

Note on Afriat's Theorem, Ichiro Obara, September 29, 2009

This note is based on Fostel, Scarf, and Todd (2004).

Consider a finite data set $D = \{(p^t, x^t) \in \mathbb{R}_{++}^L \times \mathbb{R}_+^L, t = 1, \dots, T\}$. This note proves the following proposition, which is the step skipped in the class.

Proposition 1 *Suppose that a finite data set D satisfies GARP. Then there exists $\lambda^t > 0, U^t, t = 1, \dots, T$ such that*

$$U^j \leq U^i + \lambda^i p^i \cdot (x^j - x^i) \text{ for all } i, j \in \{1, \dots, T\}$$

Denote by M^T the set of $T \times T$ matrices where all diagonal elements are 0. We say that $A \in M^T$ satisfies GA if the following condition is satisfied.

$$\begin{aligned} & \text{(GA) For any } \{a_{t(n)t(n+1)}\}_{n=1}^N, \\ & \text{if } a_{t(n)t(n+1)} \leq 0 \text{ for } n = 1 \dots N - 1, \\ & \text{then } a_{t(N)t(1)} \geq 0 \end{aligned}$$

where $a_{N,N+1} = a_{N,1}$. Note that the $T \times T$ matrix where ij entry is given by $a_{ij} = p^i \cdot (x^j - x^i)$ satisfies GA if D satisfies GARP.

First we show that GA is equivalent to the following condition.

$$\begin{aligned} & \text{(GA}^* \text{) For any } \{a_{t(n)t(n+1)}\}_{n=1}^N, \\ & \text{if } a_{t(n)t(n+1)} \leq 0 \text{ for } n = 1 \dots N, \\ & \text{then } a_{t(n)t(n+1)} = 0 \text{ for } n = 1 \dots N. \end{aligned}$$

Lemma 2 *GA and GA* are equivalent.*

Proof. Suppose that GA is satisfied. If $a_{t(n)t(n+1)} \leq 0$ for all n , then any $a_{t(n)t(n+1)}$ can be regarded as the tail of a cycle (the one starting at $a_{t(n+1)t(n+2)}$) in GA. So $a_{t(n)t(n+1)} \geq 0$, hence $a_{t(n)t(n+1)} = 0$ for all n .

Conversely, suppose that GA* is satisfied. If $a_{t(n)t(n+1)} \leq 0$ for $n = 1 \dots N - 1$, then $a_{t(N)t(1)} < 0$ cannot be the case because then GA* implies $a_{t(N)t(1)} = 0$, a contradiction. Hence $a_{t(N)t(1)} \geq 0$. ■

Now we prove the above proposition.

Proof of Proposition.

The proof is based on induction. We show that we can find such $\lambda^t > 0, U^t, t = 1, \dots, T$ for every $A \in M^T$ that satisfies GA^* if we can find them for every $A \in M^{T-1}$ that satisfies GA^* . Since this is trivially true for $T = 1$, this proves that the same property holds for every T .

We start with the following lemma.

Lemma 3 *Suppose that $A \in M^T$ satisfies GA^* . Then there exists t^* such that $a_{t^*t} \geq 0$ for $t = 1, \dots, T$.*

Proof. Suppose not. Then we can construct a cycle $\{a_{t(n)t(n+1)}, n = 1, \dots, N\}$ such that $q_{t(n)t(n+1)} < 0$ for all n . This contradicts GA^* . ■

Suppose that $A \in M^T$ satisfies GA and, without loss of generality, assume that $a_{Tt} \geq 0$ for all t . Now define $(T-1) \times (T-1)$ matrix A' as follows.

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } a_{Tj} > 0 \\ \min\{a_{ij}, a_{iT}\} & \text{if } a_{Tj} = 0 \end{cases}$$

(Note that any diagonal element is zero ($a'_{ii} = a_{ii} = 0$), so $A' \in M^{T-1}$. If not, then $a'_{ii} = a_{iT} < 0$. But $a_{iT} < 0$ and $a_{Ti} = 0$ violates GA^*).

Lemma 4 *If $A \in M^T$ satisfies GA^* , then $A' \in M^{T-1}$ satisfies GA^* .*

Proof. Suppose not. Then we can find a cycle $\{a'_{t(n)t(n+1)}, n = 1, \dots, N\}$ such that $a'_{t(n)t(n+1)} \leq 0$ for all n such that at least one inequality is strict. If every a'_{ij} is a_{ij} in this cycle, then this contradicts the assumption that A satisfies GA^* . So suppose that there exists a'_{ij} within this cycle such that $a'_{ij} = a_{iT} \leq 0$ and $a_{Tj} = 0$. Then we can replace a'_{ij} with a_{iT} and a_{Tj} in the original cycle. In this way, we can eliminate such a'_{ij} and guarantee that each element of this cycle is from A . Thus again we reach a contradiction. Hence A' must satisfy GA^* . ■

Now we can complete the proof. By the inductive assumption, there exists $\lambda^t > 0, U^t, t = 1, \dots, T-1$ such that

$$U^j \leq U^i + \lambda^i a'_{ij} \text{ for all } i, j \in \{1, \dots, T-1\}.$$

.By definition of a'_{ij} , we have

$$U^j \leq U^i + \lambda^i a_{ij} \text{ for all } i, j \in \{1, \dots, T-1\}.$$

Define U^T and $\lambda^T > 0$ as follows.

$$\begin{aligned} U^T &= \min_{i \in \{1, \dots, T-1\}} \{U^i + \lambda^i a_{iT}\} \\ \lambda^T &= \max \left\{ 1, \max_{j: a_{Tj} \neq 0} \left\{ \frac{U^j - U^T}{a_{Tj}} \right\} \right\} \end{aligned}$$

We are done if we can show

$$\begin{aligned} U^T &\leq U^i + \lambda^i a_{iT} \text{ for all } i. \\ U^j &\leq U^T + \lambda^T a_{Tj} \text{ for all } j. \end{aligned}$$

The first inequalities are satisfied by definition. As for the second inequality, it follows from the definition for any j when $a_{Tj} > 0$. If $a_{Tj} = 0$, then

$$U^j \leq U^i + \lambda^i a'_{ij} = U^i + \lambda^i a_{iT} \text{ for any } i \in \{1, \dots, T-1\}.$$

by definition of a'_{ij} . Hence we get

$$\begin{aligned} U^j &\leq \min_{i \in \{1, \dots, T-1\}} \{U^i + \lambda^i a_{iT}\} \\ &= U^T \\ &= U^T + \lambda^T a_{Tj}. \end{aligned}$$

This proves that we can find such $\lambda^t > 0, U^t, t = 1, \dots, T$ for every $A \in M^T$ that satisfies GA*. Since GA* is equivalent to GA by the lemma and GA is satisfied when D satisfies GARP, the proposition is proved. ■