Econ 201B: Problem Set III

Due February 9, 2012

1. **Non-Weakly Dominated Action is Cautious Best Response**

   Consider a finite strategic game with two players. We know that a_1 is not strictly dominated if and only if a_1 is a best response to some mixed strategy $\alpha_2 \in \Delta(A_2)$. A similar result holds for actions that are not weakly dominated: a_1 is not weakly dominated if and only if a_1 is a best response to some *completely mixed* strategy, i.e. $\alpha_2 \in \triangle(A_2)$ such that $\alpha_2(a_2) > 0$ for every $a_2 \in A_2$. Prove this statement.

   Hint 1: You may follow the following steps. This approach is based on SHT.

   (a) Let a_1^* be any action that is not weakly dominated for player 1. Let $v_1(a) = u_1(a_1, a_2) - u_1(a_1^*, a_2)$. We assume that, for every a_2', there exists different a_1' such that

   $$u_1(a_1', a_2) - u_1(a_1^*, a_2) = \begin{cases} -1 \text{ for } a_2 = a_2' \\ \varepsilon \text{ for } a_2 \neq a_2' \end{cases}$$

   for some $\varepsilon > 0$. If not, we can add such auxiliary actions. Verify that a_1^* is still not weakly dominated after adding such actions for small enough $\varepsilon > 0$.

   (b) Let U_1 be the subset of $\mathbb{R}^{|A_2|}$ that can be represented by a convex combination of v_1. That is, $U_1 = \left\{ x \in \mathbb{R}^{|A_2|} | \exists \alpha_1 \in \triangle(A_1), \forall a_2 \in A_2, x(a_2) = \sum_{a_1 \in A_1} \alpha_1(a_1) v_1(a_1, a_2) \right\}$. Apply the separating hyperplane theorem to show that there exists $p^* \neq 0$ to separate $\mathbb{R}_+^{|A_2|}$ from U_1, i.e. $p^* \cdot x \leq p^* \cdot y$ for any $y \in \mathbb{R}_+^{|A_2|}$ and any $x \in U_1$

   (c) Show that $p^* \gg 0$.

   (d) Complete the proof.

Hint 2: Here is another approach, which is more in line with the hint given by O&R for Exercise 64.2. This approach is based on MMT.

(e) Let a_1^* be any action that is not weakly dominated for player 1. Define a two player zero-sum game $(\{1,2\},(A_i),(v_i))$, where $v_1(a) = u_1(a_1,a_2) - u_1(a_1^*,a_2)$ and $v_2 = -v_1$. We assume that, for every a_2', there exists different a_1' such that

$$u_1\left(a_1', a_2\right) - u_1\left(a_1^*, a_2\right) = \begin{cases} -1 \text{ for } a_2 = a_2' \\ \varepsilon \text{ for } a_2 \neq a_2' \end{cases}$$

for some $\varepsilon > 0$. If not, we can add such auxiliary actions. Verify that a_1^* is still not weakly dominated after adding such actions for small enough $\varepsilon > 0$.

(f) Show that (1) there exists a Nash equilibrium $\alpha^* \in \triangle(A_1) \times \triangle(A_2)$ and (2) α_i^* is a maxminimizer for $i = 1, 2$ (you can use the minimax theorem).

(g) Show that playing a_1^* with probability 1 is a maxminimizer for player 1 and $\alpha_2^*(a_2) > 0$ for all $a_2 \in A_2$.

(h) Complete the proof.

2. **Iterated Elimination of Dominated Actions and Nash Equilibrium**

Consider a general finite strategic game and an elimination process where all weakly dominated actions of every player are eliminated simultaneously in each step. This process stops in finite number of steps.

(a) Let α^* be a mixed strategy Nash equilibrium of the remaining strategic game. Show that α^* is a Nash equilibrium of the original strategic game.

(b) Is α^* a trembling hand perfect equilibrium? Explain.