# Repeated Games with Perfect Monitoring

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We study repeated games with perfect monitoring (and complete information). In this class of repeated games, players can observe the other players' actions directly and there is no uncertainty concerning to players' types.

## 1 Model

We present a relatively general model of repeated games first, which will be later specialized for each different repeated game with different monitoring structure. S*tage game* is a standard strategic (normal) form game $G = \{N, A, g\}$, where $N = \{1, 2, ..., n\}$ be the set of players, $A_i$ is player $i$'s finite or compact action set $\left(A = \prod_{i \in N} A_i\right)$, and $g : A \to \Re^n$ is the payoff functions of $n$ players, with $g_i(a)$ being player $i$'s payoff. We assume that $g_i(a)$ is continuous when $A_i$ is compact. *Feasible payoff set* is denoted by $V =$ convex full of $\{g(a) | a \in A\}$. Players play the stage game repeatedly over time. Time is discrete and denoted by $t = 1, 2, ....$ Player $i$ observes a signal $h_{i,t} \in Y_i$ in the end of period $t$. Player $i$'s signal is generated by $f_i : A \to \triangle Y_i$ every period given the action profile chosen in the period. Player $i$'s *period t history* is $h_i^t = (h_{i,1}, ...., h_{i,t-1})$, which is player $i$'s information accumulated by the end of period $t-1$. Let $H_i^t \left(= Y_i^{t-1}\right)$ be the set of all possible player $i$'s period $t$ history at period $t$ ($H_i^1 = \emptyset$ is null history in the beginning of the game) and $H_i = \cup_{t=1}^{\infty} H_i^t$ be all possible histories of player $i$. Player $i$'s (pure) *strategy* $\sigma_i$ is a mapping from $H_i$ to $A_i$. I assume that strategies are pure unless noted otherwise. Let $\Sigma_i$ be the set of player $i$'s strategies. Take any strategy profile $\widetilde{\sigma}$, which generates a sequence of action profiles $\left(\widetilde{a}^1, \widetilde{a}^2, ...\right)$. Then player $i'$s *discounted average payoff* from the strategy profile $\widetilde{\sigma}$ is given by $V_i(\widetilde{\sigma}) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} g_i\left(\widetilde{a}^t\right)$.[1] where $\delta$ is a common discount factor.

[1] Another way to define average payoffs would be to use a variety of limit of arithmetic average $\frac{\sum_{t=1}^{T} g\left(a^t\right)}{T}$. We don't discuss these preferences here. See F&T (p148,149) [8] and

**Perfect Monitoring**

Monitoring is *perfect* if players can observe the other players' actions directly. $(Y_i = A$ and $\Pr(f_i(a) = a) = 1$ for all $i)$. In this case, player $i'$s period $t$ history is $h_i^t = \left(a^1, ..., a^{t-1}\right)$. Since this history is *public* and shared by all the players, we just denote period $t$ history by $h^t$ without subscript.

**Equilibrium**

We use a notion of *subgame perfect equilibrium* for repeated games with perfect monitoring. For each history $h^t$, let $\sigma|_{h^t} \in \Sigma$ be a profile of continuation strategies for the subgame after $h^t$.[2] The payoff for the subgame is simply $V_i(\sigma|_{h^t}) = (1-\delta)\sum_{s=t}^{\infty}\delta^{s-t}g_i(a^s)$ where $\left(a^t, a^{t+1}, ...\right)$ is a sequence of action profiles generated by $\sigma|_{h^t}$.

**Definition 1** *$\sigma^*$ is a subgame perfect equilibrium if, for any history $h^t \in H$, $\sigma^*|_{h^t}$ is a Nash equilibrium of the subgame starting at $h^t$, i.e.,*

$$V_i(\sigma^*|_{h^t}) \geq V_i\left(\sigma_i, \sigma^*_{-i}|_{h^t}\right) \text{ for all } \sigma_i \in \Sigma_i \text{ and all } i$$

Note that the set of continuation strategies is identical to original strategy set $\Sigma_i$. Indeed every subgame is completely identical to the original game. We exploit this recursive nature of repeated games to characterize the whole set of equilibrium payoffs in the next section.

The above definition is not so useful because there are uncountable number of constraints.[3] We finish this preliminary section with one very useful proposition, which claims that we need to check only a particular class of incentive constraints. Consider the following type of deviations.

**Definition 2** *$\sigma_i^{'}$ is one-shot deviation from $\sigma_i$ if $\sigma_i'(h_t') \neq \sigma_i(h_t')$ for some $h_t' \in H_i$ and $\sigma_i'(h_t) = \sigma_i(h_t)$ for all $h_t \in H_i/\{h_t'\}$. Denote this one-shot deviation by $\sigma_i^{h_i'}$. We say $\sigma_i^{h_i'}$ is a profitable one-shot deviation for player $i$ if $V_i\left(\sigma_i^{h_i'}|_{h_i'}, \sigma_{-i}|_{h_i'}\right) > V_i\left(\sigma|_{h_i'}\right)$.*

There are only $|H|$ one-shot deviation constraints for each player, which are just countable. Thus the following claim reduces the number of incentive constraints drastically.

Rubinstein [11].

