Bayesian Nash Equilibrium

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February 1, 2012

Bayesian Game

- We like to model situations where each party holds some private information. For example,
  - A bidder does not know other bidders' values in auction.
  - A trader has some insider information about a recent technological innovation by some firm.
  - ...

Bayesian Game

Bayesian Game

A **Bayesian Game** consists of

- **player:** a finite set N
- **state:** a set Ω
- **action:** a set A_i for each $i \in N$
- **type:** a set T_i for each $i \in N$
- **belief:** a function $p_i : T_i \to \Delta\left(\Omega \times \prod_{j \neq i} T_j\right)$ for each $i \in N$
- **payoff:** a function $u_i : A \times \Omega \to \Re$ for each $i \in N$.

Bayesian Game

- Interpretation:
  - Ω is a set of possible **states of nature** that determine all physical setup of the game (payoffs).
  - T_i is the set of i's private **types** that encode player i's information/knowledge (ex. private signal).
  - p_i is player i's **interim belief** about the state and the other players' types.
- We are already familiar with the basic idea of BG: correlated equilibrium and variety of interpretations of mixed strategy (Harsanyi's purification argument etc.) are some special cases. BG introduces incomplete information into games in a very flexible way.

Bayesian Game

- Bayesian games are often described more simply by eliminating Ω as follows.
  - Interim belief on T_{-i}: $p_i(t_{-i}|t_i) := \sum_{\omega\in\Omega} p_i(\omega, t_{-i}|t_i)$
  - Payoff on $A \times T$: $u_i(a,t) := \sum_\omega u_i(a,\omega)p_i(\omega|t)$ where $p_i(\omega|t) = \frac{p_i(\omega,t_{-i}|t_i)}{\sum_{\omega\in\Omega} p_i(\omega,t_{-i}|t_i)}$.
- In this formulation, type encodes both payoff and belief.

Example: Auction

- Consider the standard model of first price auction where each bidder only knows her value and have a belief about the other bidders' values. This is a simple Bayesian game where
  - the set of players (bidders) is N
  - the set of states is $V_1 \times ..., \times V_n$
  - the set of actions for bidder i is $A_i = \Re_+$
  - the set of types for bidder i is V_i
  - bidder i's interim belief is $p_i(v_{-i}|v_i)$.
  - bidder i's payoff is $u_i(b, v) = \mathbf{1}(b_i \geq \max_{j \neq i} b_j)(v_i - b_i)$.

Common Prior Assumption (CPA)

- We almost always assume that all the interim beliefs are derived from the same prior. That is, we assume **common prior**.

Common Prior Assumption

A Bayesian Game $(N, \Omega, (A_i), (T_i), (p_i), (u_i))$ satisfies the **common prior assumption** if there exists $p \in \Delta(\Omega \times \prod_{i \in N} T_i)$ such that $p_i(\omega, t_{-i}|t_i), t_i \in T_i, i \in N$ are all conditional distributions derived from p.

- This assumption is not entirely convincing, but it is a useful one.

Bayesian Nash Equilibrium

- Player i's strategy s_i is a mapping from T_i to A_i. Let S_i be the set of player i's strategies. It is like a contingent plan of actions.
- Given $s = (s_1, ..., s_n)$, player i's interim expected payoff for type t_i is

$$E\left[u_i\left(\left(s_i(t_i), s_{-i}(\widetilde{t}_{-i})\right), \widetilde{\omega}\right)|t_i\right]$$
$$:= \sum_{\omega\in\Omega}\sum_{t_{-i}\in T_{-i}} u_i\left(\left(s_i(t_i), s_{-i}(t_{-i})\right), \omega\right) p_i(\omega, t_{-i}|t_i)$$

- A strategy profile $s = (s_1, ..., s_n)$ is a **Bayesian Nash Equilibrium** if for every $i \in N$, s_i assigns an optimal action for each t_i that maximizes player i's interim expected payoff.

Bayesian Nash Equilibrium

Here is a formal definition.

Bayesian Nash Equilibrium

$s^* = (s_1^*, ..., s_n^*) \in S$ is a **Bayesian Nash Equilibrium** if

$$E\left[u_i(\left(s_i^*(t_i), s_{-i}^*(\widetilde{t}_{-i})\right), \widetilde{\omega})|t_i\right]$$
$$\geq E\left[u_i(\left(a_i, s_{-i}^*(\widetilde{t}_{-i})\right), \widetilde{\omega})|t_i\right]$$

holds for every $a_i \in A_i$ and $t_i \in T_i$, for every $i \in N$.

Comments.

- A Bayesian Nash equilibrium can be regarded as a Nash Equilibrium of some appropriately defined strategic game.
  - One interpretation is to regard each type as a distinct player and regard the game as a strategic game among such $\sum_i |T_i|$ players (cf. definition in O&R). Then a BNE can be regarded as a NE of this strategic game.
  - Suppose that there exists common prior p. Let $U_i(s) = E\left[u_i((s_i(\widetilde{t}_i), s_{-i}(\widetilde{t}_{-i})), \widetilde{\omega})\right]$ be player i's ex ante expected payoff given $s \in S$. Then a BNE s^* is a NE of strategic game $(N, (S_i), (U_i))$.

