The Maximum Efficient Equilibrium Payoff in the Repeated Prisoners' Dilemma[*]

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The Maximum Efficient Equilibrium Payoff in the Repeated Prisoners' Dilemma

by
George J. Mailath, Ichiro Obara, and Tadashi Sekiguchi

Abstract

We describe the maximum efficient subgame perfect equilibrium payoff for a player in the repeated Prisoners' Dilemma, as a function of the discount factor. For discount factors above a critical level, every efficient, feasible, individually rational payoff profile can be sustained. For an open and dense subset of discount factors below the critical value, the maximum efficient payoff is not an equilibrium payoff. When a player cannot achieve this payoff, the unique equilibrium outcome achieving the best efficient equilibrium payoff for a player is eventually cyclic. There is an uncountable number of discount factors below the critical level such that the maximum efficient payoff is an equilibrium payoff. *Journal of Economic Literature* classification numbers C72, C73. Keywords: Repeated prisoners' dilemma, repeated games, equilibrium payoffs, perfect monitoring.

1. Introduction

While the discounted repeated Prisoners' dilemma is one of the most intensively studied games, little is known about the set of subgame perfect equilibrium payoffs for a wide range of discount factors. It is known that for low values of the discount factor, only the minmax payoff vector is an equilibrium payoff vector, while the folk theorem asserts that for large values of the discount factor, every feasible and individually rational payoff vector is an equilibrium payoff vector. In this paper, we describe the maximum efficient subgame perfect equilibrium payoff for a player for the intermediate values of the discount factor.

Sorin (1986) gives a complete characterization of the set of equilibrium payoffs of the repeated Prisoners' Dilemma for low discount factors—in particular, he shows that for discount factors strictly below a critical value, the only Nash equilibrium payoff of the repeated game is the myopic Nash equilibrium payoff. Sorin (1986) also calculates the set of equilibrium payoffs at the critical value of the discount factor.[1] No structure on equilibrium payoffs is given for larger values of the discount factor, the region of discount factors that is our focus. van Damme (1991, Section 8.4), building on Sorin (1986), shows that at the critical value of the discount factor, every feasible and individually rational payoff vector is an equilibrium payoff vector *when* players' mixed strategies are observable. We make the more standard assumption that players' mixed strategies are not observable.[2]

An important assumption in our paper is that public correlation devices are *not* available. Such correlation devices considerably simplify the analysis. For example, it is known that the equilibrium payoff set with public correlation is monotonic with respect to the discount factor (see, for example, Abreu *et al.* (1990, Theorem 6) and footnote 10). However, as we will see, the efficient equilibrium payoff set is nonmonotonic in the absence of such correlation devices.[3] The complete characterization of the equilibrium payoff correspondence for the repeated Prisoners' Dilemma when such a correlation device is available is given in Stahl (1991). Building on an insight of Abreu (1986), Cronshaw and Luenberger (1994) describe the set of *symmetric* equilibrium payoffs for general repeated symmetric games using a scalar equation to solve for the maximal level of deterrence, and then describing the best and worst equilibrium payoffs as a function of this deterrence level. Their analysis requires either a public correlation device, or a convex set of pure actions (such as in a Cournot quantity setting game).

Our findings are as follows. First, and not surprisingly, there is a critical value of the discount factor such that for discount factors above this value, any efficient payoff vector that is individually rational is an equilibrium payoff. We can immediately conclude that for patient players, the best equilibrium for one player is the one that minimizes the other player's payoff, subject to the individual rationality constraint. Denote this value for player i by v_i^*.

For other discount factors, not all individually rational and efficient payoffs are equilibrium payoffs. For an open and dense subset of discount factors below the critical value,[4] v_i^* is not an equilibrium payoff for player i. On the other hand, the set of discount factors below the critical value for which the maximum equilibrium payoff for player i is v_i^* is uncountable. Thus, the maximum equilibrium payoff does not exhibit monotonicity with respect to the discount factor.

If the discount factor is such that v_i^* is not an equilibrium payoff for player i, the (unique) best efficient equilibrium outcome for player i is eventually cyclic: after some finite history, play follows a cycle. We also show that the best efficient equilibrium payoff is sometimes different from the maximum of *all* equilibrium payoffs for a player (the remark at the end of Section 3). On the other hand, when v_i^* is an equilibrium payoff for player i, various types of outcomes are consistent with being the best equilibrium, among which are acyclic outcomes.

2. Preliminary analysis

We study the Prisoners' Dilemma $g : \{C, D\}^2 \to \Re^2$, where g is described in the following bimatrix:

		Player 2: C	Player 2: D
Player 1	C	$1, 1$	$-1, 2$
	D	$2, -1$	$0, 0$

While we have chosen to work with a particular version of the Prisoners' Dilemma for clarity, our results hold for any Prisoners' Dilemma.[5] The set of individually rational and feasible payoffs is denoted V^*. Our interest lies in equilibrium payoffs on the Pareto boundary of this set. Without loss of generality, we restrict attention to the boundary $B = \left\{(v_1, v_2) : v_1 = \frac{3}{2} - \frac{v_2}{2},\ v_2 \in [0, 1]\right\}$ (see Figure 1).

We base our analysis on self-generation (Abreu *et al.* (1990)). A pair (α, w), where α is a (possibly mixed) action profile and $w : A \to V^*$ is a specification of continuation payoffs, is *admissible* if α is a Nash equilibrium of the game with payoffs $(1 - \delta) g(a) + \delta w(a)$. A payoff vector v is *decomposable* with respect to an action profile α and continuation values w if the pair (α, w) is admissible and has value v.

Equilibrium payoffs on B require player 2 to always play C, since they can only be achieved as convex combinations of the action profiles CC and DC. The issue

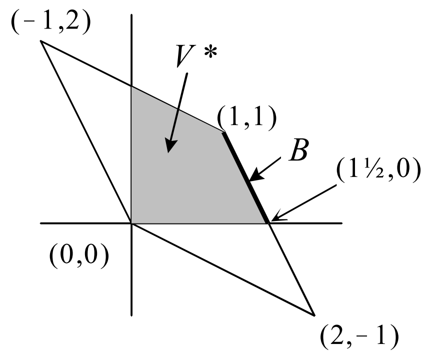

Figure 1: The sets V^* and B.

is to determine how often player 1 can play D in equilibrium. While increasing the proportion of periods that player 1 plays D increases player 1's payoff, it simultaneously decreases player 2's payoff. We need to ensure that player 2's payoff still provides the incentive for player 2 to always play C. The profile we describe specifies the most severe punishment possible: any deviation by either player results in the grim outcome of DD forever. Note that player 1's incentive constraint is not an issue. The incentive constraint for player 2 in period $t \geq 0$ is:[6]

$$\begin{aligned} (1-\delta)\, g_2\left(a_1^t, C\right) + \delta v_2^{t+1} &\geq (1-\delta)\, g_2\left(a_1^t, D\right) + \delta \times 0 \\ \Longleftrightarrow \quad & \delta v_2^{t+1} \geq 1-\delta \Longleftrightarrow v_2^{t+1} \geq \frac{1-\delta}{\delta}. \end{aligned}$$

Thus, as long as the continuation value (in periods $t \geq 1$) to player 2 exceeds $(1-\delta)/\delta$, player 2 will play C.

Denote by V_2^C the set of payoffs for player 2 that can be decomposed using CC and a payoff $w_2 \in [(1-\delta)/\delta,\ 1]$ (any continuation payoff smaller than $(1-\delta)/\delta$ is inconsistent with 2 playing C):

$$\begin{aligned} v_2 \in V_2^C \Longleftrightarrow \quad \exists w_2 \in \quad & [(1-\delta)/\delta,\ 1] \\ & \text{s.t. } v_2 = (1-\delta)\, g_2(CC) + \delta w_2 = (1-\delta) + \delta w_2. \end{aligned} \tag{1}$$

Thus, $V_2^C = [2-2\delta, 1]$.

Denote by V_2^D the set of payoffs for player 2 that can be decomposed using

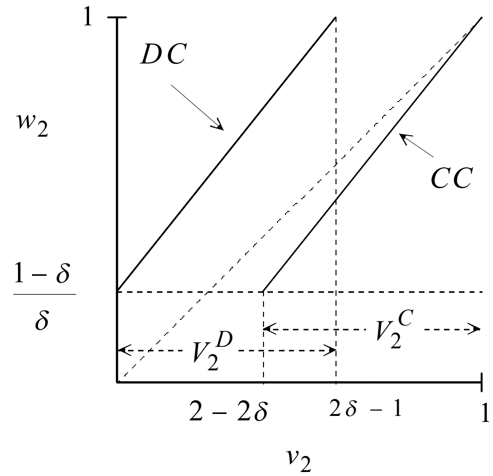

Figure 2: Self-generating sets for $\delta \geq \frac{3}{4}$. This is drawn for $\delta = \frac{4}{5}$. Any v_2 can be decomposed into a current action profile and continuation value w_2.

DC and a payoff $w_2 \in [(1-\delta)/\delta,\ 1]$:

$$v_2 \in V_2^D \iff \exists w_2 \in [(1-\delta)/\delta,\ 1] \text{ s.t. } v_2 = (1-\delta)\, g_2(DC) + \delta w_2 = \delta w_2 - (1-\delta). \tag{2}$$

Thus, $V_2^D = [0, 2\delta - 1]$.

