# Preference and Utility

Ichiro Obara

UCLA

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# Preference Relation

# Preference

- We study a classical approach to consumer behavior: we assume that consumers choose the bundle of commodities/goods that they like most given their budget.
- We need to make this "like most" more precise.

### Preference Relation

**Preference relation** on $X$ is a subset of $X \times X$. When $(x, y)$ is an element of this set, we say $x$ is preferred to $y$ and denote $x \succeq y$.

- We usually use $\succeq$ to denote a preference relation.
- $X$ can be any set. For consumer problems, $X$ is typically $\Re^L_+$.

# Preference

Some basic properties of preference relations:

- $\succeq$ on $X$ is **complete** if either $x \succeq y$ or $y \succeq x$ for any $x, y \in X$
- $\succeq$ on $X$ is **transitive** if $x \succeq y$ and $y \succeq z$ imply $x \succeq z$ for any $x, y, z \in X$.

Are they reasonable?

# Preference

**Some critique of transitivity**

1. How much sugar do you need for a cup of coffee? You are indifferent between no sugar and one grain of sugar, one grain of sugar and two...But are you indifferent between no sugar and 10 spoons of sugar?
2. Framing problem.

# Preference

We almost always assume these properties. So let's give them some name.

## Rational Preference

$\succeq$ on $X$ is **rational** if it is complete and transitive.

From now on, we only consider rational preferences most of the time.

## Remark

- We can derive two other preference relations from a preference relation.

### Strict Preference

**Strict preference relation** $\succ$ is defined by $x \succ y \Leftrightarrow \{x \succeq y \text{ and } y \not\succeq x\}$

### Indifference

**Indifference** $\sim$ is defined by $x \sim y \Leftrightarrow \{x \succeq y \text{ and } y \succeq x\}$.

- From a rational preference, we can derive a strict preference that satisfies **asymmetry** and **negative transitivity**. On the other hand, we can derive a rational preference from a strict preference that satisfies these properties.

# Preference

There are many other properties we assume from time to time. Let $X$ be a subset of $\Re^L_+$.

- $\succeq$ on $X$ is **locally nonsatiated** if for every $x \in X$ and $\epsilon > 0$, there exists $y \in X$ such that $\|y - x\| < \epsilon$ and $y \succ x$.
- $\succeq$ on $X$ is **monotone** (resp. **strongly monotone**) if $x \gg y$ (resp. $x > y$) implies $x \succ y$ for any $x, y \in X$.
- $\succeq$ on $X$ is **continuous** if both the **upper contour set** $\mathcal{U}(x) = \{y \in X : y \succeq x\}$ and the **lower contour set** $\mathcal{L}(x) = \{y \in X : x \succeq y\}$ are (relatively) closed for any $x \in X$ (equivalently, if $x_n \to x \in X, y_n \to y \in X$ and $x^n \succeq y^n$, then $x \succeq y$).

# Preference

- $\succeq$ on $X$ is **convex** if $\mathcal{U}(x)$ is convex for any $x \in X$.
- $\succeq$ on $X$ is **strictly convex** if $y \succeq x$ and $z \succeq x$ and $y \neq z$ imply $\alpha y + (1-\alpha)z \succ x$ for any $\alpha \in (0,1)$.
- $\succeq$ on $X = \Re_{+}^{L}$ is **homothetic** if $x \sim y \to \alpha x \sim \alpha y$ for any $\alpha \geq 0$.

# Utility Representation

# Utility Representation

It is usually more convenient to work with **utility functions** rather than preferences.

**Definition: Representation of Preference**

$\succeq$ is represented by a **utility function** $u : X \to \Re$ if $x \succeq y \Leftrightarrow u(x) \geq u(y)$ for all $x, y \in X$.

# Utility Representation

Once a preference is represented by a utility function, then we can formulate the consumer problem as a constrained optimization problem:

$$\max_{x \in X} u(x) \; s.t. \; p \cdot x \leq w,$$

or equivalently,

$$\max_{x \in B(p,w)} u(x)$$

, which may be easily solved analytically or numerically.

# Utility Representation

**Examples of Utility Functions**

- Cobb-Douglas utility function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ for $\alpha \in (0, 1)$.
- Quasi-linear utility function: $u(x, m) = v(x) + m$.
- Leontief utility function: $u(x_1, x_2) = \min\{x_1, x_2\}$.

# Utility Representation

**When can a rational preference be represented by a utility function?**

- Consider the easiest case: $X$ is a finite set. Clearly every rational preference on $X$ can be represented by some utility function (Try to prove this *formally*).

# Utility Representation

**When can a rational preference be represented by a utility function?**

- What if $X$ is a countable set? For example, this is the case if no good is divisible ($X = Z_+^L$). We can still obtain a representation as follows.
  - Let $X_n = \{x_1, ..., x_n\}$ for $n = 1, 2, ...$.
  - For each $n$, we can find $u_n$ to satisfy $x \succeq y \Leftrightarrow u_n(x) \geq u_n(y)$ for any $x, y \in X_n$. In fact, we can keep the same $u_n$ in each step (i.e. $u_n(x) = u_{n+1}(x) = ...$ for any $x \in X_n$).
  - For each $x \in X$, define $u(x)$ by $u(x) := u_n(x)$ by taking any large $n$. It can be easily verified that (1) $u$ is well-defined and (2) $u$ represents $\succeq$.

