Applications and Extensions

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1 Risk Sharing without Commitment

1.1 Model

- Consumer: $i = 1, 2.$
- State: $\{z_1, z_2, ..., z_S\}$, follows i.i.d. Prob$(Z = z_s) = \Pi_s$.
- Income: $Y_1(z_s) = y_s$, $0 < y_1 < y_2 < ... < y_S < 1$ and $Y_2(z_s) = 1 - y_s$ (no aggregate uncertatinty). Income is observable, nonstorable, has to be consumed within the period.
- After consumers observe a realization of their income, they transfer the income between them. Denote transfer at period t by TR_t. Public history $h^t \in H$ is a sequence of past realizations of income and transfer, and current income realization, $(h^t = (y_1, TR_1, y_2, TR_2, ..., y_{t-1}, TR_{t-1}, y_t))$.
- Consumption: $(c_{1,t,s}, c_{2,t,s}) \in [0,1] \times [0,1]$ and $c_{1,t,s} + c_{2,t,s} = 1$.
- Consumption plan: $C : H \to [0,1]$. $(C(h_t) = c_{1,t,s}, (c_{2,t,s} = 1 - c_{1,t,s}))$.
- Utility $u_1(c) = u_2(c) = u(c)$, $u' > 0$, $u'' < 0$. $\lim_{c\to 0} u'(c) = \infty$ and $V(C) = E\sum_{t=1}^{\infty} \delta^{t-1} u(c_t)$.

After a realization of income, they redistribute their income every period according to the consumption plan they agreed on, but they cannot commit to their consumption plan. Each consumer can deny redistribution any period and consume his or her own income (Then, no transfer happens in that period). The driving force of this model is a tension between risk sharing and incentive constraint associated with the absence of commitment. Even if there is a benefit from risk sharing in the future by following C, you might want to consume your own income instead of giving some of them to the other consumer if the realization of your income is very high.

Let $V_{1,aut} = \frac{1}{1-\delta}\sum_{s=1}^{S} \Pi_s u(y_s)$ and $V_{2,aut} = \frac{1}{1-\delta}\sum_{s=1}^{S} \Pi_s u(1 - y_s)$ be their total discounted utility when they just consume their endowments. We assume $V_{1,aut} = V_{2,aut} = V_{aut}$ for simplicity.

Subgame perfect allocation is an allocation for which there is no incentive for each consumer to deviate from the consumption plan at any period. For example, consuming one's own income every period is a subgame perfect allocation. Remember that we can focus on the harshest punishment to support any equilibrium without loss of generality. So, an allocation is a subgame perfect allocation if and only if they are punished by receiving (V_{aut}, V_{aut}) after any deviation.

Lemma 1 $\{(c_{1,t,s}, c_{2,t,s})\}_{t,s}$ *is a subgame perfect allocation if and only if for all* s *and* t

$$\begin{aligned} u(c_{1,t,s}) + \delta E_t \sum_{\tau=1}^{\infty} \delta^{\tau-1} u(c_{1,t+\tau}) &\geq u(y_s) + \delta V_{aut} \\ u(c_{2,t,s}) + \delta E_t \sum_{\tau=1}^{\infty} \delta^{\tau-1} u(c_{2,t+\tau}) &\geq u(1-y_s) + \delta V_{aut} \end{aligned}$$

1.2 Perfect Risk Sharing

The optimal risk sharing allocation is an allocation for which two consumer's marginal rate of substitution is equal over every state and time. Clearly, the optimal risk sharing allocations are

$$(c_{1,t,s}, c_{2,t,s}) = (c, 1-c) \text{ for all } s \text{ and } t$$

Such allocation is a subgame perfect allocation if it is an "individually rational" allocation and consumers are patient enough.

