## Note on Afriat's Theorem, Ichiro Obara, September 29, 2009

This note is based on Fostel, Scarf, and Todd (2004).

Consider a finite data set  $D = \{(p^t, x^t) \in \mathbb{R}_{++}^L \times \mathbb{R}_{+}^L, t = 1, ..., T\}$ . This note proves the following proposition, which is the step skipped in the class.

**Proposition 1** Suppose that a finite data set D satisfies GARP. Then there exists  $\lambda^t > 0, U^t, t = 1, ..., T$  such that

$$U^{j} \leq U^{i} + \lambda^{i} p^{i} \cdot \left(x^{j} - x^{i}\right) \text{ for all } i, j \in \{1, ..., T\}$$

Denote by  $M^T$  the set of  $T \times T$  matrices where all diagonal elements are 0. We say that  $A \in M^T$  satisfies GA if the following condition is satisfied.

(GA) For any 
$$\{a_{t(n)t(n+1)}\}_{n=1}^{N}$$
,  
if  $a_{t(n)t(n+1)} \leq 0$  for  $n = 1...N - 1$ ,  
then  $a_{t(N)t(1)} \geq 0$ 

where  $a_{N,N+1} = a_{N,1}$ . Note that the  $T \times T$  matrix where ij entry is given by  $a_{ij} = p^i \cdot (x^j - x^i)$  satisfies GA if D satisfies GARP.

First we show that GA is equivalent to the following condition.

(GA<sup>\*</sup>) For any 
$$\{a_{t(n)t(n+1)}\}_{n=1}^{N}$$
,  
if  $a_{t(n)t(n+1)} \leq 0$  for  $n = 1...N$ ,  
then  $a_{t(n)t(n+1)} = 0$  for  $n = 1...N$ .

## **Lemma 2** GA and $GA^*$ are equivalent.

**Proof.** Suppose that GA is satisfied. If  $a_{t(n)t(n+1)} \leq 0$  for all n, then any  $a_{t(n)t(n+1)}$  can be regarded as the tail of a cycle (the one starting at  $a_{t(n+1)t(n+2)}$ ) in GA. So  $a_{t(n)t(n+1)} \geq 0$ , hence  $a_{t(n)t(n+1)} = 0$  for all n.

Conversely, suppose that GA\* is satisfied. If  $a_{t(n)t(n+1)} \leq 0$  for n = 1...N - 1, then  $a_{t(N)t(1)} < 0$  cannot be the case because then GA\* implies  $a_{t(N)t(1)} = 0$ , a contradiction. Hence  $a_{t(N)t(1)} \geq 0$ .

Now we prove the above proposition.

## **Proof of Proposition.**

The proof is based on induction. We show that we can find such  $\lambda^t > 0, U^t, t = 1, ..., T$  for every  $A \in M^T$  that satisfies GA\* if we can find them for every  $A \in M^{T-1}$  that satisfies GA\*. Since this is trivially true for T = 1, this proves that the same property holds for every T.

We start with the following lemma.

**Lemma 3** Suppose that  $A \in M^T$  satisfies  $GA^*$ . Then there exists  $t^*$  such that  $a_{t^*t} \ge 0$  for t = 1, ..., T.

**Proof.** Suppose not. Then we can construct a cycle  $\{a_{t(n)t(n+1)}, n = 1, ..., N\}$  such that  $q_{t(n)t(n+1)} < 0$  for all n. This contradicts  $GA^*$ .

Suppose that  $A \in M^T$  satisfies GA and, without loss of generality, assume that  $a_{Tt} \geq 0$  for all t. Now define  $(T-1) \times (T-1)$  matrix A' as follows.

$$a'_{ij} = \begin{cases} a_{ij} \text{ if } a_{Tj} > 0\\ \min\{a_{ij}, a_{iT}\} \text{ if } a_{Tj} = 0 \end{cases}$$

(Note that any diagonal element is zero  $(a'_{ii} = a_{ii} = 0)$ , so  $A' \in M^{T-1}$ . If not, then  $a'_{ii} = a_{iT} < 0$ . But  $a_{iT} < 0$  and  $a_{Ti} = 0$  violates GA<sup>\*</sup>).

**Lemma 4** If  $A \in M^T$  satisfies  $GA^*$ , then  $A' \in M^{T-1}$  satisfies  $GA^*$ .

**Proof.** Suppose not. Then we can find a cycle  $\left\{a'_{t(n)t(n+1)}, n = 1, ..., N\right\}$  such that  $a'_{t(n)t(n+1)} \leq 0$  for all n such that at least one inequality is strict. If every  $a'_{ij}$  is  $a_{ij}$  in this cycle, then this contradicts the assumption that A satisfies GA<sup>\*</sup>. So suppose that there exists  $a'_{ij}$  within this cycle such that  $a'_{ij} = a_{iT} \leq 0$  and  $a_{Tj} = 0$ . Then we can replace  $a'_{ij}$  with  $a_{iT}$  and  $a_{Tj}$  in the original cycle. In this way, we can eliminate such  $a'_{ij}$  and guarantee that each element of this cycle is from A. Thus again we reach a contradiction. Hence A' must satisfy GA<sup>\*</sup>

Now we can complete the proof. By the inductive assumption, there exists  $\lambda^t > 0, U^t, t = 1, ..., T - 1$  such that

$$U^{j} \leq U^{i} + \lambda^{i} a'_{ij}$$
 for all  $i, j \in \{1, ..., T-1\}$ .

.By definition of  $a'_{ij}$ , we have

$$U^{j} \leq U^{i} + \lambda^{i} a_{ij}$$
 for all  $i, j \in \{1, ..., T-1\}$ .

Define  $U^T$  and  $\lambda^T > 0$  as follows.

$$U^{T} = \min_{i \in \{1, \dots, T-1\}} \left\{ U^{i} + \lambda^{i} a_{iT} \right\}$$
$$\lambda^{T} = \max \left\{ 1, \max_{j:a_{Tj} \neq 0} \left\{ \frac{U^{j} - U^{T}}{a_{TJ}} \right\} \right\}$$

We are done if we can show

$$U^{T} \leq U^{i} + \lambda^{i} a_{iT} \text{ for all } i.$$
  

$$U^{j} \leq U^{T} + \lambda^{T} a_{Tj} \text{ for all } j.$$

The first inequalities are satisfied by definition. As for the second inequality, it follows from the definition for any j when  $a_{Tj} > 0$ . If  $a_{Tj} = 0$ , then

$$U^{j} \leq U^{i} + \lambda^{i} a'_{ij} = U^{i} + \lambda^{i} a_{iT}$$
 for any  $i \in \{1, ..., T-1\}$ 

by definition of  $a'_{ij}$ . Hence we get

$$U^{j} \leq \min_{i \in \{1,...,T-1\}} \left\{ U^{i} + \lambda^{i} a_{iT} \right\}$$
$$= U^{T}$$
$$= U^{T} + \lambda^{T} a_{Tj}.$$

This proves that we can find such  $\lambda^t > 0, U^t, t = 1, ..., T$  for every  $A \in M^T$  that satisfies GA\*. Since GA\* is equivalent to GA by the lemma and GA is satisfied when D satisfies GARP, the proposition is proved.