Belief Persistence and the Disposition Effect†

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Abstract

We consider a model where agents have heterogeneous beliefs about state persistence. Equilibrium trading behavior is ordered if agents can be ranked according to the degree of disposition effect (i.e. they buy when prices fall and sell when prices rise) that they exhibit. We show that trading behavior is ordered if and only if beliefs in the population can be ordered via a single parameter measuring persistence. Agents who believe that states are less persistent exhibit the disposition effect while those who believe that states are more persistent exhibit the opposite behavior (i.e. a form of the house-money effect).

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1 Introduction

One of the most robust findings in behavioral finance is the disposition effect. This refers to the tendency of individual investors to oversell stocks that have gone up in value (i.e. price) and to undersell stocks that gone down. Although suboptimal, this behavior has been well-documented in many situations and in various markets around the world.\(^1\) Moreover, similar observations have also been recorded in other contexts, such as the housing market or in the exercise of executive stock options.\(^2\)

In this paper, we consider a belief-based explanation for the disposition effect. In particular, we study a model of heterogeneous beliefs where beliefs differ only along a single dimension measuring persistence. For example, some agents may believe that earnings information from this quarter will be highly correlated with earnings information from the next quarter (high persistence). On the other hand, others may believe that the earnings information from both quarters are completely uncorrelated (no persistence). In equilibrium, agents who believe in the least persistence decrease their holdings of the stock when prices rise and increase their holdings when prices fall; in essence, they exhibit the disposition effect. On the other hand, those who believe in the most persistence exhibit the opposite behavior; they employ a trading strategy based on stock price momentum and exhibit a form of the house-money effect.\(^3\)

To be concrete, suppose that institutional investors pay close attention to certain financial indicators after each earnings release which they believe to be highly correlated with future earnings information. Individual investors on the other hand, believe these indicators to be noisy and ignore them when placing their trades. In equilibrium, our model predicts that individual investors will exhibit the disposition effect while institutional investors will engage in momentum trading.\(^4\)

Much of the existing research has focused on providing a preference-based explanation to

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\(^1\) Studies by Odean [15] in the U.S., Grinblatt and Keloharju [10] in Finland and Feng and Seasholes [8] in China confirm that the disposition effect is a global phenomenon that transcends both institutional and cultural divisions.


\(^3\) The house-money effect refers to the tendency of gamblers to increase their bets after a gain and to decrease their bets after a loss (see Thaler and Johnson [20]).

\(^4\) This is somewhat consistent with the empirical evidence suggesting that the disposition effect is much more pronounced for investors who are financially less sophisticated (see Dhar and Zhu [6] for example).
rationalize the disposition effect. Our results show that by introducing belief heterogeneity in a model with equilibrium trading, we can obtain behavior that exhibits the disposition effect while still retaining the standard assumption of expected utility preferences. While our approach does not insist that a belief-based equilibrium model is the only explanation for the disposition effect, it does suggest that in many cases, a careful study of the disposition effect should also take into account investor beliefs and how they interact with equilibrium forces. We leave the more practical exercises of testing different explanations of the disposition effect as avenues for future research.

Formally, we consider a simple two-period model where agents form beliefs about the state space $S$ in each period. For example, $S$ could represent all the financial information available after each earnings release. We assume that agents share the same correct prior about $S$, but have different beliefs about the transition probabilities between the two time periods. We model these beliefs as Markov kernels on $S$. For example, some agents may believe that state realizations in the two periods are highly correlated (i.e. high persistence) while others may believe that there is little correlation (i.e. low persistence). We say that the distribution of beliefs in a population has a persistence representation iff each agent’s belief corresponds to a Markov kernel

$$K = \lambda \overline{K} + (1 - \lambda) K$$

where $K$ is the constant kernel, $\overline{K}$ is some other fixed kernel and $\lambda \in [0, 1]$. Thus, all agents in the population can be parametrized by some $\lambda \in [0, 1]$ measuring each agent’s belief of state persistence. For example, an agent with $\lambda = 0$ believes in zero persistence. In the case where $S$ is finite, $K$ is simply the $\lambda$-mixture of the two matrices $\overline{K}$ and $K$. Note that this is not a model of asymmetric information; after the first period, all agents observe the realized state $s \in S$ but form their own individual beliefs about what that means for second period state realizations.

We consider an equilibrium where agents trade claims to two securities: a risky “stock” that yields a payoff contingent on the realization of the state $s \in S$ and a risk-free “bond” that yields the same payoff regardless of which state is realized. To simplify the exposition,

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5 See Barberis and Xiong [2], Strahilevitz, Odean and Barber [19] and Barberis and Xiong [3] for explanations based on prospect theory, emotional regret and realization utility respectively.
we assume that all agents have the same endowment in each state and that they share the same CARA utility index. An equilibrium is ordered iff all agents can be ranked according to the degree of disposition effect they exhibit. Our first main result shows that any equilibrium under a persistence representation is ordered. In this case, those who believe in the least persistence (i.e. smallest $\lambda$) exhibit the disposition effect while those who believe in the most persistence (i.e. largest $\lambda$) exhibit the opposite behavior (i.e. momentum trading or a form of the house-money effect). Thus, in our model, the disposition effect emerges as equilibrium behavior induced by belief heterogeneity.

We then study a special case where beliefs are Gaussian. There are two distinct groups in the population: the first group believes that states are independent and identically distributed (i.i.d.) while the second believes that states are correlated. This CARA-Gaussian setup is easily tractable and allows for simple expressions for both prices and trading strategies. In equilibrium, a simple inequality characterizes when stock prices will increase (or decrease) and when agents in the first group (i.e. those who believe in no persistence) will decrease (or increase) their stock holdings. We then proceed to analyze some of the comparative statics. We show that increasing the proportion of agents in the second group (i.e. those who believe in persistence) will inflate prices in “good” states (i.e. states where prices rise) and deflate prices in “bad” states (i.e. states where prices fall). Thus, introducing belief heterogeneity in our model does not uniformly increase prices and may result in greater dispersion (i.e. higher volatility) of stock prices.

Lastly, we consider the uniqueness properties of the persistence representation. In general, beliefs in a population are not uniquely specified given equilibrium prices and strategies. However, by varying the payoffs of the stock and observing the resulting equilibrium behavior, we can completely identify the distribution of beliefs in a population. We show that without loss of generality, beliefs in a population has a persistence representation if and only if any equilibrium is ordered by the same disposition ranking over agents. In other words, for any other representation, there exists some stock where the disposition ranking is reversed in some states. One agent exhibits a greater disposition effect than another agent under one equilibrium but not in another. Thus, the persistence representation is the only distribution of beliefs that permits a consistent disposition ranking of agents in all equilibria. This is

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6 This approach is similar in spirit to that of Savage [16] where one varies the state-contingent payoffs (i.e. “acts”) to uniquely identify beliefs under individual decision-making.
a complete characterization of the persistence representation. It also allows us to equate observable equilibrium prices and trading strategies with the unobservable beliefs of agents in a population.

This paper is related to a long literature on the disposition effect. Shefrin and Statman [18] first used the term the “disposition effect” to describe this behavior. Odean [15] provided the first comprehensive study of the disposition effect and ruled out various explanations including portfolio re-balancing, trading costs and tax considerations. Other studies include Grinblatt and Keloharju [10] in Finland and Feng and Seasholes [8] in China. Weber and Camerer [21] demonstrated that the effect is robust even in experimental studies where subjects behave in a manner inconsistent with Bayesian updating. Barberis and Xiong [2] illustrated that under certain parametric assumptions, the prospect theory of Kahneman and Tversky [13] could explain the disposition effect although under other assumptions the theory predicts the opposite effect. Other preference-based explanations include emotional regret by Strahilevitz, Odean and Barber [19] and realization utility by Barberis and Xiong [3]. In contrast, our belief-based model can be viewed as the “dual” approach to addressing the disposition effect, analogous to how the dual theory of Yaari [22] addresses violations of expected utility under individual decision-making.