[2] $\sigma|_{h^t}$ is defined as $\sigma|_{h^t}(\phi) = \sigma\left(h^t\right)$ and $\sigma|_{h^t}(h^s) = \sigma\left(\left(h^t, h^s\right)\right)$.

[3] There are $|\Sigma_i| \times |H|$ incentive constraints for player $i$ in the definition Note that $\Sigma_i$ is not countable.

**Proposition 3** *(One-Shot Deviation Principle): $\sigma$ is a subgame-perfect equilibrium if and only if there is no profitable one-shot deviation from a strategy profile $\sigma$.*

**Proof.** (Sketch) "Only if" follows from the definition of SPE. So we just need to prove "if" part. First, if there is no profitable one-shot deviation, then there is no profitable finite-period deviation. This follows from a simple induction argument. Second, if there is profitable deviation which is different from $\sigma_i$ for infinite number of periods, then a finite truncation of such deviation must be still profitable if you take a window of finite periods large enough.[4] This is a contradiction, thus there should be no profitable infinite-periods deviation either. ■

# 2 Characterization of Equilibrium Payoff Set

## 2.1 Dynamic Programming Approach

In general, there are many equilibria in repeated games. For example, a cooperative outcome can be supported in Repeated prisoners' dilemma, but playing always $(D, D)$ is also a subgame perfect equilibrium. Indeed, this multiplicity of equilibria is the reason why the cooperative outcome can be ever supported in subgame perfect equilibrium. We can look at each equilibrium one by one, but it is more useful to look at all equilibria (in precise, the set of all equilibrium payoffs) at the same time. The results of this Section are from Abreu, Pearce and Stacchetti [4].

As we observed before, any subgame is identical to the original game. Hence, any continuation strategy profile of a subgame perfect equilibrium is an equilibrium profile of the original game, that is, if $E(\delta)$ is the set of equilibrium payoffs of a repeated game, then for any subgame perfect equilibrium $\sigma^*$,

$$V(\sigma^*|_{h^t}) \in E(\delta)$$

holds for any history $h^t$. This fact motivates to define the following one shot game. Fix a subgame perfect equilibrium $\sigma^*$ and let $w$ be a mapping from $A$ to $W \subset \Re^n$ defined by $w(a) = V(\sigma^*|_a)$ and $g_w(a) = (1-\delta) g(a) + \delta w(a)$. This defines a strategic form game $\{N, A, g_w\}$. Let $a^{1*}$ be the equilibrium action profile in the first period of $\sigma^*$. Then, clearly $a^{1*}$ should be also a Nash equilibrium of $\{N, A, g_w\}$.

Now, we explore this observation more formally.

[4] Difference in payoffs of any two strategy is negligible in very far future because of discounting.

**Definition 4** *For any $W \subset \Re^n$, a pair $(a, w(\cdot))$ is admissible with respect to $W \subset \Re^n$ if (1) $w(a) \in W$ for all $a \in A$ and (2) $a$ is a Nash equilibrium of the strategic form game $\{N, A, g_w\}$.*

**Definition 5** *For any $W \subset \Re^n$, $B(W, \delta) =$*
*$\{v | \exists \ (a, w(\cdot))$ admissible w.r.t. $W$ such that $v = (1-\delta) g(a) + \delta w(a) \ \}$*

Our observation can be stated more formally with these definitions.

**Lemma 6** $E(\delta) \subset B(E(\delta), \delta)$

Now we need a little bit more work to show the opposite inclusion.

**Lemma 7** *If $W \subset \Re^n$ is bounded and $W \subset B(W, \delta)$, then $B(W, \delta) \subset E(\delta)$*

**Proof.** (sketch); Take any point $v \in B(W, \delta)$. By definition, there is an admissible pair $a^1(v) \in A$, and $w^1(\cdot|v) : A \to W$ such that $v = (1-\delta) g(a^1(v)) + \delta w^1(a^1(v)|v)$. Since $W \subset B(W, \delta)$, for every $w^1(a^1), a^1 \in A$, we can find an admissible pair $(a^2(a^1), w^2(\cdot|a^1))$ such that $w^1(a^1) = (1-\delta) g(a^2(a^1)) + \delta w^2(a^2(a^1)|a^1)$. In this way, we can find an admissible pair $(a^t(h^t), w^t(\cdot|h^t))$ such that $w^t(h^t) = (1-\delta) g(a^t(h^t)) + \delta w^{t+1}(a^t(h^t)|h^t)$ for every history $h^t = (a^1, a^2, ...., a^{t-1})$ for $t = 2, 3, ....$(where $w^t(h^t) = w^t(a^{t-1}|h^{t-1})$ for $h^t = (h^{t-1}, a^t)$).

Define a strategy profile $\sigma$ by $\sigma(h^t) = a^t(h^t)$ for $t = 1, 2, ...$ Then you can check that $v = V(\sigma)$ and $w^t(h^t) = V(\sigma(h^t))$ for all $h^t$ because $W$ is bounded. There is no profitable one-shot deviation from $\sigma$ by construction. Hence, $\sigma$ is a subgame perfect equilibrium by one-shot deviation principle. ■

Such $W$ to satisfy $W \subset B(W, \delta)$ is called a *self-generating set*.

From the above two lemmas, we obtain the following theorem.

