Repeated Games with Imperfect Public Monitoring

Ichiro Obara

January 11, 2003

1 Why Imperfect Public Monitoring?

In this note, we study an another information structure, *imperfect public monitoring*. In this class of model, players cannot observe the other players' actions directly, but can observe imperfect and public signals about them. There are at least two reasons why this information structure is worth special attention. First, it is simply a more reasonable assumption than perfect monitoring in some settings. Second, the equilibrium behavior can be significantly different from the equilibrium behavior in models with perfect monitoring, and they sometimes have significant economic implications. The following example illustrates these points.

Example

This is an (overly) simplified version of the dynamic quantity competition model in Green and Porter [4]. Consider two firms 1 and 2 producing the same product. Each firm can either "collude" (produce small amount of goods) or "not collude" (produce a large amount of goods). Each firm's profit is determined by its choice and a realization of common price. If both firms choose "collude", the price is "high" with probability $1 - p^0$ and "low" with probability p^0, if only one firm chooses "collude", the price is "high" with probability $1 - p^1$ and "low" with probability p^1, and finally, if both firms choose " not collude", the price is "high" with probability $1 - p^2$ and "low" with probability p^2. We assume that $p^2 > p^1 > p^0$, that is, low price is likely to realize when more firms choose " not collude". A firm's action is not observable, but price is a public information, hence this is a model of imperfect public monitoring. Each firm's strategy is just a mapping from past realizations of prices to $\{$"*collude*", "*not collude*"$\}$.

Suppose that stage game expected payoffs are given by the following matrix.

	C	NC
C	4,4	-1,6
NC	6, -1	1,1

First notice that the usual trigger strategy may constitute an equilibrium when players are very patient. Suppose that both players play (C, C) until a low price is observed, and play (NC, NC) forever once a low price is observed. Each firm's incentive constraint is

$$\begin{aligned} (1-\delta)\,6+\delta\left(\left(1-p^1\right)V^*+p^1V_*\right) &\leq (1-\delta)\,4+\delta\left(\left(1-p^0\right)V^*+p^0V_*\right) \\ (or\ \ (1-\delta)\,2 &\leq \delta\left(p^1-p^0\right)\left(V^*-V_*\right)) \\ where\ V^* &= (1-\delta)\,4+\delta\left(\left(1-p^0\right)V^*+p^0V_*\right) \\ and\ V_* &= 1 \end{aligned}$$

You can check that if $\frac{p^1}{p^0} > \frac{5}{3}$ (random price is informative enough) and players are patient enough, then the above constraints can be satisfied.

What is interesting in this trigger strategy equilibrium is that (NC, NC) is played on the equilibrium path. This contrasts with models with perfect monitoring. If firms play the above stage game over time with perfect monitoring, (NC, NC) may not be observed at all (if they play the best symmetric collusive equilibrium.). Indeed , the most important feature of this model is that *NC has to be played on the equilibrium path in any nontrivial equilibrium.* This is because firms have to punish themselves after a realization of low price to support any degree of collusion, which requires firms to play more NC following a realization of low price.

This might affect the way you interpret a dynamics of price observed in some particular market. Suppose that there is enough reason to believe that firms are playing the best strongly symmetric collusive equilibrium. If you believe that the true model is one with perfect monitoring, then an episode of "price war" would be taken as a proof that there is no collusion in the market. If firms are playing the best strongly collusive equilibrium, the price has to be always high on average and has to follow a stationary process. On the other hand, if you believe that the true model is one with imperfect public monitoring, you would not be able to reach that conclusion. On the contrary, you might take an episode of "price war" as a proof of collusion if such regime-switching is a part of the best strongly symmetric collusive equilibrium.[1]

Remark 1 *Since a punishment has to occur as an equilibrium phenomenon, it is not efficient to use the strongest punishment after a realization of "bad" signal unlike repeated games with perfect monitoring. So, players might want to use a punishment which is minimum, but enough to keep players' incentive to collude. For example, firms might be interested in a punishment strategy such as "play (NC, NC) for T periods and go back to (C, C)" rather than the grim trigger strategy in the above example.*

[1] This suggests that it is important to characterize the equilibrium behavior in the best strongly symmetic equilibrium, in particular, the equilibrium behavior in the punishment phase. Abreu, Pearce, and Stacchetti [1] shows that the price process in the best strongly symmetric equilibrium follows a certain Markov process.

