t^i dt$, where $\lambda > 0$ is a parameter. We assume that each firm faces a technological constraint that limits its investment in R&D to at most 1 unit of effort at any point in time t. Thus, $k_t^i \in [0, 1]$. One can interpret this constraint as an extreme form of diminishing marginal returns to R&D. If the project was indeed viable, and a firm exerted the maximum feasible level of R&D effort ($k = 1$), then $\frac{1}{\lambda}$ would be the expected time until a breakthrough occurs.

R&D effort is costly, and the total cost $C(k_t^i)$ of R&D effort is assumed to be an identical linear function for each firm, $C\left(k_t^i\right) = \alpha k_t^i$, where $\alpha > 0$ denotes the marginal cost of R&D effort.[4] This cost could either be a direct cost of R&D effort, or an opportunity cost of redeploying scarce internal resources to support the focal R&D effort.

The achievement of a breakthrough is assumed to be "big news" and visible to all firms competing in the R&D race. As time passes and a breakthrough has not occurred, firms become more pessimistic about the viability of the project. Let $p\left(t\right)$ denote firms' posterior belief about the project's viability at date t. If no breakthrough occurs, $p(t)$ adjusts downward according to Bayes rule:

$$p\left(t + dt\right) = \frac{p\left(t\right)\left(1 - \lambda \sum_{i=1}^{N} k_t^i dt\right)}{1 - p\left(t\right) + p\left(t\right)\left(1 - \lambda \sum_{i=1}^{N} k_t^i dt\right)}. \tag{1}$$

In the numerator of (1), $1 - \lambda \sum_{i=1}^{N} k_t^i dt$ is the conditional probability that there is no breakthrough within

[4] The linearity of the cost function is essential for being able to solve for the equilibrium investment level in closed form. The basic intuition underlying the results does not depend on the linearity of the cost function.

a time interval $[t, t+dt)$, given that the project is viable.[5] The denominator of (1) is the unconditional probability that there is no breakthrough within the time interval $[t, t+dt)$. It can be shown that

$$\frac{dp}{dt} = \lim_{dt \to 0} \frac{p(t+dt) - p(t)}{dt} = -\lambda \sum_{i=1}^{N} k_t^i p(t)(1 - p(t)). \tag{2}$$

This rate of belief updating is independent of its starting state, so we may rewrite it as

$$dp = -\lambda \sum_{i=1}^{N} k_t^i p(1-p)\, dt. \tag{3}$$

The solution concept is Markov Perfect Equilibrium, with each firm's common posterior belief p being the payoff-relevant state variable and equation (3) representing the law of motion for the state variable. Investment behavior and firm value functions are thus conditioned on p.

We assume that the firm that wins the race to achieve the R&D breakthrough earns a present discounted value of profit given by $\Pi > 0$. The present discounted value of profit to each of the $N-1$ remaining firms is assumed to be $\theta\Pi$, where $\theta \in [0,1]$. If $\theta = 0$, the R&D race is winner-take-all; if $\theta > 0$, the breakthrough has positive spillovers for the non-successful firms in the R&D race. The discounted present value of the social benefit from the new product is given by $CS + \Pi + (N-1)\theta\Pi$, where CS is the benefit of the breakthrough to consumers that the discovering and non-discovering firms cannot capture.[6] For later use, let $\rho \equiv \frac{\Pi + (N-1)\theta\Pi}{CS + \Pi + (N-1)\theta\Pi} \in (0,1]$ be the appropriability ratio, i.e., the share of the total benefit captured by firms and thus $1-\rho$ is the share of the total benefit that accrues to consumers. There are thus two potential justifications for government intervention in the market for R&D: the presence of R&D spillovers across firms (when $\theta > 0$) and the imperfect appropriability of the benefit from the breakthrough (when $\rho < 1$). Throughout the analysis, we assume that $\frac{\lambda[CS + \Pi + (N-1)\theta\Pi]}{\alpha} > 1$, which implies that the social benefit-cost ratio of a viable R&D project exceeds 1.[7]

We assume that as long as the breakthrough has not yet occurred, the government provides financial

[5] The probability of no success within time dt for all firms i is $e^{-\lambda \sum_{i=1}^{N} k_t^i dt}$, whose first order approximation is $1 - \lambda \sum_{i=1}^{N} k_t^i dt$.

[6] Throughout the analysis, we assume that Π, CS, and θ do not depend on N. In other words, we assume that the structural conditions that determine post-breakthrough profit, consumer surplus, and spillovers are independent of the number of firms engaged in competition to achieve the breakthrough itself. We thus have in mind a situation in which post-breakthrough profits and consumer surplus are shaped by forces such as patent policy, market entry, and imitation that may be very different from the forces that resulted in N competitors appearing at the "starting line" of the R&D race.

[7] Let $\widetilde{T}$ be the random time to discovery for a project that is certain to be viable. With hazard rate λ and flat-out investment at any point in time, $\widetilde{T}$ is an exponential random variable with parameter λ. The *ex ante* expected social benefit of a viable R&D project would be $[CS + \Pi + (N-1)\theta\Pi]\, E(e^{-r\widetilde{T}})$, which equals $\frac{\lambda[CS+\Pi+(N-1)\theta\Pi]}{\lambda + r}$. The *ex ante* expected cost of a viable

support to firms actively engaged in R&D effort. In effect, the government has made a commitment to support an on-going program of research carried out by private firms. The commitment is assumed to be open-ended in that the funding could potentially continue until the breakthrough occurs. An example of such a commitment is the U.S. government's funding of the SEMATECH consortium during the 1980s and 1990s aimed at discovering new technologies for semiconductor manufacturing.

The government's R&D subsidy policy is represented by an array of three instruments (z, s, ϕ) that form a schedule $S(k_t^i|z, s, \phi)$ that determines the funding flow a firm receives at each instant in time prior to a breakthrough:

$$S(k_t^i|z, s, \phi) = \begin{cases} s + \phi\alpha(k_t^i - z) & \text{if } k_t^i \geq z, \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

In this schedule:

- $z \in [0, 1]$ is the minimum R&D effort mandated by the government at each instant in time in order for the firm to be eligible for any funding, and thus αz is the minimum mandated spending on R&D.

- $s \in [\alpha z, \infty)$ is the baseline amount of funding the firm receives, provided it satisfies the mandate. We require that $s \geq \alpha z$ so that at any point in time the firm would prefer to adhere to the mandate and accept the associated funding, rather than rejecting it. If $s = \alpha z$, the government exactly reimburses the firm its mandated R&D effort, while if $s > \alpha z$, the firm receives a subsidy in excess of its minimum mandated R&D expenditure. In this latter case, the firm could (in principle) spend some of its government funds on other activities besides the focal R&D project (e.g., it could fund other R&D projects). We thus refer to $s - \alpha z$ as the *unrestricted component* of its R&D subsidy.

- $\phi \in [0, 1]$ is the matching rate: the additional funding the firm receives for every additional dollar of R&D spending undertaken above the mandated level.

We assume throughout that the government does not know the firms' prior belief p_0 and thus cannot infer the posterior belief $p(t)$. This rules out the possibility that the government can write a fine-tuned

R&D project would be $\int_0^\infty \alpha \left(\int_0^t e^{-r\tau} d\tau\right) \lambda e^{-\lambda t} dt$ which can be shown to equal $\frac{\alpha}{r+\lambda}$. The *ex ante* benefit cost ratio is thus

$$\frac{\lambda\left[CS + \Pi + (N-1)\theta\Pi\right]}{\alpha}.$$

"forcing contract" in which it ties the parameters of the subsidy function to the posterior belief p in such a way that it replicates the first-best investment policy.

The policy instruments (z, s, ϕ) embrace three interesting special cases:

- If $z = 0$, $s = 0$, and $\phi \in (0, 1]$, then a firm receives a *pure matching subsidy*: for every αk dollars of R&D investment, the government "matches" the firm's R&D spending by providing a subsidy of $\phi\alpha k$. In other words, the government reimburses a fraction ϕ of the firm's actual R&D costs, with the firm, in effect, bearing a fraction $(1 - \phi)$ of those costs.

- If $z > 0$, $s = \alpha z$, and $\phi = 0$, then a firm receives an *earmarked subsidy*: it receives a subsidy of αz dollars, provided that its R&D effort satisfies the mandate z. The subsidy s just compensates the firm for the cost αz associated with obeying the mandate, and the firm cannot use the funds for anything other than the focal R&D project.

- If $z = 0$, $s > 0$, and $\phi = 0$, then a firm receives a *pure unrestricted subsidy*: it receives a no-strings-attached grant of s until the R&D breakthrough is achieved.

The pure matching subsidy and the earmarked subsidy are particularly noteworthy, not only because they have counterparts in practice, but also because they represent two alternative incentive tools for promoting R&D. The pure matching subsidy is a price-based approached that lowers the firm's perceived marginal cost of R&D effort. The earmarked subsidy is a quantity-based approach that ensures a minimum level of R&D effort z in exchange for a subsidy payment αz.

3 Monopoly R&D

This section considers the case of a single firm engaged in the search to achieve the R&D breakthrough, i.e., $N = 1$. Faced with a subsidy policy (z, s, ϕ), the firm's Bellman equation is[8]

$$V(p) = \max_{k \in [z,1]} \left[(s - \alpha\phi z - \alpha(1-\phi)k)dt + \lambda kpdt\Pi + (1 - \lambda kpdt)\, e^{-rdt} V(p + dp) \right], \tag{5}$$

[8] Implicit in this formulation is that the firm accepts the subsidy and agrees to abide by the requirement that $k \geq z$, i.e., that it invests at least z units in R&D. It is conceivable that if the level of subsidy (represented by s or ϕ) is sufficiently low and z is sufficiently high, the firm would find the minimum R&D restriction too onerous and would prefer to refuse the subsidy altogether. A sufficient condition for the subsidy to be acceptable to the firm is $s \geq \alpha z$ because the firm would then receive at least as much from the government in subsidy as it is required to spend. For the special case of an earmarked subsidy, $z = \frac{s}{\alpha}$, and thus this sufficient condition holds. It also (trivially) holds for the case of a matching subsidy since in that case $s = z = 0$.

where r is the firm's discount rate. The firm's profit flow at an instant in time dt is $s - \alpha(1-\phi)k$. With probability $\lambda kpdt$, a breakthrough will take place within the interval $[t, t+dt)$, which gives the firm the prize Π from discovering the new product. With probability $1 - \lambda kpdt$, no breakthrough occurs within the interval $[t, t+dt)$, and the firm's value become $e^{-rdt}V(p+dp)$. Using standard arguments, the firm's value function can be shown to be implicitly defined by the following differential equation:[9]

$$(1+r)V(p) = s - \alpha z + \max_{k\in[z,1]} \left\{ \begin{array}{c} [\lambda kp\Pi + (1-\lambda kp)V(p)] \\ -\alpha(1-\phi)(k-z) - \lambda kp(1-p)V'(p) \end{array} \right\}. \tag{6}$$

The firm's value consists of four components:

1. $s - \alpha z$ is the flow benefit from the unrestricted component of the subsidy;
2. $\alpha(1-\phi)(k-z)$ is the net-of-subsidy total cost of k units of R&D effort;
3. $[\lambda kp\Pi + (1-\lambda kp)V(p)]$ is the expected benefit from k units of R&D effort (with probability λkp the firm achieves the breakthrough and gets Π and with probability $(1-\lambda kp)$ the firm gets the continuation value $V(p)$);
4. $\lambda kp(1-p)V'(p)$ is the expectation, with k units of R&D effort, of the option value of waiting that is foregone if the firm achieves a breakthrough.

In the Appendix we prove:

Proposition 1 *Given a subsidy policy (z, s, ϕ), a monopolist's optimal R&D strategy is as follows:*[10]

If $\frac{\alpha(1-\phi)}{\lambda\left[\Pi - \frac{(s-\alpha z)}{r}\right]\frac{r}{(r+\lambda z)}} < 1$

$$k_1(p) = \left\{ \begin{array}{ll} 1 & \text{if } p \in [p_1, 1] \\ z & \text{if } p \in [0, p_1] \end{array} \right., \tag{7}$$

where p_1 solves

$$\lambda p_1 \left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda z}\right) - (1-\phi)\alpha = 0, \tag{8}$$

[9] The approach is to form a Taylor expansion of $V(p)$ (ignoring the higher order terms which disappear as dt goes to 0), which gives

$$V(p+dp) = V(p) + V'(p)\,dp = V(p) - \lambda kp(1-p)V'(p)\,dt.$$

Substituting this into (5), taking limits as $dt \to 0$, and simplifying yields (6).

[10] We use the subscript "1" to denote the monopoly case, and the subscript "N" to denote the N-firm case.

which implies

$$p_1 = \frac{\alpha(1-\phi)}{\lambda\left[\Pi - \left(\frac{s-\alpha z}{r}\right)\right]\left(\frac{r}{r+\lambda z}\right)}. \tag{9}$$

The firm's value function is

$$V_1(p) = \begin{cases} \frac{s-(1-\phi)\alpha-\alpha\phi z}{r} + \frac{\lambda}{r+\lambda}\left(\Pi - \left[\frac{s-(1-\phi)\alpha-\alpha\phi z}{r}\right]\right)p + B_1 p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & \text{if } p \in [0, p_1] \\ \frac{s-\alpha z}{r} + \frac{\lambda z}{r+\lambda z}\frac{r\Pi-(s-\alpha z)}{r}p & \text{if } p \in [p_1, 1] \end{cases},$$

where

$$B_1 = \frac{\lambda\alpha(1-z)(1-\phi)}{r(r+\lambda)}\left(\frac{1-p_1}{p_1}\right)^{-\frac{r}{\lambda}}.$$

If $\frac{\alpha(1-\phi)}{\lambda\left[\Pi - \frac{(s-\alpha z)}{r}\right]\frac{r}{(r+\lambda z)}} \geq 1$,

$$k_1(p) = z \text{ for all } p \in [0,1], \tag{10}$$

which implies $p_1 = 1$, *and the firm's value function is*

$$V_1(p) = \frac{s-\alpha z}{r} + \frac{\lambda z}{r+\lambda z}\frac{r\Pi - (s-\alpha z)}{r}, \text{ for all } p \in [0,1].$$

According to Proposition 1, the monopolist's optimal investment decision $k_1(p)$ is a "bang-bang" rule: k either equals the minimum required level z, or the maximum feasible level 1, depending on whether the posterior belief p is greater or less than p_1. We refer to p_1 as the *abandonment threshold*. It is the point at which the firm abandons a "flat out" commitment to the R&D program. A monopolist following the optimal policy would thus behave in one of two ways. If the firm is sufficiently optimistic about the project's viability *ex ante* so that $p_0 > p_1$, the firm would begin by exerting R&D maximum effort (i.e., $k = 1$). As time passes without a breakthrough, the firm would become more pessimistic about the viability of the project, and once its posterior p falls to the abandonment threshold p_1, the firm would switch from "flat out" investment (i.e., $k = 1$), to the minimum level mandated by the government in exchange for the subsidy (i.e., $k = z$). As long as a breakthrough does not occur, the firm would persist with the minimum mandated level. By contrast, if the firm is sufficiently pessimistic about the viability of the project *ex ante* so that $p_0 \leq p_1$, it would exert the minimum mandated level z of R&D effort and would never invest more than this.