Note that for $\delta < \frac{1}{2}$, V_2^C and V_2^D are both empty and so any action profile in which player 2 plays C is not admissible. In fact, it is easy to show that for $\delta < \frac{1}{2}$, the only equilibrium payoff is $(0,0)$. Moreover, for $\delta \geq 1/2$, grim trigger is an equilibrium. Thus, the best symmetric equilibrium payoff is $(0,0)$ for $\delta < 1/2$ and $(1,1)$ for $\delta \geq 1/2$.

If $V_2^C \cup V_2^D = [0,1]$ (which is implied by $\delta \geq \frac{3}{4}$), on the other hand, every payoff on the segment $\left\{(v_1, v_2) : v_1 = \frac{3}{2} - \frac{v_2}{2},\ v_2 \in [0,1]\right\}$ can be supported as an equilibrium payoff in the first period. This is illustrated in Figure 2. These last two observations imply our first result. For $\delta \geq 1/2$, let $\bar{v}_1(\delta)$ be the maximum of player 1's payoff in any equilibrium with payoffs on B, given discount factor δ.

Lemma 1. *For $\delta < \frac{1}{2}$, $(0,0)$ is the only subgame perfect equilibrium payoff. For $\delta \geq \frac{3}{4}$, $\bar{v}_1(\delta) = \frac{3}{2}$.*

The analysis of the case $\delta \in \left(\frac{1}{2}, \frac{3}{4}\right)$ is more complicated, and is the concern of this paper. Suppose $\delta < \frac{3}{4}$. Since $V_2^C \cap V_2^D = \emptyset$, there is a unique action and continuation payoff vector corresponding to any decomposable payoff vector on the efficient frontier. The following *value dynamic* (from (2) and (1)) is thus a useful tool to describe continuation payoffs:

$$v_2^{t+1} = \begin{cases} \frac{v_2^t}{\delta} + \frac{1-\delta}{\delta}, & \text{if } v_2^t \leq 2\delta - 1, \\ \frac{v_2^t}{\delta} - \frac{1-\delta}{\delta}, & \text{otherwise} \end{cases} \tag{3}$$

For $\delta \geq \frac{1}{2}$, the value dynamic (3) describes a function from $[-1, 1]$ into $[-1, 1]$, and so the dynamic is well-defined for any choice of $v_2^0 \in [-1, 1]$. We say a sequence $\{v_2^t\}_{t=0}^{\infty}$ is *generated by* v *(under* δ*)* if (3) holds for all t, starting with $v_2^0 = v$.

Associated with the value dynamic (3) is the *outcome path*, $\pi = \{\pi_t\}_{t=0}^{\infty}$, where

$$\pi_t = \begin{cases} DC, & \text{if } v_2^t \leq 2\delta - 1, \\ CC, & \text{otherwise.} \end{cases} \tag{4}$$

We also call π the *outcome path generated by* v. Note that if $\{v_2^t\}_{t=0}^{\infty}$ and $\{\pi_t\}_{t=0}^{\infty}$ are generated by v, then for all t,

$$v_2^t = (1-\delta)\, g_2(\pi_t) + \delta v_2^{t+1}. \tag{5}$$

It is convenient to define player 2's payoff of the repeated game as a function of an outcome path and δ. When an outcome path π and δ are given, we define

$$h_2^T(\pi; \delta) = (1-\delta) \sum_{\tau=T}^{\infty} \delta^{\tau-T} g_2(\pi_\tau),$$

i.e., $h_2^T(\pi; \delta)$ is player 2's continuation payoff under π from period T on, given δ. When $T = 0$, we will often write $h_2(\pi; \delta)$ rather than $h_2^0(\pi; \delta)$.

Lemma 2. *Fix $\delta \in [\frac{1}{2}, 1)$ and $v \in [-1, 1]$. The outcome path generated by v has value v and so achieves the payoff vector $\left(\frac{3-v}{2}, v\right)$. If the sequence $\{v_2^t\}_{t=0}^{\infty}$ generated by $v \in [0, 1]$ under δ satisfies $v_2^t \geq \frac{1-\delta}{\delta}$ for all $t \geq 1$, then the payoff vector $\left(\frac{3-v}{2}, v\right)$ is an equilibrium payoff vector.*

Proof. Let $\{v_2^t\}_{t=0}^{\infty}$ and π be generated by v. Iteratively applying (5) yields

$$v_2^t = (1-\delta) \sum_{\tau=t}^{T-1} \delta^{\tau-t} g_2(\pi_\tau) + \delta^{T-t} v_2^T \tag{6}$$

for any $t \geq 0$ and any $T > t$. Since the sequence $\{v_2^t\}_{t=1}^{\infty}$ generated by v is bounded, taking $T \to \infty$ in (6) gives

$$v_2^t = h_2^t(\pi; \delta) \tag{7}$$

for any $t \geq 0$.

Suppose the sequence $\{v_2^t\}_{t=0}^{\infty}$ generated by $v \in [0,1]$ under δ satisfies $v_2^t \geq \frac{1-\delta}{\delta}$ for all $t \geq 1$. Consider the strategy profile in which π is played on the path and any deviation is punished by the Nash reversion. Then $h_2^t(\pi, \delta) = v_2^t \geq \frac{1-\delta}{\delta}$ for all $t \geq 1$ ensures that player 2 has no incentive to deviate from the path. Player 1 also has no profitable deviation because her continuation payoff from any period is greater than player 2's continuation payoff, which proves that π is an equilibrium outcome and so $\left(\frac{3-v}{2}, v\right)$ is an equilibrium payoff. ■

Next we show that the path generated by $v \in [0,1]$ is the unique equilibrium path that achieves the equilibrium payoff $\left(\frac{3-v}{2}, v\right)$ when $\delta < \frac{3}{4}$.

Lemma 3. *Fix $\delta < \frac{3}{4}$ and let π be the path generated by $v \in [0,1]$. If a pure outcome path $\mu \neq \pi$ achieves $\left(\frac{3-v}{2}, v\right)$, then $h_2^{T+1}(\mu; \delta) < \frac{1-\delta}{\delta}$ where T is the smallest $t \geq 0$ such that $\mu_t \neq \pi_t$.*

Proof. For any $t \geq 1$, we have

$$v = h_2(\mu; \delta) = (1-\delta) \sum_{\tau=0}^{t-1} \delta^{\tau} g_2(\mu_\tau) + \delta^t h_2^t(\mu; \delta) \tag{8}$$

By the definition of T, if $T \geq 1$, (6) and (8) imply that $h_2^t(\mu; \delta) = v_2^t$ for any $t \leq T$. If $T = 0$, we trivially have $h_2^T(\mu; \delta) = v_2^T = v$.

If $\pi_T = CC$, then $2\delta - 1 < v_2^T = h_2^T(\mu; \delta)$. Since $\mu_T = DC$,

$$h_2^{T+1}(\mu; \delta) = \frac{1}{\delta} h_2^T(\mu; \delta) + \frac{1-\delta}{\delta} > 1. \tag{9}$$

However, since μ achieves $\left(\frac{3-v}{2}, v\right)$ and therefore consists of CC and DC only, we must have $h_2^{T+1}(\mu; \delta) \leq 1$, a contradiction.

Suppose then that $\pi_T = DC$, in which case $h_2^T(\mu; \delta) = v_2^T \leq 2\delta - 1$ by (4). Since $\mu_T = CC$,

$$h_2^{T+1}(\mu; \delta) = \frac{1}{\delta} h_2^T(\mu; \delta) - \frac{1-\delta}{\delta} \leq \frac{3\delta - 2}{\delta} < \frac{1-\delta}{\delta}, \tag{10}$$

where the last inequality follows from $\delta < \frac{3}{4}$. ■

Lemmas 2 and 3 imply that, for any equilibrium with the payoff vector $\left(\frac{3-v}{2}, v\right)$, there is a unique pure outcome path π (and hence continuation payoffs $\{v_2^t\}_{t=0}^{\infty}$) that achieves this equilibrium payoff, and that this π is described by (4). Note that we restrict attention to pure strategies; we justify this in Section 2.1. The following result is immediate from Lemmas 2 and 3.

Proposition 1. *Fix $\delta < \frac{3}{4}$ and $v \in [0,1]$. The payoff vector $(\frac{3-v}{2}, v)$ is an equilibrium payoff if and only if the sequence $\{v_2^t\}_{t=0}^{\infty}$ generated by v under δ satisfies $v_2^t \geq \frac{1-\delta}{\delta}$ for all $t \geq 1$.*

Proposition 1 provides a characterization of all efficient equilibrium payoffs when $\delta < \frac{3}{4}$. In general, it is not possible to determine whether a payoff vector $(\frac{3-v}{2}, v)$ is an equilibrium payoff vector for an arbitrary discount factor δ by examining $\left\{v_2^t\right\}_{t=0}^{T}$ for any finite T. However, an approximation result is possible.