2 Model

In the following, we introduce some notations and facts particular to repeated games with imperfect public monitoring. All the other details of the model follows a general model we introduced before.

Imperfect Public Monitoring

In repeated games with imperfect public monitoring, players' information is a stochastic public signal and the distribution of public signal depends on the action profile chosen by the players ($f(a) \in \triangle Y$). Typically, we assume that the support of the public signal distribution is constant across A, that is, support $f(a) = Y$ for all $a \in A$ (full support assumption).

Remark 2 *(Difficult) If full support is assumed, then any Nash equilibrium outcome (not strategy itself) is a sequential equilibrium outcome. In precise, for any Nash equilibrium, there exists a sequential equilibrium which generates the equivalent outcome distribution. Hence there is no discrepancy between Nash equilibrium payoff set and sequential equilibrium payoff set. The key point is that a player always believes that the other players are on the equilibrium path whether she has ever deviated or not. Since sequential rationality is satisfied on the equilibrium path in Nash equilibrium, if all players adopt the same strategies off the equilibrium path as on the equilibrium path, that whole strategy profile is a sequential equilibrium with the same outcome distribution. (Also see the footnote 4)*

Public history and Public Strategy

Player i's history consists of her own action and a realization of the public signal in the past. Let's call a sequence of her own past actions player i's *private history* and denote it by $h_i'^t = \left(a^1, a^2, ..., a^{t-1}\right) \in A^{t-1}$. A sequence of realization of the public signal is called *public history* and denoted simply by $h^t = \left(y^1, y^2, ..., y^{t-1}\right) \in Y^{t-1}$. Hence player i's t period history is $h_i^t = \left(h^t, h_i'^t\right)$.

It is usually the case that players can observe their payoffs. If payoff is observable, a player may be able to know what the other players might have played, which conflicts with the notion of imperfect public monitoring. The most common interpretation is to treat $g(a)$ as a expected payoff profile rather than a realized payoff profile (introduce $r_i(a_i, y)$ as a (realized) payoff of player i when a_i is played and y is observed so that $g_i(a) = \sum_y f(y|a)\, r_i(a_i, y)$).[2] In this way, we can guarantee that players can observe their (realized) payoffs, but they do not get any additional information from their realized payoffs.[3] Player

[2] This formulation is consistent with the partnership example in the first section.

[3] Alternatively, you may assume that $g(a)$ is the players' payoff given the action profile a, but players cannot access to their payoffs until the end of the game.

i's (pure) *strategy* σ_i is still a mapping from H_i to A_i. We come back to the issue of mixed strategy in the end. A strategy is called *public strategy* if it only depends on a public history.

Perfect Public Equilibrium (PPE)

Let σ^* be a profile of public strategies. We define a class of perfect Bayesian equilibrium based on public strategies.

Definition 3 *σ^* is a perfect public equilibrium if, for any public history $h^t \in Y^{t-1}$, $\sigma|_{h^t}^*$ forms a Nash equilibrium from that period on, that is,*

$$V_i\left(\sigma^*|_{h^t}\right) \geq V_i\left(\sigma_i, \sigma_{-i}^*|_{h^t}\right) \text{ for all } \sigma_i \in \Sigma_i \text{ and all } i$$

Although public strategy and PPE are somewhat restricted, they can support many payoffs. First, it is known that for any pure strategy sequential equilibrium (hence any pure strategy Nash equilibrium), there exists a PPE which generates the same equilibrium outcome (distribution).[4][5] So, it is without loss of generality to restrict attention to public strategies and PPE if we are only interested in pure strategy equilibria. Second, we can prove a Folk Theorem in PPE; under a certain assumption on information structure ($f(y|a)$), any feasible and individually rational (interior) payoff can be supported in PPE as long as players are patient enough.