The abandonment threshold p_1 comes from a "marginal cost equals marginal benefit condition" (8). The marginal cost of an additional unit of R&D effort above the minimum threshold is $(1-\phi)\alpha$. The matching component of the subsidy thus reduces the firm's marginal cost. The firm's marginal benefit equals $\lambda p \left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda z}\right)$. The marginal benefit consists of two components: (a) the incremental increase in the likelihood of a breakthrough, λp and (b) the net prize to the firm if a breakthrough occurs, $\left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda z}\right)$. The firm's net prize is the present value of profits from a breakthrough, Π, minus the present value of the fungible portion of the subsidy, $\frac{s-\alpha z}{r}$, which the firm foregoes if it achieves a breakthrough. The net prize is further "deflated" by the term $\frac{r}{r+\lambda z}$ which is less than 1 when $z > 0$; as z increases the extent of this deflation increases, and the firm's marginal benefit falls.[11] As we discuss below, this term reflects the impact of option value the firm foregoes if it achieves a breakthrough.

Given (9), we can immediately determine the incentive properties of the various policy instruments.

Proposition 2 *(a) Holding s and z fixed, an increase in the matching rate ϕ decreases the monopolist's abandonment threshold, thus expanding the range over which firm invests "flat out" in excess of the mandated minimum; (b) Holding ϕ and z fixed, an increase in the baseline subsidy s (which thus increases the unrestricted portion of the subsidy $s - \alpha z$) increases the monopolist's abandonment threshold, thus contracting the range over which the firm invests "flat out" in excess of the mandated minimum; (c) Holding ϕ fixed and the unrestricted portion of the subsidy $s - \alpha z$ fixed, an increase in the mandated minimum z increases the monopolist's abandonment threshold, thus contracting the range over which the firm invests "flat out" in excess of the mandated minimum.*

Increasing the matching rate ϕ strengthens the firm's incentive to persist with discovery efforts, while increases in the unrestricted component of the subsidy $s - \alpha z$ or the minimum mandate z (with the unrestricted portion of the subsidy being held fixed in the latter case) weaken those incentives. The unrestricted component of the subsidy is a drag on R&D incentives because by investing more heavily

[11] Formally, it can be shown that $\frac{r}{r+\lambda z}$ is the arc elasticity of the expected present value of dollar with respect to z. Specifically:

$$\frac{r}{r+\lambda z} = \frac{\frac{E(e^{-r\tilde{T}}|k=1)-E(e^{-r\tilde{T}}|k=z)}{E(e^{-r\tilde{T}}|k=1)}}{\frac{1-z}{1}}, \tag{11}$$

where $\widetilde{T}$ denotes the time to discovery; $E(e^{-r\widetilde{T}}|k=1) = \int_0^\infty e^{-rt}\lambda e^{-\lambda t}dt = \frac{\lambda}{\lambda+r}$ is the expected present value of \$1 when the firm invests flat out; and $E(e^{-r\widetilde{T}}|k=z) = \int_0^\infty e^{-rt}\lambda z e^{-\lambda z t}dt = \frac{\lambda z}{\lambda z+r}$ is the expected present value of \$1 when the firm invests at level $z \in [0,1]$. These expressions arise because, conditional on the project being viable, if the investment effort is a constant k, then discovery time is an exponetial random variable with parameter λk. Thus, $\frac{r}{r+\lambda z}$ represents (approximately) the percentage change in the expected value of \$1 worth of a prize per one pecent change in R&D effort above the minimum level. When $z = 0$, this elasticity equals 1, As the mandated minimum increases, this elasticity decreases.

and accelerating the expected time to a breakthrough, the firm brings to an end more quickly the benefit flow $s - \alpha z$ it gets from the fungible portion of its funding. Increases in the fungible funding magnify this negative consequence of achieving a breakthrough.

Increasing the minimum mandate z (holding the fungible portion of the subsidy fixed) is also a drag on R&D incentives for a related, but slightly different, reason. When $k = z$, the firm, in effect, receives a fully-funded option from the government: the government is paying for the R&D investment z, but the firm receives the benefit Π if a discovery is made. A firm faces a trade-off between accelerating the time to breakthrough at its own cost or retaining the free option from the government. A larger z increases the option value and thus reduces a firm's incentives to use its own resources at its own cost.

The insight from Proposition 2 can be reinforced by considering the special cases of a pure matching subsidy, an earmarked subsidy, and a pure unrestricted subsidy and comparing the outcome in those cases to the case in which the firm does not receive a subsidy. Using (9), when the firm does not receive a subsidy, its abandonment threshold is

$$p_1^{NO} = \frac{\alpha}{\lambda \Pi}. \tag{12}$$

The abandonment thresholds p_1^M, p_1^E, and p_1^U for a pure matching subsidy, earmarked subsidy, and unrestricted subsidy are respectively

$$\begin{aligned} p_1^M &= \frac{(1-\phi)\,\alpha}{\lambda \Pi}, \\ p_1^E &= \frac{\alpha}{\lambda \Pi \left(\frac{r}{r + \lambda z} \right)}. \\ p_1^E &= \frac{\alpha}{\lambda \left[\Pi - \frac{s}{r} \right]}. \end{aligned} \tag{13}$$

Proposition 3 *(a) $p_1^M < p_1^{NO}$, i.e., with the pure matching subsidy, the firm persists with the R&D project for a longer duration than it would have in the absence of a subsidy. If $p_0 \in [p_1^M, p_1^{NO})$, a pure matching subsidy induces the firm to invest in a project that a non-subsidized firm would not have. In this case, the matching subsidy stimulates additional private R&D spending that would not have otherwise occurred; (b) $p_1^E > p_1^{NO}$, i.e., with an earmarked subsidy, the firm stops funding R&D from its own resources earlier than it would have without a subsidy. If $p_0 \in [p_1^{NO}, p_1^E]$ then net-of-subsidy R&D spending by a firm receiving an earmarked subsidy would be zero, while R&D spending by a non-subsidized firm would be positive. Thus, an* *earmarked subsidy crowds out private R&D spending; (c)* $p_1^E > p_1^{NO}$*, i.e., with a pure unrestricted subsidy, the firm persists with the R&D project for a shorter duration than it would have in the absence of a subsidy. If* $p_0 \in [p_1^{NO}, p_1^U]$ *a pure unrestricted subsidy induces the firm to shut down investment in a project that a non-subsidized firm would have continued to fund. Thus, an unrestricted subsidy crowds out private R&D spending.*

We can now summarize the incentive properties of subsidies on R&D investment behavior. Increases in the matching rate ϕ will stimulate private R&D activity, while increases in the unrestricted portion of the subsidy $s - \alpha z$ (holding z fixed) suppresses private R&D activity. An increase in the minimum mandate z (holding $s - \alpha z$ fixed) also suppresses private R&D activity, but at the same time, it ensures that a minimum level of R&D activity will be undertaken.

3.1 Socially Optimal R&D Subsidy Policies: $N = 1$

We now consider the subsidy policy that would be chosen by a planner seeking to maximize expected social welfare. As a benchmark, the first-best R&D investment policy is[12]

$$k^*(p) = \begin{cases} 1 & \text{if } p \in [p^*, 1] \\ 0 & \text{if } p \in [0, p^*] \end{cases},$$

where

$$p^* = \frac{\alpha}{\lambda[CS + \Pi]} < 1.$$

The first-best policy involves an abandonment threshold equal to the reciprocal of the social benefit-cost ratio $\frac{\lambda[CS+\Pi]}{\alpha}$.

However, as discussed above, we assume that the government cannot direct the firm to follow this policy by fiat or a forcing contract and must instead rely on subsidies to provide incentives to the firm. The subsidy policies considered here are thus inherently second best. Throughout the analysis, we assume that the subsidy is funded from revenues from broad-based taxes that do not materially affect the firm's incentive to invest R&D. However, we do allow for the possibility of a shadow cost of public funds $\gamma \geq 0$. That is, a subsidy S to the firm entails a transfer of S from taxpayers plus a social cost γS, where $\gamma \geq 0$.

[12] This policy is derived as part of the proof of Proposition 4.

We begin by deriving the expression for expected social welfare induced by the firm's optimal investment rule $k_1(p)$ derived in Proposition 1. To do so, we note that if the firm invests $k_1(p)$ units in the R&D project for the time interval $[t, t+dt)$ when the belief about the project's viability is p, then the social welfare schedule $W_1(p)$ is given recursively as follows:

$$\begin{aligned} W_1(p) &= -\alpha k_1(p)dt - \gamma\left[s + \phi\alpha(k_1(p) - z)\right]dt \\ &\quad + \lambda k_1(p)pdt\left[CS + \Pi\right] + (1 - \lambda k_1(p)pdt)e^{-rdt}W_1(p + dp). \end{aligned}$$

This recursion can be transformed into the following differential equation:

$$0 = -\alpha k_1(p) - \gamma\left[s + \phi\alpha(k_1(p) - z)\right] + \lambda k_1(p)p\left(CS + \Pi\right) - \left(r + \lambda k_1(p)p\right)W(p) - \lambda k_1(p)p\left(1 - p\right)W'(p). \tag{14}$$

In the Appendix, the solution to this differential equation is shown to be:

$$W_1(p) = \begin{cases} -\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right) + \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p + B_W p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & p \in [p_1, 1] \\ -\left(\frac{\alpha z+\gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right]p & p \in [0, p_1] \end{cases}, \tag{15}$$

where p_1 is given by Proposition 1, and B_W makes the welfare schedule continuous at $p = p_1$.[13] Note that if there is no minimum R&D mandate, i.e., $z = 0$, then for posterior beliefs less than or equal to the abandonment threshold p_1, social welfare is 0.

Ex ante social welfare, denoted by $EW_1(z, s, \phi)$, is found by evaluating $W_1(p)$ at the prior belief p_0. We express it as

$$EW_1(z, s, \phi) \equiv W(p_0) = \begin{cases} \Psi(B_W(p_1(z,s,\phi), z, s, \phi), z, s, \phi) & p_0 \in [p_1(z,s,\phi), 1] \\ -\left(\frac{\alpha z+\gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right]p_0 & p_0 \in [0, p_1(z,s,\phi)] \end{cases} \tag{16}$$

where

$$\Psi(B_W, z, s, \phi) \equiv \left\{ \begin{array}{c} -\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right) \\ +\frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p \\ +B_W p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}} \end{array} \right\}. \tag{17}$$

[13] Continuity of the welfare schedule follows because at $p = p_1$, the firm is indifferent between $k = 0$ and $k = 1$.

$$p_1(z,s,\phi) \equiv \min \left\{ \frac{\alpha(1-\phi)}{\lambda\left[\Pi - \left(\frac{s-\alpha z}{r}\right)\right]\left(\frac{r}{r+\lambda z}\right)}, 1 \right\}. \tag{18}$$

$$B_W(p_1,z,s,\phi) \equiv \frac{\left\{ \begin{array}{c} \frac{\alpha(1-z)}{r} + \frac{\gamma\phi\alpha(1-z)}{r} - \\ \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right] p_1 \\ +\frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right] p_1 \end{array} \right\}}{p_1\left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{r}}}.$$

The social planner's problem in choosing a subsidy policy is

$$\max_{z\in[0,1],s\geq\alpha z,\phi\in[0,1]} EW_1(z,s,\phi) \tag{19}$$

Throughout, we let $(z^{**}, s^{**}, \phi^{**})$ denote the solution to this problem.

When $\gamma = 0$, a subsidy is a pure transfer between taxpayers and the firm. In this case, as the following proposition shows, there is a simple way to achieve the first-best outcome: use a pure matching subsidy whose matching rate is 1 minus the appropriability ratio. No mandated minimum level of R&D is necessary, and the policy is robust to all priors.

Proposition 4 *Under monopoly, if a subsidy is a pure transfer between consumers and firms, i.e., $\gamma = 0$, then the optimal subsidy policy is a pure matching subsidy, i.e., $z^{**} = 0$, $s^{**} = 0$, with a matching rate $\phi^{**} = 1 - \rho$ (where recall $\rho = \frac{\Pi}{\Pi + CS}$). This policy, which is independent of the prior belief p_0, induces the firm to choose an investment policy $k_1(p) = k^*(p)$ and thus achieves the first-best level of ex ante welfare for any prior belief p_0.*

Proof. See Appendix ■

When there is a positive shadow cost of public funds, the first-best policy cannot be implemented through subsidies. This is because a positive shadow cost of public funds creates a trade-off between inducing more R&D and incurring higher social costs due to the subsidy. Still, as we show in the following sequence of propositions and numerical computations, for a wide range of circumstances, the optimal subsidy policy is a pure matching subsidy. To begin, we show that at an optimal policy, the unrestricted component of the subsidy is set equal to 0., i.e., $s = \alpha z$.

Proposition 5 *Suppose a subsidy entails a positive shadow cost of public funds, i.e., $\gamma > 0$. If the optimal subsidy policy involves a matching rate $\phi^{**} \in (0,1)$, then $s^{**} = \alpha z^{**}$,i.e., the subsidy policy does not involve an unrestricted component.*

Proof. See Appendix ■

Proposition 5 results directly from the poor incentive properties of the unrestricted component $s - \alpha z$ of the subsidy: as we saw above, increases in $s - \alpha z$ crowd out private incentives for R&D.

Could the optimal policy involve a minimum mandated level of R&D activity, i.e., $z > 0$? A minimum mandate ensures that a minimum level of R&D effort is expended. Of course, the government could induce the voluntary provision of R&D effort with a large enough matching rate, but with a shadow cost of public funds, that approach might be quite expensive. Thus, a potential advantage of a minimum mandate is that it may be a less expensive way of inducing R&D effort than a matching subsidy. Of course, the mandate is also socially costly, since $s = \alpha z$. *A priori*, then, neither subsidy tool appear to have an inherent advantage over the other, and it seems plausible that an optimal subsidy policy when social funds are costly could involve a mix of subsidy instruments. However, our analysis suggests that a minimum mandate is never optimal. We begin by providing sufficient conditions under which the second-best optimal policy is a pure matching policy, and using those condition, we characterize the nature of that policy and the behavior that is induced.