This paper also fits in the large literature on heterogeneous beliefs. Harrison and Kreps [11] introduce belief heterogeneity to obtain speculative pricing. Morris [14] relates belief heterogeneity with short-term IPO overpricing while Scheinkman and Xiong [17] study heterogeneous beliefs with Gaussian learning. Eyster and Piccione [7] consider a model where agents have incomplete knowledge but are all convinced about their own “theories” about how states transition. In all these models, risk-neutrality and the absence of short-selling of the risky security imply that belief heterogeneity generates inflated prices. In contrast, our model allows for both risk-aversion and short-selling. As a result, the implications on prices are more subtle as demonstrated in our Gaussian special case.
2 The Persistence Representation

Let $S$ be a Polish space and let $\Pi$ be the set of all probability measures on $(S, \mathcal{F})$. Consider a simple two-period model $S \times S$. We assume that all agents share the same prior belief $p \in \Pi$ on $S$ at time 0. However, after the realization of some $s \in S$ at time 1, agents update their beliefs differently and may have various posterior beliefs about the realization of $s \in S$ at time 2. For example, $S$ could represent earnings information about a company where agents differ in their beliefs about how correlated earnings are over time.

Formally, we model these conditional beliefs as Markov kernels $K : S \times \mathcal{F} \rightarrow [0, 1]$ that have $p$ as the invariant measure.

**Definition.** The Markov kernel $K : S \times \mathcal{F} \rightarrow [0, 1]$ is $p$-invariant iff $K_s$ is absolutely continuous with respect to $p$ and for all $A \in \mathcal{F}$

$$p(A) = \int_S p(ds) K_s(A)$$

If $K$ is $p$-invariant, then each $K_s$ has a density $\kappa_s$ with respect to $p$. In what follows, almost surely (a.s.) always mean almost surely with respect to the measure $p$.

We say that a $p$-invariant Markov kernel exhibits persistence iff the conditional density of the same state occurring is greater than unity.

**Definition.** $K$ exhibits persistence iff $\kappa_s(s) \geq 1$ a.s.

For example, if $S$ is finite, then persistence implies that $K_s\{s\} \geq p\{s\}$ for all $s \in S$. In other words, after observing the realization of $s \in S$ at time 1, all agents increase their beliefs about $s \in S$ occurring again at time 2.

Let $K$ denote the set of all $p$-invariant kernels that exhibit persistence. Since we are interested in beliefs that differ only in the degree of persistence they exhibit, we only consider kernels in $K$. One extreme example is the constant kernel $K \in K$ such that $K_s = p$ for all $s \in S$. Thus, $K$ corresponds to believing that realizations of $s \in S$ in both time periods are completely independent and identically distributed (i.i.d.). This is an example of zero persistence.

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7 This is without loss of generality since we only consider measures that are absolutely continuous with respect to $p$. 
Consider a population with heterogeneous beliefs about persistence. Formally, we model this as a probability measure \( \mu \) on \( \mathcal{K} \) with finite support. Each \( K \in \mathcal{K} \) such that \( \mu \{ K \} > 0 \) represents the belief of an agent (or group of agents with the same belief) in the population. A persistence representation is a linear one-dimensional parametrization of the degree of persistence in the population.

**Definition.** \( \mu \) has a persistence representation iff there is some \( K \in \mathcal{K} \) such that for all \( \mu \{ K \} > 0 \), there is some \( \lambda \in [0, 1] \) where a.s.

\[
K = \lambda K + (1 - \lambda) \overline{K}
\]

Under a persistence representation, each agent’s belief \( K \in \mathcal{K} \) is characterized by a parameter \( \lambda \in [0, 1] \) which specifies the level of persistence of \( S \) over time. For example, if \( \lambda = 0 \), then \( K = \overline{K} \) is the constant kernel and there is no persistence. Note that \( \mu \) is completely characterized by a scalar distribution on \([0, 1]\) representing persistence. Hence, beliefs in the population are heterogeneous only along this dimension measuring the persistence level of realizations of \( S \). Note that the model is completely agnostic as to what beliefs should be, that is, the true distribution of states are irrelevant. Moreover, there is no asymmetric information. For example, an agent with belief \( \overline{K} \) also observes realizations of \( S \) in time \( 1 \) but believes they are irrelevant and chooses to ignore them.

We end this section with two examples of persistence representations.

**Example 1.** Let \( S = \{ s_1, s_2, s_3 \} \), \( p = \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \) and \( \overline{K} \) be the identity matrix. Note that \( p\overline{K} = p \) so \( \overline{K} \) is \( p \)-invariant. Also, for all \( s \in S \),

\[
\frac{\overline{K}_s \{ s \}}{p \{ s \}} = 3 \geq 1
\]

so \( \overline{K} \in \mathcal{K} \). In fact, \( \overline{K} \) is the identity kernel representing full persistence. Let \( K := \frac{1}{2} K + \frac{1}{2} \overline{K} \) and

\[
\mu := \frac{1}{3} \delta_K + \frac{1}{3} \delta_K + \frac{1}{3} \delta_{\overline{K}}
\]

Thus, \( \mu \) is a persistence representation with equal masses on three beliefs of increasing persistence: \( \overline{K}, K \) and \( \overline{K} \).
**Example 2** (Gaussian Case). Let $S = \mathbb{R}$ and suppose that an agent believes that the joint distribution on $S \times S$ is Gaussian (or normal). Let $p$ be a Gaussian distribution with mean $m$ and variance $\sigma^2$. If we let $\tau$ be the correlation coefficient, then for $s \in S$, $K_s$ is Gaussian with mean $m (1 - \tau) + \tau s$ and variance $(1 - \tau^2) \sigma^2$. Thus,

$$
p(ds') \kappa_s(s') = K_s(ds') = ds' \frac{1}{\sigma \sqrt{2\pi (1 - \tau^2)}} e^{-\frac{(s' - m(1 - \tau) - \tau s)^2}{2(1 - \tau^2)}\sigma^2}.
$$

Note that

$$
p(ds') = ds' \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(s' - m)^2}{2\sigma^2}}.
$$

Hence, for $s = s'$, we have

$$
\kappa_s(s) = \frac{1}{\sqrt{1 - \tau^2}} e^{\frac{(s - m)^2}{1 + \tau^2}} \geq 1
$$

iff $\tau \geq 0$. Thus, $K \in \mathcal{K}$. Let $\overline{K} = K$ and $\mu := \frac{1}{2}\delta_{\overline{K}} + \frac{1}{2}\delta_K$. This represents a population where half of the agents believe that realizations of $S$ over time are completely uncorrelated while the other half of agents believe that realizations of $S$ over time are correlated with correlation $\tau \geq 0$.

**3 Trading Equilibrium**

We now consider the trading equilibrium for a population with beliefs distributed according to $\mu$. Let $I \subset \mathbb{N}$ be finite and for each $i \in I$, let $K^i \in \mathcal{K}$ be such that $\mu \{K^i\} > 0$. Thus, each $i \in I$ represents an agent (or group of agents) with belief $K^i \in \mathcal{K}$ about the realization of $S$ in both time periods.

Let $Z_0$ denote the set of all bounded and measurable $z : S \to \mathbb{R}_+$. We interpret each $z \in Z_0$ as a security that gives payoff $z(s)$ if $s \in S$ is realized. For example, $1 \in Z_0$ and we call this risk-free security that pays one unit in every $s \in S$ a “bond”. Let $Z$ denote the set of all securities that have non-zero variance.\(^8\) We call any $z \in Z$ a “stock” and note that $1 \not\in Z$.