3 Characterization

In repeated games with imperfect public monitoring, we can still find a recursive structure similar to what we find in repeated games with perfect monitoring. *Since players use public strategies, the continuation strategy after any realization of public signal needs to be a PPE of the original game.*[6]

[4] Fix any pure strategy sequential equilibrium σ^*. For each history $h_i^t = \left(h^t, h_i'^t\right)$, let $h_i^{t*} = \left(\left(h^t, h_i'^{t*}\right)\right)$ be a history which shares the same public history as h_i^t, but realizes on the equilibrium path with σ^*. Define σ' by $\sigma_i'\left(h_i^t\right) = \sigma_i^*\left(h_i^{t*}\right)$. This strategy σ' is a public strategy profile which coincides with σ^* on the equilibrium path. Note that, because of the full support assumption, every player always believes that the other players have never deviated from the equilibrium path. This implies that (1): σ_i' is the best response after every plyer i's history given σ_{-i}^*, (2): σ_{-i}^* and σ_{-i}' are equivalent strategy profiles to player i. So, σ' is a perfect public equilibrium which generates the same outcome distribution as σ^*.

[5] This depends on the assumption of full support. If the supports of the public signal are different for different action profiles, then a pure strategy sequential equilibrium may support a payoff which cannot be supported by a (pure strategy) perfect public equilibrium. If a public signal which should not be observed on the equilibrium path is observed, players might have to infer the other players' private actions in the past to form a belief about their continuation strategies (because strategies are not necessarily public for pure strategy SE). Such belief could be strange enough to generate some strange behavior off the equilibrium path so that the equilibrium outcome which cannot be generated by PPE may emerge.

[6] This recursive method was originally developped for repeated games with imperfect public montoring by [2]. But, we applied this method from the beginnng to repeated games with perfect monirtoring for pedagogical reason.

We can use almost exactly the same logic as the one with perfect monitoring. Let w be a mapping from Y to $\Re^n$ and define g_w by $g_w = (1-\delta)\, g(a) + \delta \sum_y f(y|a)\, w(y)$. For the sake of completeness, we repeat the following definition in this setting.

Definition 4 *For any $W \subset \Re^n$, a pair $(a, w(\cdot))$ is admissible with respect to $W \subset \Re^n$ if (1) $w(a) \in W$ for all $a \in A$ and (2) a is a Nash equilibrium of the game $\{N, A, g_w\}$.*

Definition 5 *For any $W \subset \Re^n$, $B(W, \delta) =$*
$$\left\{ v | \exists \ (a, w(\cdot)) \ \textit{admissible w.r.t. } W \textit{ such that } v = (1-\delta)\, g(a) + \delta \sum_y f(y|a)\, w(y) \ \right\}$$

It is easy to verify that almost all the results with perfect monitoring (Lemma 5 - Proposition 12) carry through to this setting.

Bang-Bang Property of PPE

Let $extW$ be the set of extreme points of coW (the convex hull of W). It is the set of the points in W which cannot be represented as a linear combination of different points in coW. Then, we have the following result.

Proposition 6 *Let $W \in \Re^n$ be compact. If a public randomization device is available, $B(W, \delta) = B(extW, \delta)$.*

This is simply because any point in coW is some linear combination of extreme points of coW ($coW = co(extW)$), therefore the public randomization device can induce an appropriate randomization over the points on $extW$ to generate any point $w \in coW$ after any realization of public signal. We have two comments. First, this result also holds for games with perfect monitoring. Second, this result is similar to "simple strategy" in the sense that one can restrict attention to only the extreme points of the equilibrium payoff set to support any outcome (remember that $E(\delta)$ is compact, hence $extE(\delta) \subset E(\delta)$)

In the original paper by Abreu, Pearce and Stacchetti [2], the range (Y) of a public signal is a subset of finite dimensional euclidean space and $f(\cdot|a)$ is a density function on Y derived from an absolutely continuous measure on Y (and σ algebra of Y). Their results are more remarkable in such settings.

