Subgame Perfect Equilibrium

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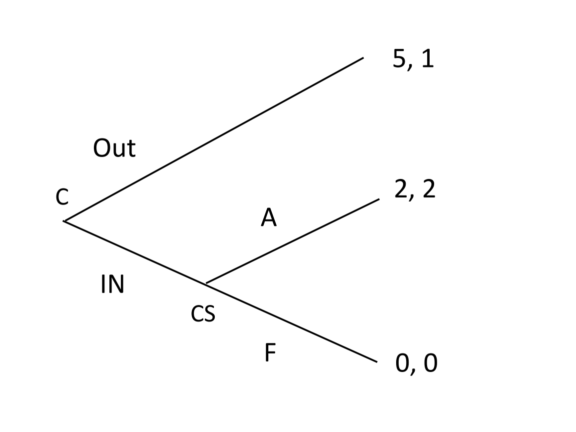
Backward Induction

Chain Store Game:



Backward Induction

- There are two NE: $(Out, F), (In, A)$
- One may argue that (Out, F) is not a reasonable NE in the chain store game:
  - If C chose "In", then A would be the best response for CS.
  - If C expects this, then C would choose "In".

Backward Induction

- This logic can be generalized to general finite horizon extensive games with perfect information.
- **Backward induction** is the following procedure.
  - Let $L < \infty$ be the maximum length of all histories.
  - Find all nonterminal histories of $L-1$ length and assign an optimal action there. Eliminate unreached L-length terminal histories and regard other L-length terminal histories as $L-1$-length terminal histories.
  - Find all nonterminal histories of $L-2$ length and assign an optimal action there. Eliminate unreached $L-1$-length terminal histories and regard other $L-1$-length terminal histories as $L-2$-length terminal histories.

- Backward induction selects (In, A) in the chain-store game.
- Let's apply BI to Stackelberg game with linear inverse demand function $P(Q) = 12 - Q$ and linear cost $3q_i$.
  - Given firm 1's production q_1, firm 2 solves $\max_{q_2 \in \Re_+} (12 - q_1 - q_2)q_2 - 3q_2$. So $q_2^*(q_1) = \frac{9-q_1}{2}$.
  - After eliminating firm 2's non-optimal response, firm 1's problem is $\max_{q_1 \in \Re_+} (12 - q_1 - \frac{9-q_1}{2})q_1 - 3q_1$. So $q_1^* = 4.5$.
  - Hence we derive the strategy profile $(q_1^*, q_2^*(\cdot))$ and the outcome profile $(4.5, 2.25)$ from BI.
- Note that the level of firm 1's production is different from its NE production level $(= 3)$.

Subgame Perfect Equilibrium

- We introduce a new notion of equilibrium for extensive games with perfect information to formalize and generalize the intuition behind BI.

- At any history, the "remaining game" can be regarded as an extensive game on its own. It is called a **subgame** after the history.

Subgame

The subgame of the extensive game with perfect information $(N, H, P, (V_i))$ that follows $h \in H/Z$ is the extensive game $(N, H|_h, P|_h, (V_i|_h))$ that satisfies the following conditions (with $(h, \emptyset) = h$).

- $h' \in H|_h \Leftrightarrow (h, h') \in H$.
- $P|_h(h') = P(h, h')$ for any $h' \in H|_h$.
- $V_i|_h(h') = V_i(h, h')$ for any terminal history $h' \in Z|_h \subset H|_h$.

- One example of subgame is the original game itself.

- Note that a strategy specifies a strategy for any subgame.
- For any strategy $s_i \in S_i$, denote the **continuation strategy** after $h \in H/Z$ by $s_i|_h$, which is a strategy for $(N, H|_h, P|_h, V_i|_h)$ that satisfies:
  - $s_i|_h(h') := s_i(h, h')$ for any h' such that $(h, h') \in H/Z$.
- Let $S_i|_h$ be the set of all strategies for player i for $(N, H|_h, P|_h, V_i|_h)$.

- A strategy profile is a **subgame perfect equilibrium** if after any nonterminal history it constitutes a Nash equilibrium.

Subgame Perfect Equilibrium

For a finite extensive game with perfect information $(N, H, P, (V_i))$, a strategy profile $s^* \in S$ is a **subgame perfect equilibrium** if

$$V_i|_h(O(s^*|_h)) \geq V_i|_h(O(s_i, s^*_{-i}|_h))$$

for any $s_i \in S_i|_h$, any $i \in N$ and any $h \in H/Z$.

- Note that every SPE is a NE.

One-Shot Deviation Principle

- To verify that some strategy profile is a SPE, we need to check a very large number of **incentive constraints**.
- It turns out that we can focus on a subset of incentive constraints for finite horizon extensive games with perfect information.
- A strategy $s_i' \in S_i|_h$ at $h \in H/Z$ for $i = P(h)$ is called **one-shot deviation** from s_i if $s_i|_h$ and s_i' prescribes a different action only at the initial history (i.e. $s_i'(\emptyset) \neq s_i(h)$ and $s_i'(h') = s_i(h, h')$ for any $h' \neq \emptyset$ such that $(h, h') \in H/Z$).