Comments.

- Suppose that there are finite actions and finite types for each player. In this case, the whole game can be regarded as a finite strategic game (in either interpretation). In this setting, we can allow each type to randomize over actions as we did in mixed strategy NE. Then we can define a mixed strategy BNE and it follows immediately from the existence of MSNE that there exists a MSBNE.

Example 1: Second Price Auction

- Suppose that $v = (v_1, ..., v_n)$ is generated by common prior $p \in \Delta([0,1]^n)$ in second price auction.
- Since it is a dominant action to bid one's true value, bidders' beliefs are not relevant for their decision. Hence $b^* = (b_1^*, ..., b_n^*)$, where $b_i^*(v_i) = v_i$, is a BNE.

Example 2: First Price Auction

- Assume that v_i is i.i.d. across bidders and follow the uniform distribution on $[0, 1]$.
- We look for a symmetric BNE $(b^*, ..., b^*)$ in first price auction.
- We use "guess and verify method": we assume $b(v) = \theta v$ for some θ, then verify that this strategy is in fact optimal against itself for some θ.

Example 2: First Price Auction

- Bidder i's expected payoff with value $v_i \in [0,1]$ and bid $b_i \in [0,1]$ is

$$\begin{aligned}\Pr(win|b_i)(v_i - b_i) &= \Pr\left(\max_{j\neq i} v_j \leq \frac{b_i}{\theta}\right)(v_i - b_i) \\ &= \left(\frac{b_i}{\theta}\right)^{n-1}(v_i - b_i)\end{aligned}$$

- The first order condition is:

$$\frac{n-1}{\theta}\left(\frac{b_i}{\theta}\right)^{n-2}(v_i - b_i) - \left(\frac{b_i}{\theta}\right)^{n-1} = 0$$

- $\frac{n-1}{n}v_i$ maximizes the payoff given v_i (independent of θ).
- Hence $b^*(v) = \frac{n-1}{n}v$ is the optimal bid for each bidder when all the other bidders are using it. That is, it is a symmetric BNE for every bidder to follow $b^*(v) = \frac{n-1}{n}v$.

Example 2: First Price Auction

- Next we consider general cumulative distribution function F (still i.i.d. is assumed).
- Assume that the symmetric BNE b^* is strictly increasing and differentiable.
- The expected payoff for type v bidder is

$$\Pr(win|b)(v-b) = F(b^{*-1}(b))^{n-1}(v-b)$$

- The type v bidder's optimal bid $\widehat{b}(v)$ is obtained from the following first order condition:

$$\frac{(n-1)f(b^{*-1}(\widehat{b}(v)))F(b^{*-1}(\widehat{b}(v)))^{n-2}}{b^{*\prime}(b^{*-1}(\widehat{b}(v))}(v-\widehat{b}(v)) \\ -F(b^{*-1}(\widehat{b}(v)))^{n-1} = 0$$

Example 2: First Price Auction

- Since $\widehat{b}(v) = b^*(v)$ in equilibrium, this differential equation simplifies to

$$\frac{\left(F(v)^{n-1}\right)'}{b'(v)}(v - b(v)) - F(v)^{n-1} = 0.$$

- Solving this, we obtain
  - $$b^*(v) = \frac{\int_0^v \left(F(x)^{n-1}\right)' x dx}{F(v)^{n-1}}.$$

    or, by integration by parts,

    $$b^*(v) = v - \frac{\int_0^v F(x)^{n-1} dx}{F(v)^{n-1}}$$
  - We can verify that (1) this is in fact strictly increasing and differentiable and (2) second order condition is satisfied.

Example 2: First Price Auction

Bayesian Nash equilibrium for the first price auction

It is a Bayesian Nash equilibrium for every bidder to follow the strategy $b(v) = v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}}$ for the first price auction with i.i.d. private value.

Example 3: Cournot Competition with Private Cost

- Consider a Cournot model where each firm's cost is private information and drawn from $[0, 1]$ according to the same CDF F independently. Let $\overline{c}$ be the average cost.
- Assume that the inverse demand function is $p(Q) = 3 - Q$.
- Let's try to find a symmetric BNE $(q^*, ..., q^*)$.

Example 3: Cournot Competition with Private Cost

- Firm i's expected profit when its cost is c_i and q_i is produced is

$$\pi_i(q_i, q^*) = E\left[(3 - q_i - (n-1)q^*(\widetilde{c}) - c_i)\, q_i\right]$$

- From FOC, we obtain

$$q_i(c_i) = \frac{3 - (n-1)E\left[q^*(\widetilde{c})\right] - c_i}{2}$$

- Taking the expectation and imposing symmetry, we have $E\left[q^*(\widetilde{c})\right] = \frac{3-\overline{c}}{n+1}$.
- Hence $(q^*, ..., q^*)$, where $q^*(c) = \frac{3-\overline{c}}{n+1} - \frac{c-\overline{c}}{2}$, is the symmetric BNE.