First, a public randomization is not required in such setting.

Theorem 7 *Let $W \in \Re^n$ be compact. Then $B(W, \delta) = B(extW, \delta)$*

We immediately obtain the following corollary

Corollary 8 *Let $W \in \Re^n$ be compact. Then $B(W, \delta) = B(coW, \delta)$*

which implies the monotonicity of the equilibrium payoff set.

Corollary 9 *For $0 < \delta^1 < \delta^2 < 1$, $E(\delta^1) \subset E(\delta^2)$.*

Proof. [7]

$$\begin{aligned} E\left(\delta^1\right) &= B\left(E\left(\delta^1\right), \delta^1\right) \text{ (Theorem 7)} \\ &= B\left(coE\left(\delta^1\right), \delta^1\right) \\ &\subset B\left(coE\left(\delta^1\right), \delta^2\right) \text{ (Proposition 12)} \\ &= B\left(E\left(\delta^1\right), \delta^2\right) \end{aligned}$$

So, $E\left(\delta^1\right) \subset E\left(\delta^2\right)$ by Lemma 6. ■

This theorem says that any equilibrium payoff can be generated by using only extreme points of the equilibrium payoff set as continuation values. So, any equilibrium CAN have a so-called bang-bang structure.

What is even more surprising is another theorem, which says that such extreme points, in particular efficient payoff profiles, have to have a bang-bang structure, i.e. continuation values of $extE(\delta)$ are NECESSARILY from $extE(\delta)$. The proof of this theorem and the last theorem are a bit involved, so we skip them.

4 Folk Theorem

4.1 A Partnership Game Example

The following is a simplified example of the partnership game in Radner, Myerson and Maskin [8]. Consider two individuals involved in a team production. The model is as follows.

- Action: $A_i = \{e, s\}$ (effort or shirk)
- Outcome: $Y = \{\overline{y}, \underline{y}\}$ (good or bad).

  Revenues are $\pi(\overline{y}) = 12, \pi(\underline{y}) = 0$
- Distribution: $\begin{cases} f(\underline{y}|ee) = \frac{1}{3} \\ f(\underline{y}|es) = f(\underline{y}|se) = \frac{2}{3} \\ f(\underline{y}|ss) = \frac{3}{4} \end{cases}$
- Payoff: $g_i(a) = E_a \frac{\pi(y)}{2} - c(a_i), a_i \in \{e, s\}$

  where $c(e) = 3$, $c(e) = 0$.

  Then the expected payoff matrix is a good old PD;

	e	s
e	1,1	-1,2
s	2, -1	0,0

[7] All these theorem and lemma refer to the ones with perfect monitoring.

Let (v_1, v_2) is the equilibrium payoff profile which maximizes $v_1 + v_2$ (Remember that $E(\delta)$ is compact).

Claim 10 $v_1 + v_2 \leq 1$ *independent of* $\delta \in (0, 1)$.

Proof. Suppose that $v_1 + v_2 > 1$. Both players have to play C with positive probability in the first period to support (v_1, v_2) (why?). Let m_i be such probability for $i = 1, 2$. Then

$$v_i = (1-\delta) g_i(e, m_j) + \delta \left\{ m_j \left(\frac{2}{3} w_i(\overline{y}) + \frac{1}{3} w_i(\underline{y}) \right) + (1 - m_j) \left(\frac{1}{3} w_i(\overline{y}) + \frac{2}{3} w_i(\underline{y}) \right) \right\} \tag{1}$$

where $w_i(y)$ is player i's continuation payoff after a realization of y.