Proposition 6 *Suppose a subsidy entails a positive shadow cost of public funds, i.e., $\gamma > 0$. Suppose, further, that the optimal subsidy policy involves a matching rate $\phi^{**} \in (0,1)$ and is such that $p_1^{**} \equiv p_1(z^{**}, s^{**}, \phi^{**}) < p_0$. Then the following are sufficient conditions for $z^{**} = 0$: (a) γ sufficiently close to 0.(b) $\gamma \geq \frac{\lambda(CS+\Pi)}{\alpha} - 1$; (c) $\gamma < \frac{\lambda(CS+\Pi)}{\alpha} - 1$ and $r > \lambda$. Furthermore, under conditions (a), (b), or (c) $p_1^{**} > p^* = \frac{\alpha}{\lambda(CS+\Pi)}$ and $\phi^{**} < 1 - \rho$, i.e., the abandonment threshold is greater than the first-best abandonment threshold, and the subsidy rate is less than the subsidy rate $1 - \rho$ that induces the first-best outcome in the absence of a shadow cost.*

Proof. See Appendix. ■

The sufficient conditions in this proposition establish a wide set of circumstances under which the optimal subsidy policy is a pure matching policy, i.e., $z = 0$. Moreover, under these conditions, there is less subsidization than there would be if there was no shadow cost of public funds, and the R&D effort that is induced by the optimal subsidy scheme is less than first-best level. However, the proposition leaves open the question of whether there are other parameter conditions under which it is optimal for $z > 0$. To explore this question, we have done numerical calculations summarized in the two tables below. The table shows the optimal subsidy policy (z^{**}, ϕ^{**})) for various values of p_0 (the prior belief) and $\rho = \frac{\Pi}{CS+\Pi}$ (the appropriability ratio).

The parameter values for these calculations are:

- $r = 0.03$ (i.e. a real interest rate of 3 percent annually).
- $\lambda = 0.1$, which implies that the expected time to discovery, conditional on the project being viable and maximum R&D effort, is 10 years.
- $\alpha = 0.2$.
- $CS + \Pi = 15$ (upper table), or $CS + \Pi = 45$ (lower table), so that the social benefit-cost ratio $\frac{\lambda[CS+\Pi]}{\alpha}$ is 7.5 (upper table) or 22.5 (lower table).
- $\gamma = 0.2$. (i.e., a \$1 subsidy entails a net shadow cost of public funds of \$0.20).

These parameter values are empirically plausible. The social benefit-cost ratios of 7.5 and 22.5 are broadly consistent with empirical evidence on the social rate of return of innovations.[14] The shadow cost of funds of 0.2 is also empirically plausible.[15] Moreover, except possibly for condition (a), the parameter values in these calculations *do not* satisfy the sufficient conditions of Proposition 6.

The tables below show the optimal policy as a vector (z, ϕ). For this set of calculations, a positive value of z is never optimal, even in those cases in which $\phi = 0$.[16] Thus, the calculations provide additional circumstances under which a pure matching policy is optimal under monopoly, and thus strongly suggest that in the case of monopoly, a matching rate (the "price instrument") strictly dominates a minimum

[14] To make this point, consisder in the context of our model a viable investment whose time to discovery is the average time of $\frac{1}{\lambda}$. The discounted cost equals $\frac{\alpha}{r}(1 - e^{-\frac{r}{\lambda}})$ and the discounted social benefit is $(CS + \Pi)\, e^{-\frac{r}{\lambda}}$. The realized social rate of return on the R&D investment is the r that solves $\frac{\alpha}{r}(1 - e^{-\frac{r}{\lambda}}) = (CS + \Pi)\, e^{-\frac{r}{\lambda}}$. When $\lambda = 0.1, \alpha = 0.2$, and $CS + \Pi = 15$, the realized social rate of return on a project that took its expected time to complete ($\frac{1}{\lambda}$ =10 years) would be 32.3 percent. When $\lambda = 0.1, \alpha = 0.2$, and $CS + \Pi = 45$, the realized social rate of return would be 46.6 percent. Jones and Williams (1998), in reviewing the empirical literature on the social rate of return to R&D, write: "Estimates of the social return average about 27 percent when only R&D from one's own industry is included and average nearly 100 percent when the broadest concept of return ... is employed" (p. 1129).

[15] Sandmo (1998) suggests that a plausible estimate of the shadow cost of public funds is in the range of 0.1 to 0.2.

[16] The optimal policies were determined using a grid search over values of (z, ϕ) in $[0, 1] \times [0, 1]$. The table reports the results of calculations with a grid of 0.1. Making the grid finer does not change the result and does not uncover cases in which $z > 0$.

mandate (the "quantity instrument") as a policy tool for eliciting higher levels of R&D. The calculations reported here are only a subset of the calculations we have conducted, and we have yet to find an example in which the optimal $z > 0$.[17]

$CS + \Pi = 15$

ρ \ p_0	0.1	0.3	0.5	0.7	0.9
0.1	$(0,0)$	$(0,0.9)$	$(0,0.9)$	$(0,0.9)$	$(0,0.8)$
0.2	$(0,0)$	$(0,0.8)$	$(0,0.7)$	$(0,0.7)$	$(0,0.6)$
0.3	$(0,0)$	$(0,0.6)$	$(0,0.6)$	$(0,0.6)$	$(0,0.3)$
0.4	$(0,0)$	$(0,0.5)$	$(0,0.5)$	$(0,0.4)$	$(0,0)$
0.5	$(0,0)$	$(0,0.4)$	$(0,0.3)$	$(0,0.2)$	$(0,0)$
0.6	$(0,0)$	$(0,0.3)$	$(0,0.2)$	$(0,0)$	$(0,0)$
0.7	$(0,0)$	$(0,0.2)$	$(0,0)$	$(0,0)$	$(0,0)$
0.8	$(0,0)$	$(0,0.1)$	$(0,0)$	$(0,0)$	$(0,0)$
0.9	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

$CS + \Pi = 45$

ρ \ p_0	0.1	0.3	0.5	0.7	0.9
0.1	$(0,0.9)$	$(0,0.9)$	$(0,0.9)$	$(0,0.9)$	$(0,0.8)$
0.2	$(0,0.8)$	$(0,0.7)$	$(0,0.7)$	$(0,0.6)$	$(0,0.2)$
0.3	$(0,0.6)$	$(0,0.6)$	$(0,0.5)$	$(0,0.4)$	$(0,0)$
0.4	$(0,0.5)$	$(0,0.4)$	$(0,0.3)$	$(0,0)$	$(0,0)$
0.5	$(0,0.4)$	$(0,0.3)$	$(0,0.1)$	$(0,0)$	$(0,0)$
0.6	$(0,0.3)$	$(0,0.1)$	$(0,0)$	$(0,0)$	$(0,0)$
0.7	$(0,0.2)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
0.8	$(0,0.1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
0.9	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

[17] This suggests that it may be possible to prove analytically that $z > 0$ is never optimal. However, we have been unable to prove such a result. As one will see from the proof of Proposition 6, in the Apendix, the expressions for the derivatives of EW_1 with respect to z and ϕ are very complex. Ruling out the optimality of $z > 0$ using first-order conditions would require analytical characterization of a higher-order polynomial. Perhaps there is non-calculus based proof, but we have been unable to find it. The basic problem, intuitively, is that the minimum mandate z is not inherently dominated by ϕ since, as noted, by increasing z above 0, the government can alwasy ensure some R&D and this can be socially valuable.

It may seem surprising that a minimum R&D mandate is not optimal. After all, a mandate is the most direct mechanism for achieving a minimum level of R&D effort that might not otherwise be provided. Indeed, when $z > 0$, the probability of a breakthrough is 1, conditional on the project being viable. That is, with a mandate, the social commitment to the project is absolute and will inevitably lead to a breakthrough if a breakthrough is possible. Since the monopolist firm underinvests in R&D relative to the social optimum ($p_1^{NO} > p^*$), a minimum mandate might appear to be a potentially useful policy tool to counteract this underinvestment.

However, the matching rate ϕ dominates the mandate z for two reasons. First, ϕ has superior incentive properties. As we saw above, increasing the minimum mandate z results in a crowding out of private R&D effort, while, by contrast, increasing the matching rate ϕ stimulates private R&D effort. Second, in those circumstances in which it is socially desirable to prolong the R&D effort (i.e., when $k^*(p) = 1$ for nearly all $p > 0$, or equivalently, $p^* \approx 0$), which are the circumstances in which a minimum R&D mandate z might potentially be most desirable, the planner can nearly replicate the outcome with a minimum R&D mandate by setting ϕ sufficiently close to 1. Thus, the advantages of earmarking can be nearly replicated by a matching subsidy.

The calculations suggest two other features of the optimal subsidy policy. First, consistent with Proposition 6, the optimal matching rate in all of the calculations is less than $1-\rho$, the matching rate that implements the first-best solution when the shadow cost of public funds is zero. For example, when $CS + \Pi = 15$ and $\rho = 0.6$, the optimal matching rates ϕ^* are 0, 0.3, 0.2, 0, and 0 for $p_0 = 0.1, 0.3, 0.5, 0.7$, and 0.9, respectively, which are all less than $1 - \rho = 0.4$. Second, there are circumstances under which no subsidy is optimal, i.e., $z^* = 0, \phi^* = 0$. Broadly this occurs under two sets of circumstances: (a) if the social benefit-cost ratio is relatively low ($CS + \Pi = 15$) and the prior belief p_0 about the viability of the project is low, it will not be worthwhile to subsidize the project; (b) if the appropriability ratio ρ is sufficiently high, it will not be worthwhile subsidizing the project. In this case, given the shadow cost of public funds, it is better to rely solely on the firm's private incentives to engage in R&D effort, which are rather strong when it can appropriate a significant share of the surplus.

4 N-Firm Oligopoly R&D

In some cases, subsidies are provided to all firms within an industry that makes R&D investments, as, for example, when firms receive tax credits for R&D expenditures. And so we now consider the case in which N firms compete to achieve the R&D breakthrough. As we will see, the N-firm case is not a simple multiple of one-firm case. This is because an individual firm will behave strategically in choosing its R&D investment strategy. Strategic behavior among firms introduces the possibility of free-riding, which in turn influences how subsidy policy shapes incentives.

4.1 Equilibrium Investment Strategy

We begin by considering a firm's optimal investment strategy given the government's R&D subsidy policy. An individual firm's value function is given by the recursion

$$V^i(p) = \max_{k^i \in [z,1]} \left[\begin{array}{c} \left(s - \alpha\phi z - \alpha(1-\phi) k^i\right) dt + \lambda p k^i dt \Pi + \lambda p K^{-i} dt \theta \Pi \\ + \left(1 - \lambda p \left(k^i + K^{-i}\right) dt\right) e^{-rdt} V^i(p + dp) \end{array} \right], \tag{20}$$

where $K^{-i} = \sum_{j \neq i} k^j$ is the sum of the R&D investments by firm i's rivals. We can rewrite the value function in (20) as a differential equation:

$$(1+r)V^i(p) = s - \alpha z + \max_{k^i \in [z,1]} \left\{ \begin{array}{c} \left[\lambda k^i p \Pi + \lambda p K^{-i} \theta \Pi + (1 - \lambda\left(k^i + K^{-i}\right) p) V^i(p)\right] \\ -\alpha(1-\phi)(k-z) - \lambda\left(k^i + K^{-i}\right) p (1-p) V^{i\prime}(p) \end{array} \right\} \tag{21}$$

As in the case of $N = 1$, a firm's value has four components. Two of these components are identical to the monopoly case; the other two differ due to the presence of competitors who can also achieve the breakthrough:

1. $s - \alpha z$ is the flow benefit from the unrestricted component of its subsidy;

2. $\alpha(1-\phi)(k-z)$ is the net-of-subsidy total cost of k^i units of R&D effort;

3. $\left[\lambda k^i p \Pi + \lambda p K^{-i} \theta \Pi + (1 - \lambda\left(k^i + K^{-i}\right) p) V^i(p)\right]$ is the expected benefit to a firm from its own k^i units of R&D effort and its rivals' collective R&D effort K^{-i} (with probability $\lambda k^i p$ the firm achieves the breakthrough and gets Π; with probability $\lambda p K^{-i} \theta \Pi$ a rival firm achieves the breakthrough, in

which case the firm gets a payoff $\theta\Pi$, and with probability $1-\lambda\left(k^i+K^{-i}\right)p$ no breakthrough occurs and the firm receives its continuation value $V^i(p)$.

4. $\lambda\left(k^i+K^{-i}\right)p(1-p)V^{i\prime}(p)$ is the expectation, with k^i+K^{-i} units of R&D effort in total, of the option value of waiting that is foregone if some firm achieves a breakthrough.

Throughout, we focus on the symmetric equilibrium, i.e., $k^1=\ldots=k^N=k$. In the Appendix we prove:

Proposition 7 *Define*

$$\begin{aligned}
V_N^H(p) &= \frac{s-\alpha(1-\phi)-\alpha\phi z}{r}+\frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha(1-\phi)+\alpha\phi z\right)}{r(r+\lambda N)}p+B_H p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}}, \\
V_N^M(p) &= \frac{\lambda\Pi-(1-\phi)\alpha}{\lambda}-B_M^n(1-p)+\frac{(1-\phi)\alpha(1-p)}{\lambda}\ln\frac{1-p}{p}, \\
V_N^L(p) &= \frac{s-\alpha z}{r}+\frac{\lambda N z\left[\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha z\right]}{r(r+\lambda N z)}p.
\end{aligned}$$

Let

$$p_N=\frac{\alpha(1-\phi)}{\lambda\left(\left[\Pi-\frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda N z}\right)+(N-1)(1-\theta)\Pi\left(\frac{\lambda z}{r+\lambda N z}\right)\right)}; \tag{22}$$

let q_N satisfy

$$\frac{rV_N^M(q_N)-s+\alpha\phi z}{(N-1)(\alpha(1-\phi)-\lambda q_N(1-\theta)\Pi)}=1. \tag{23}$$

Finally, define

$$\overline{\theta}(z,s)\equiv 1-\left[\frac{\Pi-\left(\frac{s-\alpha z}{r}\right)}{\Pi}\right]\left[\frac{r}{r+\lambda z}\right]\in[0,1]. \tag{24}$$

(i) If the spillover θ exceeds the critical level $\overline{\theta}(z,s)$, there exists a unique symmetric Markov perfect equilibrium with the following strategy:[18]

$$k_N(p)=\begin{cases} 1 & \text{if } p\geq q_N \\ \frac{rV_N^M(p)-s+\alpha\phi z}{(N-1)(\alpha(1-\phi)-\lambda p(1-\theta)\Pi)} & \text{if } p_N>p>q_N \\ z & \text{if } p\leq p_N \end{cases}$$

[18] Recall that the subscript N denotes the N-firm case.

and an individual firm's value function is

$$V_N(p) = \begin{cases} \frac{s-\alpha(1-\phi)-\alpha\phi z}{r} + \frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha(1-\phi)+\alpha\phi z\right)}{r(r+\lambda N)}p + B_H p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}} & \text{if } p \geq q_N \\ \frac{\lambda\Pi-(1-\phi)\alpha}{\lambda} - B_M(1-p) + \frac{(1-\phi)\alpha(1-p)}{\lambda}\ln\frac{1-p}{p} & \text{if } p_N < p < q_N \\ \frac{s-\alpha z}{r} + \frac{\lambda N z\left[\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha z\right]}{r(r+\lambda N z)}p & \text{if } p \leq p_N \end{cases}$$

where B_M satisfies $V_N^M(p_N) = V_N^L(p_N)$ and B_H satisfies $V_N^H(q_N) = V_N^M(q_N)$.