Proposition 2. *Fix $\delta < \frac{3}{4}$ and $v \in [0,1]$. For all $\varepsilon > 0$, there exists T such that if the sequence $\{v_2^t\}_{t=0}^{\infty}$ generated by v under δ satisfies $v_2^t \geq \frac{1-\delta}{\delta}$ for all $1 \leq t \leq T$, then there is an efficient equilibrium with payoffs within ε of the payoff vector $(\frac{3-v}{2}, v)$.*

Proof. Let π be the outcome generated by δ, and for any T define the path π^T by

$$\pi_t^T = \begin{cases} \pi_t, & \text{if } t \leq T, \\ CC, & \text{if } t > T. \end{cases}$$

For T sufficiently large, the payoff from π^T will be within ε of $(\frac{3-v}{2}, v)$. Moreover, if $v_2^t \geq \frac{1-\delta}{\delta}$ for all $1 \leq \delta \leq T$, then π^T is clearly an equilibrium outcome path, since $h_2^t(\pi^T;\delta) \geq h_2^t(\pi;\delta)$ for all t, and $h_2^t(\pi^T;\delta) = 1$ for all $t \geq T+1$. ■

For $\delta = \frac{1}{2}$, $\frac{1-\delta}{\delta} = 1$, and so $v_2 = 1$ is a fixed point of the value dynamic (3). Thus, $(1,1)$ is an equilibrium payoff vector (implied by CC in every period). In addition, $(\frac{3}{2}, 0)$ is also an equilibrium payoff vector, since the payoff sequence generated by 0 is $\{0,1,1,1,\dots\}$ (the associated outcome path has DC in the initial period, followed by CC forever). However, any other payoff vector $\left(\frac{3-v}{2}, v\right)$, where $v \in (0,1)$, is not an equilibrium payoff: the payoff sequence generated by $v \in (0,1)$ satisfies $v_2^1 = 2v - 1 < v < 1 = \frac{1-\delta}{\delta}$ by (3). To sum up, we have two equilibrium payoff vectors in the region B, and $\bar{v}_1(\frac{1}{2}) = \frac{3}{2}$.

Consider now $\delta \in (\frac{1}{2}, \frac{1}{\sqrt{2}})$ (see Figure 3). The inequality $\delta < 1/\sqrt{2}$ is equivalent to $2\delta - 1 < (1-\delta)/\delta$. Observe that, as for $\delta = \frac{1}{2}$, the outcome paths

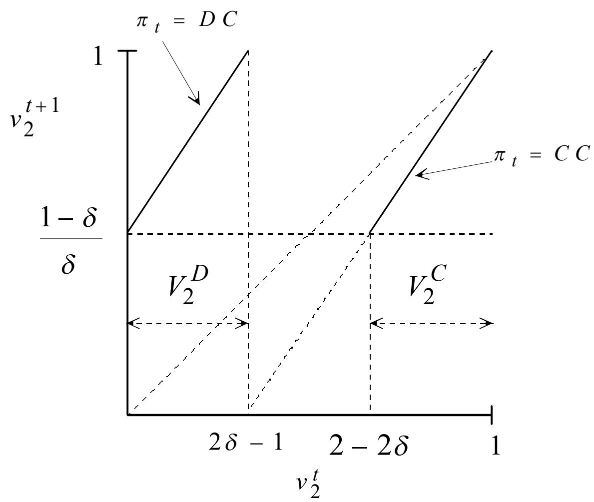

Figure 3: The dynamic for $\delta \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$. This is drawn for $\delta = \frac{2}{3}$.

CC^∞ and DC, CC^∞ are equilibrium outcome paths, so that the payoff vectors $(1,1)$ and $(2-\delta,\ 2\delta-1)$ are equilibrium payoffs.[7] Moreover, no other payoff in B is an equilibrium payoff. Consider first $v \in (2\delta-1, 1)$. The sequence generated by v is decreasing (with the decrements increasing in magnitude as v falls) as long as $v_2^t > 2\delta - 1$ (see Figure 3), and so there exists T such that $v_2^T \leq 2\delta - 1$. Since $\delta < 1/\sqrt{2}$, that implies $v_2^T < \frac{1-\delta}{\delta}$. Thus $\left(\frac{3-v}{2}, v\right)$ is not an equilibrium outcome. Next, suppose $v < 2\delta - 1$. Then, from the value dynamic (3), $v_2^1 \in [\frac{1-\delta}{\delta}, 1) \subset (2\delta-1, 1)$. Since we have just seen that no element of $(2\delta-1, 1)$ is an efficient equilibrium payoff for player 2, $\left(\frac{3-v}{2}, v\right)$ is again not an equilibrium outcome. Therefore, for all $\delta \in (\frac{1}{2}, \frac{1}{\sqrt{2}})$, $\bar{v}_1(\delta) = 2 - \delta$.

It is also straightforward to show that $\bar{v}_1(\frac{1}{\sqrt{2}}) = \frac{3}{2}$, because the path generated by 0 is $(0, \sqrt{2}-1, 1, 1, \dots)$, with associated outcome path DC, DC, CC^∞. Therefore $(\frac{3}{2}, 0)$ is an equilibrium payoff when $\delta = 1/\sqrt{2}$ (see Figure 4). Moreover, for $\delta = 1/\sqrt{2}$, there are a countable number of equilibrium payoff vectors in B: any path of the form $(DC)^x$, $(CC)^t$, DC, $(CC)^\infty$, where $x \in \{0,1\}$ and t is a nonnegative integer, is an equilibrium outcome path. The path DC, CC, CC, DC, $(CC)^\infty$ is illustrated in Figure 4.

We have thus proved the following proposition.

Proposition 3. *Suppose $\delta \geq \frac{1}{2}$. For $\delta \notin (\frac{1}{\sqrt{2}}, \frac{3}{4})$, the maximum efficient equilib-*

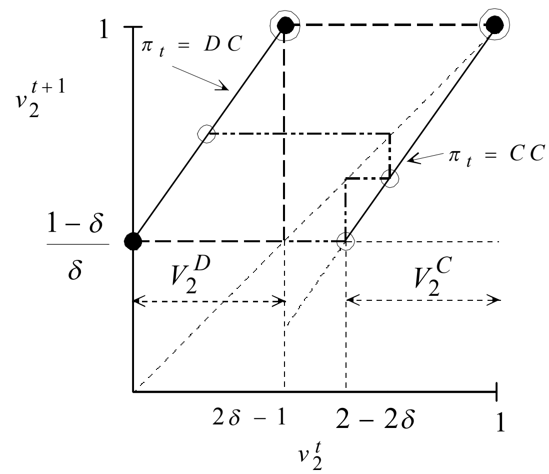

Figure 4: The dynamic for $\delta = 1/\sqrt{2}$. The equilibrium path DC, DC, $(CC)^\infty$ (indicated by solid circles) yields a payoff $v_2 = 0$. The equilibrium path DC, CC, CC, DC, $(CC)^\infty$ is indicated by hollow circles.

rium payoff for player i is

$$\bar{v}_i(\delta) = \begin{cases} 2 - \delta, & \text{if } \delta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}), \text{ and} \\ \frac{3}{2}, & \text{if } \delta = \frac{1}{\sqrt{2}} \text{ or } \delta \geq \frac{3}{4}. \end{cases}$$

Things are more complicated when $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$, because there are too many equilibria to describe explicitly. This multiplicity is due to the ability of the dynamic to revisit both V_2^C and V_2^D, without violating the requirement of $v_2^t \geq \frac{1-\delta}{\delta}$ (see Figure 5).

The remaining case of $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$ is considered in the next section. It will be helpful to introduce the following terminology:

Definition 1. *An equilibrium is* wonderful *if it has payoff* $\left(\frac{3}{2}, 0\right)$. *A discount factor δ is* wonderful *if there is a wonderful equilibrium for that discount factor.*

Thus, we have seen that any $\delta \geq \frac{3}{4}$ is wonderful, while the only wonderful discount factors less than or equal to $\frac{1}{\sqrt{2}}$ are $\frac{1}{2}$ and $\frac{1}{\sqrt{2}}$.

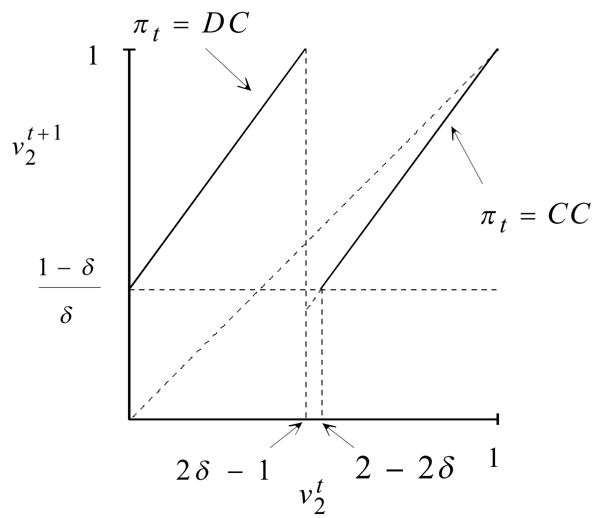

Figure 5: The dynamic for $\delta \in \left(\frac{1}{\sqrt{2}}, \frac{3}{4}\right)$. This is drawn for $\delta = 0.74$.

2.1. The role of mixed strategies

We argue here that allowing for mixed strategy equilibria does not change the set of equilibrium payoffs in B. Since payoffs in B can only be convex combinations of $g(DC)$ and $g(CC)$, player 2 must always play C, and player 2's continuation value cannot exceed 1. In order for player 1 to randomize in some period between C and D, he must be indifferent between C and D, so

$$(1-\delta) + \delta v_1^C = (1-\delta)\,2 + \delta v_1^D,$$

where v_1^a is player 1's continuation value after action a. Moreover, on B, we have $v_2^a = 3 - 2v_1^a$, so that

$$v_2^D = v_2^C + \frac{2(1-\delta)}{\delta} \geq \frac{3(1-\delta)}{\delta}.$$

For $\delta \geq \frac{3}{4}$, every payoff in B can be achieved in a pure strategy equilibrium, and so mixing is redundant. For $\delta < \frac{3}{4}$, the above inequality implies $v_2^D > 1$, which is impossible if the continuation values are to lie in B.