Incentive constraint for player i is

$$(1-\delta) \leq \delta \frac{w_i(\overline{y}) - w_i(\underline{y})}{3} \tag{2}$$

Since (1) can be rewritten as

$$v_i = (1-\delta) g_i(e, m_j) + \delta \left\{ w_i(\overline{y}) - (m_j + 2(1 - m_j)) \frac{w_i(\overline{y}) - w_i(\underline{y})}{3} \right\} \tag{3}$$

(2) implies that

$$v_i \leq (1-\delta) g_i(e, m_j) + \delta w_i(\overline{y}) - (m_j + 2(1 - m_j))(1-\delta)$$

Summing up these two equations to get

$$v_1 + v_2 \leq (1-\delta)(g_1(e, m_2) + g_2(e, m_1)) + \delta (w_1(\overline{y}) + w_2(\overline{y})) - (4 - m_1 - m_2)(1-\delta)$$

Since $g_1(e, m_2) + g_2(e, m_1) \leq 2$ and $w_1(\overline{y}) + w_2(\overline{y}) \leq v_1 + v_2$, we finally obtain

$$v_1 + v_2 \leq 2 - (4 - m_1 - m_2) \leq 0$$

which is a contradiction. ■

This result implies that we cannot obtain a folk theorem in this particular model . The reason is that players have to use an inefficient continuation payoff after a realization of $\underline{y}$ because of the symmetric information structure

Remark 11 *Radner [7] shows that the efficient payoff profile* $(1, 1)$ *can be achieved when players do not discount future* $(\delta = 1)$. *So, this example also shows a discontinuity of* $E(\delta)$ *at* $\delta = 1$.

4.2 Fudenberg, Levine, and Maskin [3]

In the above example, it is critical that there is only one signal ($\underline{y}$) which is informative about players' deviations. Now suppose that there are two public signals $y_i, i = 1, 2$ such that y_i is more informative about player i's deviation, then players may be able to punish player i by transferring the continuation payoff from player i to player j when y_i is observed. This is more efficient punishment because it is based on a transfer of utility, not a waste of it.

Let's modify the above model as follows.

- Outcome: $Y = \{\overline{y}, y_1, y_2\}$

  Revenues are $\pi(\overline{y}) = 12, \pi(y_i) = 0, i = 1, 2$

- Distribution:

  $$\begin{cases} f(\overline{y}|ee) = \frac{2}{3} \\ f(y_1|ee) = f(y_2|ee) = \frac{1}{6} \end{cases}$$

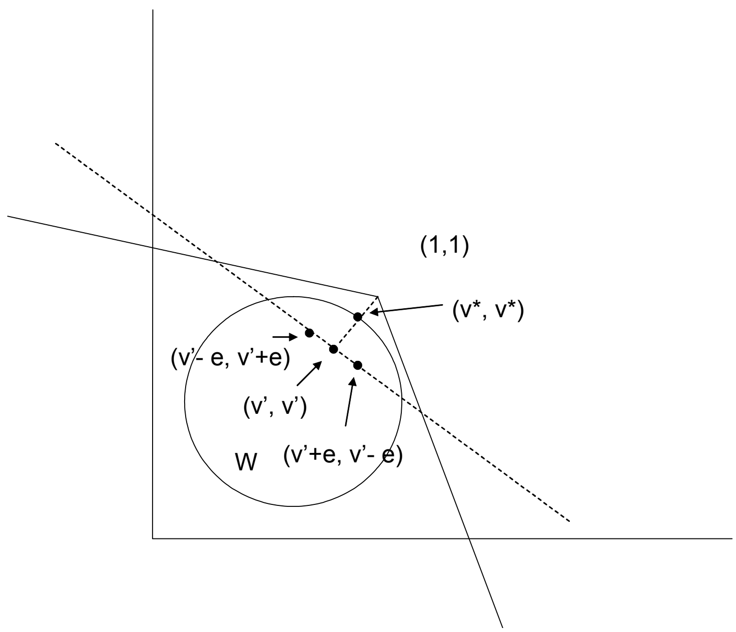
  $$\begin{cases} f(\overline{y}|se) = \frac{1}{3} \\ f(y_1|se) = \frac{1}{2} \\ f(y_2|se) = \frac{1}{6} \end{cases}$$

  $$\begin{cases} f(\overline{y}|es) = \frac{1}{3} \\ f(y_1|es) = \frac{1}{6} \\ f(y_2|es) = \frac{1}{2} \end{cases}$$

  $$\begin{cases} f(\overline{y}|es) = \frac{1}{4} \\ f(y_1|ss) = \frac{3}{8} \\ f(y_2|ss) = \frac{3}{8} \end{cases}$$

  The expected payoff matrix is the same as before.

