Econ 201A: Problem Set VII

Due December 2, 2010

1. **Unbinding Constraints in Programming Problem**

   Consider the programming problem $\max_{x \in \mathbb{R}^L} f(x)$ s.t. $x \in X$, where X is a closed and convex set in $\mathbb{R}^L$. Suppose that you added another constraint $g(x) \geq 0$ to the problem, derived an optimal solution x^*, and found that this additional constraint is not really binding at x^*.

   (a) Can you conclude that x^* is also a solution for $\max_{x \in \mathbb{R}^L} f(x)$ s.t. $x \in X$? If not, find a counterexample (a graphical example would suffice).

   (b) Suppose that f is either strictly quasi-concave or pseudo-concave and g is continuous. Is x^* a solution for $\max_{x \in \mathbb{R}^L} f(x)$ s.t. $x \in X$ in this case?

   (c) Consider the consumer utility maximization problem with truncated consumption set ($\max_{x_i \in X_i} u_i(x_i)\ s.t.\ p \cdot x_i \leq p \cdot e_i,\ x_i \leq \overline{r}$). Let $\overline{x}_i(p)$ be a solution for this problem. Apply the above result to show that $\overline{x}_i(p)$ is also a solution for the standard utility maximization problem if $\overline{x}_i(p) \ll \overline{r}$. Also do the same exercise with the profit maximization problem with truncated production set ($y_j \leq \overline{y}$).

2. **Boundary Behavior of Walrasian Demand**

   Consider the following utility maximization problem for consumer i given $p_n \in \mathbb{R}^L_{++}$ and $e_i = (1, ..., 1)$:

$$\max_{x_i \in \mathbb{R}^L_{++}} \sum_{\ell=1}^{L} \sqrt{x_{i,\ell}}$$
$$\text{s.t. } p_n \cdot x_i \leq p_n \cdot e_i.$$

   (a) Suppose that $L = 2$. Consider a sequence of prices $\{p_n\}_{n=1}^{\infty} \subset \mathbb{R}^L_{++}$ that converges to some $p^* (\neq 0) \in \mathbb{R}^L_{+}$ such that $p^*_1 = 0$. Either prove that $\lim_{n \to \infty} x_{i,1}(p_n, p_n \cdot e_i) = \infty$ holds for any such sequence or find a counterexample where $\lim_{n \to \infty} x_{i,1}(p_n, p_n \cdot e_i) < \infty$.

   (b) Suppose that $L = 3$. Answer the same question as (a).

3. **Joint Maximization Approach to Equilibrium Existence**

The goal of this problem is to illustrate another approach to the existence of competitive equilibrium. We focus on pure exchange economy, but allow demand correspondences unlike other proofs ("Excess demand approach") we went over in the class.

We make the following assumptions.

- $X_i = \mathbb{R}_+^L$
- $\succeq_i$ is rational and continuous (hence can be represented by a continuous utility function u_i).
- $\succeq_i$ is locally nonsatiated.
- $x_i' \succ_i x_i \Longrightarrow \alpha x_i' + (1-\alpha) x_i \succ_i x_i$ for any $\alpha \in (0,1)$.
- $e_i \gg 0$.

Answer the following questions.

(a) Consider the utility maximization problem with truncated consumption set $\{x_i \in X_i | x_i \leqq \overline{r}\}$, where $\overline{r} \gg r = \sum_{i=1}^{I} e_i$. Show that the solution $\overline{x}_i(p)$ is nonempty, convex-valued and upper hemicontinuous in the price simplex $\triangle$.

(b) Consider some hypothetical agent, call him an "auctioneer", who maximizes the value of excess demand $p \cdot \left(\sum_{i=1}^{I} x_i - r \right)$ with respect to $p \in \triangle$ given any allocation $x \in \overline{A} = \left\{ x \in \mathbb{R}_+^{L \times I} | \sum_{i=1}^{I} x_i \leq \overline{r} I \right\}$. Let $p(x) \subset \triangle$ be the set of solutions given $x \in \overline{A}$. Show that $p(x)$ is nonempty, convex-valued and upper hemicontinuous in A.

(c) Define $\phi : \overline{A} \times \triangle \rightrightarrows \overline{A} \times \triangle$ by $\phi(x,p) := (\overline{x}(p), p(x))$, where $\overline{x}(p) = (\overline{x}_1(p), ..., \overline{x}_I(p))$. Since a product of uhc correspondences is upper hemicontinuous, ϕ is an upper hemicontinuous correspondence on $\overline{A} \times \triangle$. Show that there exists $(x^*, p^*) \in A \times \triangle$ (where A is the set of feasible allocations) such that $(x^*, p^*) \in \phi(x^*, p^*)$

(d) Finish the proof of existence by showing that (x^*, p^*) is a competitive equilibrium. This last step involves the following two small steps:

   i. show $\overline{x}_i(p^*) = x_i(p^*)$.

   ii. show $\sum_{i=1}^{I} x_i^* \leq r$, and $\sum_{i=1}^{I} x_{i,\ell}^* = r_\ell$ if $p_\ell^* > 0$.