It is instructive to compare our situation (where mixed strategies are unobservable) with the situation where they are observable. Suppose $\delta \geq \frac{1}{2}$ and consider the payoff vector $(\frac{3-v}{2}, v)$ for $v \in [0,1]$. If mixed strategies are observable, then the following strategy profile is an equilibrium and has payoff $(\frac{3-v}{2}, v)$: In the first period, player 1 plays C with probability $\frac{v}{2(1-\delta)}$ and D with the complementary probability, and player 2 plays C for sure. Thereafter, on the equilibrium path, in every period, player 1 plays C with probability $\frac{1}{2\delta}$ and D with the complementary probability, and player 2 plays C for sure. After a deviation (which here means player 1 not randomizing as required or player 2 not playing C), both players player D in every subsequent period. The randomization of player 1 after the initial period ensures that the equilibrium continuation payoff to player 2 is $\frac{1-\delta}{\delta}$, and so player 2 is (just) willing to play C in every period. It is straightforward to verify that player 1 has no incentive to deviate. Note that there is no requirement (nor can there be) that player 1 be indifferent between C and D.

3. The Set of Nonwonderful Discount Factors

Now we consider $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$. We do not attempt to derive the whole set of equilibrium payoff vectors in B explicitly. Rather, we describe the equilibrium in B that maximizes player 1's payoff for any nonwonderful δ. We also show that the set of nonwonderful δ is open.

We start with some preliminary results.

Lemma 4. *Suppose* $\left\{v_2^t\right\}_{t=0}^{\infty}$ *and* $\{\pi_t\}_{t=0}^{\infty}$ *are generated by* 0 *under some* $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$. *Define* $T(\delta) \equiv \max\left\{T : v_2^t \geq (1-\delta)/\delta,\ t = 1, \ldots, T\right\}$. *Then,* $T(\delta) \geq 3$ *and the first four periods of the outcome path are given by* DC, DC, CC, *and* CC. *Moreover, for* $1 \leq t \leq T(\delta)$, *if* $\pi_t = DC$ *then* $\pi_{t+1} = \pi_{t+2} = CC$.

Proof. The first two claims are immediate. To prove the last, note that for $1 \leq t \leq T(\delta)$, if $\hat{\pi}_t = DC$ then $\frac{1-\delta}{\delta} \leq \hat{v}_2^t \leq 2\delta - 1$ and so $\hat{v}_2^{t+1} = \frac{\hat{v}_2^t + 1 - \delta}{\delta} \geq \frac{1-\delta^2}{\delta^2} > 2\delta - 1$, where the last inequality is implied by $\delta \leq \frac{3}{4}$. Since $\hat{v}_2^{t+1} \geq \frac{1-\delta^2}{\delta^2}$, $\hat{v}_2^{t+2} = \frac{\hat{v}_2^{t+1} - 1 + \delta}{\delta} \geq \frac{1 - 2\delta^2 + \delta^3}{\delta^3} > 2\delta - 1$ where again the last inequality is implied by $\delta \leq \frac{3}{4}$. ■

While not needed for what follows, it is not hard to verify that in fact $T(\delta) \geq 5$, and that $\pi_4 = CC$. Moreover, if $\delta < 0.74763$, $\pi_5 = CC$ (note that $1/\sqrt{2} \approx 0.70711$). However, for $\delta \in (0.74636, 0.74763)$, $v_2^6 < \frac{1-\delta}{\delta}$, and so no $\delta \in (0.74636, 0.74763)$ is wonderful.

Lemma 5. *A discount factor* $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$ *is not wonderful if and only if there exists a finite path* $\{\mu_t\}_{t=0}^{T}$, $T \geq 3$ *, satisfying:*

$$-\sum_{t=0}^{k} \delta^t g_2(\mu_t) \geq \delta^k \tag{11}$$

for $k \in \{0, 1, 2, \cdots, T-2\}$,

$$-\sum_{t=0}^{T-1} \delta^t g_2(\mu_t) > \frac{\delta^T (2\delta - 1)}{1-\delta} \tag{12}$$

and

$$-\sum_{t=0}^{T} \delta^t g_2(\mu_t) < \delta^T. \tag{13}$$

Proof. A discount factor δ is not wonderful if and only if $\{v_2^t\}_{t=0}^{\infty}$ generated by 0 satisfies, for some T, $v_2^t \geq (1-\delta)/\delta$ for $t = 1, ..., T$, and $v_2^{T+1} < (1-\delta)/\delta$. Using (6), the first inequality is equivalent to (11) and the second inequality is equivalent to (13) for the path generated by 0. Moreover, $v_2^T > 2\delta - 1$, since otherwise $v_2^{T+1} \geq (1-\delta)/\delta$ from the value dynamic (3), which in turn is (12) for the path generated by 0. Lemma 4 implies $T \geq 3$. So, if δ is not wonderful, the finite path $\{\mu_t\}_{t=0}^{T(\delta)}$ generated by 0 satisfies (11), (12), and (13).

Conversely, suppose that the conditions (11), (12), and (13) hold for some $\{\mu_t\}_{t=0}^{T}$. Note that $\mu_0 = \mu_1 = DC$ (evaluate (11) at $k = 0$ and 1 and use $\delta > \frac{1}{\sqrt{2}} > \frac{1}{2}$). Moreover, (12) implies

$$-\sum_{t=0}^{T} \delta^t g_2(\mu_t) > \delta^T \left(\frac{2\delta - 1}{1-\delta} - 1 \right) > 0, \tag{14}$$

where the first inequality follows from $g_2(\mu_T) \leq 1$, and the second from $\delta > 1/\sqrt{2}$.

Let $\pi' = \{\pi'_t\}_{t=0}^{\infty}$ be the path generated by $w = -\frac{(1-\delta)}{\delta^{T+1}} \sum_{t=0}^{T} \delta^t g_2(\mu_t)$. Define the path ρ as

$$\rho_t = \begin{cases} \mu_t, & \text{if } t \leq T, \\ \pi'_{t-T-1}, & \text{if } t > T. \end{cases}$$

Note that, by construction, $h_2(\rho; \delta) = 0$, and that (11) and (12) imply $h_2^t(\rho; \delta) \geq \frac{1-\delta}{\delta}$ for all $t \in \{1, 2, \cdots, T\}$.

Suppose that δ is wonderful, and let π be the wonderful equilibrium outcome path. Then, by Lemma 3, π is generated by 0. Observe that the actions in the first T periods of π and ρ coincide. [If not, Lemma 3 implies that there exists $t \in \{1, 2, ..., T\}$ such that $h_2^t(\rho; \delta) < \frac{1-\delta}{\delta}$, which is a contradiction.] Hence, $h_2^T(\pi; \delta) = h_2^T(\rho; \delta)$, and from (12), $h_2^T(\pi; \delta) = h_2^T(\rho; \delta) > 2\delta - 1$. By (4), we then have $\pi_T = CC$. Moreover, $h_2^T(\rho; \delta) = (1-\delta)\, g_2(\mu_T) + \delta w < (1-\delta)(g_2(\mu_T) + 1)$ (since (13) implies $w < \frac{(1-\delta)}{\delta}$). The inequality $h_2^T(\rho; \delta) > 2\delta - 1$ then requires $\mu_T = CC$. But now π and ρ also agree in period T, and so $h_2^{T+1}(\pi; \delta) = h_2^{T+1}(\rho; \delta) = w < \frac{1-\delta}{\delta}$, contradicting the assertion that π is an equilibrium outcome path. Consequently, δ is not wonderful. ∎

Lemma 6. *Suppose* $\left\{\hat{v}_2^t\right\}_{t=1}^{\infty}$ *and* $\hat{\pi}$ *are generated by* 0 *under* $\hat{\delta} \in [\frac{1}{\sqrt{2}}, \frac{3}{4})$.

1. *If*

$$f_k(\hat{\pi}; \delta) = \sum_{\tau=0}^{k} \delta^{\tau} g_2(\hat{\pi}_{\tau}),$$

then $\partial f_k(\hat{\pi}; \delta)/\partial \delta > 0$ *for all* $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4}]$ *and any* k *such that* $2 \leq k \leq T(\hat{\delta})$.

2. *Suppose* $T(\hat{\delta}) = \infty$. *Then* $h_2(\hat{\pi}; \delta) < 0$ *for all* $\delta \in (\frac{1}{\sqrt{2}}, \hat{\delta})$ *and* $h_2(\hat{\pi}; \delta) > 0$ *for all* $\delta \in (\hat{\delta}, \frac{3}{4}]$.

Proof. Since

$$\frac{\partial f_k(\hat{\pi}; \delta)}{\partial \delta} = \sum_{\tau=1}^{k} \tau \delta^{\tau-1} g_2(\hat{\pi}_{\tau}),$$

we just need to prove $\sum_{\tau=1}^{k} \tau \delta^{\tau-1} g_2(\hat{\pi}_{\tau}) > 0$. For $\ell \geq 1$, define $s_{\ell} = \sum_{\tau=\ell}^{k} \tau \delta^{\tau-1} g_2(\hat{\pi}_{\tau})$, and recall from Lemma 4 that $\hat{\pi}_1 = DC$ and $\hat{\pi}_2 = CC$. For $k = 2$, $s_1 = -1 + 2\delta > 0$. For $k \geq 3$,

$$s_1 = -1 + 2\delta + s_3 > s_3, \tag{15}$$

because $\delta > \frac{1}{2}$. For any $t > 2$ such that $\hat{\pi}_t = DC$, we have (again from Lemma 4) $t \geq 4$ and $\hat{\pi}_{t-1} = CC$. Therefore, for such t,

$$(t-1)\delta^{t-2} g_2(\hat{\pi}_{t-1}) + t\delta^{t-1} g_2(\hat{\pi}_t) = (t-1)\delta^{t-2} - t\delta^{t-1},$$

which is non-negative, since $\delta \leq \frac{3}{4} \leq \frac{t-1}{t}$. Hence $s_3 \geq 0$, and (by (15)) $s_1 = \partial f_k(\hat{\pi};\delta)/\partial\delta > 0$, which proves the first part of the lemma.