Let W be the set of perfect public equilibrium payoff. We illustrate how to support the best symmetric equilibrium payoff (v^*, v^*). Players use (v', v'), $(v' - \varepsilon, v' + \varepsilon)$ and $(v' + \varepsilon, v' - \varepsilon)$ as continuation payoffs after $\overline{y}$, y_1, and y_2 respectively as shown in the figure below.



The incentive constraint is as follows

$$(1-\delta)\,2+\delta\left(v'-\frac{1}{3}\varepsilon\right)\leq(1-\delta)+\delta v'$$

or

$$(1-\delta)\leq\frac{\delta}{3}\varepsilon$$

If δ is very large $\left(\delta\geq\frac{3}{3+\varepsilon}\right)$ given ε or the payoff variations (ε) are very large given δ, this inequality is satisfied and both players obtain $(1-\delta)+\delta v'$. Note that, after any realization of public signal, there is no efficiency loss because the sum of two players' continuation payoffs are always $2v'$. This is due to the informational asymmetry we introduced; players can distinguish each player's deviation statistically.

When players are not patient, ε might be too big to push $(v'-\varepsilon,v'+\varepsilon)$ or $(v'+\varepsilon,v'-\varepsilon)$ outside of W. However, ε can be chosen small enough to keep $(v'-\varepsilon,v'+\varepsilon)$ or $(v'+\varepsilon,v'-\varepsilon)$ inside of W when δ is large. The similar argument applies to every boundary point of W. Take any boundary point w^* and a tangent hyperplane at w^*(just a line this case). Then, w^* can be generated by (1) an action profile whose payoff is on the opposite side of the tangent hyper plane and (2) continuation payoffs on a hyperplane parallel to the tangent plane. As $\delta\to 1$, continuation payoffs on the hyperplane stay inside of W, hence $w^*\in B\,(W,\delta)$. Finally, it is fairly easy to show that any interior point is contained in $B(W,\delta)$ as $\delta \to 1$. So, we can conclude that $W \subset B(W,\delta)$ when players are patient enough.[8]

We finish this section by presenting their formal result. Players are allowed to use a mixed action profile $\alpha \in \prod_i \triangle A_i$. Interpret $f(\cdot|\alpha) = \sum_{a \in A} f(\cdot|a)\,\alpha(a)$ as a $|Y|$ dimensional column vector in the following definition and theorem.

Definition 12 *α has individual full-rank for player i if*

$$rank\left\{f\left(\cdot|a_i', \alpha_{-i}\right)|a_i' \in A_i\right\} = |A_i|$$

(there is no linear dependence among $|A_i|$ vectors.)

Definition 13 *α has pairwise full-rank for player i and j if*

$$rank\left\{f\left(\cdot|a_i', \alpha_{-i}\right), f\left(\cdot|a_j'', \alpha_{-j}\right)|a_i' \in A_i, a_j'' \in A_j\right\} = |A_i| + |A_j| - 1$$

Remark 14 *Pairwise full rank condition is related to statistical distinguishability of each player's deviation. Since there has to be at least one linear dependency among $f\left(\cdot|a_i', \alpha_{-i}\right), f\left(\cdot|a_j'', \alpha_{-j}\right)$, $|A_i| + |A_j| - 1$ is the maximum possible rank of such vectors.*

Theorem 15 *If (1) the individual full rank condition is satisfied for every player at every action profile, (2) for each pair $(i,j)\,(j \neq i)$, players there exists an action profile which has pairwise full-rank for i and j, and (3) the dimension of V^* is equal to the number of players, then for any closed set $W \subset intV^*$, there exists a $\underline{\delta}$ such that $W \subset E(\delta)$ for all $\delta \in (\underline{\delta}, 1)$.*

5 Mixed Strategies and Private Strategies

We have so far focused on public strategy; a strategy which only depends on past realizations of public signals. Now we study what would happen if we allow more general strategies. In general, strategy can depend on past realizations of one's own action. We call such strategy *private strategy.*

Remember that for any pure strategy equilibrium, there exists an outcome equivalent PPE. This means that *it is necessary to use a mixed strategy to generate something more than what PPE can generate.* So, player i's strategy is now given by $\sigma_i : H_i \to \triangle A_i$. Formally, player i's strategy σ_i is private if there is a public history h^t and two private histories $h_i'^t$ and $h_i''^t$ such that $\sigma_i\left(h^t, h_i'^t\right) \neq \sigma_i\left(h^t, h_i''^t\right)$.