(ii) If the spillover θ is below the critical level $\overline{\theta}(z,s)$, there exists a unique symmetric Markov perfect equilibrium with the following strategy:

$$k_N(p) = \begin{cases} 1 & \text{if } p \geq p_N \\ z & \text{if } p \leq p_N \end{cases}$$

with the value function

$$V_N(p) = \begin{cases} \frac{s-\alpha(1-\phi)}{r} + \frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha(1-\phi)\right)}{r(r+\lambda N)}p + B_H^* p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}} & \text{if } p > p_N \\ \frac{s-\alpha z(1-\phi)}{r} + \frac{\lambda N z\left[\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha z(1-\phi)\right]}{r(r+\lambda N z)}p & \text{if } p \leq p_N. \end{cases},$$

where B_H^* satisfies $V_N^H(p_N) = V_N^L(p_N)$.

Proof. See Appendix. ■

Panel (a) of Figure 1 illustrates the equilibrium investment policy when the spillover parameter θ exceeds the critical level $\overline{\theta}(z,s) \equiv 1 - \left[\frac{\Pi-\left(\frac{s-\alpha z}{r}\right)}{\Pi}\right]\left[\frac{r}{r+\lambda z}\right]$. In contrast to monopoly, in which the equilibrium investment policy is "bang-bang," there is a range of beliefs (p_N, q_N) over which $k_N(p) \in (z,1)$. As before, we refer to p_N as the abandonment threshold. Once the posterior belief reaches p_N, a firm chooses the minimum required R&D effort, $k_N(p) = z$. We refer to q_N as the slowdown threshold. Once the posterior belief falls to q_N, firms start to slow down their research efforts by reducing k below 1.

The equilibrium in this case involves $k_N(p) \in (z,1)$ because of a free-rider problem. The free-rider problem arises because, in essence, the firm can achieve a positive payoff $\theta\Pi$ from spillover even if it loses the R&D competition, a phenomenon that does not arise under monopoly. In particular, when $\theta > \overline{\theta}(z,s)$ and $p \in (p_N, q_N)$, given that all other firms invest "flat out," it will be optimal for a firm to reduce its R&D investment below the maximum level. On the other hand, though, given that all other firms invest at the

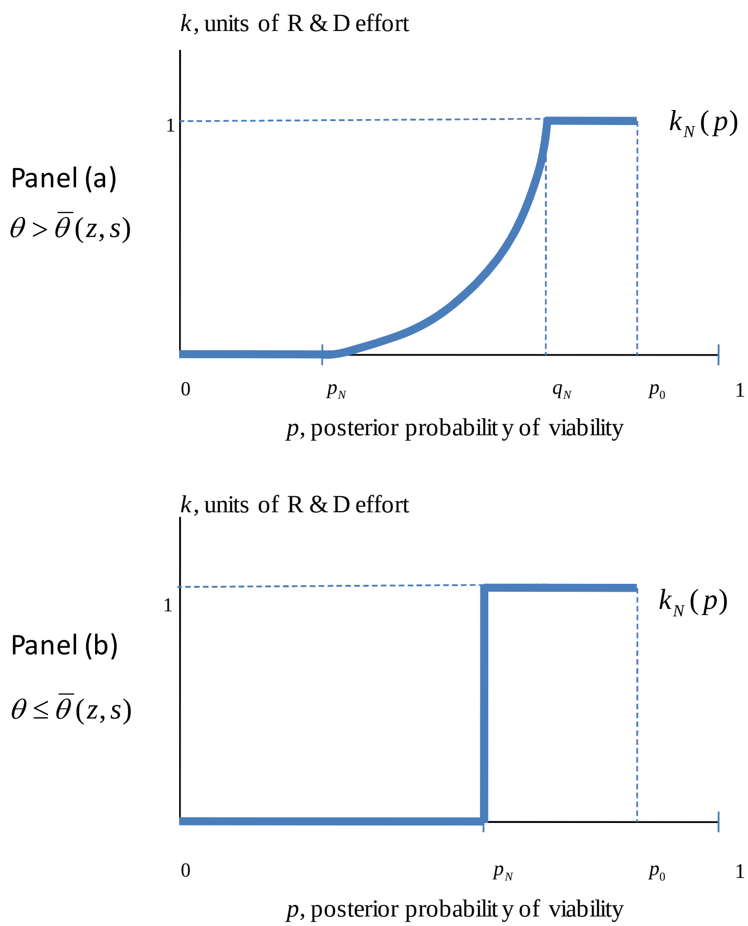

Figure 1: Equilibrium Investment Policy

minimum level, it will be optimal for a firm to invest "flat out" in R&D. The "concession" to the free-rider problem that is made in equilibrium is that for $p \in (p_N, q_N)$, all firms reduce k to a positive number less than 1. The equilibrium value of $k_N(p)$ is such that when a firm's $N-1$ competitors invest $k_N(p)$, it is indifferent in investing among all $k \in (z, 1)$ and thus chooses $k_N(p)$ in a symmetric equilibrium.

By contrast, when the spillover parameter is less than $\overline{\theta}(z, s)$, the free-rider problem does not arise, and as shown in panel (b) of Figure 1, the equilibrium investment policy is "bang-bang," as in the case of a monopoly. However, the abandonment threshold p_N does not correspond to the monopoly threshold. In fact, when the free-rider problem is absent, for any given subsidy policy (z, s, ϕ) there is unambiguously more investment with N firms than with a monopolist. This is because from (9) and (22),

$$\frac{p_1}{p_N} = \frac{\lambda \left(\left[\Pi - \frac{(s-\alpha z)}{r} \right] \left(\frac{r}{r+\lambda N z} \right) + (N-1)(1-\theta) \Pi \left(\frac{\lambda z}{r+\lambda N z} \right) \right)}{\lambda \left[\Pi - \left(\frac{s-\alpha z}{r} \right) \right] \left(\frac{r}{r+\lambda z} \right)} > 1.$$

Thus R&D competition with no free riding increases the provision of R&D effort relative to that in a monopoly.

Because $\overline{\theta}$ depends on z and s, whether or not free riding arises in equilibrium depends on the subsidy policy. As the next proposition shows, if there is no unrestricted funding or minimum investment mandate, free riding will always arise irrespective of ϕ.

Proposition 8 *If $N > 1$, $\theta > 0$, and $z = s = 0$, then $\overline{\theta}(z, s) = 0$ and the free-rider problem always arises.*

This result tells us that a necessary condition for avoiding the free-rider problem is to establish a mandated minimum level of R&D ($z > 0$) or provide positive baseline funding ($s > 0$), or both. To understand why, recall from the discussion of monopoly that a subsidy with an unrestricted component $s - \alpha z$ creates an implicit loss for the firm when it achieves a breakthrough. By the same token, an unrestricted component creates an implicit loss when *another firm* achieves a breakthrough. Thus, the unrestricted component of the subsidy offsets part of the gain the firm receives when another firm makes the discovery, thereby reducing the temptation to free ride. A subsidy policy with a minimum mandate also creates an implicit loss for the firm when another firm achieves the breakthrough, but for a different reason. To see why note that when $k = z$, the firm is receiving an option (the possibility that it, or another firm, achieves a breakthrough) that is fully paid for by the government. When another firm achieves a breakthrough, this government-funded option goes away, creating an implicit loss that can offset some of the gains from free riding. Thus, in contrast to the monopoly case, in which increases in s and z had unambiguously adverse affects on the provision of R&D, Proposition 8 suggests that s and/or z may have potentially beneficial incentive effects by mitigating the extent of free riding behavior by firms.

Each of the policy choices affects investment incentives through the entire equilibrium strategy $k_N(p)$. These effects cannot be determined analytically, but we can determine the impact of z, s, and ϕ on the abandonment threshold p_N:

Proposition 9 *(a) Holding s and z fixed, an increase in the matching rate ϕ decreases the N-firm equilibrium abandonment threshold, thus expanding the range over which firm invests in excess of the mandated minimum; (b) Holding ϕ and z fixed, an increase in the baseline subsidy s (which thus increases the fungible portion of the subsidy $s - \alpha z$) increases the N-firm equilibrium abandonment threshold, thus contracting the range over which the firm invests in excess of the mandated minimum.*

Proof. The result follows immediately from the expression for the abandonment threshold (22). ■

Propositions 8 and 9 hint at an interesting tension involving the baseline subsidy s. On the one hand, it can crowd out private R&D investment, by contracting the range over which the firm invests in excess of the mandated minimum. On the other hand, it can counteract the free rider problem. In the next section, we will see that in designing an optimal subsidy scheme, a social planner can exploit this tension to balance the firm's incentives so that the first-best outcome can be attained.

To summarize the impact of subsidies in the N firm case, if s and z are sufficiently large, the free-rider problem will not arise in equilibrium. However, changes in ϕ have no impact on free riding. As in the monopoly case, increases in ϕ decrease the abandonment threshold p_N (thus expanding the range of private funding of R&D), while decreases in s increase the abandonment threshold. Unlike the monopoly case, changes in z have an ambiguous impact on the abandonment threshold. The impact of z, s, and ϕ on the slowdown threshold q_N and the equilibrium investment policy $k_N(p)$ more generally cannot be determined analytically.

4.2 Socially Optimal R&D Subsidy Policies: $N > 1$

With N firms, the first-best R&D investment policy solves the social planner's problem:

$$W^*(p) = \max_{k\in[0,1]} \left[-\alpha Nkdt + \lambda Nkpdt\left[CS + \Pi + (N-1)\theta\Pi\right] + (1-\lambda Nkpdt)e^{-rdt}W^*(p+dp)\right]. \qquad (25)$$

This policy is a straightforward extension of the first-best solution for the monopoly case:[19]

$$k^*(p) = \begin{cases} 1 & \text{if } p \in [p^*, 1] \\ 0 & \text{if } p \in [0, p^*] \end{cases},$$

where

$$p^* = \frac{\alpha}{\lambda[CS + \Pi + (N-1)\theta\Pi]} < 1.$$

As in the case of monopoly, we want to characterize the subsidy policy $(z^{**}, s^{**}, \phi^{**})$ that maximizes *ex ante* social welfare, $EW_N(z, s, \phi) = W_N(p_0)$, where $W_N(p)$ is the welfare schedule given by the recursion

$$\begin{aligned} W_N(p) &= -\alpha Nk_N(p)dt - \gamma N\left[s + \phi\alpha(k_N(p) - z)\right]dt \\ &\quad + \lambda k_N(p)pdt\left[CS + \Pi + (N-1)\theta\Pi\right] + (1 - \lambda k_N(p)pdt)e^{-rdt}W_1(p+dp). \end{aligned}$$

The welfare schedule can be transformed into the following differential equation:

$$\begin{aligned} 0 &= -\alpha Nk_N(p) - \gamma N\left[s + \phi\alpha(k_N(p) - z)\right] + \lambda Nk_N(p)p\left[CS + (1 + (N-1)\theta)\Pi\right] \\ &\quad - (r + \lambda Nk_N(p)p)W(p) - \lambda Nk_N(p)p(1-p)W'(p). \end{aligned} \qquad (26)$$

When $k_N(p) = 1$ (i.e., $p > q_N$), the solution to this differential equation is

$$\begin{aligned} W_N(p) &= -\frac{\alpha N}{r} - \frac{N\gamma\left[s + \alpha\phi(1-z)\right]}{r} \\ &\quad + \frac{\lambda N\left(r\left[CS + (1 + (N-1)\theta)\Pi\right] + N\alpha + N\gamma\left[s + \alpha\phi(1-z)\right]\right)}{r(r + \lambda N)}p + B_W^H p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}}, \end{aligned}$$

[19] It is straightforward to transform this problem to one that is identical to the social planner's problem under monopoly (discussed in the proof of Proposition 4 using the transformations $\alpha' = N\alpha$, $\lambda' = N\lambda$, and $\Pi' = \Pi + (N-1)\theta\Pi$.

where B_W^H is a constant. When $k(p) = z$ (i.e., $p < p_N$), the solution to this differential equation is

$$W_N(p) = -\frac{N\alpha z}{r} - \frac{N\gamma s}{r} + \frac{\lambda N z\left(r\left[CS + (1 + (N-1)\theta)\Pi\right] + \alpha N z + N s\gamma\right)}{r(r + \lambda N z)} p.$$

When $k_N(p) \in (z, 1)$, the differential equation (26) does not have a closed form solution. However, we will be able to numerically compute the welfare schedule using interpolation method.

We now show that when there is no shadow cost of public funds ($\gamma = 0$), the first-best outcome can be attained. However, unlike the monopoly case, the policy that attains the first-best is not a pure matching policy. It involves combining a matching policy with an unrestricted subsidy.

Proposition 10 *With N firms, if a subsidy is a pure transfer between consumers and firms, i.e., $\gamma = 0$, then the optimal subsidy policy is $z^{**} = 0$; $\phi^{**} = \frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$; and $s^{**} = r\theta\Pi$. This policy, which is independent of the prior belief p_0, induces each firm to choose an investment policy $k_1(p) = k^*(p)$ and thus achieves the first-best level of ex ante welfare for any prior belief p_0. If there are no spillovers ($\theta = 0$), the optimal subsidy policy for N firms is identical to that for a monopoly.*

Proof. See Appendix ■

In the absence of a shadow cost of public funds, the subsidy policy that implements the first-best solution has an intuitively appealing form. The firm receives an unrestricted subsidy s that equals the flow equivalent of the spillover benefits $\theta\Pi$ that it would have received had another won the R&D competition. This ensures that the only way that a firm can improve its payoff is by winning the R&D competition, thus eliminating the free-rider problem and making the firm's goal to win the competition. Though a positive value of s eliminates the free rider problem, it does not fully align the private marginal benefit of R&D with the social marginal benefit of R&D. By choosing the matching rate ϕ to equal the fraction of social surplus $\frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$ that is not internalized, private and social marginal benefits are aligned.

As in the case of monopoly, the first-best solution cannot be implemented when there is a positive shadow cost of public funds. In this case, analytical results (even sufficient conditions) cannot be obtained, and so we illustrate the optimal subsidy policy using numerical calculations. The parameters used in those calculations are the same as those used in the monopoly calculations above, with the exception that N is fixed at 4 firms. The optimal R&D subsidy policies for the case of $CS + \Pi = 15$ are reported in Table 2 attached to the end of the paper. As in the monopoly, case, there is a set of parameter values for which no subsidy is optimal. This generally occurs when p_0 is small or ρ is large. In addition, given the chosen parameters, for θ less than 0.8, it is not optimal to have either a minimum mandate or an unrestricted component to the subsidy for most cases. As a result, the optimal subsidy induces free riding.[20]. For θ greater or equal to 0.8, there are a few cases involve very small minimum mandate and an unrestricted subsidy component. They are not significant enough to allow us to conclude that optimal subsidy policy require minimum mandate, although they suggest that when spillover is large, it may be necessary to introduce a small amount of earmark subsidy to induce more investment.