\(^8\) That is, the variance of $z$ with respect to the measure $p$. Note that this is equivalent to requiring that $z$ is not constant a.s..
Fix some stock $z \in Z$. Consider a trading equilibrium where at time 0, agents trade claims to both the stock and bond. We let $s_0 \notin S$ represent the initial state at time 0 and $S_0 := \{s_0\} \cup S$. The price density be given by a measurable function $\psi : S_0 \to \mathbb{R}^2$. For $s \in S_0$, we interpret $\psi_0(s)$ and $\psi_1(s)$ as the price densities for the bond and stock respectively. Note that $\psi(s_0)$ is the price vector for both the stock and the bond at time 0.

Let $\Theta$ denote the set of all measurable functions $\theta : S_0 \to \mathbb{R}^2$. Each $\theta \in \Theta$ represents a trading strategy. For $s \in S_0$, we interpret $\theta_0(s)$ and $\theta_1(s)$ as the holdings for the bond and stock respectively. As before, $\theta(s_0)$ represents the initial holdings for both the stock and the bond at time 0.

Given a price $\psi$, the pricing functional $\Psi : \Theta \to \mathbb{R}$ is given by

$$\Psi(\theta) := \psi(0) \cdot \theta(0) + \int_S p(ds) \psi(s) \cdot \theta(s)$$

for all $\theta \in \Theta$. Hence, $\Psi(\theta)$ specifies the total price for executing the trading strategy $\theta \in \Theta$. We assume that agents have constant endowment 1 in all states. Thus, the budget set given price $\psi$ is

$$B(\psi) := \{\theta \in \Theta \mid \Psi(\theta) \leq \Psi(1)\}$$

Let $u$ be a CARA utility index with constant risk aversion $\rho > 0$ and let $\delta \in (0, 1)$ denote the discount rate. We let $z := (1, z)$ denote the asset vector. The utility of an agent with belief $K \in \mathcal{K}$ is given by the function $U_K : \Theta \to \mathbb{R}$ where

$$U_K(\theta) := \int_S p(ds) u(\theta(s_0) \cdot z(s)) + \delta \int_S p(ds) \int_S K_s(ds') u(\theta(s) \cdot z(s'))$$

Thus, an agent with belief $K \in \mathcal{K}$ and chooses trading strategy $\theta \in \Theta$ obtains utility $U_K(\theta)$. We say $\theta \in B(\psi)$ is optimal for $K \in \mathcal{K}$ iff $U_K(\theta) \geq U_K(\theta')$ for all $\theta' \in B(\psi)$. For $i \in I$, we let $\theta^i \in B(\psi)$ be optimal for $K^i \in \mathcal{K}$.

An allocation is a probability $\nu$ on $\Theta$. We let $(\psi, \nu)$ denote the price and allocation pair. We now define an equilibrium given a stock $z$ and a distribution of beliefs $\mu$ as follows.

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9 The CARA utility index is given by $u(x) = -e^{-\rho x}$ for some $\rho > 0$. 

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Definition. \((\psi, \nu)\) is an equilibrium for \((z, \mu)\) iff \(\nu \{ \theta^i \} = \mu \{ K^i \}\) for all \(i \in I\) and a.s.

\[
\sum_{i \in I} \nu \{ \theta^i \} \theta^i (s) = 1
\]

Thus, in an equilibrium, all agents elect their optimal strategies and all claim markets clear. Note that since \(\mu\) has finite support, \(\nu\) must also have finite support on \(\Theta\).

Given a price \(\psi\), we can consider the normalized price density \(\bar{\psi} : S_0 \to \mathbb{R}\) such that for all \(s \in S_0\)

\[
\bar{\psi} (s) := \frac{\psi_1 (s)}{\psi_0 (s)}
\]

The normalized price density gives the relative price of the stock to the bond in every realization of \(s \in S\). We interpret this as the interest-adjusted price of the stock.

We now address the behavioral characteristics of an equilibrium. Let \(\succeq\) be a complete binary relation on \(I\).

Definition. An equilibrium \((\psi, \nu)\) is ordered by \(\succeq\) iff a.s. the following are equivalent:

1. \(i \succeq j\)
2. \(\bar{\psi} (s_0) \geq \bar{\psi} (s)\) iff \(\theta^i_1 (s) \geq \theta^j_1 (s)\)

We say an equilibrium is ordered iff it is ordered by some \(\succeq\). To illustrate this definition, consider two agents \(i\) and \(j\) such that \(i \succeq j\). Now, in all states where the normalized price decreases (increases), agent \(i\) has greater (less) stock holdings than agent \(j\). In other words, agent \(i\) exhibits a greater disposition effect than agent \(j\). If an equilibrium is ordered, then this ranking on \(I\) is complete. In other words, all agents can be ranked according to the degree of disposition effect that they exhibit. Note that this is a behavioral characterization of the equilibrium that is completely observable.

Theorem 1 below asserts that every \(\mu\) with a persistence representation has an ordered equilibrium.

Theorem 1. If \(\mu\) has a persistence representation, then any equilibrium of \((z, \mu)\) is ordered by some \(\succeq\). Moreover, \(\lambda^i \leq \lambda^j\) iff \(i \succeq j\) for all \(\{i, j\} \subset I\).

Proof. See Appendix.

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\(^{10}\) Lemma A1 in the Appendix ensures that we can always define \(\bar{\psi}\) without loss of generality.
In an equilibrium where \( \mu \) has a persistence representation, agents who exhibit the greatest disposition effect are exactly those who have beliefs that exhibit the least persistence. This is because at time 0, all agents share the same prior on \( S \) so they all hold the same amount of the stock. In states where the price of the stock increases (decreases), the agent with the least persistent belief holds the most (least) amount of stock in equilibrium. Market clearing ensures that this agent exhibits the disposition effect.

On the other hand, agents with beliefs that have the greatest persistence exhibit the opposite of the disposition effect. They increase stock holdings when prices rise and decrease holdings when prices fall. Thus, they trade based on stock price momentum. Note that this is exactly the house-money effect if we consider the limit case where prices are constant.\(^{11}\) Agents who believe in any state persistence increase holdings of the stock after “good” realizations of \( s \in S \) and decrease their holdings after “bad” realizations.

4 Special Case: Gaussian Beliefs

In this section, we consider a special case where beliefs are Gaussian. Let \( S = \mathbb{R} \) and assume that \( p \) is Gaussian with mean \( m \) and variance \( \sigma^2 > 0 \). Recall from Example 2 that if we let \( \tau \in (-1, 1) \) be the correlation coefficient between the two periods, then for every \( s \in S \), \( K_s \) is Gaussian with mean \( m (1 - \tau) + \tau s \) and variance \( (1 - \tau^2) \sigma^2 \).

\textbf{Definition.} \( K \in \mathcal{K} \) is \textit{Gaussian} iff there is some \( \tau \geq 0 \) such that \( K_s \) is a Gaussian distribution with mean \( m (1 - \tau) + \tau s \) and variance \( (1 - \tau^2) \sigma^2 \).

Note that the constant kernel \( K \in \mathcal{K} \) is Gaussian with \( \tau = 0 \). Suppose that the measure \( \mu \) only puts strictly positive mass on the constant kernel \( K \) and some other Gaussian kernel \( K \in \mathcal{K} \). We call such a \( \mu \) \textit{simple Gaussian}.