Suppose $T(\hat{\delta}) = \infty$ (so that $\hat{\pi}$ is an equilibrium outcome path). For any δ satisfying $h_2(\hat{\pi};\delta) = 0$ (i.e., $f_\infty(\hat{\pi};\delta) = 0$),

$$\frac{\partial h_2(\hat{\pi};\delta)}{\partial\delta} = (1-\delta)\frac{\partial f_\infty(\hat{\pi};\delta)}{\partial\delta} - f_\infty(\hat{\pi};\delta) > 0.$$

Thus, $h_2(\hat{\pi};\delta)$ can only equal zero for one value of δ, implying the desired result. ■

Lemma 7. *Fix $\delta_0 \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$ and $(\frac{3-v}{2}, v)$ an equilibrium payoff vector for δ_0. Let π be the outcome path generated by v under δ_0, and set $\delta_1 = \min\{\delta \leq \delta_0 : \pi$ is an equilibrium outcome path for all $\delta' \in [\delta, \delta_0]\}$. Then, δ_1 is wonderful. If, in addition, $\pi_0 = DC$ and $h_2^t(\pi;\delta_0) \geq h_2^1(\pi;\delta_0)$ for all $t \geq 1$, then π is the wonderful equilibrium path at δ_1.*

Proof. Define $w = \inf_{t\geq 1} h_2^t(\pi;\delta_1)$. If $w > (1-\delta_1)/\delta_1$, then π is a strict equilibrium path at δ_1.[8] Since for all $t \geq 1$, the function $h_2^t(\pi;\delta)$ satisfies the Lipschitz condition with constant $2/(1-\delta)$, we can lower δ_1 slightly, while keeping π an equilibrium path, contradicting the definition of δ_1. Hence, $w = (1-\delta_1)/\delta_1$. By compactness of equilibrium payoffs, there exists an equilibrium outcome path ρ with payoffs $(\frac{3-w}{2}, w)$ under δ_1. Since $w = (1-\delta_1)/\delta_1$, the path starting with DC and then playing ρ is a wonderful equilibrium path at δ_1. Hence, δ_1 is wonderful.

Next, assume $h_2^t(\pi;\delta_0) \geq h_2^1(\pi;\delta_0)$ for all $t \geq 1$. Suppose first $h_2^t(\pi;\delta_1) < h_2^1(\pi;\delta_1)$ for some $t \geq 1$. Then the continuation path starting in period t must be different from that starting in period 1, and so Lemma 3 implies $h_2^t(\pi;\delta_0) > h_2^1(\pi;\delta_0)$. Thus there exists $\delta_2 \in (\delta_1, \delta_0)$ such that $h_2^t(\pi;\delta_2) = h_2^1(\pi;\delta_2)$. However, since π is an equilibrium path at δ_2 (by definition of δ_1), so too are the paths starting in period t and in period 1, contradicting Lemma 3. Hence $h_2^t(\pi;\delta_1) \geq h_2^1(\pi;\delta_1)$ for all $t \geq 1$, which implies $h_2^1(\pi;\delta_1) = w = (1-\delta_1)/\delta_1$. Finally, if $\pi_0 = DC$, $h_2(\pi;\delta_1) = 0$, which proves that π is wonderful at δ_1. ■

We are now ready to state and prove the main result of this section: if no wonderful equilibrium exists, only cyclic behavior, namely, a path eventually ending in a cycle, forms the best efficient equilibrium outcome for a player. In other words, other behavior is only consistent with the best efficient equilibrium if it is wonderful. One virtue of the result is that we are able to provide the best efficient equilibrium explicitly and to describe the range of discount factors for which the same path continues to be the best efficient equilibrium.

Definition 2. *A path* $\pi = \{\pi_t\}_{t=0}^{\infty}$ *is* eventually n-cyclic, *where n is a positive integer, if*

1. *there exists $T \geq 1$ such that for all $s > T$ and $t = kn + s$ for some integer k, $\pi_t = \pi_s$, and*

2. *the above property does not hold for any $n' < n$.*

Proposition 4. *If $\delta_0 \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$ is not wonderful, then the best efficient equilibrium outcome for player 1 under δ_0 is a path π^* that is eventually n-cyclic with $n \neq 1$. Moreover, there exists a half-open interval $[\delta_1, \delta_2)$ containing δ_0 such that*

1. *for any $\delta' \in [\delta_1, \delta_2)$, π^* is the best efficient equilibrium outcome for player 1, being wonderful if and only if $\delta' = \delta_1$; and*

2. *$\delta_2 < \frac{3}{4}$ is wonderful, and the corresponding wonderful equilibrium outcome is eventually 1-cyclic.*

Proof. Let π and $\{v_2^t\}_{t=0}^{\infty}$ be the path and the sequence generated by 0 under δ_0. Since δ_0 is not wonderful, $T(\delta_0) < \infty$. From Lemma 4, $T(\delta_0) \geq 3$. We write T for $T(\delta_0)$ in this proof.

Definition of δ_1 and δ_2:

As in the proof of Lemma 5, we have at $\delta = \delta_0$ the following three inequalities:

$$-\sum_{t=0}^{k} \delta^t g_2(\pi_t) \geq \delta^k \tag{16}$$

for any $k \in \{0, 1, 2, \ldots, T-2\}$,

$$-\sum_{t=0}^{T-1} \delta^t g_2(\pi_t) > \frac{\delta^T(2\delta - 1)}{1-\delta} \tag{17}$$

and

$$-\sum_{t=0}^{T} \delta^t g_2(\pi_t) < \delta^T. \tag{18}$$

We define δ_1 and δ_2 as the unique discount factors satisfying

$$-\sum_{t=0}^{T} \delta_1^t g_2(\pi_t) = \delta_1^T \tag{19}$$

and

$$-\sum_{t=0}^{T-1} \delta_2^t g_2(\pi_t) = \frac{\delta_2^T (2\delta_2 - 1)}{1-\delta_2}. \tag{20}$$

We first verify that $\delta_1$and δ_2 are well-defined. Note that Lemma 6 applies to the left hand sides of (17) and (18). Thus the left hand sides of (17) and (18) are strictly decreasing in δ on $(\frac{1}{\sqrt{2}}, \frac{3}{4})$, while the right hand side of (17) and (18) are clearly increasing in δ.

The inequality in (18) is reversed at $\delta = 1/\sqrt{2}$, since

$$\begin{aligned}\delta^T + \sum_{t=0}^{T} \delta^t g_2(\pi_t) = \delta^T - 1 - \delta + \sum_{t=2}^{T} \delta^t g_2(\pi_t) \\ \text{(from } \pi_0 = \pi_1 = DC) \\ \leq \quad \delta^T - 1 - \delta + \sum_{t=2}^{T} \delta^t \\ < \quad \frac{\delta^2}{1-\delta} - 1 - \delta = 0,\end{aligned}$$

using $\delta = 1/\sqrt{2}$ in the last line. Therefore, $\delta_1 \in (\frac{1}{\sqrt{2}}, \delta_0)$ is well-defined. Note also that (18) holds for all $\delta > \delta_1$.

Turning to δ_2, note that $\delta = 3/4$ implies

$$\begin{aligned}\frac{\delta^T(2\delta-1)}{1-\delta} + \sum_{t=0}^{T-1} \delta^t g_2(\pi_t) &= 2\delta^T + \sum_{t=0}^{T-1} \delta^t g_2(\pi_t) \\ &= \delta^T + \sum_{t=0}^{T} \delta^t g_2(\pi_t) > 0,\end{aligned}$$

where the first equality follows from $\delta = \frac{3}{4}$, the second from $\pi_T = CC$ (because $v_2^T > 2\delta - 1$, recall the first paragraph of the proof of Lemma 5), and the last inequality from (18). Thus, $\delta_2 \in (\delta_0, \frac{3}{4})$ is also well-defined. The above argument also shows that (17) holds for all $\delta \in (\delta_1, \delta_2)$.

No $\delta \in (\delta_1, \delta_2)$ is wonderful:

Recall that (17) and (18) hold for all $\delta \in (\delta_1, \delta_2)$. Thus, if (16) holds at all $k \in \{0, 1, 2, \cdots, T-2\}$ for any $\delta \in (\delta_1, \delta_2)$, Lemma 5 implies the desired result. Lemma 4 implies that (16) always holds for $k = 0$ and 1. Since Lemma 6 applies to the left hand side of (16) for any $k \in \{2, \ldots, T-2\}$, it suffices to show that (16) is true for all $k \in \{2, \ldots, T-2\}$ at δ_2 (note that the right hand side of (16) is always increasing).