5 R&D Consortium

An alternative arrangement to non-cooperative R&D with N competing firms is an N-firm research consortium. Faced with a subsidy policy (z, s, ϕ) the research consortium solves the following problem:

$$V(p) = \max_{k\in[z,1]} \left[(s - \alpha\phi z - \alpha(1-\phi)k)dt + \lambda N kpdt \left(\frac{\Pi + (N-1)\theta\Pi}{N} \right) + (1 - \lambda N kpdt) e^{-rdt} V(p+dp) \right],$$

This can be transformed into the following differential equation

$$rV(p) = s - \alpha\phi z + \max_{k\in[z,1]} \left\{ -\alpha(1-\phi)k + \lambda N kp \left[\left(\frac{\Pi + (N-1)\theta\Pi}{N} \right) - V(p) - (1-p)V'(p) \right] \right\}.$$

Like a monopolist, the research consortium's optimal R&D policy $k_N^C(p)$ is bang-bang:

$$k_N^C(p) = \begin{cases} 1 & \text{if } p \geq p_N^C \\ z & \text{if } p \leq p_N^C \end{cases},$$

where the optimal stopping threshold p_N^C is given by

$$p_N^C = \frac{\alpha(1-\phi)}{\lambda \left[\Pi + (N-1)\theta\Pi - \frac{s-\alpha z}{r} \right] \left(\frac{r}{r+\lambda z} \right)}.$$

[20] In other calculations not reported here, we find that the shadow cost of public funds must be sufficiently close to 0 before an unrestricted component of the subsidy becomes optimal. In these cases, $z = 0$ continues to be optimal.

The research consortium's optimal investment plan induces the following expected social welfare:

$$W_N^c(p) = \begin{cases} -\frac{\alpha N}{r} - \frac{N\gamma[s+\alpha\phi(1-z)]}{r} + \frac{\lambda N(r[CS+(1+(N-1)\theta)\Pi]+N\alpha+N\gamma[s+\alpha\phi(1-z)])}{r(r+\lambda N)}p + B_W^c p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}} & \text{if } p \geq p_N^C \\ -\frac{N\alpha z}{r} - \frac{N\gamma s}{r} + \frac{\lambda N z(r[CS+(1+(N-1)\theta)\Pi]+\alpha N z+Ns\gamma)}{r(r+\lambda N z)}p & \text{if } p < p_N^C \end{cases}.$$

As with a monopoly, in an N-firm research consortium, the first-best level of welfare can be attained when there is no shadow cost of public funds.

Proposition 11 *Under a research consortium,*

(i) if a subsidy is a pure transfer between consumers and firms, i.e., $\gamma = 0$, then the optimal subsidy policy is a pure matching subsidy, i.e., $z^{**} = 0$, $s^{**} = 0$, with a matching rate $\phi^{**} = 1-\rho$ (where recall $\rho = \frac{\Pi+(N-1)\theta\Pi}{\Pi+(N-1)\theta\Pi+CS}$). This policy, which is independent of the prior belief p_0, induces the the firm to choose an investment policy $k_N^C(p) = k^*(p)$ and thus achieves the first-best level of ex ante welfare for any prior belief p_0;

(ii) if a subsidy entails a positive shadow cost of public funds, i.e., $\gamma > 0$. Suppose, further, that the optimal subsidy policy involves a matching rate $\phi^{**} \in (0,1)$ and is such that $p_N^{C*} \equiv p_N^C(z^{**}, s^{**}, \phi^{**}) < p_0$. Then the following are sufficient conditions for $z^{**} = 0$: (a) γ sufficiently close to 0. (b) $\gamma \geq \frac{\lambda(CS+\Pi+(N-1)\theta\Pi)}{\alpha} - 1$; (c) $\gamma < \frac{\lambda(CS+\Pi+(N-1)\theta\Pi)}{\alpha} - 1$ and $r > \lambda$. Furthermore, under conditions (a), (b), or (c) $p_N^{C*} > p^* = \frac{\alpha}{\lambda(CS+\Pi+(N-1)\theta\Pi)}$ and $\phi^{**} < 1-\rho$, i.e., the abandonment threshold is greater than the first-best abandonment threshold, and the subsidy rate is less than the subsidy rate $1-\rho$ that induces the first-best outcome in the absence of a shadow cost.

Proof. *The proof is is directly analogous to the proof of Proposition (4) in the Appendix and is thus omitted.* ∎

When there is a zero shadow cost of public funds, the government is indifferent between subsidizing N non-cooperative firms or N firms organized into a research consortium, provided that in each case the matching rate and unrestricted subsidy are appropriately chosen as indicated in Propositions (10) and (11).

If there is a positive shadow cost of public funds, part (ii) of Proposition 11 implies that for a large range of parameter values, the optimal policy is a pure matching policy. The intuition is very similar to the monopoly case. This is because a research consortium coordinate its R&D efforts to achieve the maximum joint profit $\Pi + (N-1)\theta\Pi$, which replaces the monopoly profit Π in the monopoly case.**??**

We note from part (i) of Proposition 11 that the matching rate $\frac{CS}{\Pi+(N-1)\theta\Pi+CS}$ needed to attain the first-best outcome under a research consortium is less than the matching rate $\frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$ needed to attain the first-best outcome under non-cooperative research. Further, with a research consortium attaining the first best outcome entails no unrestricted subsidy, while an unrestricted subsidy is required under non-cooperative research. Thus, when there is no shadow cost of public funds, attaining the first best with a research consortium involves a smaller overall subsidy than attaining the first best with N non-cooperative firms. This suggests that the research consortium may have an advantage over non-cooperative research when the shadow cost of public funds is positive. To determine whether *ex ante* welfare is higher under an optimally subsidized research consortium or an optimally subsidized non-cooperative firms, we turn to calculations. We use the same set of parameter values we used in the N-firm calculations in the previous section. The tables below show two numbers: the maximum level of expected welfare with N-firm non-cooperative research, given optimal subsidy and the maximum level of expected welfare with an N-firm research consortium, given optimal subsidization.

6 Conclusions

In this paper we introduce a two-armed bandit model to study the government's optimal policy to subsidize private R&D programs bearing both "if" and "when" uncertainty. Surprisingly, if there is no shadow cost of public funding, the government can design a subsidy policy to achieve the first best welfare outcome even if it has no knowledge of underlying R&D's project's viability. If the subsidy is awarded to one firm only, the optimal subsidy scheme is a pure matching policy; if the subsidy benefits multiple firms, the optimal policy is a combination of matching and unrestricted subsidy. However, when there is positive shadow cost of public funding, then first best outcome is not achievable. If the shadow cost is small, the optimal subsidy policy is very similar to the one without shadow cost. For large cost of public funding, the optimal subsidy policy is a pure matching policy for both the one-firm and multiple-firm case.

7 Appendix

Proof of Proposition 1:

Because the objective function is linear in k, the firm's optimal investment decision $k_1(p)$ is a "bang-

bang" rule. This implies that it either sets k equal to the minimum required level z, or its maximum feasible level 1, i.e.,

$$k_1(p) = \begin{cases} 1 & \text{if } p > p_1 \\ z & \text{if } p < p_1 \end{cases},$$

where p_1 is the "abandonment threshold": once the firm's posterior belief about the project's viability falls to p_1, the firm switches from "flat-out" investment (i.e., $k = 1$), to investing the minimum feasible level z required by the government (i.e., $k = z$). If $k = z$, the general solution to the equation (6) is [21]

$$V_L(p) = \frac{s - \alpha z}{r} + \frac{\lambda z}{r + \lambda z} \frac{r\Pi - (s - \alpha z)}{r} p.$$

If $k = 1$, the general solution to the equation (6) is

$$V_H(p) = \frac{s - (1-\phi)\alpha - \alpha\phi z}{r} + \frac{\lambda}{r + \lambda}\left(\Pi - \left[\frac{s - (1-\phi)\alpha - \alpha\phi z}{r}\right]\right) p + B_1 p \left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}},$$

where B_1 is a constant. This constant and the abandonment threshold p_1 and the constant B_1 are determined by the value matching and smooth pasting conditions for $V_L(p)$ and $V_H(p)$:

$$V_L(p_1) = V_H(p_1). \tag{27}$$

$$V_L'(p_1) = V_H'(p_1). \tag{28}$$

These yield a solution for p_1 and B_1 given by:

$$p_1 = \frac{\alpha(1-\phi)(r + \lambda z)}{\lambda(r\Pi - s + \alpha z)}. \tag{29}$$

$$B_1 = \frac{\lambda\alpha(1-z)(1-\phi)}{r(r+\lambda)}\left(\frac{1-p_1}{p_1}\right)^{-\frac{r}{\lambda}}. \tag{30}$$

Algebraic analysis reveals that p_1 can be expressed as the solution to:

$$\frac{\lambda}{r+\lambda}\left[\Pi - \frac{(s - \alpha z)}{r}\right] p_1 = \frac{r}{r+\lambda}\left[\frac{\alpha(1-\phi)(1-z)}{r}\right] + \frac{\lambda z}{r + \lambda z}\left[\Pi - \frac{(s-\alpha z)}{r}\right] p_1. \tag{31}$$

[21] Note that there is a nonlinear term $B_0 p \left(\frac{1-p}{p}\right)^{\frac{r+\lambda z}{\lambda z}}$ to the general soluction of this differential equation, but is dropped because $V(0)$ needs to be finite and thus implies $B_0 = 0$. In fact, $V(0)$ represents the value of the firm when the R&D project is destined to fail.

If $\alpha, \phi, r, \lambda, z, s$ and Π are such that $\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi - s+\alpha z)} < 1$, then over the meaningful range of possible beliefs, $p \in [0,1]$, the value function has two pieces:

$$V_1(p) = \begin{cases} V_H(p) = \frac{s-(1-\phi)\alpha-\alpha\phi z}{r} + \frac{\lambda}{r+\lambda}\left(\Pi - \left[\frac{s-(1-\phi)\alpha-\alpha\phi z}{r}\right]\right)p + B_1 p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & \text{if } p \in [p_1, 1] \\ V_L(p) = \frac{s-\alpha z}{r} + \frac{\lambda z}{r+\lambda z}\frac{r\Pi-(s-\alpha z)}{r}p & \text{if } p \in [0, p_1] \end{cases}.$$

If $\alpha, \phi, r, \lambda, z, s$ and Π are such that $\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi - s+\alpha z)} \geq 1$, then over the meaningful range of possible beliefs the value function consists only of the $V_L(p)$ piece:

$$V_1(p) = V_L(p) = \frac{s-\alpha z}{r} + \frac{\lambda z}{r+\lambda z}\frac{r\Pi - (s-\alpha z)}{r}p \text{ for all } p \in [p_1, 1]. \tag{32}$$

■

Derivation of the Welfare Schedule $W_1(p)$:

As Proposition 1 shows, if $\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi - s+\alpha z)} < 1$, the monopoly firm either invests $k = 1$ or $k = z$ in the R&D project. If $k_1(p) = 1$ (which occurs if $p > p_1$), the solution to the differential equation in (**??**) is:

$$W_1(p) = -\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right) + \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p + B_W p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}}.$$

If $k_1(p) = z$ (which occurs if $p < p_1$), the solution to the differential equation in (**??**) is:[22]

$$W_1(p) = -\left(\frac{\alpha z+\gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right]p.$$

Thus,

$$W_1(p) = \begin{cases} -\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right) + \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p + B_W p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & p \in [p_1, 1] \\ -\left(\frac{\alpha z+\gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right]p & p \in [0, p_1] \end{cases}, \tag{33}$$

Because at p_1, the firm is indifferent between $k = 1$ and $k = 0$, the welfare schedule is continuous at p_1. Thus, the constant B_W equates the upper piece of $W_1(p)$ and the lower piece. Straightforward algebra

[22] As usual, we drop the nonlinear term by requiring the value function to be finite at $p = 0$.

establishes

$$B_W = B_W(p_1, z, s, \phi) = \frac{\left\{ \begin{array}{c} \frac{\alpha(1-z)}{r} + \frac{\gamma\phi\alpha(1-z)}{r} + \\ \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right] p_1 \\ -\frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right] p_1 \end{array} \right\}}{p_1 \left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{r}}}$$

If $\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi - s+\alpha z)} \geq 1$, the monopoly firm invests $k = z$ in the R&D project for all p, and thus

$$W_1(p) = -\left(\frac{\alpha z + \gamma s}{r}\right) + \frac{\lambda z}{r + \lambda z}\left[CS + \Pi + \left(\frac{\alpha z + \gamma s}{r}\right)\right] p.$$

Proof of Proposition 4:

To prove the proposition, we will show that the first-best level of welfare can be attained by setting $s = z = 0$ and choosing $\phi = 1 - \rho$. The first-best R&D policy $k^*(p)$ solves the social planner's problem:

$$W^*(p) = \max_{k\in[0,1]} \left[-\alpha k dt + \lambda k p dt \left[CS + \Pi\right] + (1 - \lambda k p dt)e^{-rdt} W^*(p + dp)\right]. \tag{34}$$

Using an approach similar to that used to prove Proposition 1, the first-best R&D policy is given by:

$$k^*(p) = \begin{cases} 1 & \text{if } p \in [p^*, 1] \\ 0 & \text{if } p \in [0, p^*] \end{cases}, \tag{35}$$

where

$$p^* = \frac{\alpha}{\lambda(\Pi + CS)}.^{23}$

is the first-best abandonment threshold.

Now, when $\gamma = 0$,

$$W_1(p) = -\alpha k_1(p)dt + \lambda k_1(p)pdt\left[CS + \Pi\right] + (1 - \lambda k_1(p)pdt)e^{-rdt} W_1(p + dp),$$

and because of (34), for any arbitrary subsidy policy, it must be the case that $W_1(p) \leq W^*(p)$, and

[23] Because we have assumed that the social benefit-cost ratio for a viable project exceeds 1, we have $\frac{\lambda[CS+\Pi]}{\alpha} > 1$, and thus $p^* < 1$. This implies that there must some exist some set of prior beliefs for which flat-out R&D investment would occur under the socially optimal policy.

in particular $EW_1(z,s,\phi) = W_1(p_0) \leq W^*(p_0)$. However, if we can find a subsidy policy such that $k_1(p) = k^*(p)$,then $W_1(p) = W^*(p)$ and in particular, $EW_1(z,s,\phi) = W_1(p_0) = W^*(p_0)$ for any prior belief p_0. A subsidy policy that implements the first-best investment policy must therefore maximize expected *ex ante* welfare when $\gamma = 0$.