**Theorem 8** $E(\delta) = B(E(\delta), \delta)$

This means that $E(\delta)$ is a "fixed point" of the set valued operator $B(\cdot, \delta)$. But, $E(\delta)$ is not the only set to satisfy this equation. For example, any unique Nash equilibrium payoff profile $v^*$ satisfies $v^* = B(v^*, \delta)$. $\Re^n$ also trivially satisfies it as well. More precisely, $E(\delta)$ is the maximal set in $V$ which satisfies $E(\delta) = B(E(\delta), \delta)$.

**Remark.** All the subsets of $V$ forms a lattice with respect to $\cup$ and $\cap$ and a partial order induced by inclusion. Since $B(\cdot, \delta)$ is a monotone operator on this lattice as claimed below, (1) there exists a maximal fixed point, (2) a minimal fixed point, and (3) all fixed points form a complete lattice by Tarski's fixed point theorem. The maximal fixed point is given by $\cup_{W\subset V:W\subset B(W,\delta)}W$ (union of all self-generating sets). Since $E(\delta)\subset\cup_{W\subset V:W\subset B(W,\delta)}W$ by Theorem 8 and $\cup_{W\subset V:W\subset B(W,\delta)}W\subset E(\delta)$ by Lemma 7, $E(\delta)$ is the maximal fixed point of $B(\cdot,\delta)$.

It is easy to show that $E(\delta)$ is compact.

**Proposition 9** $E(\delta)$ *is compact*

**Proof.** I assume that $A$ is finite. The proof for compact $A$ is left as an exercise. Since $E(\delta)$ is bounded, we just need to show that it is closed. I sketch two proofs. The first, more direct one, is as follows. (1) Pick $v_n\in E(\delta)$ and $v^*$ such that $\lim_n v_n\to v^*$. Let $\sigma_n\in\Sigma$ be a subgame perfect equilibrium to support $v_n$. Then you can find a subsequence of $\{\sigma_n\}$ (how?) which converges to some $\sigma^*\in\Sigma$ at every history. In the limit, $\sigma^*$ achieves $v^*$ and every one-shot deviation constraint is satisfied for $\sigma^*$. Thus $\sigma^*$ is a subgame perfect equilibrium by one-shot deviation principle, hence $v^*\in E(\delta)$. (2) we show $\overline{E(\delta)}\subset B\left(\overline{E(\delta)},\delta\right)$, which implies $\overline{E(\delta)}\subset E(\delta)$ (definition of closedness). Again pick $v_n\in E(\delta)$ and $v^*$ such that $\lim_n v_n\to v^*$. Let $(a_n,w_n(\cdot))$ be an admissible pair with respect to $E(\delta)$ to achieve $v_n$. Since $A$ is a finite set and $\overline{E(\delta)}$ is compact, $\{(a_n,w_n(\cdot))\}_n$ has a subsequence which converges to $(a^*,w^*(\cdot))$ which is admissible with respect to $\overline{E(\delta)}$ and achieves $v^*$. Therefore $v^*\in B\left(\overline{E(\delta)},\delta\right)$ ■

**Exercise.** Prove this proposition when $A_i, i=1,...,n$ are compact sets in two ways ((1) and (2)). (hint: use *simple strategies* for (1) (the definition below))

**Algorithm to compute** $E(\delta)$

Here we assume that $A_i$ is finite. We need two useful lemmas about operator $B$.

**Lemma 10** *(monotonicity) If* $W'\subset W''$, *then* $B(W',\delta)\subset B(W'',\delta)$

**Lemma 11** *(compactness)* $B(W,\delta)$ *is compact if* $W$ *is compact.*

The proof of these lemmas is left for your exercise (The first one is obvious and the proof of the second one is identical to the proof (2) above). They can be used to derive a useful algorithm to obtain the "fixed point" $E(\delta)$. Let $W^0$ be any compact set to include $E(\delta)$ such that $B\left(W^0, \delta\right) \subset W^0$. For example, the whole feasible payoff set $V$ can be taken as $W^0$.

**Theorem 12** $E(\delta) = \lim_{t\to\infty} B^t\left(W^0, \delta\right)$.

**Proof.** By assumption, $E(\delta) \subset W^0$. Applying $B(\cdot, \delta)$ to both sides, we obtain $E(\delta) \subset B\left(W^0, \delta\right) = W^1 \subset W^0$ by Monotonicity Lemma and Theorem 8. Then by induction, $W^t = B^t\left(W^0, \delta\right), t = 1, 2, ...$ satisfies

$$E(\delta) \subset ... \subset W^t .... \subset W^2 \subset W^1 \subset W^0$$

Since $\left\{W^t\right\}_t$ are all compact sets by Compactness Lemma, $\cap_t W^t = \lim_{t\to\infty} W^t$ is compact (and nonempty because $E(\delta)$ is nonempty). Clearly $E(\delta) \subset \cap_t W^t$. On the other hand, for any $v \in \cap_t W^t$, there exist admissible pairs $\left(a^t, w^t\right), t = 0, 1, 2, ...$ such that $w^t : A \to W^t$ and $v = (1-\delta) g\left(a^t\right) + \delta w^t\left(a^t\right)$. Since each $W^t$ is compact, we can take a converging subsequence of $\left(a^t, w^t\right) \to (a^*, w^*)$ such that $(a^*, w^*)$ is admissible with respect to each $W^t$, hence also admissible with respect $\cap_t W^t$, and $v = (1-\delta) g(a^*) + \delta w^*(a^*)$. Thus, we obtain $\cap_t W^t \subset B\left(\cap_t W^t, \delta\right)$, which implies $\cap_t W^t \subset E(\delta)$ by Lemma 7 ■

**Remark**. Since $E(\delta)$ is an nonempty intersection of compact sets, compactness of $E(\delta)$ follows from this theorem, too.