Example 4: Double Auction (Chatterjee and Samuelson 1983)

- One buyer and one seller.
- Buyer's value and and seller's value is independently and uniformly distributed on $[0, 1]$.
- Buyer's payoff is $v_b - p$ when trading at price p, 0 otherwise. Seller's payoff is $p - v_s$ when trading at price p, 0 otherwise.
- Buyer's strategy is $p_b : [0, 1] \to [0, 1]$ (price offer) and seller's strategy $p_s : [0, 1] \to [0, 1]$ (asking price).
- Trade occurs at price $\frac{p_b + p_s}{2}$ only if $p_b \geq p_s$. Otherwise no trade.

Example 4: Double Auction

- Look for a linear BNE $p_b^*(v_b) = a_b + c_b v_b, p_s^*(v_s) = a_s + c_s v_s$.
- Given this linear strategy by the seller, the buyer's problem is

$$\max_{p_b} \left[v_b - \frac{1}{2} \left\{ p_b + \frac{a_s + p_b}{2} \right\} \right] \frac{p_b - a_s}{c_s}$$

- From the first order condition, we obtain

$$p_b(v_b) = \frac{2}{3} v_b + \frac{1}{3} a_s$$

Example 4: Double Auction

- Seller's problem is

$$\max_{p_s} \left[\frac{1}{2} \left\{ p_s + \frac{p_s + a_b + c_b}{2} \right\} - v_s \right] \frac{a_b + c_b - p_s}{c_b}$$

- From the first order condition, we obtain

$$p_s(v_s) = \frac{2}{3} v_s + \frac{1}{3} (a_b + c_b)$$

Example 4: Double Auction

- Matching coefficients, we get $a_b = 1/12, a_s = 1/4, c_b = c_s = 2/3$.
- Hence we find one BNE:

$$
\begin{aligned}
p_b^*(v_b) &= \frac{2}{3} v_b + \frac{1}{12} \\
p_s^*(v_s) &= \frac{2}{3} v_s + \frac{1}{4}
\end{aligned}
$$

Example 4: Double Auction

Remark

- An allocation is efficient if trade occurs whenever $v_b \geq v_s$. So this BNE does not generate an efficient allocation.
- In this uniform distribution environment, however, this BNE is the most efficient one. Thus it is impossible to achieve the efficient allocation by any BNE in double auction.
- This inefficiency result is very general. The efficient allocation cannot be achieved by ANY Bayesian Nash Equilibrium in ANY mechanism.

Example 5: Global Game (Carlson and van Damme 1993)

- There are two investors $i = 1, 2$. Each investor chooses whether to purchase risky asset (RA) or safe asset (SA).
- Each investor's payoff depends on the state of economy $\theta \in \Re$ and the other investor's decision.
- **Information Structure:**
  - θ is "uniformly distributed" on $\Re$.
  - Investor i observes a private signal $t_i = \theta + \epsilon_i$, where ϵ_i follow $N(0, \sigma)$.
  - Given t_i, investor i believes that θ_i is distributed according to $N(t_i, \sigma)$ and t_{-i} is distributed according to $N(t_i, \sqrt{2}\sigma)$.

Example 5: Global Game

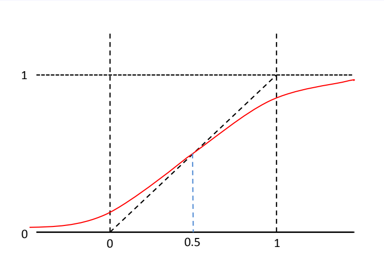
- The payoff matrix given each realization of θ is as follows.

	RA	SA
RA	θ,θ	$\theta-1,0$
SA	$0, \theta-1$	$0,0$

- Note that, if θ is publicly known,
  - RA is a dominant action when $\theta > 1$.
  - SA is a dominant action when $\theta < 0$
  - Both (RA, RA) and (SA, SA) is a NE when $0 \leq \theta \leq 1$.

Example 5: Global Game

- Let $b(x)$ be a solution for $t - \Phi\left(\frac{x-t}{\sqrt{2}\sigma}\right) = 0$ given x. The interpretation of $b(x)$ is that $b(x)$ should be an investor's optimal cutoff point (i.e. choose SA if t is below $b(x)$ and RA if t is above $b(x)$) when the other investor's cutoff point is x.
- $b(x)$ looks as follows. It is strictly increasing and crosses the 45 degree line at $x = 0.5$.



Example 5: Global Game

- We can show that this is the unique BNE by applying iterated elimination of strictly dominated actions for each type as follows.
  - For $t_i < 0, i = 1, 2$, RA is strictly dominated. So delete RA for every t_i strictly below 0.
  - Then we can delete RA for every $t_i < b(0)(> 0)$ for $i = 1, 2$ (why?).
  - We can delete RA for every t_i below $b^2(0) < b^3(0) < ......$
  - So we can delete RA for every $t_i < 0.5$ in the limit.
  - Similarly we can delete SA for every t_i above $b^2(0) > b^3(0) > .....,$ hence for every $t_i > 0.5$ in the limit.