\textbf{Definition.} \( \mu \) is \textit{simple Gaussian} iff there is some Gaussian \( K \in \mathcal{K} \) and \( \alpha \in [0, 1] \) such that

\[ \mu = (1 - \alpha) \delta_K + \alpha \delta_K \]

\[^{11}\text{For example, by assuming that } \mu \{K\} \to 1.\]
Thus, a population with a simple Gaussian $\mu$ consists of agents ($\alpha$ proportion) who believe in that $S$ is correlated with coefficient $\tau$ while the rest ($1 - \alpha$ proportion) believe that there is no correlation. Note that $\mu$ is trivially a persistence representation. Moreover, since mixtures of Gaussian distributions are in general not Gaussian, any $\mu$ that has a persistence representation and only puts weight on Gaussian beliefs must be simple Gaussian.

Now, suppose the stock is given by $z(s) = s$. In other words, payoffs are increasing in $s \in S$. The Proposition below characterizes the equilibrium prices and strategies for the stock.

**Proposition 1.** Let $\mu$ be simple Gaussian and $\theta$ be the equilibrium strategy for $K \in K$. Then for all $s \in S$,

$$\bar{\psi}(s) = \frac{\alpha \tau (s - (1 - \tau) m) + (1 - \tau^2) (m - \rho \sigma^2)}{1 - (1 - \alpha) \tau^2}$$

$$\theta_1(s) = \frac{\rho \sigma^2 + (s - m) (1 - \alpha) \tau}{\rho (1 - (1 - \alpha) \tau^2) \sigma^2}$$

**Proof.** See Appendix. \qed

This immediately implies the following corollary below.

**Corollary 1.** Let $\mu$ be simple Gaussian and $\theta$ be the equilibrium strategy for $K \in K$. Then the following are equivalent for all $s \in S$.

1. $s \geq m - \tau \rho \sigma^2$
2. $\bar{\psi}(s) \geq \bar{\psi}(s_0)$
3. $\theta_1(s) \leq \theta_1(s_0)$

**Proof.** See Appendix. \qed

Corollary 1 provides a very precise illustration of Theorem 1. A realization $s \in S$ at time 1 is considered “good” for the stock iff $s \geq m - \tau \rho \sigma^2$ and the price density increases. In this case, agents who believe that is no persistence ($\tau = 0$) decrease their holdings of the stock. Vice-versa, a realization $s \in S$ at time 1 is considered “bad” for the stock iff $s \leq m - \tau \rho \sigma^2$, the price density decreases and the agents who believe in no persistence increase their holdings. This is exactly the disposition effect in this special case with Gaussian beliefs.

Corollary 2 below summarizes some comparative statistics of pricing in this model.
Corollary 2. If $\mu$ is simple Gaussian, then

1. $\tilde{\psi}$ is increasing in $\alpha$ iff $s \geq m - \tau \rho \sigma^2$
2. $\tilde{\psi}$ is decreasing in $\rho$

Proof. Follows directly from Proposition 1.

Thus, as more agents believe in persistence, the stock prices increase under “good” states and decrease under “bad” states. In other words, increasing the relative mass of agents in the population who believe in persistence results in prices that are more dispersed. This is in contrast to many other models of heterogeneous beliefs where introducing belief heterogeneity uniformly increases prices. In our model, risk aversion and the absence of any short-sale constraints result in a more subtle interaction between belief heterogeneity and prices. Note that on the other hand, increasing risk aversion uniformly lowers stock prices.

5 Characterization and Uniqueness

In this section, we consider the uniqueness properties of persistence representations. First, note that we can also view each kernel $K \in \mathcal{K}$ as an operator $K : \Pi \to \Pi$ where for any $q \in \Pi$, $K(q) \in \Pi$ is the measure that satisfies

$$(K(q))(A) = \int_S q(ds) K_s(A)$$

for all $A \in \mathcal{F}$. Since $K$ is $p$-invariant, $p$ is a fixed point of this operator. We say $K$ is generic iff the operator is injective. If $S$ is finite, then this is equivalent to requiring that all $K_s$ are linearly independent. Thus, in this case, an agent with a generic kernel possesses conditional beliefs with enough persistence that they span the entire probability simplex. For example, the matrix corresponding to the kernel $\overline{K}$ in Example 1 is invertible so $\overline{K}$ is generic. We say $\mu$ is generic iff it puts strictly positive mass on some generic kernel.

Definition. $\mu$ is generic iff $\mu \{ K \} > 0$ for some generic $K \in \mathcal{K}$. 

If $\mu$ is generic, then there is at least some non-trivial proportion of agents in the population we exhibit enough variation and persistence in beliefs. The following is an example of a non-generic $\mu$.

**Example 3.** Let $S = \{s_1, s_2, s_3\}$, $p = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$ and

$$K := \begin{bmatrix}
\frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}$$

Note that $pK = p$ so $K$ is $p$-invariant. Also, for all $s \in S$,

$$\frac{K_s \{s\}}{p \{s\}} \in \left\{2, 1, \frac{3}{2}\right\}$$

so $K$ is persistent and $K \in \mathcal{K}$. However, note that $K$ is not an invertible matrix so it is not generic. This is because an agent with belief $K$ does not update her beliefs if $s_2$ occurs. If we let

$$\mu := \frac{1}{2} \delta_K + \frac{1}{2} \delta_K$$

then $\mu$ is not generic.

Fix some $z \in Z$ and consider the equilibrium $(\psi, \nu)$ for some $(z, \mu)$. Clearly, there are cases where we can find some other $\mu'$ such that $(\psi, \nu)$ is also the equilibrium for $(z, \mu')$. In other words, $\mu$ is not unique given its equilibrium. However, suppose we were able to vary the security $z$ and observe the equilibria for $(z, \mu)$ for each $z \in Z$. We say all the equilibria of $\mu$ are ordered by some $\succeq$ iff $\succeq$ ranks all agents by the degree of disposition effect they exhibit for all $z \in Z$.

**Definition.** The equilibria of $\mu$ are ordered by $\succeq$ iff a.s. for all $z \in Z$, the following are equivalent:

1. $i \succeq j$
2. $\bar{\psi}(0) \succeq \bar{\psi}(s)$ iff $\theta_i^1(s) \geq \theta_1^1(s)$
As before, we say that equilibria of $\mu$ are ordered iff it is ordered by some $\succeq$. Theorem 2 below asserts that persistence representations are the only representations that ensure that all equilibria of $\mu$ are ordered by the same $\succeq$.

**Theorem 2.** The equilibria of a generic $\mu$ are ordered iff $\mu$ has a persistence representation.

**Proof.** See Appendix.

Thus, if $\mu$ is generic, then observing a consistent disposition ranking $\succeq$ for all equilibria completely characterizes persistence representations. For any other representation of $\mu$, we can find some $z \in Z$ such that the disposition ranking is violated. This allows us to completely identify the unobservable $\mu$ with observable characteristics of equilibrium prices and strategies. By varying the stock payoffs $z \in Z$, we can uniquely pin down the distribution of beliefs in the population.

## 6 Conclusion

We introduce a model where beliefs are heterogeneous along a single dimension measuring persistence. In equilibrium, agents can all be ordered by the degree of disposition effect they exhibit. In particular, those who hold the most persistent beliefs exhibit the disposition effect while those who hold the least persistent beliefs engage in momentum trading (a form of the house-money effect). Although we only consider a simple two-period trading model, our results could be generalized to a dynamic infinite-period setup with Markov trading strategies.
References


Appendix A

In this appendix, we prove the main results for the model.

Lemma (A1). If $(\psi, \nu)$ is an equilibrium for $(z, \mu)$, then $\psi_0(s_0) > 0$ and $\psi_0(s) > 0$ a.s.