This theorem still does not offer an explicit method to compute $E(\delta)$. Here is one algorithm to compute $E(\delta)$ (Judd, Yeltekin, and Conklin [9]). Start with $\widehat{W}^0 = V$. At step $t$, find a outer (inner) polytope approximation $\widetilde{W}^{t-1}$ which contains (is contained in) $\widehat{W}^{t-1}$. A systematic way to do this is to pick a finite points on the boundary of $\widehat{W}^{t-1}$ which is largest with respect to a set of preselected finite directions and use them to define an outer (inner) polytope approximation. Apply $B(\cdot, \delta)$ to $\widetilde{W}^{t-1}$ to obtain $\widehat{W}^t$ (this step is computationally easy because $\widetilde{W}^{t-1}$ is a polytope) and go to step $t+1$. You can show that the sequence $\left\{\widetilde{W}^t\right\}_t$ is a decreasing sequence and $\cap_t \widetilde{W}^t$ provides an upper bound and a lower bound of $E(\delta)$ depending on whether you use an outer polytope approximation or an inner polytope approximation. If the polytopes at each step is a good approximation of $\widehat{W}^t$ (if the number of preselected finite directions is large), then the resulting upper estimate and lower estimate of $E(\delta)$ gets very close and you can obtain a good idea of the boundary of $E(\delta)$.

If $E(\delta)$ is one dimensional, then you may be able to compute $E(\delta)$ with pen and pencil. Try the following exercise. Also see the application in the next subsection and Cronshaw and Luenberger [5].

**Exercise.** *Strongly symmetric subgame-perfect equilibrium* is a subgame-perfect equilibrium in which every player plays the same action at every history. Let $E^{SSSPE}(\delta)$ be the strongly symmetric SPE payoff set of the repeated game of the following PD game. For which $\delta$ does $E^{SSSPE}(\delta)$ coincides with all symmetric feasible payoffs $\{(x,x) : x \in [0,1]\}$?

| | $C$ | $D$ |
|---|---|---|
| $C$ | $1,1,$ | $-1,2$ |
| $D$ | $2,-1$ | $0,0$ |

**Comparative Statics in $\delta$**

We have fixed $\delta$ so far. How $E(\delta)$ would change as players become more patient? Intuitively, we should be able to support more payoffs as players become more patient. This conjecture is partially justified by the following proposition.

**Proposition 13** *If $W$ is convex, then $W \subset B(W,\delta)$ implies that $W \subset B(W,\delta')$ for $\delta' \in (\delta, 1)$.*

**Proof.** Take any $v \in W$ and an admissible pair $(a, w(\cdot))$ to generate $v$. Define $w^{\delta'}(\cdot)$ as follows;

$$w^{\delta'}(a) = \frac{(1-\delta')\delta}{(1-\delta)\delta'} w(a) + \frac{\delta' - \delta}{(1-\delta)\delta'} v$$

Since $w^{\delta'}(a)$ is a positive linear combination of $w(a)$ and $v$, $w^{\delta'}(a) \in W$. It is also straightforward to check all the incentive constraints are satisfied, hence $(a, w^{\delta'}(\cdot))$ is admissible w.r.t. $W$, and $(a, w^{\delta'}(\cdot))$ generates $v$. So, $W \subset B(W, \delta')$. ■

**Remark.** This implies that $E(\delta)$ is "weakly expanding" in $\delta$ when it is convex. Note that if there is a public randomization device, it convexifies the equilibrium payoff set, hence the equilibrium payoff set should be weakly increasing. (Stahl [13]).

If there is no public randomization device, then $E(\delta)$ may not be convex, and indeed may not be monotonically expanding as $\delta \to 1$. Consider the following example by Sorin [12].

| | $L$ | $R$ |
|---|---|---|
| $U$ | $1,0$ | $0,0$ |
| $D$ | $0,0$ | $0,1$ |

Suppose that $\delta = \frac{1}{8}$. Then $\left(\frac{7}{8}, \frac{1}{8}\right)$ can be achieved by playing $(U, L)$ first and playing $(D, R)$ forever. However this cannot be achieved with $\delta = \frac{1}{4}$. Note that $(U, L)$ must be played in the first period. Then each player's continuation payoff must be $\frac{1}{2}$ to achieve $\left(\frac{7}{8}, \frac{1}{8}\right)$. But whoever gets one in the next period gets more than $\frac{3}{4}$. Of course $(U, R)$ or $(D, L)$ must not be played because they are inefficient.