- For finite extensive games with perfect information, we just need to check the incentive constraints with respect to one-shot deviations.

One-Shot Deviation Principle

For a finite extensive game with perfect information $(N, H, P, (V_i))$, a strategy profile $s^* \in S$ is a **subgame perfect equilibrium** if and only if

$$V_i|_h(O(s^*|_h)) \geq V_i|_h(O(s_i, s^*_{-i}|_h))$$

for any one shot deviation $s_i \in S_i|_h$ from $s^*_i|_h$ for $i = P(h)$ at any $h \in H/Z$.

- The proof is omitted as we prove a more general version of this result.

Comments:

- For any finite horizon extensive game with perfect information (ex. Chess),
  - the set of subgame perfect equilibria is exactly the set of strategy profiles that can be found by BI.
  - there always exists a subgame perfect equilibrium.
  - there exists the unique subgame perfect equilibrium for generic games (i.e. $V_i(z) \neq V_i(z')$ for any $z \neq z'$ for any $i \in N$).

Continuity at Infinity

- In general, one-shot deviation principle does not hold for infinite horizon games.
- We say that a game satisfies **continuity at infinity** if payoffs in very far future are not important.

Continuity at Infinity

$(N, H, P, (V_i))$ satisfies **continuity at infinity** if for any $\epsilon > 0$ there exists T such that

$$\left| V_i(z) - V_i(z') \right| < \epsilon$$

for any $z, z' \in Z$ with the same initial T-length histories and for any $i \in N$.

Comments:

- Every finite horizon extensive game satisfies continuity at infinity.
- Consider infinite horizon games where player i's payoff takes the following form:

$$V_i(z) = \sum_{t=0}^{\infty} \delta^t g_i(a^t)$$

for $z = (a^1, a^2, ...,)$. Suppose that g_i is bounded. Then this infinite horizon extensive game satisfies continuity at infinity.

One-Shot Deviation Principle

Suppose that $(N, H, P, (V_i))$ satisfies continuity at infinity. Then a strategy profile $s^* \in S$ is a **subgame perfect equilibrium** if and only if

$$V_i|_h(O(s^*|_h)) \geq V_i|_h(O(s_i, s^*_{-i}|_h))$$

for any one shot deviation $s_i \in S_i|_h$ from $s^*_i|_h$ for $i = P(h)$ at any $h \in H/Z$.

Proof.

- The "only if" part follows from the definition.
- For the "if" part:
  - If there is no one-shot profitable deviation, then there is no finite-period profitable deviation.
  - If there is no finite-period profitable deviation, then there cannot be any infinite-period profitable deviation because of continuity at infinity.

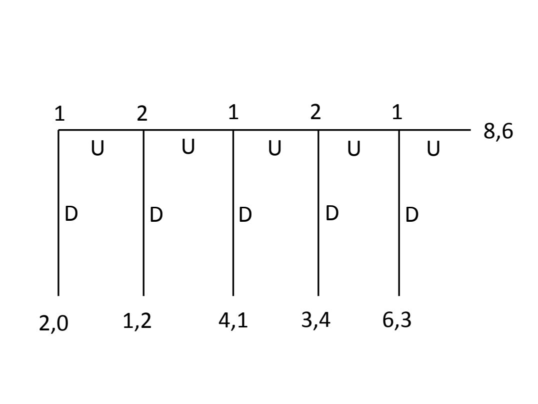
□

Example: Chain-Store game

- Suppose that the chain store plays the chain-store game sequentially with K potential competitors in K different cities.
- In market k, competitor C_k chooses either "In" or "Out" given the histories in the previous $k-1$ markets. Then CS chooses whether to accommodate or fight. The payoff of CS is the sum of its payoffs in all K markets.
- The unique subgame perfect equilibrium is that every competitor always enters and the chain store always accommodates.
- There are many other Nash equilibria.

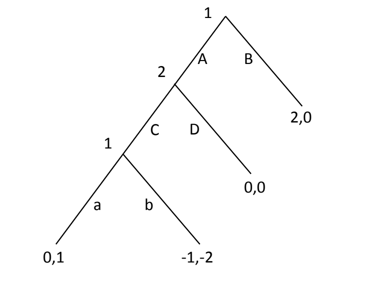
Example: Centipede game

- Consider the following game with two players. What is the subgame perfect equilibrium?



Interpretation of Strategy

- In the following game, the unique SPE is that player 1's strategy is *Ba* and player 2's strategy is *C*.



- Why does player 1 need to choose between *a* and *b* when playing *B*?

- Player 1 cannot determine player 2's behavior without specifying his behavior after (A, C).
- Thus one interpretation of a strategy off the equilibrium path is that they are just beliefs: player 1 believes that player 2 believes that player 1 plays a after (A, C) in this case.
- Another interpretation of a strategy off the equilibrium path is that it is an actual plan in noisy environments: (A, C) sometimes happen because people always make mistakes or some unexpected things always happen.