Proof. Consider $K \in K$ such that $\mu \{K\} > 0$ and let $\theta \in \Theta$ be optimal for $K$. First, suppose $\psi_0(s_0) \leq 0$ and consider $\hat{\theta} \in \Theta$ such that $\hat{\theta}(s) = \theta(s)$ for all $s \in S$ and $\hat{\theta}(s_0) = (\theta_0(s_0) + \varepsilon, \theta_1(s_0))$ for some $\varepsilon > 0$. Now,

$$
\Psi(\hat{\theta}) = \psi_0(s_0) \varepsilon + \Psi(\theta) \leq \Psi(1)
$$

so $\hat{\theta} \in B(\psi)$. Since $u$ is CARA,

$$
U_K(\hat{\theta}) - U_K(\theta) = \int_S p(ds) [u(\theta(s) \cdot z(s) + \varepsilon) - u(\theta_0(s_0) \cdot z(s))]
$$

$$
= (e^{-\rho \varepsilon} - 1) \int_S p(ds) u(\theta(s) \cdot z(s))
$$

Since $z$ is bounded, $U_K(\hat{\theta}) > U_K(\theta)$ contradicting the fact that $\theta$ is optimal. Hence, $\psi_0(s_0) > 0$.

Define

$$
E := \{s \in S \mid \psi_0(s) \leq 0\}
$$

and suppose $p(E) > 0$. Let $\hat{\theta} \in B(\psi)$ be such that $\hat{\theta}(s) = \theta(s)$ if $s \in \{s_0\} \cup (S \setminus E)$ and $\hat{\theta}(s) = (\theta_0(s) + \varepsilon, \theta_1(s))$ if $s \in E$ for some $\varepsilon > 0$. Now,

$$
\Psi(\hat{\theta}) = \Psi(\theta) + \varepsilon \int_E p(ds) \psi_0(s) \leq \Psi(1)
$$

so $\hat{\theta} \in B(\psi)$. Again, as $u$ is CARA,

$$
U_K(\hat{\theta}) - U_K(\theta) = \delta(e^{-\rho \varepsilon} - 1) \int_E p(ds) \int_S K_s(ds') u(\theta(s) \cdot z(s'))
$$

Let

$$
\zeta(s) := \int_S K_s(ds') u(\theta(s) \cdot z(s'))
$$

and note that since $z$ is bounded, $\zeta(s) < 0$ for all $s \in S$. For $\eta > 0$, let

$$
E_\eta := \{s \in E \mid \zeta(s) < -\eta\}
$$
Now, for all $\eta > 0$,
\[
\int_E p(ds) \zeta(s) \leq -p(E_\eta) \eta
\]
Suppose $p(E_\eta) = 0$ for all $\eta > 0$. As $\eta \to 0$, $E_\eta \nearrow E$ so $p(E_\eta) \to p(E) > 0$ a contradiction. Thus, $\exists \eta > 0$ such that $p(E_\eta) > 0$ so $\int_E p(ds) \zeta(s) < 0$. Hence, $U_K(b) > U_K(a)$ again contradicting the optimality of $\theta$. We thus have $\psi_0(s) > 0$ a.s.. □

Lemma A1 ensures that we can define the normalized price density $\tilde{\psi} : S_0 \to \mathbb{R}$ such that
\[
\tilde{\psi}(s) := \frac{\psi_1(s)}{\psi_0(s)}
\]
For ease of notation, we let $\mathbb{E}^q$ denote the expectation operator with respect to the measure $q \in \Pi$. For $q = K_i^s$, we let $\mathbb{E}_s^i := \mathbb{E}^{K_i^s}$. Let $\Pi_0$ be the set of probability measures on $S$ absolutely continuous with respect to $p$. Note that by definition, $K \in \mathcal{K}$ implies $K_s \in \Pi_0$ for all $s \in S$. Fix $z \in Z$ and let $\xi : \Pi_0 \times \mathbb{R} \to \mathbb{R}$ be such that
\[
\xi(q, a) := \frac{\mathbb{E}^q [u(az) z]}{\mathbb{E}^q [u(az)]}
\]
Note that since $z$ is bounded and $u$ is CARA, $\mathbb{E}^q [u(az)] < 0$ so $\xi$ is well-defined.

**Lemma (A2).** Fix $z \in Z$.

1. $\xi$ is strictly decreasing in $a$

2. $\lim_{a \to -\infty} \xi(q, a) = \sup_{s \in S} z(s)$ and $\lim_{a \to -\infty} \xi(q, a) = \inf_{s \in S} z(s)$

3. If $\xi(q, a) = \xi(r, a)$, then $\xi(\lambda q + (1 - \lambda) r, a) = \xi(q, a)$ for all $\lambda \in [0, 1]$.

**Proof.** Fix $z \in Z$. We prove the lemma in order.

1. Since $u$ is CARA,
\[
\frac{\partial u(az)}{\partial a} = u'(az) z = -\rho u(az) z
\]
As $z$ is bounded, by Theorem 16.8 of Billingsley [4], $\xi$ is differentiable in $a$ and
\[
\frac{\partial \xi}{\partial a} = -\rho \frac{\mathbb{E}^q [u(az) z^2]}{\mathbb{E}^q [u(az)]} + \rho \left( \frac{\mathbb{E}^q [u(az) z]}{\mathbb{E}^q [u(az)]} \right)^2
\]
For $a \in \mathbb{R}$, define $q_a \in \Pi$ such that
\[
q_a(A) := \frac{\mathbb{E}^q [\mathbf{1}_A u(az)]}{\mathbb{E}^q [u(az)]}
\]
for all $A \in \mathcal{F}$. Now,
\[
\frac{\partial \xi}{\partial a} = \rho \left( \mathbb{E}^q [z] \right)^2 - \rho \mathbb{E}^{q_0} [z^2] = -\rho \mathbb{E}^{q_0} \left[ (z - \mathbb{E}^{q_0} [z])^2 \right] \leq 0
\]

If the last inequality is an equality, then $z = \mathbb{E}^{q_0} [z]$ a constant $q_0$-a.s.. Note that since $q \in \Pi_0$, $p(A) = 0$ implies $q(A) = 0$ which implies $q_0(A) = 0$. Hence, $q_0 \in \Pi_0$ for all $a \in \mathbb{R}$ so $z$ is non-constant $q_0$-a.s.. Thus, $\xi$ must be strictly decreasing in $a$.

(2) Since $z$ is measurable, we can approximate $z$ by a sequence of increasing simple functions. In other words, $z = \lim_n z_n$ where the $z_n$ are increasing and

\[
z_n = \sum_t c^n_t 1_{A^n_t}
\]

for $A^n_t \in \mathcal{F}$. Now,

\[
\frac{\mathbb{E}^q [u (az_n) z_n]}{\mathbb{E}^q [u (az_n)]} = \frac{\sum_t q (A^n_t) u (ac_t^n) c^n_t}{\sum_t q (A^n_t) u (ac_t^n)} = \frac{\sum_t q (A^n_t) e^{-\rho ac_t^n} c^n_t}{\sum_t q (A^n_t) e^{-\rho ac_t^n}}
\]

Clearly, if $\bar{c}_t^n = \sup_{s \in S} z_n (s)$, then

\[
\lim_{a \to \infty} \frac{\mathbb{E}^q [u (az_n) z_n]}{\mathbb{E}^q [u (az_n)]} = \bar{c}_t^n = \sup_{s \in S} z_n (s)
\]

By dominated convergence (see Theorem I.4.16 of Cinlar [5]),

\[
\xi (q, a) = \frac{\mathbb{E}^q [u (az)]}{\mathbb{E}^q [u (az)]} = \lim_n \frac{\mathbb{E}^q [u (az_n) z_n]}{\mathbb{E}^q [u (az_n)]}
\]

Hence,

\[
\lim_{a \to \infty} \xi (q, a) = \lim_n \lim_{a \to \infty} \frac{\mathbb{E}^q [u (az_n) z_n]}{\mathbb{E}^q [u (az_n)]} = \lim_n \sup_{s \in S} z_n (s) = \sup_{s \in S} \lim_n z_n (s) = \sup_{s \in S} z (s)
\]

The case for $\lim_{a \to -\infty} \xi (q, a) = \inf_{s \in S} z (s)$ is symmetric.