Nonmonotonicity can be really extreme (Mailath, Obara and Sekiguchi [10]). Take the PD game in the above exercise. Consider the best equilibrium payoff $v_2(\delta)$ for player 2 among all the efficient payoff profiles. $v_2(\delta)$ is 1.5 when $\delta \in \left[\frac{2}{3}, 1\right)$. No efficient profile is supported when $\delta \in \left(0, \frac{1}{2}\right)$. In $\left(\frac{1}{2}, \frac{2}{3}\right]$, $v_2(\delta)$ is downward sloping almost everywhere and there are uncountably many discontinuity points (within $\left(\frac{1}{2}, \frac{2}{3}\right]$).

However we will show later that $E(\delta)$ expands to its natural limit as $\delta \to 1$.

**Simple Strategy** (Abreu [2])

Since $E(\delta)$ is compact, for each player $i$, there exists a SPE payoff $\underline{w}_i$ which minimizes player $i$'s payoff among $E(\delta)$. Let $Q_i$ be the equilibrium path to generate $\underline{w}_i$. Take any equilibrium strategy $\sigma^*$ and let $Q_0$ be the equilibrium path of the equilibrium $\sigma^*$. Now consider a following *simple strategy* $\sigma(Q_0, Q_1, ..., Q_n)$ based on the $n$ path $Q_0, Q_1, ..., Q_n$;

(1) Play $Q_0$ if there is no deviation or there are simultaneous deviations by more than one player. If there is a unilateral deviation by player $i$, then go to (2) (i).

(2)(i) Play $Q_i$ if there is no deviation or there are simultaneous deviations by more than one player. If there is a unilateral deviation by player $j$, then go to (2) (j).

This is a well defined strategy, generates the same payoff as $\sigma^*$, and still a subgame perfect equilibrium (because all the punishments in $\sigma^*$ are replaced by the harshest punishments $\underline{w}_i, i = 1, ..., n$). This means that every equilibrium payoff can be supported by simple strategies and we can restrict our attention to $\underline{w}_i, i = 1, ..., n$ as a mean of punishment without loss of generality. Remember that $\underline{w}_i, i = 1, ..., n$ are critical numbers to determine the shape of the whole equilibrium payoff set. The smaller the possible punishments $\underline{w}_i, i = 1, ..., n$ are, the more payoffs you can support.

## 2.2 Application to Dynamic Oligopoly Problem

This subsection studies a dynamic Cournot competition model from Abreu [1]. This and related class of dynamic oligopoly models have been an important application of repeated games and, at the same time, a source of idea behind theoretical development of repeated games.

We focus on strongly symmetric equilibrium: equilibrium in which all firms produce the same output after every history. This means that the dimension of $E(\delta)$ is at most 1. This fact simplifies computation of an equilibrium payoff set. Furthermore, we just need to focus on the best and the worst equilibrium payoffs in order to find the best equilibrium payoff. A pair of the best payoff and the worst payoff turns out to be a fixed point of two simple mappings.

There are $n$ firms with payoff functions $\pi_i(\mathbf{q}) = \left(p\left(\sum_j q_j\right) - c\right) q_i$, where $\mathbf{q} = (q_1, ..., q_n)$ is a profile of quantity chosen by the $n$ firms, $c$ is a constant marginal cost, and $p : \Re^+ \to \Re^+$ is a strictly decreasing and continuous inverse demand function such that $p(0) > c$ and $\lim_{q\to\infty} p(q) < c$. We assume that the monopoly output and the Cournot equilibrium output is unique. Let $Q_m$ be the monopoly output and $x^m = \frac{Q_m}{n}$, and denote Cournot equilibrium output for each firm by $x^{cn}$. Let $\pi_i(q)$ be firm $i$'s payoff when every firm chooses $q$ and $\pi_i^*(q') = \max_{q_i \in [0,\infty]} \pi_i\left(q_i, q'_{-i}\right)$ (the maximum payoff when deviating from a symmetric profile $q'$). Figure 1 shows a typical $\pi_i^*$ and $\pi_i$.

For each discount factor, there is a certain level of $\overline{q}(\delta)$ such that any $q > \overline{q}$ cannot be chosen in any equilibrium. This is because if $q'$ is large enough, then $\pi_i(q', q_{-i})$ becomes very low for any $q_{-i}$ so that the discounted average payoff is negative even if firm $i$ becomes a monopoly from the next period. Thus we can restrict firms' choice sets to a compact set $[0, \overline{q}]$ without loss of generality. We omit subscript $i$ from now on unless necessary because the game is symmetric.

For $V' \geq V''$, let

$$f\left(V'', V'\right) = \max_{q \in [0,\overline{q}], V \in [V'', V']} (1-\delta)\,\pi(q) + \delta V \qquad (1)$$
$$s.t.\ \ (1-\delta)\left(\pi^*(q) - \pi(q)\right) \leq \delta\left(V - V''\right)$$

and

$$g\left(V'', V'\right) = \min_{q \in [0,\overline{q}], V \in [V'', V']} (1-\delta)\,\pi(q) + \delta V \qquad (2)$$
$$s.t.\ \ (1-\delta)\left(\pi^*(q) - \pi(q)\right) \leq \delta\left(V - V''\right)$$

We can apply the algorithm of Theorem 12. Let's start with $W^0 = [0, \pi(x^m)]$, which is a feasible set of symmetric payoffs. Note that $f(0, \pi(x^m))$ and $g(0, \pi(x^m))$ provides the upper end and the lower end of compact set $W^1 = B\left(W^0, \delta\right)$.