Suppose, then, that (16) does not hold for some $k \in \{2, \ldots, T-2\}$ at δ_2. This implies the existence of $\delta_3 \in [\delta_0, \delta_2)$ such that (16) is true for all $k \in \{2, \ldots, T-2\}$ at δ_3, with an equality for some $\hat{k} \in \{2, \ldots, T-2\}$. Since (17) and (18) hold at δ_3, δ_3 is not wonderful by Lemma 5. Now consider the path $\hat{\rho}$ that starts with DC and then cycles through $\{\pi_t\}_{t=1}^{\hat{k}}$. We have

$$h_2(\hat{\rho}; \delta_3) = -(1-\delta_3) + \frac{1-\delta_3}{1-\delta_3^{\hat{k}}} \sum_{\tau=1}^{\hat{k}} \delta_3^{\tau} g_2(\pi_\tau), \tag{21}$$

because $\hat{\rho}_0 = DC$. Since (16) holds at $\hat{k}$ with equality, (21) implies $h_2(\hat{\rho}; \delta_3) = 0$. Furthermore, since (16) holds at any $k \leq \hat{k}$, it follows that

$$h_2^k(\hat{\rho}; \delta_3) = -\frac{1-\delta_3}{\delta_3^k} \sum_{\tau=0}^{k-1} \delta_3^{\tau} g_2(\pi_\tau) \geq \frac{1-\delta_3}{\delta_3}$$

for any $k = 2, ..., \hat{k}$. Since for any $t > \hat{k}$, $h_2^t(\hat{\rho}; \delta_3) = h_2^k(\hat{\rho}; \delta_3)$ for some $k \leq \hat{k}$, $\hat{\rho}$ is a wonderful equilibrium path, a contradiction. Thus, (16) holds for all $k \in \{2, \cdots, T-2\}$ at δ_2, and therefore at any $\delta \in (\delta_1, \delta_2)$. Hence no discount factor $\delta \in (\delta_1, \delta_2)$ is wonderful.

Definition of π^*: for $t \leq T$, $\pi_t^* = \pi_t$; and for $t = kT + s$, for positive integers k and $1 \leq s \leq T$, $\pi_t^* = \pi_s$. Thus, π^* is an eventually T-cyclic path starting with DC and then cycling through $\{\pi_t\}_{t=1}^{T}$.

The path π^* is an equilibrium path for any $\delta \in [\delta_1, \delta_2]$, being wonderful at δ_1:

Since $\pi_0^* = DC$, we have

$$h_2(\pi^*; \delta) = -(1-\delta) + \frac{1-\delta}{1-\delta^T} \sum_{\tau=1}^{T} \delta^{\tau} g_2(\pi_\tau).$$

Thus (19) implies $h_2(\pi^*; \delta_1) = 0$, while (18) implies $h_2(\pi^*; \delta) > 0$ for any $\delta \in (\delta_1, \delta_2]$. Now fix $\delta \in [\delta_1, \delta_2]$. Since (16) holds at any $0 \leq k = t-1 \leq T-2$, we obtain (as above)

$$h_2^t(\pi^*; \delta) \geq -\frac{1-\delta}{\delta^t} \sum_{\tau=0}^{t-1} \delta^{\tau} g_2(\pi_\tau) \geq \frac{1-\delta}{\delta} \tag{22}$$

for any $1 \le t \le T-1$. Next, (22) is valid at $t = T$, too, because (17) and $\delta > 1/\sqrt{2}$ implies (16) holds strictly at $k = T-1$. Finally, (22) holds for any $t > T$, because $h_2^t(\pi^*;\delta) = h_2^k(\pi^*;\delta)$ for some $k \le T$. Thus we have proved that $h_2^t(\pi^*;\delta) \ge \frac{1-\delta}{\delta}$ for any t. Since $\delta \in [\delta_1,\delta_2]$ is arbitrary, π^* is an equilibrium path for any $\delta \in [\delta_1,\delta_2]$, which is wonderful if and only if $\delta = \delta_1$.

The path π^* is the best (though not wonderful) equilibrium for any $\delta \in (\delta_1,\delta_2)$:

Suppose at some $\delta \in (\delta_1,\delta_2)$ there exists an efficient equilibrium path ρ which gives a greater payoff to player 1 than π^*. As a result, we have $h_2(\rho;\delta) < h_2(\pi^*;\delta)$. Lemma 7 implies the existence of a wonderful $\hat{\delta} \le \delta$ with ρ an equilibrium path for all $\delta' \in [\hat{\delta},\delta]$. Since no $\delta \in (\delta_1,\delta_0)$ is wonderful, $\hat{\delta} \le \delta_1$. Thus ρ is an equilibrium path at δ_1 and therefore $h_2(\rho;\delta_1) \ge 0 = h_2(\pi^*;\delta_1)$. This implies the existence of $\delta'' \in [\delta_1,\delta)$ such that $h_2(\rho;\delta'') = h_2(\pi^*;\delta'')$. Because ρ and π^* are both equilibrium paths at δ'', we have a contradiction to Lemma 3. This establishes that π^* is the best equilibrium outcome and uniqueness also follows from Lemma 3.

δ_2 is wonderful, and the corresponding wonderful outcome path is 1-cyclic:

Consider the path ρ^* defined as: $\rho_t^* = \pi_t$ for any $t \le T-1$, $\rho_T^* = DC$ and $\rho_t^* = CC$ for all $t \ge T+1$. Since

$$h_2(\rho^*;\delta_2) = (1-\delta_2)\sum_{\tau=0}^{T-1}\delta_2^\tau g_2(\pi_t) - (1-\delta_2)\delta_2^T + \delta_2^{T+1},$$

(20) implies $h_2(\rho^*;\delta_2) = 0$. Moreover, since the second inequality in (22) holds for all $1 \le t \le T$, we obtain $h_2^t(\rho^*;\delta_2) \ge \frac{1-\delta_2}{\delta_2}$ for all $1 \le t \le T$. We also have $h_2^t(\rho^*;\delta_2) = 1$ for all $t \ge T+1$. Hence $h_2^t(\rho^*;\delta_2) \ge \frac{1-\delta_2}{\delta_2}$ for any $t \ge 1$, which proves ρ^* is an equilibrium path, which is wonderful and unique (by Lemma 3). ■

The best efficient equilibrium for nonwonderful discount factors has a simple structure in that the path immediately cycles after playing DC in the first period. In other words, the best equilibrium outcome has no "frills." Proposition 4 also shows what type of equilibrium dominates the original best equilibrium when it ceases to be best at δ_2. The equilibrium plays the same as the original one until the very last period of the first phase of the cycle, then switches to DC followed by CC forever. Therefore, the equilibrium is eventually 1-cyclic and, more importantly, wonderful.

Corollary 1. *Fix $\delta \in (\frac{1}{\sqrt{2}},\frac{3}{4})$. Let π be the outcome path generated by 0 under δ.*

Let $T \equiv \max\left\{T' : v_2^t \geq (1-\delta)/\delta,\ t = 1, \ldots, T'\right\}$. *The best efficient equilibrium outcome for player* 1 *is given by* π *if* $T = \infty$ *and by* π^* *if* $T < \infty$, *where*

$$\pi_t^* = \begin{cases} \pi_t, & \text{for } t \leq T, \\ \pi_s, & \text{for } t = kT + s, \text{ for positive} \\ & \quad \text{integers } k \text{ and } 1 \leq s \leq T. \end{cases}$$

Remark. We have so far considered the best efficient equilibrium for player 1, i.e., the equilibrium which gives the greatest payoff to player 1 among equilibrium payoffs on the Pareto frontier of the feasible payoff set. We should emphasize that the best efficient equilibrium payoff is sometimes different from the maximum of *all* equilibrium payoffs for player 1.

To illustrate this possibility, fix a nonwonderful discount factor δ_0 and consider the half-open interval $[\delta_1, \delta_2)$ presented in Proposition 4. Let ρ be the wonderful equilibrium path for δ_2, which is eventually 1-cyclic. For $\varepsilon > 0$, this path is not an equilibrium path for $\delta \in (\delta_2 - \varepsilon, \delta_2)$ (Lemma 6). Since δ is not wonderful for ε small, from Proposition 4, the same outcome path is the best efficient equilibrium outcome path for player 1 for all $\delta \in (\delta_2 - \varepsilon, \delta_2)$, and so there exists $\zeta > 0$ such that player 1's payoff in the best efficient equilibrium is bounded above by $\frac{3}{2} - \zeta$ for all $\delta \in (\delta_2 - \varepsilon, \delta_2)$. Intuitively, the path ρ fails to be an equilibrium for $\delta < \delta_2$ because in period T, ρ requires DC, leading to $h_2(\rho; \delta) < 0$. In contrast, the best efficient equilibrium outcome path specifies $\pi_T = CC$. Modify ρ by replacing CC in a distant future period with CD. The modified path, denoted ρ', results in a greater payoff for player 2. So, if we consider $\delta \in (\delta_2 - \varepsilon, \delta_2)$ for $\varepsilon > 0$ sufficiently small, ρ' gives player 2 more than 0 and so ρ' is indeed an equilibrium path. Moreover, the payoff to player 1 exceeds $\frac{3}{2} - \zeta$, and so player 1 receives a greater payoff than the eventually cyclic efficient equilibrium path we consider in Proposition 4.

The above argument suggests that the full characterization of the best equilibrium for player 1 is significantly more complicated when we remove the restriction to efficient paths. Nonetheless, we conjecture that the optimality of cyclic behavior is a general phenomenon.

4. Denseness

One message of the analysis in the previous section is that the set of nonwonderful discount factors is open. The purpose of this section is to show that it is also a dense subset of $(\frac{1}{\sqrt{2}}, \frac{3}{4})$. With this result, our previous finding that the best efficient equilibrium behavior is eventually cyclic is shown to be pervasive in the discount factor. We should note here that we do not know the Lebesgue measure of the set of nonwonderful discount factors.[9]

Proposition 5. *The set of nonwonderful discount factors in* $(\frac{1}{\sqrt{2}}, \frac{3}{4})$ *is an open and dense subset of* $(\frac{1}{\sqrt{2}}, \frac{3}{4})$.