(3) Suppose $\xi (q, a) = \xi (r, a) = \xi^*$ for $\{r, q\} \subset \Pi_0$ and $a \in \mathbb{R}$. Thus,

\[
\mathbb{E}^q [u (az) (z - \xi^*)] = \mathbb{E}^r [u (az) (z - \xi^*)] = 0
\]
If we let \( q_\lambda := \lambda q + (1 - \lambda) r \) for \( \lambda \in [0, 1] \), then

\[
\mathbb{E}^{q_\lambda} [u (az) (z - \xi^*)] = 0
\]

which implies

\[
\xi^* = \frac{\mathbb{E}^{q_\lambda} [u (az) z]}{\mathbb{E}^{q_\lambda} [u (az)]} = \xi (q_\lambda, a)
\]

\( \blacksquare \)

**Lemma (A3).** Let \( q^i = \lambda q^i + (1 - \lambda) q^k \) for some \( \lambda \in (0, 1) \), and suppose

\[
\xi (q^k, a^k) = \xi (q^i, a^i) = \xi (q^i, a^i)
\]

Then \( a^k = a^i \) implies \( a^k = a^i = a^i \) and \( a^k > a^i \) implies \( a^k > a^j > a^i \).

**Proof.** Let

\[
\xi^* := \xi (q^k, a^k) = \xi (q^i, a^i) = \xi (q^i, a^i)
\]

First, suppose \( a := a^i = a^k \). Thus, from Lemma A2, \( \xi (q^i, a) = \xi^* \) and \( a = a^j \).

Now, assume \( a^k > a^i \). Again, by Lemma A2,

\[
\xi (q^k, a^i) > \xi^* > \xi (q^i, a^k)
\]

Suppose \( a^j > a^k \) so \( \xi (q^j, a^k) > \xi^* \). By continuity, we can find some \( \gamma \in (0, 1) \) such that

\[
\xi^* = \xi (\gamma q^j + (1 - \gamma) q^i, a^k) = \xi (q^k, a^k)
\]

Now, if we let \( \hat{\gamma} := \frac{\lambda}{\lambda + (1 - \gamma) (1 - \lambda)} \), then

\[
\hat{\gamma} (\gamma' q^j + (1 - \gamma') q^i) + (1 - \hat{\gamma}) q^k = \lambda q^i + (1 - \lambda) q^k = q^j
\]

Thus, by Lemma A2, \( \xi^* = \xi (q^j, a^k) \) which implies \( a^j = a^k \) a contradiction. The case for \( a^i > a^j \) is symmetric, so we have \( a^k \geq a^j \geq a^i \). Lastly, suppose \( a := a^k = a^i \) so by Lemma A2,

\[
\xi^* = \xi (q^i, a^i) > \xi (q^i, a) = \frac{\mathbb{E}^{q^i} [u (az) z]}{\mathbb{E}^{q^i} [u (az)]}
\]
By Lemma A2 and A3, we have

\[ E^q i [u(az)(z - \xi^i)] > 0 = E^q k [u(az)(z - \xi^k)] \]

\[ = E^q i [u(az)(z - \xi^i)] \]

Thus, by Lemma A2, the optimality conditions and the fact that \( a^i = a^j \) is symmetric, so \( a^k > a^j > a^i \).

**Theorem (A4)**. If \( \mu \) has a persistence representation, then any equilibrium of \((z, \mu)\) is ordered by some \( \succeq \). Moreover, \( \lambda^i \leq \lambda^j \) iff \( i \succeq j \) for all \( \{i, j\} \subset I \).

**Proof.** Let \( \mu \) have a persistence representation and \((\psi, \nu)\) be an equilibrium for \((z, \mu)\). From the optimality conditions and the fact that \( u \) is CARA, we have for all \( i \in I \),

\[
\bar{\psi}(s) = \frac{\psi_1(s)}{\psi_0(s)} = \frac{E^p [u'((\theta^i_1(s)) \cdot z)]}{E^p [u'((\theta^i_0(s)) \cdot z)]} = \frac{E^p [u((\theta^i_1(s)) z)]}{E^p [u((\theta^i_0(s)) z)]} = \xi(p, \theta^i_1(s))
\]

Thus, by Lemma A2, \( \theta^i_1(s) \) is the same for all \( i \in I \) so by market clearing, \( \theta^i_1(s) = 1 \) for all \( i \in I \). Note that by similar reasoning, we have a.s.

\[
\bar{\psi}(s) = \frac{\psi_1(s)}{\psi_0(s)} = \frac{E^i_p [u((\theta^i_1(s)) z)]}{E^i_p [u((\theta^i_0(s)) z)]} = \xi(K^i_s, \theta^i_1(s))
\]

Define \( i \succeq j \) iff \( \lambda^i \leq \lambda^j \) iff \( i \leq j \) for all \( \{i, j\} \subset I \). We show that \((z, \mu)\) is ordered by \( \succeq \).

Let \( \{1, n\} \subset I \) be such that \( 1 \leq i \leq n \) for all \( i \in I \). Let

\[
E := \{ s \in S \mid \bar{\psi}(s_0) \geq \bar{\psi}(s) \text{ and } \theta^1_1(s) < 1 \}
\]

and suppose \( p(E) > 0 \). For \( s \in E \), the Lemma A2 ensures that we can always find some \( a_s \in \mathbb{R} \) such that \( \xi(p, a_s) = \bar{\psi}(s) \). Note that we can assume \( \bar{\psi}(s) = \xi(K^i_s, \theta^i_1(s)) \) for all \( i \in I \) without loss of generality, so

\[
\xi(p, \theta^i_1(s)) > \xi(p, 1) = \bar{\psi}(0) \geq \bar{\psi}(s) = \xi(p, a_s) = \xi(K^1_s, \theta^1_1(s)) = \xi(K^m_s, \theta^0_1(s))
\]

By Lemma A2 and A3, we have \( a_s \geq 1 > \theta^1_1(s) \) so \( a_s > \theta^1_1(s) > \theta^0_1(s) \) as \( K^1_s = \lambda^1 p + (1 - \lambda^1) K^m_s \). Since this is true for all \( s \in E \), we have \( 1 > \theta^i_1(s) \) for all \( i \in I \) on a set of strictly positive measure, contradicting market clearing. Thus, \( \bar{\psi}(s_0) \geq \bar{\psi}(s) \) implies
\( \theta_i^1(s) \geq 1 \) which implies \( \theta_i^j(s) \geq \theta_i^j(s) \) for \( j \geq i \) a.s. By symmetric argument, we have \( \tilde{\psi}(s_0) \leq \tilde{\psi}(s) \) implies \( \theta_i^j(s) \leq \theta_i^j(s) \) for \( j \geq i \) a.s. Thus, \( \tilde{\psi}(s_0) \geq \tilde{\psi}(s) \) iff \( \theta_i^j(s) \geq \theta_i^j(s) \) for \( j \geq i \) a.s. so \((z, \mu)\) is ordered by \( \succeq \).

**Lemma (A5).** Suppose \( L_s = \lambda p + (1 - \lambda) K_s \) a.s. for \( \{K, L\} \subset \mathcal{K} \) and \( \lambda \in (0, 1) \). Then \( K \) is generic iff \( L \) is generic.