For some parametric example, $W^t, t = 0, 1, 2, ...$ are always convex. In this case, the operator $B$ reduces to two mappings $f$ and $g$. The equilibrium payoff set $E(\delta)$ is a closed interval and $(\min E(\delta), \max E(\delta))$ is a fixed point of $(f, g)$.

Even when $W^t, t = 0, 1, 2, ...$ are not convex, the best and worst equilibrium can be still characterized by $f$ and $g$. Let $\overline{V}_1 = f(0, \pi(x^m))$ and $\underline{V}_1 = g(0, \pi(x^m))$. Even though $W^1$ may be strictly smaller than $\left[\underline{V}_1, \overline{V}_1\right]$, we can apply $f$ and $g$ to $\left(\underline{V}_1, \overline{V}_1\right)$ to obtain $\left\{\underline{V}_t, \overline{V}_t\right\}_t$. It can be easily shown that $\overline{V}_t$ is a decreasing sequence and $\underline{V}_t$ is an increasing sequence, thus converging to some $\overline{V}$ and $\underline{V}$ respectively, which is a fixed point of $(f, g)$. Then it is easy to show that $\overline{V} = \max E(\delta)$ and $\underline{V} = \min E(\delta)$ still holds.

**Exercise.** Prove $\overline{V} = \max E(\delta)$ and $\underline{V} = \min E(\delta)$.

**Remark.**

- Note that we can just focus on two points, which can simplify the computation significantly.
- The only assumptions we have used so far are that (1) symmetry, (2) continuity of $\pi(q)$, and (3) compactness of the choice sets.

Now let's try to figure out the equilibrium behavior for $\overline{V}$ and $\underline{V}$. See the following figure.

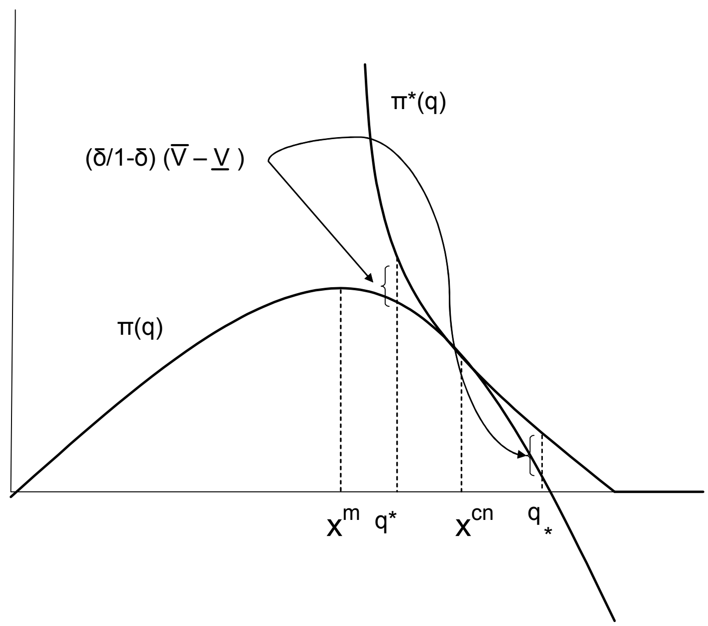


Figure 1

From this figure, it is clear that, given any $\underline{V} < \overline{V}$, the solution of (1) is $q^* \in [x^m, x^{cn}]$ which satisfies $(1-\delta)(\pi^*(q^*) - \pi(q^*)) = \delta\left(\overline{V} - \underline{V}\right)$ ($\leq$ if $q^* = x^m$) and is closest to $x^m$. Similarly, the solution of (2) turns out to be $q_*\,(\geq x^{cn})$ and $\overline{V}$ such that $(1-\delta)(\pi^*(q_*) - \pi(q_*)) = \delta\left(\overline{V} - \underline{V}\right)$ without loss of generality. This is because of the following reason. Fix any $\left(\widetilde{q}, \widetilde{V}\right)$ which satisfies (2). If $\widetilde{V} < \overline{V}$, then we can find $\left(\widetilde{\widetilde{q}}, \overline{V}\right)$ such that (1): $(1-\delta)\pi\left(\widetilde{\widetilde{q}}\right) + \delta\overline{V} = (1-\delta)\pi(\widetilde{q}) + \delta\widetilde{V}$ (because $\pi(q)$ is unbounded below) and (2) incentive constraints are satisfied.[5]

[5] (2) can be shown as follows,

$$\begin{aligned}
& (1-\delta)\pi\left(\widetilde{\widetilde{q}}\right) + \delta\overline{V} \\
= \;& (1-\delta)\pi(\widetilde{q}) + \delta\widetilde{V} \\
\geqq \;& (1-\delta)\pi^*(\widetilde{q}) + \delta\underline{V} \\
\geqq \;& (1-\delta)\pi^*\left(\widetilde{\widetilde{q}}\right) + \delta\underline{V}\ .
\end{aligned}$$

Therefore $\overline{V}$ $(\underline{V})$ can be supported by playing $q^*$ $(q_*)$ in the current period and using $\overline{V}$ and $\underline{V}$ as continuation payoffs. This means that $\overline{V}$ can be supported by a simple (symmetric) strategy $\sigma(Q^*, Q_*)$ where $Q^* = (q^*, q^*, ....)$ and $Q_* = (q_*, q^*, q^*, ....)$ (called "stick and carrot"). Similarly, $\underline{V}$ can be supported by a simple strategy $\sigma(Q_*, Q_*)$.