We just study a simple two period game based on Kandori and Obara [5] to see some possible role of private strategies.[9]. Suppose that the first stage game

[8] In precise, we show that there exists a $\underline{\delta}$ such that for any $\delta \in (\underline{\delta}, 1)$, $W \subset B(W,\delta)$. Note that the level of patience with which a particular point in W is supported might differ for different points. We can use the compactness of W to get the uniformness of $\underline{\delta}$ with respect to W.

[9] You can also find some other interesting examples of two period games in [6].

is given by

	C	D
C	3,3	-1,4
D	4, -1	0,0

In the end of the first period, either $\overline{y}$ or $\underline{y}$ is observed, and they play the second stage game,

	l	r
H	K,K	0,K
L	K, 0	0,0

In this second stage game, player i's action completely determines player j's payoff. Also note that any action profile is a Nash equilibrium and any payoff profile in $[0, K] \times [0, K]$ can be an equilibrium payoff profile in the second stage.

Let's assume the following information structure.

$$\begin{cases} p^0 = f\left(\underline{y}|CC\right) = \frac{1}{20} \\ p^1 = f\left(\underline{y}|CD\ (DC)\right) = \frac{1}{10} \\ p^2 = f\left(\underline{y}|DD\right) = \frac{8}{10} \end{cases}$$

So, $\underline{y}$ is more likely to be observed as more players play D.

The best strongly symmetric pure PPE

$$\begin{aligned} \overline{V}_{pure} &= 3 + \left(1 - p^0\right) K + p^0(K - d) \\ 1 &\leq \left(p^1 - p^0\right) d \text{ (IC)} \end{aligned}$$

where d is a degree of punishment. Since d should be minimized, IC constraint should hold with equality. Hence,

$$\begin{aligned} \overline{V}_{pure} &= 3 + K - \frac{1}{\frac{p^1}{p^0} - 1} \\ &= 2 + K \end{aligned}$$

Note that the likelihood ratio $\frac{p^1}{p^0}$ $\left(\frac{\text{probability of } \underline{y} \text{ given } (D,C)}{\text{probability of } \underline{y} \text{ given } (C,C)}\right) = 2$ is the critical factor to determine this bound.

The best strongly symmetric mixed PPE

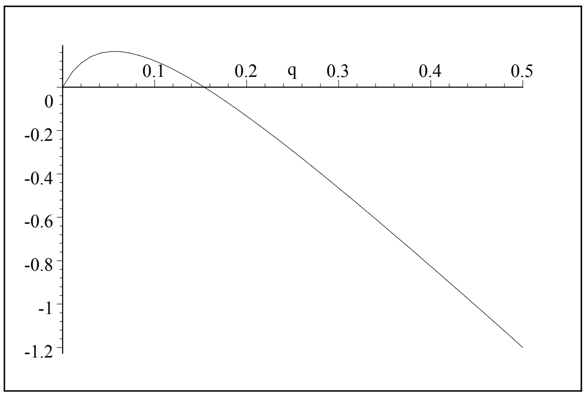
Since players are indifferent between C and D, they can mix them if they like. Then, we again have two equations.

$$\begin{aligned} \overline{V}_{mixed} &= 3\,(1-q) - q + \left[1 - \left\{(1-q)\,p^0 + qp^1\right\}\right] K + \left\{(1-q)\,p^0 + qp^1\right\}(K - d) \\ 1 &= \left\{(1-q)\left(p^1 - p^0\right) + q\left(p^2 - p^1\right)\right\} d \end{aligned}$$

where q is the probability to play D. Then,

$$\overline{V}_{mixed} = 3 - 4q + K - \frac{1}{\frac{(1-q)p^1+qp^2}{(1-q)p^0+qp^1} - 1}$$