Proposition 2 *Suppose that* $\min\left\{\frac{u(c)}{1-\delta}, \frac{u(1-c)}{1-\delta}\right\} > V_{aut}$. *Then there exists* $\underline{\delta} \in (0,1)$ *such that for any* $\delta \in (\underline{\delta}, 1)$, $\{(c_{1,t,s}, c_{2,t,s}) = (c, 1-c)$ *for all* s *and* $t\}$ *is a subgame perfect allocation.*

1.3 Efficient Allocation

Let Γ^* be the set of the subgame perfect allocation. Let V^* be the set of consumer a's total discounted payoff which can be generated by subgame perfect allocations. Efficient subgame perfect allocations are characterized by the following programming problem.

$$\begin{aligned} &P(V) = \max_{c \in \Gamma^*} E \textstyle\sum_{t=1}^{\infty} \delta^{t-1} u(1-c_{1,t}) \\ &s.t.\ E \textstyle\sum_{t=1}^{\infty} \delta^{t-1} u(c_{1,t}) \geq V \end{aligned}$$

Note that

- Γ^* is convex and $V^* = [V_{aut}, V_{max}]$ for some V_{max}.

- The solution to the above problem is unique because of convexity of Γ^* and strict concavity of u.

  It is not difficult to see that $P(V)$ satisfies the following functional equation.

$$P(V) = \max_{c_s \in [0,1], V_s \in [V_{aut}, V_{max}]} \sum_{s \in S} \Pi_s \{u(1-c_s) + \delta P(V_s)\}$$

$$\begin{aligned} s.t. \sum_{s \in S} \Pi_s \{u(c_s) + \delta V_s\} &\geq V & (\eta) \\ u(c_s) + \delta V_s &\geq u(y_s) + \delta V_{aut} & (\lambda_{1,s}\Pi_s) \\ u(1-c_s) + \delta P_s(V_s) &\geq u(1-y_s) + \delta V_{aut} & (\lambda_{2,s}\Pi_s) \end{aligned}$$

Instead of choosing a whole consumption plan, consumer 2 is choosing a current consumption plan and consumer 1's subgame perfect continuation values after all states.

Note also that

- $P(V)$ is decreasing, strictly concave, and continuously differentiable in (V_{aut}, V_{max}) (by Benveniste-Scheinkman Theorem).

Since this is a concave programming problem, K-T condition characterizes the optimal solution.

$$\begin{aligned} (1+\lambda_{2,s}) P'(V_s) + \lambda_{1,s} + \eta &= 0 \text{ if } V_s \in (V_{aut}, V_{max}) \\ &\geq \text{ if } V_s = V_{max} \\ &\leq \text{ if } V_s = V_{aut} \end{aligned} \tag{1}$$

$$(\lambda_{1,s} + \eta) u'(c_s) - (1+\lambda_{2,s}) u'(1-c_s) = 0 \tag{2}$$

Combining these conditions together, we obtain

$$\begin{aligned} \frac{u'(1-c_s)}{u'(c_s)} &= -P'(V_s) \text{ if } V_s \in (V_{aut}, V_{max}) \\ &\geq -P'(V_s) \text{ if } V_s = V_{max} \\ &\leq -P'(V_s) \text{ if } V_s = V_{aut} \end{aligned} \tag{3}$$

This shows that there is a monotone relationship between c_s and V_s (V_s is nondecreasing in c_s).

By envelope theorem, we can also obtain

$$P'(V) = -\eta \tag{4}$$

We now classify this problem into three cases according to whether each constraint is binding or not.

Case 1: $\lambda_{1,s} > 0, \lambda_{2,s} = 0$

(1) and (4) implies that $V < V_s$ or $V = V_s = V_{max}$. c_s and V_s are determined by $u(c_s) + \delta V_s = u(y_s) + \delta V_{aut}$ and (3). Since $V_s > V_{aut}$, it should be the case that $c_s < y_s$.

Case 2: $\lambda_{1,s} = 0, \lambda_{2,s} > 0$

(1) and (4) implies that $V > V_s$ or $V = V_s = V_{aut}$. c_s and V_s are determined by $u(1 - c_s) + \delta V_s = u(1 - y_s) + \delta V_{aut}$ and (3). This time $c_s \geq y_s$ should hold.