Extensive Games with Exogenous Uncertainty

- It is easy to incorporate random public signal into an extensive game.
- **Extensive game with perfect information and chance moves** $(N, H, P, f_c, (V_i))$ is an extensive game with perfect info + exogenous randomization.
  - P maps nonterminal histories to $N \bigcup \{c\}$. Let H_c be the set of histories such that $P(h) = c$.
  - For any $h \in H_c$, $f_c(h) \in \Delta(A(h))$ assigns an action $a \in A(h)$ with probability $f_c(a|h)$.

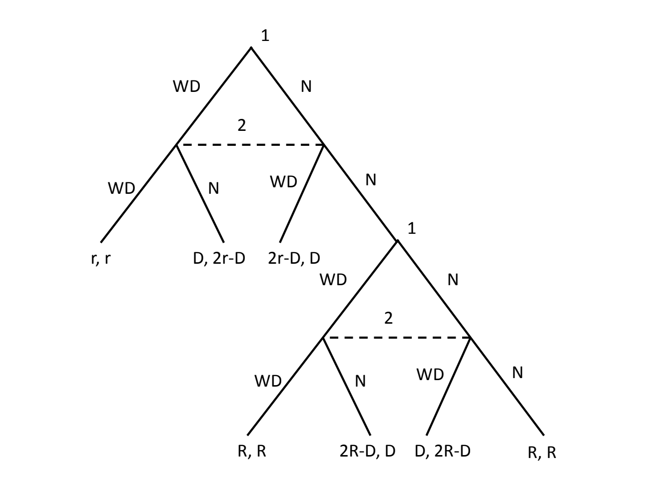
Extensive Games with Simultaneous Moves

- Another extension is to allow for simultaneous moves.
- **Extensive game with perfect information and simultaneous moves** $(N, H, P, (V_i))$ is an extensive game with perfect info where players may move simultaneously.
  - History H is a sequence of action profiles.
  - P maps nonterminal histories to a subset of N.
  - $A(h)$ is a product set: $A(h) = \prod_{i \in P(h)} A_i(h)$.

Example: Bank Runs

- Two investors with deposit $D > 0$ each at some bank, invested in some project.
- There are two periods. Each investor decides whether to withdraw her deposit in each period. If any investor withdraws her deposit in the first period, then the project is terminated and the game ends.
- The payoffs of the investors are as follows (assume $0.5D < r < D < R$):
  - each investor receives r if both investors withdraw in the 1st period.
  - if one investor withdraws and the other decides not to withdraw in the 1st period, then this investor secures D and the other investor receives the rest $(= 2r - D)$.
  - if both investors do not withdraw in the first period, then the game moves to the 2nd period and the project matures to generate $2R$.
  - each investor receives R if both investors withdraw or do not withdraw in the 2nd period.
  - if one investor withdraws and the other decides not to in the 2nd period, then this investor receives all the profits except for the other investor's original investment D.

- This game can be represented as follows.



- Find all SPE.

Location Model with Quadratic Transportation Cost

- Two restaurants A, B are opening their restaurant on $[0,1]$ on which customers are uniformly distributed.
- A customer $i \in [0,1]$ pays transportation cost $t(x-i)^2$ to go to any restaurant located at $x \in [0,1]$ and enjoys utility V by consuming one meal .
- The game proceeds as follows.
  - In the first period, restaurants choose locations x_A, x_B simultaneously
  - In the second period, restaurants choose prices p_A, p_B simultaneously. Customer i buys one meal from restaurant A if and only if $V - p_A - t(x_A - i)^2 \geq \max\left\{V - p_B - t(x_B - i)^2, 0\right\}$. The same for restaurant B.

- Under a regulation that the price must be $\overline{p}$, this is essentially one-period location game. The equilibrium would be $(x_A, x_B) = (0.5, 0.5)$ if V is large enough.
- What is the subgame perfect equilibrium of this two period game?

War of Attrition

- Two players are fighting with each other.
- In period $t = 0, 1, 2, \ldots$, the players decide whether to fight or concede simultaneously. The game ends as soon as any player concedes.
- The cost of fighting is $c > 0$. When player i fights and player $-i$ concedes in period t, player i receives a reward $V > 0$ in period t without incurring cost c.
- Player i's payoff is
  - $-\sum_{t=0}^{T-1} \delta^t c$ if i concedes in period T.
  - $-\sum_{t=0}^{T-1} \delta^t c + \delta^T V$ if i fights and $-i$ concedes in period T.
- Is there any stationary subgame perfect equilibrium?

SPE and IEWDS

- Write down an extensive game as a strategic game.
- Notice that BI for extensive games with perfect information corresponds to a particular way of eliminating weakly dominated strategies.

Burning Money

- Here is an interesting example.
  - Players play the following game.

	B	C
B	3,1	0,0
C	0,0	1,3

  - Player 1 can burn $1 publicly before playing this game.
- Write down an extensive game. Find all SPE.
- Write down a reduced form strategic game. Apply IEWDS. What do you