Note that if $s = z = 0$,

$$p_1(z,s,\phi) = \frac{\alpha(1-\phi)}{\lambda(\Pi + CS)}$$

A matching rate given by $\phi = 1 - \frac{\Pi}{\Pi + CS} = 1 - \rho$, along with $s = z = 0$, ensures that $p_1(z,s,\phi) = p^*$, and thus implements the maximum level of expected welfare for any prior belief p_0. Therefore, $z = 0, s = 0$, $\phi = 1 - \frac{\Pi}{\Pi + CS}$ is the optimal policy.■

Proof of Proposition 5:

Suppose, to the contrary, that the optimal subsidy policy entails $s > \alpha z$. If the subsidy policy occurs in the range where $p_0 \in [0, p_1(z,s,\phi))$, then $EW_1(z,s,\phi) = -\left(\frac{\alpha z + \gamma s}{r}\right) + \frac{\lambda z}{r + \lambda z}\left[CS + \Pi + \left(\frac{\alpha z + \gamma s}{r}\right)\right]p$, which is strictly decreasing in s, and expected welfare can be increased by decreasing s, contradicting the presumed optimality of policy. Suppose, then, the subsidy policy occurs in the range where $p_0 \in [p_1(z,s,\phi), 1]$. Then $EW_1(z,s,\phi) = \Psi(B_W(p_1(z,s,\phi), z, s, \phi), z, s, \phi)$, and the first-order conditions for an optimal $s > \alpha z$ and a $\phi > 0$ are

$$\frac{\partial EW_1}{\partial \phi} = \frac{\partial \Psi}{\partial B_W}\left[\frac{\partial B_W}{\partial p_1}\frac{\partial p_1}{\partial \phi} + \frac{\partial B_W}{\partial \phi}\right] + \frac{\partial \Psi}{\partial \phi} = 0 \tag{36}$$

$$\frac{\partial EW_1}{\partial s} = \frac{\partial \Psi}{\partial B_W}\left[\frac{\partial B_W}{\partial p_1}\frac{\partial p_1}{\partial s} + \frac{\partial B_W}{\partial s}\right] + \frac{\partial \Psi}{\partial s} = 0 \tag{37}$$

or equivalently

$$\frac{-\left[\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial \phi} + \frac{\partial \Psi}{\partial \phi}\right]}{\frac{\partial p_1}{\partial \phi}} = \frac{-\left[\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial s} + \frac{\partial \Psi}{\partial s}\right]}{\frac{\partial p_1}{\partial s}} \tag{38}$$

We know from Proposition 2 that $\frac{\partial p_1}{\partial \phi} < 0$, while $\frac{\partial p_1}{\partial s} < 0$. Thus, $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial \phi} + \frac{\partial \Psi}{\partial \phi}$ and $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial s} + \frac{\partial \Psi}{\partial s}$ have opposite signs. Now,

$$\frac{\partial \Psi}{\partial B_W} = p_0 \left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}} > 0 \tag{39}$$

$$\frac{\partial B_W}{\partial s} = \frac{\frac{\gamma p_1}{r}\left[\frac{\lambda z}{r+\lambda z} - \frac{\lambda}{r+\lambda}\right]}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} < 0 \tag{40}$$

$$\frac{\partial \Psi}{\partial s} = -\frac{\gamma}{r}\left[1 - \frac{\lambda}{r+\lambda}p_0\right] < 0. \tag{41}$$

Hence, $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial s} + \frac{\partial \Psi}{\partial s} < 0$ and we must have $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial \phi} + \frac{\partial \Psi}{\partial \phi} > 0$. Now,

$$\frac{\partial B_W}{\partial \phi} = \frac{\frac{\gamma\alpha(1-z)}{r}\left[1 - \frac{\lambda p_1}{r+\lambda}\right]}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} \tag{42}$$

$$\frac{\partial \Psi}{\partial \phi} = -\frac{\gamma\alpha(1-z)}{r}\left[1 - \frac{\lambda p_0}{r+\lambda}\right] \tag{43}$$

Thus, straightforward algebra establishes $\frac{\partial \Psi}{\partial B_W}\frac{\partial B_W}{\partial \phi} + \frac{\partial \Psi}{\partial \phi} > 0 \Longleftrightarrow$

$$\frac{\gamma\alpha(1-z)}{r}\left\{\frac{p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}}{1-\frac{\lambda p_0}{r+\lambda}} - \frac{p_1\left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}{1-\frac{\lambda p_1}{r+\lambda}}\right\} > 0. \tag{44}$$

The function $F(p) \equiv \frac{p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}}}{1-\frac{\lambda p}{r+\lambda}}$ can be shown to be a strictly decreasing function of p.[24] However, $p_1 < p_0$, which implies that the above expression is negative, not positive, and we have a contradiction. This establishes that *if* that the optimal ϕ is interior and that the optimal policy occurs within the range where $p_1(z, s, \phi) < p_0$, then we must have $s = \alpha z$.

Proof of Proposition 6:

Preliminary Steps

With $\phi^{**} \in (0, 1)$, the previous lemma implies that $s^{**} = \alpha z^{**}$. Thus, we redefine the function $\Psi(\cdot)$ to

[24] Differentiating F with respect to p yields

$$F'(p) = \left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}}\left\{\frac{1}{1-p} - \frac{\frac{r+\lambda}{\lambda}}{1-p}\right\} < 0.$$

depend on B_W, z and ϕ; we redefine $B_W(\cdot)$ to depend on p_1, z, and ϕ, and we redefine $p_1(\cdot)$ to depend on z and ϕ. This gives us:

$$\Psi(B_W, z, \phi) \equiv \left\{ \begin{array}{c} -\left(\frac{\alpha+\gamma\alpha z+\gamma\phi\alpha(1-z)}{r}\right) \\ +\frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma\alpha z+\gamma\phi\alpha(1-z)}{r}\right)\right]p \\ +B_W p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}} \end{array} \right\}$$

$$p_1(z,\phi) \equiv \frac{\alpha\left(1-\phi\right)}{\lambda\Pi\left(\frac{r}{r+\lambda z}\right)}, \tag{45}$$

$$B_W(p_1, z, \phi) \equiv \frac{\left\{ \begin{array}{c} \frac{\alpha(1-z)}{r}+\frac{\gamma\phi\alpha(1-z)}{r} \\ -\frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma\alpha z+\gamma\phi\alpha(1-z)}{r}\right)\right]p_1 \\ +\frac{\lambda z}{r+\lambda z}\left[CS+\Pi+\left(\frac{\alpha z+\gamma\alpha z}{r}\right)\right]p_1 \end{array} \right\}}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{r}}}.$$

With $\phi \in (0,1)$ and $p_1(z,s,\phi) < p_0$, ϕ satisfies the first-order condition[25]

$$\frac{\partial EW_1}{\partial \phi} = \frac{\partial \Psi}{\partial B_W}\left[\frac{\partial B_W}{\partial p_1}\frac{\partial p_1}{\partial \phi}+\frac{\partial B_W}{\partial \phi}\right]+\frac{\partial \Psi}{\partial \phi}=0,$$

where, as noted above, $\frac{\partial \Psi}{\partial B_W} = p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}$ and

$$\frac{\partial B_W}{\partial p_1} = \frac{\alpha(1-z)\left([CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]\right)}{\lambda\Pi p_1^2(1-p_1)\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}$$

$$\frac{\partial B_W}{\partial \phi} = \frac{\frac{\alpha\gamma(1-z)}{r}\left(1-\frac{\lambda}{r+\lambda}p_1\right)}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}$$

$$\begin{aligned} \frac{\partial \Psi}{\partial \phi} &= -\frac{\alpha\gamma\left(1-z\right)}{r}\left(1-\frac{\lambda}{r+\lambda}p_0\right) \\ \frac{\partial p_1}{\partial \phi} &= -\frac{\alpha}{\lambda\Pi}\frac{r+\lambda z}{r}. \end{aligned}$$

[25] Hereafter, to keep the notation a bit simpler, we drop the ** on the optimal subsidy policy.

We can thus rewrite the first-order condition for ϕ as

$$\begin{aligned}
\frac{\partial EW_1}{\partial \phi} &= p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}\left[\begin{array}{c}\frac{\alpha(1-z)\left([CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]\right)}{\lambda\Pi(1-p_1)p_1^2\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}\left(-\frac{\alpha}{\lambda\Pi}\frac{r+\lambda z}{r}\right) \\ +\frac{\frac{\alpha\gamma(1-z)}{r}\left(1-\frac{\lambda}{r+\lambda}p_1\right)}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}\end{array}\right] \\
&\quad -\frac{\alpha\gamma(1-z)}{r}\left(1-\frac{\lambda}{r+\lambda}p_0\right) \\
&= 0.
\end{aligned}$$

Because $p_1(z,\phi) \equiv \frac{\alpha(1-\phi)}{\lambda\Pi}\frac{r+\lambda z}{r}$, it follows that $-\frac{\alpha}{\lambda\Pi}\frac{r+\lambda z}{r} = -\frac{p_1}{1-\phi}$. Substituting this into the previous expression and rearranging terms, we can rewrite the first-order condition for ϕ as

$$\frac{\partial EW_1}{\partial \phi} = p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}\alpha(1-z)\left\{\frac{\frac{-1}{1-\phi}\left([CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]\right)}{\lambda\Pi(1-p_1)p_1\left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}+G(p_1)-G(p_0)\right\}=0, \tag{46}$$

where

$$G(p) \equiv \frac{\frac{\gamma}{r}\left(1-\frac{\lambda}{r+\lambda}p\right)}{p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}}}.$$

When $\gamma > 0$, the function $G(p)$ can be shown to be a strictly increasing function of p. Since $p_1 < p_0$, it follows that when $\gamma > 0$, $G(p_1) - G(p_0) < 0$. Thus, (46) implies that when $\gamma > 0$,

$$[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}] < 0, \tag{47}$$

or

$$1-\phi > \frac{\Pi(1+\gamma)}{CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}}. \tag{48}$$

Now, consider the expression for $\frac{\partial EW_1}{\partial z}$:

$$\frac{\partial EW_1}{\partial z} = \frac{\partial \Psi}{\partial B_W}\left[\frac{\partial B_W}{\partial p_1}\frac{\partial p_1}{\partial z}+\frac{\partial B_W}{\partial z}\right]+\frac{\partial \Psi}{\partial z} = 0. \tag{49}$$

It can be shown that

$$\begin{aligned}
\frac{\partial p_1}{\partial z} &= \frac{\alpha(1-\phi)}{r\Pi} \\
\frac{\partial \Psi}{\partial z} &= -\frac{\alpha\gamma\,(1-\phi)}{r}\left(1-\frac{\lambda}{r+\lambda}p_0\right) \\
\frac{\partial B_W}{\partial z} &= \frac{\frac{\alpha\gamma(1-\phi)}{r}\left(1-\frac{\lambda}{r+\lambda}p_1\right)}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} + \frac{\left[CS+\Pi+\frac{\alpha(1+\gamma)z}{r}\right]\frac{\alpha(1-\phi)}{\Pi(r+\lambda z)}}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} - \frac{\frac{\alpha(1+\gamma)}{r}\left[1-\frac{(1-\phi)\alpha z}{r\Pi}\right]}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}
\end{aligned}$$

Noting that $\frac{\partial p_1}{\partial z} = \frac{\alpha(1-\phi)}{r\Pi} = \frac{\lambda p_1}{r+\lambda z}$, and factoring out $\alpha(1-\phi)$ and $p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}$ from the components of $\frac{\partial EW_1}{\partial z}$, we can write $\frac{\partial EW_1}{\partial z}$ as

$$\frac{\partial EW_1}{\partial z} = p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}\alpha(1-\phi)\left\{\begin{array}{c} \frac{\frac{1}{1-\phi}\frac{\lambda(1-z)}{r+\lambda z}\left([CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]\right)}{\lambda\Pi(1-p_1)p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} + G(p_1) - G(p_0) \\ +\frac{\left[CS+\Pi+\frac{\alpha(1+\gamma)z}{r}\right]\frac{1}{\Pi(r+\lambda z)}}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} \\ -\frac{\frac{\alpha(1+\gamma)}{r}\frac{1}{\alpha(1-\phi)}\left[1-\frac{\alpha z(1-\phi)}{r\Pi}\right]}{p_1\left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{\lambda}}} \end{array}\right\}. \tag{50}$$

Now, given that $\frac{\partial EW_1}{\partial \phi} = 0$, we can substitute (46) into (50) to get

$$\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial \phi}=0} = \frac{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}}{p_0\left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}}}\alpha(1-\phi)\left\{\begin{array}{c} \frac{r+\lambda}{r+\lambda z}\frac{1}{1-\phi}\left(\frac{[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]}{\lambda\Pi(1-p_1)}\right) \\ +\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{\Pi(r+\lambda z)} \\ -\frac{(1+\gamma)}{r}\frac{1}{(1-\phi)}\left[1-\frac{\frac{\alpha z}{r}(1-\phi)}{\Pi}\right] \end{array}\right\}.$$

In deriving this expression, we used the fact that $\frac{\lambda(1-z)}{r+\lambda z} - 1 = \frac{r+\lambda}{r+\lambda z}$, we factored out the term $p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}$, and we noted that $-\frac{(1+\gamma)}{r}\frac{1}{(1-\phi)}\left[1-\frac{\frac{\alpha z}{r}(1-\phi)}{\Pi}\right] = -\frac{1}{r}\left[\frac{(1+\gamma)}{1-\phi} - \frac{\frac{\alpha(1+\gamma)z}{r}}{\Pi}\right]$. Thus, the sign of $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial \phi}=0}$ is the sign of the expression

$$
\begin{aligned}
H(z,\phi) \quad \equiv \quad & \frac{r+\lambda}{r+\lambda z}\frac{1}{1-\phi}\left(\frac{[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]}{\lambda\Pi(1-p_1)}\right) \\
& +\left(\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{\Pi(r+\lambda z)}\right) \\
& -\frac{1}{r}\left[\frac{1+\gamma}{1-\phi}-\frac{\frac{\alpha(1+\gamma)z}{r}}{\Pi}\right]. \qquad (51)
\end{aligned}
$$

Note that if we can establish that $H(z,\phi)<0$ for all $z\in[0,1]$, it follows that $z=0$.

Proof of (a)

We note that from (47) that $[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]<0$, so the first term in the expression for $H(z,\phi)$ in (51) is negative. Thus, to show that $z=0$, it suffices to show that the last two terms of $H(z,\phi)$ are negative or

$$
\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{\Pi(r+\lambda z)}-\frac{(1+\gamma)}{r}\frac{1}{(1-\phi)}\left[1-\frac{\frac{\alpha z}{r}(1-\phi)}{\Pi}\right]<0.
$$

Multiplying through by $\frac{\Pi(r+\lambda z)}{CS+\Pi}$ and combining terms, we can rewrite this condition as

$$
\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}-\left(\frac{r+\lambda z}{r}\right)\left[\frac{(1+\gamma)\,\Pi}{(1-\phi)\,(CS+\Pi)}-\frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}\right]<0. \qquad (52)
$$

Now, since we know from (48) that $1-\phi>\frac{\Pi(1+\gamma)}{CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}}$, it follows that

$$
\frac{1}{1-\phi}=\frac{CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}}{\Pi(1+\gamma)}-\theta, \qquad (53)
$$

where θ is a positive number. Substituting (53) into (52), and rearranging terms algebraically allows us to rewrite (52) as

$$
\left[\frac{\alpha}{\lambda\,[CS+\Pi]}-1\right]\frac{\lambda z}{r}+\frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}-\left(\frac{r+\lambda z}{r}\right)\frac{[\gamma\Pi-(1+\gamma)\theta\Pi]}{CS+\Pi}<0. \qquad (54)
$$

Now, note that since the social benefit-cost ratio $\frac{\lambda[CS+\Pi]}{\alpha}$ is assumed to exceed 1, it follows that $\left[\frac{\alpha}{\lambda[CS+\Pi]}-1\right]\frac{\lambda z}{r}<0$ for $z\in(0,1]$. Now, as $\gamma\to 0$,$\frac{r}{r+\lambda z}\frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}\to 0$, and $\theta\to 0$, so that $\left(\frac{r+\lambda z}{r}\right)\frac{[\gamma\Pi-(1+\gamma)\theta\Pi]}{CS+\Pi}\to 0$. Thus, the inequality in (54) does indeed hold, which shows that as $\gamma \to 0$, $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial \phi}=0} < 0$, establishing part (a) of the Proposition.