**Proof.** First, suppose \( K \) is generic. Let \( \{r, q\} \subset \Pi \) be such that

\[
\int_S q(ds)L_s(A) = \int_S r(ds)L_s(A)
\]

for all \( A \in \mathcal{F} \). Now,

\[
\int_S q(ds)(\lambda p(A) + (1 - \lambda) K_s(A)) = \int_S r(ds)(\lambda p(A) + (1 - \lambda) K_s(A))
\]

\[
\int_S q(ds)K_s(A) = \int_S r(ds)K_s(A)
\]

implying \( q = r \) so \( L \) is generic. If \( L \) is generic, then let \( \{r, q\} \subset \Pi \) be such that

\[
\int_S q(ds)K_s(A) = \int_S r(ds)K_s(A)
\]

\[
\int_S q(ds)(\lambda p(A) + (1 - \lambda) K_s(A)) = \int_S r(ds)(\lambda p(A) + (1 - \lambda) K_s(A))
\]

for all \( A \in \mathcal{F} \). Thus, \( r = q \) and \( K \) is generic. 

For \( \{q, r\} \subset \mathbb{R}^d \) where \( d \in \mathbb{N} \), define

\[
[q, r] := \{ q\lambda + (1 - \lambda) r \in \mathbb{R}^d \mid \lambda \in [0, 1]\}
\]

Similarly, for \( \{q, r\} \subset \Pi_0 \), define

\[
[q, r] := \{ q\lambda + (1 - \lambda) r \in \Pi_0 \mid \lambda \in [0, 1]\}
\]

**Lemma (A6).** If \( \mu \) has a persistence representation, then its equilibria are ordered.

**Proof.** Suppose \( \mu \) has a persistence representation, and for any \( z \in Z \), define \( E_z \subset S \) such that \( s \in E_z \) iff \( i \succeq j \) is equivalent to \( \tilde{\psi}(s_0) \geq \tilde{\psi}(s) \) iff \( \theta_i^j(s) \geq \theta_i^j(s) \). By Theorem A4, \( p(E_z) = 1 \). Now, let \( \bar{S} \subset S \) be the set such that \( s \in \bar{S} \) iff for all \( z \in Z \), \( i \succeq j \) is equivalent
Thus, if we let $\bar{s} = \bigcap_{z \in Z} E_z$

Let $Z^* \subset Z$ be some dense countable subset of $Z$ so

$\bigcap_{z \in Z} E_z \subset \bigcap_{z \in Z^*} E_z$

Now, let $s \in \bigcap_{z \in Z^*} E_z$. Thus, for all $z \in Z^*$, $i \geq j$ is equivalent to $\psi(s_0) \geq \bar{\psi}(s)$ iff $\theta_i^1(s) \geq \theta_j^1(s)$. By the continuity of prices and holdings, we have $i \geq j$ is equivalent to $\bar{\psi}(s_0) \geq \bar{\psi}(s)$ iff $\theta_i^i(s) \geq \theta_j^1(s)$ for all $z \in Z$. Thus, $s \in \bar{S}$ so $\bar{S}$ is measurable. Hence,

$$p(\bar{S}) = p\left(\bigcap_{z \in Z^*} E_z\right) = 1$$

\textbf{Theorem (A7).} The equilibria of a generic $\mu$ are ordered iff $\mu$ has a persistence representation.

\textbf{Proof.} Note that necessity follows from Lemma A6, so we prove sufficiency. Let $\mu$ be generic with equilibria ordered by $\succeq$. Without loss of generality, let $1 \geq i > i + 1 > n$ for all $i \in I$. Define $\bar{S} \subset S$ to be the a.s. set such that $s \in \bar{S}$ iff for all $z \in Z$, $i \geq j$ is equivalent to $\psi(s_0) \geq \bar{\psi}(s)$ iff $\theta_i^i(s) \geq \theta_j^1(s)$.

Fix $s \in \bar{S}$ and first consider $i > j > k$ for $\{i, j, k\} \subset I$. Let $\{A_1, \ldots, A_d\}$ be a measurable partition of $S$ for some $d \in \mathbb{N}$ and $c \in \mathbb{R}^d_+$. Let $z = \sum_t 1_{A_t} c_t$ for $t \in \{1, \ldots, d\}$. Note that $\theta_i^i(s) \geq \theta_j^1(s) \geq \theta_k^k(s)$ or $\theta_i^i(s) \leq \theta_j^1(s) \leq \theta_k^k(s)$. Assume the former without loss of generality. Let $q^i := K^i_s$ for all $i \in I$ and note that also without loss of generality

$$\bar{\psi}(s) = \xi(q^i, \theta_i^1(s)) = \xi(q^j, \theta_j^1(s)) = \xi(q^k, \theta_k^1(s))$$

If we let $a_c := \theta_j^1(s)$ and $\xi_c := \bar{\psi}(s)$, then by the lemma above,

$$\xi(q^i, a_c) \leq \xi_c = \xi(q^j, a_c) \leq \xi(q^k, a_c)$$

Thus,

$$\mathbb{E}^q_i [u(a_c z)(z - \xi_c)] \geq 0 = \mathbb{E}^q_i [u(a_c z)(z - \xi_c)] \geq \mathbb{E}^k_i [u(a_c z)(z - \xi_c)]$$

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If we let $v_c(t) = u(a_c c_t)(c_t - \xi_c)$ for all $t \in \{1, \ldots, m\}$, then $q^i \cdot v_c \geq q^j \cdot v_c \geq q^k \cdot v_c$ where \( \{q^i, v_c\} \subset \mathbb{R}^m \).

For $c = 1_t$, \[ q^i \cdot v_c = q^i(t) u(a_c)(1 - \xi_c) + (1 - q^i(t)) u(0)(-\xi_c) \]

If $q^i(t) = q^k(t)$, then $q^j(t) = q^i(t)$. Otherwise, we can find a $t' \in \{1, \ldots, d\}$ such that $q^i(t) > q^k(t)$ and $q^i(t') < q^k(t')$ without loss of generality. By continuity, there is some $\hat{c} = \beta 1_t + (1 - \beta) 1_{t'}$ such that $q^i \cdot v_{\hat{c}} = q^i \cdot v_c = q^k \cdot v_c$. Note that we can always find $m - 2$ such $\hat{c}$ where $v_{\hat{c}}$ are linearly independent. Since $\sum q^i(t) = 1$, we must have $q^i \in [q^i, q^k]$.

Since this is true for all $\{i, j, k\} \subset I$, we must have $q^i \in [q^1, q^n]$ for all $i \in I$. If $p \not\in [q^1, q^n]$, we can find some $c$ such that $\xi_c \geq \tilde{\psi}(s_0)$ and $E^{q^i}[u(az)(z - \xi_c)] \geq E^{p}[u(az)(z - \xi_c)]$ for all $i \in I$ where $\xi(p, a) = \xi_c$. Hence,

\[ \xi(q^i, a) \leq \xi(p, a) = \xi(q^i, \theta^i_1(s)) = \xi_c \geq \tilde{\psi}(s_0) = \xi(p, 1) \]

Thus, $\theta^i_1(s) \leq a \leq 1$ contradicting market clearing. Hence, $p \in [q^1, q^n]$.