Case 3: $\lambda_{1,s} = 0, \lambda_{2,s} = 0$

(1) and (4) implies that $V = V_s$.

Note that only these three cases are relevant as long as there exists a subgame perfect allocation which Pareto-dominates (V_{aut}, V_{aut}). It is impossible that both 1 and 2's constraints are binding for such cases.

When these constraints are binding? Suppose that $\lambda_{1,s'} > 0$ for some s'. If consumer 1's contraint is not bindinng for higher s $(\lambda_{1,s} = 0)$, then V_s has to be smaller than or equal to V by case 2 and 3. However, this contradicts to the monotonic relationship between c_s and V_s. So, $\lambda_{1,s} > 0$ for any $s > s'$. Similarly, if consumer 2's incentive constraint is binding for some s'', consumer 2's incentive constraint is binding for all s which is smaller than s'' (the state where consumer 2's income is high.)

Let's summarize the dynamics of consumption. Given V,

1. if consumers incomes are in some middle range, their consumtion profile is constantly $(c, 1-c)$ which is determined by $\frac{u'(1-c)}{u'(c)} = -P'(V)$, and their future promised utility stays the same (so, the next period looks exactly like the current period).

2. if consumer 1's income is high, then her incentive constraint is binding. Both her current consumption and future promised utility level increase.

3. if consumer 2's income is high, then his incentive constraint is binding. Both his current consumption and future promised utility level increase.

Positive correlation between income and consumption

Current consumption is affected by (1) current income and (2) promised level of future utility. Clearly, there is a positive correlation between current income and current consumption through binding incentive constraints. Past incomes also affect current consumption by changing the promised level of future utility. If income is high and incentive constraint is binding, then the promised utility from the next period increases, which raises the consumption level from the next period on. Hence, there is also a positive correlation between past income and current consumption.

Long-run distribution of consumption

- Suppose that there are some first best allocations which are subgame perfect. Let u_{FB} and u^{FB} be the lowest and the largest first best payoff which are generated by subgame perfect allocations. Then, (1) if $V \in \left(u^{FB}, V_{max}\right)$, consumer 1's constraint never binds and consumer 2's constraint binds with positive probability. Hence, either $V_s = V$ or $V_s \in \left(u^{FB}, V\right)$ with positive probability. It turns out that consumer 1's promised utility monotonically converges to u^{FB} with probability 1, (2) if $V \in (V_{aut}, u_{FB})$, consumer 2's constraint never binds and consumer 1's constraint binds with positive probability. Hence, either $V_s = V$ or $V_s \in (V, u_{FB})$ with positive probability. Then, consumer 1's promised utility converges monotonically to u_{FB} with probability 1 and (3) if $V \in \left[u_{FB}, u^{FB}\right]$, then two consumers can enjoy a first best allocation. To sum up, player 1's promised value converges to the first best level in the long run whatever its initial value is, hence the consumption also converges to the first best level. In this case, consumers' inability to commit cannot explain perpetual imperfect diversification of risk.

- Suppose that no first best allocation is subgame perfect. Then, for any $V \in [V_{aut}, V_{max}]$, either player 1's constraint or player 2's constraint binds with positive probability. Let C^* be the finite set of all the consumption profiles which are determined by $\frac{u'(1-c_s)}{u'(c_s)} = -P'(V_s)$ and consumer 1's constraint $u(c_s) + \delta V_s = u(y_s) + \delta V_{aut}$ or consumer 2's constraint $u(1-c_s) + \delta V_s = u(1-y_s) + \delta V_{aut}$. It is clear that any equilibrium consumption process is a Markov process and is absorbed in C^* in a finite time with probability 1. The limiting distribution of consumption is the same independent of player 1's initial promised value. In this case, we can generate perpetual imperfect diversification of risk.

2 Dynamic Games with State Variables