**Remark.** Every player gets a payoff below the stage game (Cournot) equilibrium payoff in the first period to achieve $\underline{V}$. Even in more general settings, this is always true when the minimum (strongly symmetric) SPE is lower than the stage game Nash equilibrium payoff.

# 3 Folk Theorem

We already know that there can be many equilibria in repeated games. But how many? A celebrated Folk Theorem says that, in general, any feasible and "reasonable" payoff profile can be supported in subgame perfect equilibrium if players are patient enough. We prove this theorem in this section. This entire section is based on [6].

We need some new notations.

- (minmax payoff): $\underline{v}_i = \min_{a_{-i}\in\prod_{j\neq i} A_j} \max_{a_i\in A_i} g_i(a_i, a_{-i})$

  We normalize $\underline{v} = (\underline{v}_1, ..., \underline{v}_n)$ to $\mathbf{0} \in \Re^n$ without loss of generality.

- (Action minmaxing player $i$): $m^i \in A$

- (Individually rational payoff set): $V^* = \{v \in V | v_i > 0 \text{ for all } i \in I\}$

- $\overline{g} = \max_i \max_{a,a'\in A} \{g_i(a) - g_i(a')\}$

We assume that players can use a public randomization device so that any payoff profile in $V$ is feasible at each period.

First note that in any SPE (indeed any Nash) equilibrium, each player can get at least her minmax payoff every period by playing her best response action.

**Proposition 14** *For any equilibrium profile $\sigma \in \Sigma$, $V_i(\sigma) \geq 0$ for all $i \in I$*

We show that almost every payoff profile above the minmax payoff profile can be supported in subgame perfect equilibrium.

**Theorem 15** *Suppose that $V^*$ has a nonempty interior in $\Re^n$. Then, for any $v \in V^*$, there exists $\underline{\delta}$ such that for all $\delta \in (\underline{\delta}, 1)$, there exists a subgame perfect equilibrium in which player i's discounted average payoff is $v_i$ for $i \in I$.*

We assume that there exists $a \in A$ to achieve $v\,(= g(a))$. This is without loss of generality when (1) a public randomization is available or (2) mixed strategy is observable or (3) action set is compact and $g$ is continuous.

The basic idea of the proof is simple. Players play $a$ every period if there is no deviation. Any unilateral deviation from $a$ is punished by a finite number of minmax action plays. The only complication arises when the other players suffer from minmaxing a deviator. To keep incentive of these players to punish the deviator, each player other than the deviator gets a "reward" after a finite number of periods minmaxing the deviator. More precisely, we use a payoff profile $\widetilde{v}^j = \left(\widetilde{v}_1^j, ..., \widetilde{v}_n^j\right) \in$ for each $j \in I$ which satisfies the following two properties: (1) $v_i \geq \widetilde{v}_i^i$ and (2) $\widetilde{v}_i^j > \widetilde{v}_i^i$ for every $i, j \neq i$. If player $j$ is the deviator, then the players receive $\widetilde{v}^j$ after minmaxing player $j$ for the finite number of periods. When player $i$ deviates from punishing the deviator $j$, she is immediately minmaxed for a finite number of periods and receive $\widetilde{v}^i$ afterwards. Since $\widetilde{v}_i^j > \widetilde{v}_i^i$, she would lose some payoff forever once the punishment phase is over. So player $i$ does not have incentive to deviate from minmaxing player $j$ if she is enough patient even though it could be costly to her in the short run. Finally, $\widetilde{v}^i$ can be easily supported exactly like $v$.

**Proof.** Take any $v \in V^*$ and $a \in A$ such that $v = g(a)$. If there is not such a pure action profile, we can use public randomization device to generate $v$ by randomizing between appropriate action profiles. We can find $\widetilde{v}^j = \left(\widetilde{v}_1^j, ..., \widetilde{v}_n^j\right) \in V^*$ for each $j \in I$ to satisfy the above properties when $V$ has an nonempty interior (how?). Again we assume that there exists $\widetilde{a}^j \in A$ such that $g\left(\widetilde{a}^j\right) = \widetilde{v}^j$. Choose an integer $T$ as follows,

$$\overline{g} < T \cdot \min_i \widetilde{v}_i^i. \tag{3}$$

Now we construct a subgame perfect equilibrium with payoff $v$. The game start in Phase I.

- Phase I: Play $a$ as long as there was no unilateral deviation from $a$ in the last period (or in the initial period). If player $i$ deviates from $a$ unilaterally, then move to phase $\text{II}_i$.

- Phase $\text{II}_i$: Play $m^i$ T periods, and go to Phase $\text{III}_i$. If any player $j \neq i$ deviates unilaterally in Phase $\text{II}_i$, then move to Phase $\text{II}_j$. Otherwise stay in Phase $\text{II}_i$.