Is there any reason to mix D in this way? Note that the likelihood ratio with D being played is $\frac{p^2}{p^1}$ ($\frac{\text{probability of } \underline{y} \text{ given } (D,D)}{\text{probability of } \underline{y} \text{ given } (C,D)}$)$= 8$, which is much higher than $\frac{p^1}{p^0} = 2$. When $q = 0$, $\overline{V}_{mixed}$ (of course) coincides with $\overline{V}_{pure}$, and as q increases from 0, the likelihood ratio goes up to reduce the last term of $\overline{V}_{mixed}$. If this monitoring effect dominates the stage game payoff reducing effect $(3 - 4q \downarrow)$, it can be the case that $\overline{V}_{mixed} > \overline{V}_{pure}$ for small q. Indeed, $\overline{V}_{mixed} - \overline{V}_{pure}$ looks as follows in this example.



So, this is an example in which mixed strategy can make a difference within PPE.

Private Strategy and Private Equilibrium

Note that public signal is informative when D is being played, but not informative when C is being played. This suggests that the punishment after $(C, \underline{y})$ is a waste of efficiency. Now consider a following strategy; (1) play C with probability $(1 - q)$ and D with probability q as in mixed PPE, but (2) punish the other player in the second stage only when $(D, \underline{y})$ is realized, otherwise play H or l. This is clearly a private strategy. We can get the equilibrium payoff by solving;

$$\begin{aligned} \overline{V}_{private} &= 3(1-q) - q + \left(1 - qp^1\right) K + qp^1(K - d) \\ 1 &= q\left(p^2 - p^1\right) d \end{aligned}$$

Hence,

$$\overline{V}_{private} = 3 - 4q + K - \frac{1}{\frac{p^2}{p^1} - 1}$$

which is clearly larger than $\overline{V}_{mixed}$ for every q. So, for example, $\overline{V}_{pure} < \overline{V}_{mixed} < \overline{V}_{private}$ around $q = 0.05$ (as long as K is large enough to generate a level of punishment we need.)

Remark 16 *After $\underline{y}$ is observed in the first stage, each player does not know the other player's continuation strategy because it depends on her private information (the realization of her action in the first stage). This does not create any problem here because the second stage game has a very peculiar payoff structure. However, it could create many technical problems if we try to apply the idea of this particular equilibrium to infinitely repeated games (or long, but finite horizon repeated games). We will come back to this point later in "Repeated Games with Private Monitoring".*

References

[1] D. Abreu, D. Pearce, and E. Stacchetti. Optimal cartel equilibria with imperfect monitoring. *Journal of Economic Theory*, 39:251–269, 1986.

[2] D. Abreu, D. Pearce, and E. Stacchetti. Toward a theory of discounted repeated games with imperfect monitoring. *Econometrica*, 58:1041–1063, 1990.

[3] D. Fudenberg, D. K. Levine, and E. Maskin. The folk theorem with imperfect public information. *Econometrica*, 62:997–1040, 1994.

[4] E. Green and R. Porter. Noncooperative collusion under imperfect price formation. *Econometrica*, 52:87–100, 1984.

[5] M. Kandori and I. Obara. Efficiency in repeated games revisited: The role of private strategies. unpublished, 2000.

[6] G. J. Mailath, S. A. Matthews, and T. Sekiguchi. Private strategies in finitely repeated games with imperfect public monitoring. CARESS Working Paper #01-10, University of Pennsylvania, 2001.

[7] R. Radner. Repeated partnership games with imperfect monitoring and no discounting. *Review of Economic Studies*, 53:43–57, 1986.

[8] R. Radner, R. Myerson, and E. Maskin. An example of a repeated partnership game with discounting and with uniformly inefficient equilibria. *Review of Economic Studies*, 53:59–69, 1986.