Proof. We only need to show denseness, since Proposition 4 immediately implies openness of the set of nonwonderful discount factors.

We need to show that if $\delta' \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$ is wonderful, any neighborhood of δ' has a nonwonderful discount factor. Let π' be the wonderful equilibrium path corresponding to δ'. We start from the observation that any neighborhood of δ' has an element δ'' that is wonderful and whose wonderful equilibrium path is eventually 1-cyclic. The observation follows immediately if the wonderful equilibrium at δ' itself is eventually 1-cyclic, so assume otherwise. For each T, define an eventually 1-cyclic path π^T as

$$\pi_t^T = \begin{cases} \pi'_t, & \text{if } t < T, \\ CC, & \text{otherwise} \end{cases} \tag{23}$$

For any T and any $t \geq 0$,

$$h_2^t(\pi^T; \delta') > h_2^t(\pi'; \delta'). \tag{24}$$

Since $h_2^t(\pi'; \delta') \geq \frac{1-\delta'}{\delta'}$ for any $t \geq 1$ (because π' is an equilibrium path), (24) shows that π^T is an equilibrium outcome for any T. In addition, we have $h_2^t(\pi^T; \delta') \geq h_2^1(\pi^T; \delta')$ for any $t > 1$, because the impact of replacing any DC in a future period with CC on $h_2^t(\pi^T; \delta')$, $1 < t < T$, is greater than that on $h_2^1(\pi^T; \delta')$ and $h_2^t(\pi^T; \delta') = 1$ for $t \geq T$. Therefore Lemma 7 applies, and for any T there exists $\delta_T < \delta'$ such that π^T is a wonderful equilibrium outcome at δ_T (recall that $\pi_0^T = DC$ because π' is wonderful.). Obviously, $\delta_T \to \delta'$ as $T \to \infty$.

Choose any neighborhood of δ'. Then there exists T such that δ_T belongs to that neighborhood. Let T' be the last period in which DC is played on π^T, and consider a finite path $\{\pi_t^T\}_{t=0}^{T'-1}$. Since π^T is wonderful at δ_T, we obtain

$$-\sum_{t=0}^{k} \delta_T^t g_2(\pi_t^T) \geq \delta_T^k \quad \forall k \in \{0, 1, \ldots, T'-2\} \tag{25}$$

and

$$-\sum_{t=0}^{T'-1} \delta_T^t g_2(\pi_t^T) = \frac{\delta_T^{T'}(2\delta_T - 1)}{1 - \delta_T}. \tag{26}$$

Choose an element of the neighborhood $\delta < \delta_T$ so that (12) holds at δ, and that

$$-\sum_{t=0}^{T'-1} \delta^t g_2(\pi_t^T) < 2\delta^{T'}. \tag{27}$$

This is possible because $2\delta_T - 1 < 2(1-\delta_T)$ and Lemma 6 applies. Lemma 6 also guarantees that 25 continues to hold. Consider a finite path $\mu = \{\mu_t\}_{t=1}^{T'}$ defined as: $\mu_t = \pi_t^T$ for any $t \neq T'$, and $\mu_{T'} = CC$. Then, (27) implies

$$-\sum_{t=0}^{T'} \delta^t g_2(\mu_t) < \delta^{T'}. \tag{28}$$

By Lemma 5, (25), (12) and (28) imply that δ is not wonderful. Since the neighborhood of δ' is arbitrarily chosen, the set of nonwonderful discount factors is dense. ■

5. The Set of Wonderful Discount Factors

So far, we have limited our attention to nonwonderful discount factors, and described the best equilibrium outcome paths for nonwonderful discount factors. Now we turn to the set of wonderful discount factors and the properties of corresponding wonderful equilibria. We start by classifying behavior consistent with wonderful equilibria.

Unlike the case of nonwonderful discount factors, where we can derive a certain type of behavior as the best equilibrium path, wonderful equilibria exhibit much more diversity. For example, a wonderful equilibrium may be eventually 1-cyclic, like the one we have seen at δ_2 of Proposition 4. Or it may be eventually n-cyclic like the one we have observed at δ_1 of Proposition 4. However, wonderful equilibrium paths need not be of the type considered there, that is, an eventually n-cyclic path with no frills. An *eventually n-cyclic path with a frill*, denoted by $\pi = (DC, \mu, \rho^\infty)$, where μ is a finite path, ρ is a different finite path, and ρ^∞ is the infinite repetition of ρ, could be a wonderful equilibrium.

To make things far more complicated, there is another type of wonderful equilibrium path, which never converges to any cycle. Consider the following path ρ:

$$\rho_t = \begin{cases} DC, & \text{if } t = 0, 1 \text{ or } t = 100^k \text{ for some integer } k, \\ CC, & \text{otherwise.} \end{cases}$$

There exists $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$ such that ρ gives player 2 the payoff of 0. Thus $h_2(\rho;\delta) = 0$, while $h_2^1(\rho;\delta) = (1-\delta)/\delta$. We also have $h_2^t(\rho;\delta) > (1-\delta)/\delta$ for any $t \geq 2$, because all DC's are located more distantly and/or more sparsely in the continuation play from period t than in the continuation play from period 1. Therefore, ρ is a wonderful equilibrium given δ, while it is not eventually cyclic.

Thus, we have several types of behavior, in addition to the one that is eventually n-cyclic without a frill: the eventually 1-cyclic path, the eventually n-cyclic path with a frill, and the nonconvergent or acyclic path (such as ρ above). One interesting fact about those wonderful equilibria is that, other than the eventually n-cyclic path without a frill, such an equilibrium fails to be the best equilibrium in a neighborhood of the discount factor that makes it wonderful. For any neighborhood of the discount factor, there exists a slightly larger discount factor where the outcome is not the best equilibrium outcome. Consider an eventually n-cyclic path with a frill, for example. As before, we write it as $\pi = \{(DC), \mu, \rho^\infty\}$, and assume it is a wonderful equilibrium path at some $\delta^* \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$. Then, the average payoff of the path μ (for player 2) is smaller than the average payoff of ρ. Therefore, if we define $\pi^n = \{(DC), (\mu, \rho^n)^\infty\}$, where ρ^n is the n-fold repetition of the finite path ρ, we obtain

$$h_2(\pi^n;\delta^*) < h_2(\pi;\delta^*) = 0. \tag{29}$$

for all n. It also follows that

$$\lim_{n\to\infty} h_2(\pi^n;\delta^*) = 0. \tag{30}$$

There exists n^* such that for $n > n^*$, $h_2(\pi^n; \frac{3}{4}) > 0$. By (29), we can show that for any $n > n^*$, there exists $\delta_n \in (\delta^*, \frac{3}{4})$ at which π^n is wonderful. In addition, π^n is an equilibrium for any $\delta \geq \delta_n$, satisfying $h_2(\pi^n;\delta) < h_2(\pi;\delta)$. Since (30) implies $\delta_n \to \delta^*$ as $n \to \infty$, we can conclude that any $\delta > \delta^*$ is greater than some δ_n and therefore π^n is a better equilibrium path for player 1 than π.

Given this classification, the next question is: What is the cardinality of the set of wonderful discount factors in $(\frac{1}{\sqrt{2}}, \frac{3}{4})$?

Lemma 8. *Let W be the set of wonderful discount factors. Suppose $\delta_2 \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$ is wonderful and the associated equilibrium outcome path π is eventually 1-cyclic. For all $\varepsilon > 0$, the set $W \cap (\delta_2, \delta_2 + \varepsilon)$ is uncountable.*

Proof. Since π is eventually 1-cyclic, there is a finite history ρ of length $T' > 1$ such that $\pi = (\rho, CC^\infty)$. Thus, $h_2^1(\pi, \delta_2) = (1-\delta_2)/\delta_2$ and $h_2^t(\pi, \delta_2) >$

$(1-\delta_2)/\delta_2$ for all $t>1$. [If $h_2^t(\pi,\delta_2)=(1-\delta_2)/\delta_2$ for some $t>1$, then the outcome path eventually follows an n-cycle, $n\geq 2$, contradiction.]

By Lemma 6, for δ sufficiently close to, but larger than δ_2, π is a strict equilibrium. Moreover, since $h_2^t(\pi,\delta)$ takes on only a finite number (in fact, $T'+1$) of distinct values, for δ sufficiently close to δ_2, $h_2^1(\pi,\delta)<h_2^t(\pi,\delta)$ for all $t>1$. Moreover, there exists $\eta>0$ such that $h_2^1(\pi,\delta)<h_2^t(\pi,\delta)$ for all $t>1$ and all $\delta\in(\delta_2,\delta_2+\eta)$.

Fix $\delta\in(\delta_2,\delta_2+\min\{\varepsilon,\eta\})$. There exists T such that the path defined as $\pi^T=(\rho,\ ((CC)^{T-1}(DC))^\infty)$ is an equilibrium path, and $h_2^1(\pi^T,\delta)<h_2^t(\pi^T,\delta)$ for all $t>1$. Note that π^T assigns DC to any period written as $mT+T'-1$ for a positive integer m.