- Phase $\text{III}_i$; Play $\widetilde{a}^i$ as long as $a^i_\varepsilon$ there was no unilateral deviation from $a$ in the last period (or in the beginning of the Phase $\text{III}_i$). If player $j$ deviates from $a$ unilaterally, then move to phase $\text{II}_j$.

We check that there is no profitable one-shot deviation for every subgame, i.e. in Phase I,II,III.

- Phase I: If player $i$ deviates from $a$, the maximum deviation gain is bounded by $\overline{g}$, while she loses $v_i$ for T periods at least (and more if $v_i > \widetilde{v}^i_i$). So her incentive constraint is satisfied if

$$\overline{g} \leqq \left(\delta + \delta^2 + ... + \delta^T\right) v_i$$

  holds. This equality is satisfied strictly every $i \in I$ if $\delta$ is large enough by (3).

- Phase $\text{II}_i$: Clearly player $i$ does not have incentive to deviate from $m^i$. If any player $j \neq i$ deviates from $m^i$, then player $j$ can gain at most $\overline{g}$ for at most $T$ periods. On the other hand, player $j$ have to lose $\widetilde{v}^i_j - \widetilde{v}^j_j$ forever after $T$ periods. So her incentive constraint is satisfied if

$$\left(1 + \delta + ... + \delta^{T-1}\right) \overline{g} \leqq \left(\delta^T + \delta^{T+1} + ...\right) \varepsilon$$

  holds. This incentive constraint holds strictly if $\delta$ is large enough.

- Phase $\text{III}_i$: the same as Phase I.

Hence, the strategy described above is a subgame perfect equilibrium in which players' discounted average payoff profile is $v$. ■

**Exercise.** Construct $\widetilde{v}^j$ to satisfy (1) and (2) when $V^*$ has a nonempty interior (hint. When $V^*$ has a nonempty interior, you can find an interior point in any small neighborhood of any point in $V^*$).

**Remark.**

- What we need for the above proof is simply a set of asymmetric payoff profiles $\widetilde{v}^i, i = 1, ..., n$ to satisfy (1) and (2). A weaker condition than nonempty interior of $V^*$ is sufficient for this. Abreu, Dutta and Smith [3] proposed a condition called NEU condition, which is equivalent to existence of such payoff profiles (given that $V^* \neq \emptyset$). NEU stands for *Non Equivalent Utility,* which means that any two players' payoffs are not equivalent (up to affine transformation).

- The following is an example from Fudenberg and Maskin [6] in which the folk theorem fails.

| | |
|---|---|
| (1,1,1) | (0,0,0) |
| (0,0,0) | (0,0,0) |

| | |
|---|---|
| (0,0,0) | (0,0,0) |
| (0,0,0) | (1,1,1) |

  Player 1 chooses a row, player 2 chooses a column, and player 3 chooses either the right game or the left game. The minmax payoff is 0 for every player. But, the equilibrium payoff cannot be less than $\frac{1}{4}$ (the one-shot mixed strategy equilibrium payoff). Note that this payoff matrix violates NEU.

- When NEU is violated, standard minmax payoff is not the right reference point. Wen [14] introduced the notion of *effective minmax* to general stage games, which coincide with the standard minmax when NEU is satisfied and show that the folk theorem always holds with respect to effective minmax.

  Player $i$'s effective minmax is defined as follows,

$$\min_{a \in A} \max_{j \in J(i)} \max_{a_j \in A_j} g_i(a)$$

  where $J(i)$ is the set of the players whose payoffs are equivalent to player $i$. In the above example, each player's effective minmax is $\frac{1}{4}$ assuming observable mix strategies (thus all mixed strategies are essentially pure strategies). Therefore the subgame perfect equilibrium payoff set coincides with $\left\{(x, x, x) : x \in \left[\frac{1}{4}, 1\right]\right\}$ in this example (in the limit as $\delta \to 1$).

- Suppose that action sets are finite, mixed strategies are unobservable, and a public randomization is not available. Then there may not exist $a$ to achieve $v$ or $\widetilde{a}_i$ to achieve $\widetilde{v}_i$. In this case, for any $v \in V$ and $\varepsilon > 0$, you can find $\underline{\delta} \in (0,1)$ and a sequence of action profiles which achieves $v$ and the continuation payoff profile after any period is within $\varepsilon$ of $v$ for $\delta \in (\underline{\delta}, 1)$. Using this fact, all the arguments based on strict individual rationality and strict incentive constraints go through if you pick a small enough $\varepsilon$ and players are enough patient. See Fudenberg and Maskin [7] for a detail.

- We have restricted attention to pure strategy. What would happen if mixed strategies are allowed? The only change is that we can use mixed minmax payoffs $\left(\min_{a_{-i}\in\prod_{j\neq i}\triangle A_j}\max_{a_i\in A_i} g_i\left(a_i,a_{-i}\right)\right)$ rather than pure minmax payoffs. Mixed minmax can be smaller than pure minmax by definition (Ex. Matching Penny). The proof of the above folk theorem is almost identical in this case. The only complication arises in Phase $\text{II}_i$ when player $j \neq i$ needs to randomize to minmax player $i$. To give her incentive to do so, you can adjust the continuation payoff after a finite number of minmaxing periods based on the realization of actions so that player $j$ has incentive to randomize. Again see Fudenberg and Maskin [7] for a detail.

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