Proof of part (b)

To prove part (b), we return to condition (52) which, if it can be established, shows that $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial \phi}=0} < 0$ for all z. Noting that $\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi} = 1 + \frac{\frac{\alpha z}{r}}{CS+\Pi} + \frac{\frac{\gamma\alpha z}{r}}{CS+\Pi}$, adding and subtracting the term $\left(\frac{r+\lambda z}{r}\right)$, and rearranging terms gives us:

$$\left[\frac{\alpha}{\lambda\left[CS+\Pi\right]} - 1\right]\frac{\lambda z}{r} + \frac{\frac{\gamma\alpha z}{r}}{CS+\Pi} + \left(\frac{r+\lambda z}{r}\right)\left[1 + \frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi} - \frac{1}{1-\phi}\frac{(1+\gamma)\Pi}{(CS+\Pi)}\right] < 0. \tag{55}$$

Now, we know that $p_1 = \frac{\alpha(1-\phi)}{\lambda\Pi}\left(\frac{r+\lambda z}{r}\right) \leq 1$, so $1-\phi \leq \frac{\lambda\Pi}{\alpha\left(\frac{r+\lambda z}{r}\right)}$, or $\frac{1}{1-\phi} \geq \left(\frac{r+\lambda z}{r}\right)\frac{\alpha}{\lambda\Pi}$. Thus, $-\frac{1}{1-\phi}\frac{(1+\gamma)\Pi}{CS+\Pi}$ $\leq -\left(\frac{r+\lambda z}{r}\right)\frac{\alpha(1+\gamma)}{\lambda(CS+\Pi)}$. This latter condition implies that

$$\begin{aligned}
&\left[\frac{\alpha}{\lambda\left[CS+\Pi\right]} - 1\right]\frac{\lambda z}{r} + \frac{\frac{\gamma\alpha z}{r}}{CS+\Pi} + \left(\frac{r+\lambda z}{r}\right)\left[1 + \frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi} - \frac{1}{1-\phi}\frac{(1+\gamma)\Pi}{(CS+\Pi)}\right] \\
< \;&\left[\frac{\alpha}{\lambda\left[CS+\Pi\right]} - 1\right]\frac{\lambda z}{r} + \frac{\frac{\gamma\alpha z}{r}}{CS+\Pi} + \left(\frac{r+\lambda z}{r}\right)\cdot\left[1 + \frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi} - \left(\frac{r+\lambda z}{r}\right)\frac{\alpha(1+\gamma)}{\lambda\left(CS+\Pi\right)}\right]
\end{aligned} \tag{56}$$

With a number of steps of algebra to rearrange the right-hand side of the above inequality, we can write the inequality as

$$\begin{aligned}
&\left[\frac{\alpha}{\lambda\left[CS+\Pi\right]} - 1\right]\frac{\lambda z}{r} + \frac{\frac{\gamma\alpha z}{r}}{CS+\Pi} + \left(\frac{r+\lambda z}{r}\right)\left[1 + \frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi} - \frac{1}{1-\phi}\frac{(1+\gamma)\Pi}{(CS+\Pi)}\right] \\
< \;& 1 - \frac{\alpha(1+\gamma)}{\lambda(CS+\Pi)} \\
\leq \;& 0,
\end{aligned}$$

where the final inequality follows from the assumption in this part of the proposition that $\gamma \geq \frac{\lambda(CS+\Pi)}{\alpha} - 1$, or equivalently, $1 - \frac{\alpha(1+\gamma)}{\lambda(CS+\Pi)} \leq 0$. This then suffices to establish (52) which, in turn, shows that $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial \phi}=0} < 0$.

Proof of (c)

To prove (c), we return to the expression for $H(z,\phi)$ in (51) , whose sign is the sign of $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial \phi}=0} < 0$. Multiplying through by $\frac{\Pi}{CS+\Pi}(r+\lambda z)$ and rearranging terms yields the expression $I(z,\phi)$ whose sign is the same as $H(z,\phi)$:

$$
\begin{aligned}
I(z,\phi) \quad \equiv \quad & \frac{r+\lambda}{1-\phi}\left(\frac{[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]}{\lambda\left(CS+\Pi\right)\left(1-p_1\right)}\right) \\
& +\frac{CS+\Pi+\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi} \\
& -\left(\frac{r+\lambda z}{r}\right)\left[\frac{(1+\gamma)\,\Pi}{(1-\phi)\left(CS+\Pi\right)}-\frac{\frac{\alpha(1+\gamma)z}{r}}{CS+\Pi}\right]
\end{aligned}
$$

Algebraically rearranging terms in $I(z,\phi)$ yields

$$
\begin{aligned}
I(z,\phi) \quad \equiv \quad & \frac{1}{1-\phi}\left[\frac{r+\lambda}{\lambda(1-p_1)}-\frac{r+\lambda z}{r}\right]\left(\frac{[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]}{CS+\Pi}\right) \\
& +\left[\frac{\alpha(1+\gamma)}{\lambda\left(CS+\Pi\right)}-1\right]\frac{\lambda z}{r}-\frac{\gamma\Pi}{(CS+\Pi)}\left(\frac{r+\lambda z}{r}\right)
\end{aligned}
\tag{57}
$$

The last term in $I(z,\phi)$ is negative and by assumption in this part of the proposition, $\gamma < \frac{\lambda(CS+\Pi)}{\alpha}-1$, which implies $\left[\frac{\alpha(1+\gamma)}{\lambda(CS+\Pi)}-1\right]\frac{\lambda z}{r}$ is negative for $z \in (0,1]$. In addition, we have the following chain of inequalities:

$$
\frac{r+\lambda}{\lambda(1-p_1)} > \frac{r+\lambda}{\lambda} > \frac{r+\lambda}{r} > \frac{r+\lambda z}{r},
$$

where the second inequality in the chain follows from the assumption in this part of the proposition that $r>\lambda$. Since we had established earlier that $[CS+\Pi(1+\gamma)+\frac{\alpha(1+\gamma)z}{r}]\phi-[CS+\frac{\alpha(1+\gamma)z}{r}]<0$, it follows that $I(z,\phi)<0$ for all z and hence $H(z,\phi)<0$, thus establishing that $\left.\frac{\partial EW_1}{\partial z}\right|_{\frac{\partial EW_1}{\partial\phi}=0}<0$.

Proof that $p_1^{**} > p^* = \frac{\alpha}{\lambda(CS+\Pi)}$ *and* $\phi^{**} < 1-\rho$*:*

Under conditions (a), (b), or (c), the optimal subsidy policy entails $z=s=0$. Now, *given* the optimal matching rate ϕ^{**} and the positive shadow cost of public funds γ, consider the social planner's most preferred investment policy i.e., the policy it would want the firm follow given that it had committed to the subsidy. That policy would solve the following problem:

$$
\begin{aligned}
W_1\left(p\right) \quad = \quad & \max_{k\in[0,1]} -\alpha k dt-\gamma\phi\alpha k dt \\
& +\lambda k p dt\left[CS+\Pi\right]+(1-\lambda k p dt)e^{-rdt}W_1\left(p+dp\right).
\end{aligned}
$$

That policy entails an abandonment threshold given by:

$$p_1^W = \frac{\alpha\left(1+\gamma\phi^{**}\right)}{\lambda\left(CS+\Pi\right)}.$$

By contrast, the abandonment threshold p_1^{**} induced by the optimal subsidy policy is

$$p_1^{**} = \frac{\alpha\left(1-\phi^{**}\right)}{\lambda\Pi}$$

Thus,

$$\begin{aligned}
\frac{p_1}{p_1^W} &= \frac{\frac{\alpha(1-\phi^{**})}{\lambda\Pi}}{\frac{\alpha(1+\gamma\phi^{**})}{\lambda(CS+\Pi)}} \\
&= \frac{\left(1-\phi^{**}\right)}{\left(1+\gamma\phi^{**}\right)}\frac{CS+\Pi}{\Pi} \\
&> \frac{1-\frac{CS}{CS+\Pi(1+\gamma)}}{1+\gamma\frac{CS}{CS+\Pi(1+\gamma)}}\frac{CS+\Pi}{\Pi} \\
&= 1,
\end{aligned}$$

where the inequality follows from the result that $\phi^{**} < \frac{CS+\frac{\alpha z(1+\gamma)}{r}}{CS+\Pi(1+\gamma)+\frac{\alpha z(1+\gamma)}{r}}$ proved in (48) above. Thus, we have: $p_1^{**} > p_1^W = \frac{\alpha(1+\gamma\phi^{**})}{\lambda(CS+\Pi)} > \frac{\alpha}{\lambda(CS+\Pi)} = p^*$. To complete the proof, note that $\phi^{**} < \frac{CS}{CS+\Pi(1+\gamma)} < \frac{CS}{CS+\Pi} = 1-\rho$.■

Proof of Proposition 7[26]

We begin by defining

$$\begin{aligned}
b\left(p, V^i\right) &= \lambda p\left[\theta\Pi - V^i - (1-p)\,V^{i\prime}\right] \\
c\left(p\right) &= \alpha\left(1-\phi\right) - \lambda p\left(1-\theta\right)\Pi.
\end{aligned}$$

The term $b\left(p, V^i\right)$ is the marginal benefit to the firm from an additional unit of investment effort by a rival firm, while $c(p)$ is the *net* marginal cost to the firm from an additional unit of R&D. Specifically, it is the net of subsidy marginal cost of effort $\alpha\left(1-\phi\right)$ minus the net marginal benefit $\lambda p\left(1-\theta\right)\Pi$ of achieving the breakthrough rather than free riding on a competitor's discovery. We can now write the Bellman equation

[26] Part (i) of the proof follows the method employed by Keller Rady and Cripps (2005).

(21) as

$$rV^i(p) - (s - \alpha\phi z) = K^{-i} b\left(p, V^i\right) + \max_{k^i \in [z,1]} \left\{k^i \left[b\left(p, V^i\right) - c(p)\right]\right\},$$

The firm's optimal investment decision is given by the following "reaction function":

$$k^i(p) = \begin{cases} 1 & \text{if } V^i(p) > \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r} K^{-i} \\ \in [z,1] & \text{if } V^i(p) = \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r} K^{-i} \\ z & \text{if } V^i(p) < \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r} K^{-i} \end{cases}. \tag{58}$$

The linearity of the value function implies that we only need to consider three cases: $k = z$, $k = 1$ and $k = \kappa \in (z, 1)$. As noted, we seek to characterize a symmetric equilibrium with investment policies $k^1(p) = \ldots = k^N(p) = k_N(p)$ and value function $V_N(p)$. When $k_N(p) = z$, the differential equation (21) is

$$\begin{aligned} rV_N(p) &= s - \alpha\phi z + \lambda p(N-1)z\left[\theta\Pi - V_N(p) - (1-p)V_N'(p)\right] \\ &\quad + z\left[\lambda p\left[\Pi - V_N(p) - (1-p)V_N'(p)\right] - \alpha(1-\phi)\right]. \end{aligned}$$

The solution to this equation, denoted by $V_{NL}(p)$, is[27]

$$V_{NL}(p) = \frac{s - \alpha z}{r} + \frac{\lambda N z\left[\frac{r(1-\theta)\Pi}{N} + r\theta\Pi - s + \alpha z\right]}{r(r + \lambda N z)} p.$$

When $k_N(p) = 1$, the differential equation (21) is

$$\begin{aligned} rV_N(p) &= s - \alpha\phi z + \lambda p(N-1)\left[\theta\Pi - V_N(p) - (1-p)V_N'(p)\right] \\ &\quad + \left[\lambda p\left[\Pi - V_N(p) - (1-p)V_N(p)\right] - \alpha(1-\phi)\right] \end{aligned}$$

The solution to this equation, denoted by $V_N^H(p)$, is

$$V_N^H(p) = \frac{s - \alpha(1-\phi) - \alpha\phi z}{r} + \frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N} + r\theta\Pi - s + \alpha(1-\phi) + \alpha\phi z\right)}{r(r+\lambda N)} p + B_H p \left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}},$$

[27] The coefficient to the nonlinear part is zero as we require finitensss for the value function at $p = 0$.

where B_H is a constant to be determined. Finally, if $k_N(p) \in (0,1)$, then from (58) $b(V_N(p), p) = c(p)$, or

$$\lambda p \left(\theta\Pi - V_N(p) - (1-p) V_N'(p)\right) - (\alpha(1-\phi) - \lambda p (1-\theta)\Pi) = 0.$$

The solution to this equation, which is denoted by $V_N^M(p)$, is

$$V_N^M(p) \equiv \frac{\lambda\Pi - (1-\phi)\alpha}{\lambda} - B_M(1-p) + \frac{(1-\phi)\alpha(1-p)}{\lambda} \ln\frac{1-p}{p},$$

where B_M is a constant to be determined. To solve for the abandonment threshold p_N at which the firm reduces its R&D effort to the minimum mandated level. The value matching and smooth pasting conditions imply that

$$\begin{aligned} V_N^M(p_N) &= V_N^L(p_N) \\ V_N^{M\prime}(p_N) &= V_N^{L\prime}(p_N), \end{aligned}$$

which gives us

$$p_N = \frac{\alpha(1-\phi)}{\lambda\left(\left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda N z}\right) + (1-\theta)\Pi\left(\frac{(N-1)\lambda z}{r+\lambda N z}\right)\right)}$$

Now, from (58), $k_N(p) \in (z,1)$ if and only if $V_{NM}(p) = \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}(N-1)k_N(p)$. This implies:

$$k_N(p) = \frac{rV_{NM}(p) - s + \alpha\phi z}{(N-1)c(p)},$$

which makes each firm is indifferent between choosing any investment level between z and 1. A necessary condition for this to be well defined is that $c(p_N)$ be positive; otherwise $k_N(p)$ will be negative. From the definition of $c(p)$ above, $c(p_N) > 0$ if and only if $p_N < \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$. Straightforward algebra establishes that this condition is equivalent to

$$\theta > \overline{\theta}(z,s) \equiv 1 - \left[\frac{\Pi - \left(\frac{s-\alpha z}{r}\right)}{\Pi}\right]\left[\frac{r}{r+\lambda z}\right].$$