Let $s=\{s_t\}_{t=1}^\infty$ be a sequence of natural numbers. Associated with s, define the set of natural numbers

$$Z(s)=\{m:m=\sum_{i=1}^t s_i \text{ for some } t\}.$$

We also define the path $\pi(s)$ as

$$\pi_t(s)=\begin{cases} DC, & \text{if } t=mT+T'-1 \text{ for } m\in Z(s),\\ \pi_t, & \text{otherwise.}\end{cases}$$

It is immediate that $h_2^t(\pi(s);\delta)\geq h_2^t(\pi^T;\delta)$ for any $t\geq 1$. Therefore $\pi(s)$ is an equilibrium path. Moreover, we obtain $h_2^t(\pi(s);\delta)\geq h_2^1(\pi(s);\delta)$ for any $t\geq 1$, because in the continuation path from period $t>1$, DC's are located more distantly and more sparsely than in the continuation path from period 1. Thus Lemma 7 applies, and we have a wonderful discount factor $\delta(s)<\delta$ at which $\pi(s)$ is the wonderful equilibrium path. Moreover, from Lemma 6, $\delta(s)>\delta_2$. Therefore, $\delta(s)\in W\cap(\delta_2,\delta)$.

Note that if we choose a different sequence s', $Z(s')\neq Z(s)$ and therefore $\pi(s')\neq\pi(s)$. Consequently, $\delta(s')\neq\delta(s)$. Since the set of all sequences of natural numbers has the power of the continuum, $W\cap(\delta_2,\delta)$ has the same power, and hence there are uncountably many discount factors in $W\cap(\delta_2,\delta)$ for any $\delta\in(\frac{1}{\sqrt{2}},\frac{3}{4})$. ■

Recall from Proposition 4 that if $(\delta_1,\delta_2)\subset(\frac{1}{\sqrt{2}},\frac{3}{4})$ is a maximal interval of nonwonderful discount factors, then δ_2 is wonderful and the associated equilibrium outcome path π is eventually 1-cyclic, so Lemma 8 applies. In the following proposition, note that both $\delta=\frac{1}{\sqrt{2}}$ and $\delta=\frac{3}{4}$ are wonderful discount factors, so that for all $\delta'\in(\frac{1}{\sqrt{2}},\frac{3}{4})$, both $W\cap[\frac{1}{\sqrt{2}},\delta')$ and $W\cap(\delta',\frac{3}{4}]$ are uncountable.

Proposition 6. *For any wonderful* $\delta \in [\frac{1}{\sqrt{2}}, \frac{3}{4}]$, *and for all* $\varepsilon > 0$, *there are uncountably many wonderful discount factors in* $(\delta - \varepsilon, \delta + \varepsilon) \cap [\frac{1}{\sqrt{2}}, \frac{3}{4}]$.

Proof. Since $\delta = \frac{1}{\sqrt{2}}$ is wonderful with associated outcome path $\pi^* = (DC, DC, CC^{\infty})$, Lemma 8 applies.

Fix $\varepsilon > 0$ and suppose $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4}]$ is wonderful. Since nonwonderful discount factors are dense in $(\frac{1}{\sqrt{2}}, \frac{3}{4})$, there is a nonwonderful discount factor δ_0 in $(\delta - \varepsilon, \delta)$, and from Proposition 4, there exists a wonderful $\delta_2 \in (\delta - \varepsilon, \delta)$ with an associated equilibrium outcome path π that is eventually 1-cyclic. Now apply Lemma 8. ■

6. Monotonicity

Our analysis in the previous sections has shown that the maximum efficient equilibrium payoff, $\bar{v}_1(\delta)$ is *not* monotonic with respect to δ, in the region $(0, \frac{3}{4})$. The analysis has also demonstrated that the set of all efficient equilibrium payoffs given δ does not exhibit monotonicity with respect to δ.[10]

However, while we do not have monotonicity of the maximum equilibrium payoff or the equilibrium payoff set, we do have monotonicity of efficient equilibrium *paths* with respect to δ, for $\delta < \frac{3}{4}$. Indeed, this monotonicity is a nice aspect of efficient equilibrium; for the set of (not necessarily efficient) equilibrium outcomes, it does not increase monotonically as δ increases.

Proposition 7. *Let* π *be an efficient equilibrium path for some* $\delta_0 \in (0, \frac{3}{4})$. *Then it is an equilibrium path for any* $\delta \in [\delta_0, \frac{3}{4})$.

Proof. Without loss of generality, we can assume that the payoff vector of π lies on B. If $\delta_0 < 1/\sqrt{2}$, either $\pi = \{(CC)^{\infty}\}$ or $\pi = \{(DC), (CC)^{\infty}\}$. It is easy to verify that π is an equilibrium path for any $\delta \geq \delta_0$. So assume $\delta_0 \in [\frac{1}{\sqrt{2}}, \frac{3}{4})$.

We start from the case where π is eventually 1-cyclic. We claim that if π is an equilibrium outcome path at some $\delta' \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$, then there exists $\varepsilon > 0$ such that π continues to be an equilibrium outcome path for $\delta \in [\delta', \delta' + \varepsilon)$. Since the set of continuation payoffs of π is finite (because it is eventually cyclic), there is a T that minimizes $h_2^t(\pi; \delta')$. Let $\mathcal{T} = \{t : h_2^t(\pi; \delta') > (1 - \delta')/\delta'\}$. Then, for all δ in a neighborhood of δ', and for all $t \in \mathcal{T}$, $h_2^t(\pi; \delta) > (1 - \delta)/\delta$ (using the finiteness of the set of continuation payoffs).

If $T \in \mathcal{T}$, i.e., $h_2^T(\pi; \delta') > (1 - \delta')/\delta'$, this then implies that π is an equilibrium path for $\delta \in [\delta', \delta' + \varepsilon)$ for ε small.

Suppose now that $T \notin \mathcal{T}$, i.e., $h_2^T(\pi;\delta') = \left(1-\delta'\right)/\delta'$. Define ρ as $\rho_0 = DC$ and $\rho_t = \pi_{T+t-1}$ for all $t \geq 1$. Then, $h_2(\rho;\delta') = 0$. Since $h_2^t(\rho;\delta') = h_2^{T+t-1}(\pi;\delta') \geq \left(1-\delta'\right)/\delta'$ for all $t \geq 1$, ρ is the wonderful equilibrium at δ'. Therefore, Lemma 6 implies that $h_2(\rho;\delta) > 0$ for $\delta > \delta'$, which is equivalent to $h_2^T(\pi,\delta) > (1-\delta)/\delta$, for $\delta > \delta'$. This proves our claim.

Since the set of discount factors for which π is an equilibrium outcome is closed, we thus have that π is an equilibrium outcome path for all $\delta \in [\delta_0, \frac{3}{4})$.

Now, suppose π is not eventually 1-cyclic. Let π^T be the path such that $\pi_t^T = \pi_t$ for $t < T$, and $\pi_t^T = CC$ for $t \geq T$. It is easily seen that if π is an equilibrium path, so is π^T for any T. Since each π^T is eventually 1-cyclic, the above argument shows that it is an equilibrium outcome for any $\delta \in (\delta_0, 3/4)$. Thus, for all $\delta \in (\delta_0, 3/4)$ and all $t \geq 1$, $h_2^t\left(\pi^T;\delta\right) \geq (1-\delta)/\delta$. Since $\lim_{T\to\infty} h_2^t\left(\pi^T;\delta\right) = h_2^t(\pi;\delta)$, we also have $h_2^t(\pi;\delta) \geq (1-\delta)/\delta$ for all $t \geq 1$, and so π is an equilibrium path for all $\delta \in [\delta_0, 3/4)$. ∎

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Notes

[1]While Sorin (1986) states his results in terms of Nash equilibrium, it is immediate that the characterization for the repeated Prisoner's Dilemma also holds for subgame perfect equilibrium.

Cave (1987) proves a similar result, as well as some results for the case where players' mixed strategies are observable, and where players are restricted to stationary strategies.

[2]The unobservability of mixed strategies alters the analysis considerably because the randomizing player must be indifferent over actions in the support of the mixed strategy (see Section 2.1).

[3]Nonmonotonicity of the Nash equilibrium payoff set is reported in Sorin (1986).

[4]We do not know if the set of such discount factors has full measure.

[5]Since we focus on the maximum efficient payoff from a player's (say 1) point of view, the other player always plays C. As a consequence the only relevant payoffs are those from CC and DC.

[6]We index time so that the repeated game starts at $t = 0$.

[7]The outcome paths CC^{∞} and DC, CC^{∞} are equilibrium outcomes for all $\delta \in \left(\frac{1}{2}, 1\right)$. This monotonicity property in terms of outcome paths is proved in Section 6.

[8]Here, and elsewhere, we say that an outcome path is a strict equilibrium path if deviating from the path results in a strictly lower payoff.

[9]Recall that there are open and dense subsets of $[0, 1]$ of arbitrarily small measure, for example complements of generalized Cantor sets (see Royden (1988, Problem 14.b, page 64)).

[10]This observation does not contradict Abreu *et al.* (1990, Theorem 6), which proves monotonicity of equilibrium payoff sets, since they assume the public signal is distributed on a subset of a finite Euclidean space, and that the distribution function has a density. In our context, this is equivalent to requiring the presence of a public correlating device.

Figure captions

Figure 1: The sets V^* and B.

Figure 2: Self-generating sets for $\delta \geq \frac{3}{4}$. This is drawn for $\delta = \frac{4}{5}$. Any v_2 can be decomposed into a current action profile and continuation value w_2.

Figure 3: The dynamic for $\delta \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$. This is drawn for $\delta = \frac{2}{3}$.

Figure 4: The dynamic for $\delta = 1/\sqrt{2}$. The equilibrium path DC, DC, $(CC)^\infty$ (indicated by solid circles) yields a payoff $v_2 = 0$. The equilibrium path DC, CC, CC, DC, $(CC)^\infty$ is indicated by hollow circles.

Figure 5: The dynamic for $\delta \in \left(\frac{1}{\sqrt{2}}, \frac{3}{4}\right)$. This is drawn for $\delta = 0.74$.