Suppose $q^i \not\in [q^1, q^n]$. By a standard Separating Hyperplane Theorem (see Theorem 5.61 of Aliprantis and Border [1]), we can find some measurable $\zeta : S \to \mathbb{R}$ such that

\[ E^{q^i}[\zeta] \not\in \left[ E^{q^i}[\zeta], E^{q^k}[\zeta] \right] \]

Since $\zeta$ is measurable, it is the limit of a sequence of increasing simple functions. Hence, $\zeta = \lim_t \zeta_t$ where

\[ \zeta_t = \sum_t b^t_1 a^t_l \]

Now, for each $\zeta_t$, we have

\[ E^{q^i}[\zeta_t] = \sum_t b^t_1 q^i(A^t_l) = \lambda \sum_t b^t_1 q^i(A^t_l) + (1 - \lambda) \sum t b^t_1 q^k(A^t_l) \]

so $E^{q^i}[\zeta_t] \in \left[ E^{q^i}[\zeta_t], E^{q^k}[\zeta_t] \right]$ for all $\zeta_t$. By monotone convergence, we must have $E^{q^i}[\zeta] \in \left[ E^{q^i}[\zeta], E^{q^k}[\zeta] \right]$ a contradiction. By similar argument, $p \in [q^1, q^n]$.

Thus, we have $p \in [K^1_s, K^n_s]$ a.s.. Hence, we have a.s.

\[ p(ds') = \lambda_s K^1_s(ds') + \lambda_s K^n_s(ds') = p(ds') \kappa^1_s(s') \lambda_s + p(ds') \kappa^n_s(s')(1 - \lambda_s) \]
Hence, we have \( \kappa_s^1(s) \lambda_s + \kappa_s^n(s) (1 - \lambda_s) = 1 \) a.s.. Since \( \{K^1, K^n\} \subset \mathcal{K} \) we have \( \kappa_s^1(s) \geq 1 \) and \( \kappa_s^n(s) \geq 1 \) so \( \lambda_s \in \{0, 1\} \) a.s.

Now, consider \( K^i \in [p, K^n] \) so for all \( A \in \mathcal{F} \),

\[
p(A) = \int_S p(ds) K^i_s(A) = \int_S p(ds) \left( \lambda^i_s p(A) + (1 - \lambda^i_s) K^n_s(A) \right) \]

\[
= p(A) \int_S p(ds) \lambda^i_s + p(A) - \int_S p(ds) K^n_s(A) \lambda^i_s
\]

Note that if \( \lambda^i_s = 0 \) a.s. then \( K^i_s = p \) a.s.. Hence, suppose the \( \lambda^i_s > 0 \) on some set of strictly positive \( p \)-measure. We can now define \( p^i \in \Pi \) such that

\[
p^i(A) := \int_A \frac{p(ds) \lambda^i_s}{\int_S p(ds) \lambda^i_s}
\]

so

\[
p(A) = \int_S p^i(ds) K^n_s(A) = \int_S p(ds) K^n_s(A)
\]

Since \( \mu \) is generic, by Lemma A5, \( K^n \) is generic. Hence, \( p = p^i \) so by the Radon-Nikodym Theorem (Theorem I.5.11 of Cinlar [5]), we have

\[
1 = \frac{dp^i}{dp} = \frac{\lambda^i_s}{\int_S p(ds) \lambda^i_s}
\]

a.s. so \( \lambda^i_s = \lambda^i \) a.s.. The case for \( K^i \in [p, K^1] \) is symmetric, so we have \( \lambda^i_s = \lambda^i \) for all \( i \in I \). Thus, \( \mu \) has a persistence representation. \( \square \)

**Appendix B**

In this appendix, we prove the results for Gaussian model. Let \( z \in Z \) be such that \( z(s) = s \).

**Lemma (B1).** Let \( q \in \Pi \) be Gaussian distributed with mean \( m \) and variance \( \sigma^2 > 0 \). Then

\[
\mathbb{E}^q [u(az)] = u \left( ma - \frac{\sigma^2}{2} a^2 \right)
\]

\[
\mathbb{E}^q [u(az) z] = u \left( ma - \frac{\sigma^2}{2} a^2 \right) \left( m - a \rho \sigma^2 \right)
\]
Proof. A simple computation yields
\[
\mathbb{E}^q [u(az)] = -e^{-\rho a (m - a\rho \sigma^2)}
\]
\[
\mathbb{E}^q [u(az) \bar{z}] = -e^{-\rho a (m - a\rho \sigma^2)} (m - a\rho \sigma^2)
\]
The result then follows from the definition of \( u \).

\[\square\]

Proposition (B2). Let \( \mu \) be simple Gaussian and \( \theta \) be the equilibrium strategy for \( K \in \mathcal{K} \). Then for all \( s \in S \),
\[
\bar{\psi}(s) = \frac{\alpha \tau (s - (1 - \tau) m) + (1 - \tau^2) (m - \rho \sigma^2)}{1 - (1 - \alpha) \tau^2}
\]
\[
\theta_1(s) = \frac{\rho \sigma^2 + (s - m) (1 - \alpha) \tau}{\rho (1 - (1 - \alpha) \tau^2) \sigma^2}
\]

Proof. Let \( K \in \mathcal{K} \) be Gaussian and \( \theta \) be optimal for \( K \). Fix \( s \in S \) and let \( a = \theta_1(s) \). Now, by Lemma B1,
\[
\bar{\psi}(s) = \frac{\mathbb{E}^{K_s} [u(az) \bar{z}]}{\mathbb{E}^{K_s} [u(az)]}
\]
\[
= m (1 - \tau) + \tau s - \rho \sigma^2 (1 - \tau^2)
\]
Thus, we have
\[
\theta_1(s) = a = \frac{m (1 - \tau) + \tau s - \bar{\psi}(s)}{\rho \sigma^2 (1 - \tau^2) \sigma^2}
\]
Note that if \( \tau = 0 \), then \( \theta_1(s) = \frac{m - \bar{\psi}(s)}{\rho \sigma^2} \). Since \( \theta_1(s_0) = 1 \) by market clearing, we have
\[
\bar{\psi}(s_0) = m - \rho \sigma^2
\]
Also by market clearing,
\[
1 = \alpha \theta_1^0(s) + (1 - \alpha) \theta_1^0(s) = \alpha \frac{m (1 - \tau) + \tau s - \bar{\psi}(s)}{\rho (1 - \tau^2) \sigma^2} + (1 - \alpha) \frac{m - \bar{\psi}(s)}{\rho \sigma^2}
\]
Hence
\[
\bar{\psi}(s) = \frac{\alpha \tau (s - (1 - \tau) m) + (1 - \tau^2) (m - \rho \sigma^2)}{1 - (1 - \alpha) \tau^2}
\]
Substituting the formula for \( \bar{\psi} \) yields
\[
\theta_1(s) = \frac{\rho \sigma^2 + (s - m) (1 - \alpha) \tau}{\rho (1 - (1 - \alpha) \tau^2) \sigma^2}
\]
Corollary (B3). Let \( \mu \) be simple Gaussian and \( \theta \) be the equilibrium strategy for \( K \in \mathcal{K} \). Then the following are equivalent for all \( s \in S \)

1. \( s \geq m - \tau \rho \sigma^2 \)
2. \( \bar{\psi}(s) \geq \bar{\psi}(s_0) \)
3. \( \theta_1(s) \leq \theta_1(s_0) \)

**Proof.** From Proposition B2, we have \( \bar{\psi}(s) \geq \bar{\psi}(s_0) \) iff

\[
\alpha \tau (s - (1 - \tau)m) + (1 - \tau^2)(m - \rho \sigma^2) \geq (1 - (1 - \alpha) \tau^2)(m - \rho \sigma^2)
\]

\[
s \geq m - \tau \rho \sigma^2
\]

so (1) and (2) are equivalent. Since it also follows readily that \( \theta_1(s) \geq 1 \) iff \( s \geq m - \tau \rho \sigma^2 \) we have (1), (2) and (3) are all equivalent. \( \square \)