# Utility Representation

You can find a continuous utility function when a preference is (rational and) continuous.

**Theorem (Debreu)**

Let $X \subset \Re_{+}^{L}$ be closed and convex. A rational preference $\succeq$ on $X$ is continuous if and only if there exists a continuous utility function $u : X \rightarrow \Re$ that represents $\succeq$.

**Note**: Closedness and convexity of $X$ can be dropped.

## Sketch of Proof

- "if" is trivial. We prove "only if" in the following.
- Let $B_n = \left\{x \in \Re^n_+ | \, \|x\| \leq n\right\}$. Since $B_n$ is compact, there exists the least preferred element $\underline{x}_n$ in it.
- Define a utility function on $\mathcal{U}(\underline{x}_n)$ as follows.
  - Let $u_n(x) = \min_{y \in \mathcal{U}(x)} \|y - \underline{x}_n\|$ on $\mathcal{U}(\underline{x}_n)$.
  - Then $u_n$ represents $\succeq$ on $\mathcal{U}(\underline{x}_n)$ (use convexity).
- We can adjust $u_1$,$u_2$,...in such a way that $u_m$ coincides with $u_n$ on $\mathcal{U}(\underline{x}_n)$ for all $m \geq n$. Define $u$ on $X$ by $u(x) := \lim u_n(x)$. Then $u$ represents $\succeq$ on $X$.
- We skip continuity (this follows from "Gap Theorem").

**Remark.**

- What if continuity is dropped? Can a plain rational preference $\succeq$ be always represented by some $u$? NO.
- The following rational preference is not continuous and cannot be represented by any utility function.
  - **Lexicographic Preference on $\Re_+^2$**
    For any $x, y \in \Re_+^2$, $x \succeq y$ if and only if either (1) $x_1 > y_1$ or (2) $x_1 = y_1$ and $x_2 \geq y_2$.
  - **Proof.** Define a function $f$ from $\Re_+$ to $\mathbf{Q}$ (rational number) by associating each $x$ with $f(x) \in \mathbf{Q}$ such that $u(x, 1) < f(x) < u(x, 2)$. Then a different rational number is assigned to different $x$, a contradiction.

## Properties of Preferences in terms of Utilities

- $\succeq$ on $X$ is **locally nonsatiated** $\Leftrightarrow$ for every $x \in X$ and $\epsilon > 0$, there exists $y \in X$ such that $\|y - x\| < \epsilon$ and $u(y) > u(x)$.
- $\succeq$ on $X$ is **monotone** (resp. **strongly monotone**) $\Leftrightarrow$ $x \gg y$ (resp. $x > y$) implies $u(x) > u(y)$ for any $x, y \in X$.
- $\succeq$ on $X$ is **convex** $\Leftrightarrow$ $u$ is quasi-concave, i.e. $u(y) \geq u(x)$ and $u(z) \geq u(x)$ imply $u\left(\alpha y + (1-\alpha)z\right) \geq u(x)$ for any $\alpha \in [0,1]$.
- $\succeq$ on $X$ is **strictly convex** $\Leftrightarrow$ $u$ is strictly quasi-concave, i.e. $u(y) \geq u(x)$ and $u(z) \geq u(x)$ with $y \neq z$ imply $u\left(\alpha y + (1-\alpha)z\right) > u(x)$ for any $\alpha \in (0,1)$.
- $\succeq$ is **homothetic** $\Leftrightarrow$ $u(\alpha x) = u(\alpha y)$ for any $\alpha \geq 0$ and $x, y \in X$ such that $u(x) = u(y)$.

# Ordinal Property and Cardinal Property

- Let $f : \Re \to \Re$ be any *strictly increasing* function. Then $u(x)$ and $f(u(x))$ represents the same preference because $u(x) \geq u(y) \Leftrightarrow f(u(x)) \geq f(u(y))$.
  - Ex. $u(x) = x^{0.5}$ and $\log u(x) = 0.5 \log x$.
- The properties (of utilities) that are preserved under any such monotone transformation are **ordinal properties**. The properties that are not are **cardinal properties**.
- Example.
  - Monotonicity and quasi-concavity are ordinal properties.
  - Concavity and decreasing marginal utility are cardinal properties.

# Ordinal Property and Cardinal Property

**Comment.**

- If we regard a utility function as merely one convenient representation of the underlying preference, then we should be careful to make sure that results do not depend on a particular representation and particular cardinal properties.
- On the other hand, if we know that representation does not affect a result (ex. optimization), then we had better use a more convenient representation with nicer cardinal properties (Example. Some quasi-concave utility function can be transformed into a concave utility function. This doesn't change the preference, whereas concave functions are easier to use than quasi-concave functions).