Since that $k_N(p) \to \infty$ as $p \to \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$, there exists q_N such that $k(q_N) = 1$. In this case, the value function is thus:

$$V_N(p)=\begin{cases} V_N^H(p) & p\in[q_N,1] \\ V_N^M(p) & p\in(p_N,q_N) \\ V_N^L(p) & p\in[0.p_N] \end{cases}.$$

(ii) If $\theta\leq\overline{\theta}(z,s)$, we will show that each firm's optimal strategy is either to invest "flat out" in R&D effort by setting $k=1$ when $p>p_N$ and invest at the minimum level z, when $p\leq p_N$. To establish the latter, suppose all other firms besides i are investing the minimum level z, so that $K^{-i}=(N-1)\,z$. To show that firm i's best response is also to invest z when $p\leq p_N$, then from (58), we must establish that $V_N(p)<\frac{s-\alpha\phi z}{r}+\frac{c(p)}{r}(N-1)\,K^{-i}$, or equivalently $V_{NL}(p)<\frac{s-\alpha\phi z}{r}+\frac{c(p)}{r}(N-1)\,z$, when $p\leq p_N$. Using the expressions for $V_{NL}(p)$ and $c(p)$, and some straightforward algebraic manipulations, $V_{NL}(p)<\frac{s-\alpha\phi z}{r}+\frac{c(p)}{r}(N-1)\,z$ can be shown to be equivalent to $\lambda\left(\left[\Pi-\frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda Nz}\right)+(1-\theta)\,\Pi\left(\frac{(N-1)\lambda z}{r+\lambda Nz}\right)\right)p\leq\alpha(1-\phi)$, and given the expression for p_N above, this indeed is true for $p\leq p_N$. Now assume $p>p_N$ and every other firm invests $k=1$. We need to show that firm i's best response is to invest $k=1$. Suppose not. Then, due to the linearity of the problem, the only case that the firm will invest $z<k<1$ is when $V^i=\frac{s}{r}+\frac{c(p)}{r}K^{-i}$, but this implies $p_N<\frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$. However, $\theta\leq\overline{\theta}(z,s)$ implies $z\geq\frac{r\theta\Pi-s}{\lambda(1-\theta)\Pi-\alpha}$, which, in turn, implies $p_N\geq\frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$, a contradiction. The uniqueness also follows from the fact that the investment problem is linear in k so firms will not coordinate in investing in a lower level $\kappa\in(z,1)$ because if that is the case, it then implies $b\left(p,V_i^n\right)>c\left(p\right)$, then the firm will invest $k=1$ rather than $\kappa<1$.[28] In this case the value function is

$$V_N(p)=\begin{cases} V_N^H(p) & p\in[p_N,1] \\ V_N^L(p) & p\in[0.p_N] \end{cases}.$$

Proof of Proposition 10

We employ the same logic as in the proof of Proposition (4): when $\gamma=0$ it suffices to show that if we can induce $k_N(p)=k^*(p)$ through an appropriate choice of $(z,s,\ \phi)$, then that subsidy policy must indeed maximize *ex ante* welfare $EW_N(z,s,\phi)$. Now, recall that under the equilibrium policy there is no free riding if and only if

$$\theta\leq\overline{\theta}(z,s)=1-\left[\frac{\Pi-\left(\frac{s-\alpha z}{r}\right)}{\Pi}\right]\left[\frac{r}{r+\lambda z}\right],.$$

[28] For a similar proof of this result, see Besanko and Wu (2008).

When $z = 0$, $s = r\theta\Pi$, then it can be verified that $\overline{\theta}(z, s) = \theta$, which is just enough to eliminates the free-rider problem. The equilibrium investment policy in this case is

$$k_N(p) = \begin{cases} 1 & \text{if } p \geq p_N(0, r\theta\Pi, \phi) \\ 0 & \text{if } p \leq p_N(0, r\theta\Pi, \phi), \end{cases},$$

where

$$p_N(0, r\theta\Pi, \phi) = \frac{\alpha(1-\phi)}{\lambda(1-\theta)\Pi}.$$

By setting $\phi = \frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$, we can make $p_N(0, r\theta\Pi, \phi) = p_N^*$. Thus, the first-best investment policy can be induced by setting $z = 0$ and using a combination of unrestricted funding $s = r\theta\Pi$ and a matching rate $\phi = \frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$.■

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Table 2: Optimal R&D Subsidy Policy for N Noncooperative Firms

θ=0.1

ρ	p_0 = 0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.91)	(0,0,0.91)	(0,0,0.91)	(0,0,0.90)
0.2	(0,0,0)	(0,0,0.83)	(0,0,0.82)	(0,0,0.81)	(0,0,0.77)
0.3	(0,0,0)	(0,0,0.74)	(0,0,0.72)	(0,0,0.70)	(0,0,0.61)
0.4	(0,0,0)	(0,0,0.65)	(0,0,0.63)	(0,0,0.59)	(0,0,0.42)
0.5	(0,0,0)	(0,0,0.57)	(0,0,0.53)	(0,0,0.47)	(0,0,0.20)
0.6	(0,0,0)	(0,0,0.48)	(0,0,0.44)	(0,0,0.35)	(0,0,0)
0.7	(0,0,0)	(0,0,0.40)	(0,0,0.33)	(0,0,0.22)	(0,0,0)
0.8	(0,0,0)	(0,0,0.31)	(0,0,0.23)	(0,0,0.08)	(0,0,0)
0.9	(0,0,0)	(0,0,0.23)	(0,0,0.12)	(0,0,0)	(0,0,0)

θ=0.2

ρ	p_0 = 0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.93)	(0,0,0.93)	(0,0,0.93)	(0,0,0.92)
0.2	(0,0,0)	(0,0,0.86)	(0,0,0.86)	(0,0,0.86)	(0,0,0.83)
0.3	(0,0,0)	(0,0,0.80)	(0,0,0.79)	(0,0,0.78)	(0,0,0.72)
0.4	(0,0,0)	(0,0,0.73)	(0,0,0.72)	(0,0,0.70)	(0,0,0.59)
0.5	(0,0,0)	(0,0,0.66)	(0,0,0.65)	(0,0,0.61)	(0,0,0.44)
0.6	(0,0,0)	(0,0,0.60)	(0,0,0.57)	(0,0,0.52)	(0,0,0.29)
0.7	(0,0,0)	(0,0,0.53)	(0,0,0.50)	(0,0,0.43)	(0,0,0.11)
0.8	(0,0,0)	(0,0,0.47)	(0,0,0.42)	(0,0,0.32)	(0,0,0)
0.9	(0,0,0)	(0,0,0.40)	(0,0,0.34)	(0,0,0.23)	(0,0,0)

θ=0.3

ρ	p_0 = 0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.94)	(0,0,0.95)	(0,0,0.94)	(0,0,0.94)
0.2	(0,0,0)	(0,0,0.89)	(0,0,0.89)	(0,0,0.89)	(0,0,0.87)
0.3	(0,0,0)	(0,0,0.84)	(0,0,0.83)	(0,0,0.83)	(0,0,0.79)
0.4	(0,0,0)	(0,0,0.79)	(0,0,0.78)	(0,0,0.76)	(0,0,0.70)
0.5	(0,0,0)	(0,0,0.73)	(0,0,0.72)	(0,0,0.70)	(0,0,0.60)
0.6	(0,0,0)	(0,0,0.68)	(0,0,0.66)	(0,0,0.63)	(0,0,0.49)
0.7	(0,0,0)	(0,0,0.63)	(0,0,0.61)	(0,0,0.56)	(0,0,0.37)
0.8	(0,0,0)	(0,0,0.58)	(0,0,0.55)	(0,0,0.49)	(0,0,0.24)
0.9	(0,0,0)	(0,0,0.52)	(0,0,0.49)	(0,0,0.42)	(0,0,0.10)

θ=0.4

ρ	p_0 = 0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.96)	(0,0,0.96)	(0,0,0.76)	(0,0,0.95)
0.2	(0,0,0)	(0,0,0.91)	(0,0,0.91)	(0,0,0.91)	(0,0,0.90)
0.3	(0,0,0)	(0,0,0.86)	(0,0,0.87)	(0,0,0.85)	(0,0,0.84)
0.4	(0,0,0)	(0,0,0.83)	(0,0,0.82)	(0,0,0.81)	(0,0,0.77)
0.5	(0,0,0)	(0,0,0.78)	(0,0,0.77)	(0,0,0.76)	(0,0,0.70)
0.6	(0,0,0)	(0,0,0.74)	(0,0,0.73)	(0,0,0.71)	(0,0,0.62)
0.7	(0,0,0)	(0,0,0.70)	(0,0,0.68)	(0,0,0.66)	(0,0,0.53)
0.8	(0,0,0)	(0,0,0.66)	(0,0,0.64)	(0,0,0.60)	(0,0,0.44)
0.9	(0,0,0)	(0,0,0.61)	(0,0,0.59)	(0,0,0.55)	(0,0,0.34)

θ=0.5

ρ	p_0 = 0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.96)	(0,0,0.85)	(0,0,0.96)	(0,0,0.96)
0.2	(0,0,0)	(0,0,0.93)	(0,0,0.93)	(0,0,0.93)	(0,0,0.92)
0.3	(0,0,0)	(0,0,0.89)	(0,0,0.89)	(0,0,0.89)	(0,0,0.88)
0.4	(0,0,0)	(0,0,0.86)	(0,0,0.85)	(0,0,0.85)	(0,0,0.82)
0.5	(0,0,0)	(0,0,0.82)	(0,0,0.81)	(0,0,0.80)	(0,0,0.77)
0.6	(0,0,0)	(0,0,0.78)	(0,0,0.78)	(0,0,0.77)	(0,0,0.71)
0.7	(0,0,0)	(0,0,0.75)	(0,0,0.74)	(0,0,0.73)	(0,0,0.65)
0.8	(0,0,0)	(0,0,0.72)	(0,0,0.70)	(0,0,0.68)	(0,0,0.58)
0.9	(0,0,0)	(0,0,0.68)	(0,0,0.66)	(0,0,0.64)	(0,0,0.50)

θ=0.6

ρ	p_0 = 0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.97)	(0,0,0.97)	(0,0,0.97)	(0,0,0.97)
0.2	(0,0,0)	(0,0,0.94)	(0,0,0.95)	(0,0,0.94)	(0,0,0.94)
0.3	(0,0,0)	(0,0,0.91)	(0,0,0.93)	(0,0,0.91)	(0,0,0.90)
0.4	(0,0,0)	(0,0,0.88)	(0,0,0.88)	(0,0,0.88)	(0,0,0.86)
0.5	(0,0,0)	(0,0,0.61)	(0,0,0.85)	(0,0,0.85)	(0,0,0.82)
0.6	(0,0,0)	(0,0,0.82)	(0,0,0.82)	(0,0,0.81)	(0,0,0.78)
0.7	(0,0,0)	(0,0,0.79)	(0,0,0.79)	(0,0,0.78)	(0,0,0.73)
0.8	(0,0,0)	(0,0,0.77)	(0,0,0.76)	(0,0,0.75)	(0,0,0.68)
0.9	(0,0,0)	(0,0,0.73)	(0,0,0.72)	(0,0,0.71)	(0,0,0.63)

θ=0.7

ρ	p_0 = 0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.98)	(0,0,0.98)	(0,0,0.98)	(0,0,0.97)
0.2	(0,0,0)	(0,0,0.95)	(0,0,0.95)	(0,0,0.95)	(0,0,0.95)
0.3	(0,0,0)	(0,0,0.93)	(0,0,0.93)	(0,0,0.93)	(0,0,0.92)
0.4	(0,0,0)	(0,0,0.91)	(0,0,0.91)	(0,0,0.90)	(0,0,0.89)
0.5	(0,0,0)	(0,0,0.88)	(0,0,0.88)	(0,0,0.88)	(0,0,0.86)
0.6	(0,0,0)	(0,0,0.86)	(0,0,0.86)	(0,0,0.85)	(0,0,0.83)
0.7	(0,0,0)	(0,0,0.84)	(0,0,0.83)	(0,0,0.83)	(0,0,0.79)
0.8	(0,0,0)	(0,0,0.82)	(0,0,0.81)	(0,0,0.80)	(0,0,0.76)
0.9	(0,0,0)	(0,0,0.79)	(0,0,0.79)	(0,0,0.77)	(0,0,0.72)

θ=0.8

ρ	p_0 = 0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.98)	(0,0,0.98)	(0,0,0.85)	(0,0,0.98)
0.2	(0,0,0)	(0,0,0.96)	(0,0,0.96)	(0,0,0.96)	(0,0,0.96)
0.3	(0,0,0)	(0.016,0.0034,0.93)	(0,0,0.94)	(0.01,0.002,0.94)	(0,0,0.94)
0.4	(0,0,0)	(0.019,0.0038,0.91)	(0,0,0.93)	(0,0,0.92)	(0,0,0.92)
0.5	(0,0,0)	(0,0,0.90)	(0,0,0.91)	(0,0,0.90)	(0,0,0.89)
0.6	(0,0,0)	(0.016,0.0032,0.87)	(0.015,0.0030,0.88)	(0,0,0.89)	(0,0,0.87)
0.7	(0,0,0)	(0.013,0.0026,0.59)	(0.013,0.0026,0.87)	(0.013,0.026,0.86)	(0,0,0.85)
0.8	(0,0,0)	(0,0,0.83)	(0,0,0.85)	(0,0,0.84)	(0,0,0.82)
0.9	(0,0,0)	(0,0,0.82)	(0,0,0.83)	(0,0,0.82)	(0,0,0.79)

θ=0.9

ρ	p_0 = 0.1	0.3	0.5	0.7	0.9
0.1	(0,0,0)	(0,0,0.94)	(0,0,0.99)	(0,0,0.99)	(0,0,0.99)
0.2	(0,0,0)	(0.035,0.0071,0.96)	(0.012,0.0024,0.97)	(0,0,0.97)	(0,0,0.97)
0.3	(0,0,0)	0,0,0.84)	(0,0,0.96)	(0.013,0.026,0.96)	(0,0,0.95)
0.4	(0,0,0)	(0,01,0.0022,0.93)	(0,0,0.95)	(0,0,0.94)	(0.03,0.006,0.94)
0.5	(0,0,0)	(0.17,0.0034,0.75)	(0,0,0.93)	(0.013,0.0026,0.93)	(0,0,0.93)
0.6	(0,0,0)	(0,0,0.89)	(0,0,0.92)	(0.01,0.002,0.92)	(0,0,0.91)
0.7	(0,0,0)	(0,0,0.61)	(0.015,0.0031,0.90)	(0,0,0.90)	(0,0,0.89)
0.8	(0,0,0)	(0.02,0.004,0.61)	(0,0,0.89)	(0,0,0.89)	(0,0,0.88)
0.9	(0,0,0)	(0,0,0.83)	(0,0,0.88)	(0.01,0.002,0.87)	(0,0,0.86)

Note: 1. Parameters: r=0.03, λ=0.1, α=0.2, and γ=0.2.
2.The optimal policy is presented as a vector (z,s,ϕ).