Repeated Choice:

A Theory of Stochastic Intertemporal Preferences*

Jay Lu — UCLA

Kota Saito — Caltech

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Abstract

We provide a repeated-choice foundation for stochastic choice. We obtain necessary and sufficient conditions under which an agent's observed stochastic choice can be represented as a limit frequency of the agent's choice over time. In the representation, the agent repeatedly chooses today's consumption and tomorrow's continuation menu, aware that future preferences will evolve according to a subjective ergodic *utility process*. Using our model, we demonstrate how not taking into account the agent's preference for early (late) resolution of uncertainty would lead an analyst to underestimate (resp., overestimate) the agent's risk aversion. Estimation of preferences can be performed by the analyst without explicitly modeling continuation problems (i.e. stochastic choice is *independent of continuation menus*) if and only if the utility pro-

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cess takes on the *standard* additive and separable form. Applications include dynamic discrete choice models even when agents have non-standard intertemporal preferences.

1 Introduction

Modeling choice behavior as stochastic is common across many economic applications. In many of these applications, stochasticity is interpreted as a result of unobserved heterogeneity in a population of agents (henceforth, the "population interpretation"). On the other hand, the psychological origins of stochastic choice point to a single agent interpretation.[1] There, stochasticity is interpreted as a result of one agent making choices from the same decision problem repeatedly (henceforth, the "individual interpretation"). The literature on stochastic choice, however, has mostly taken such choice frequencies as given without considering when such a repeated-choice interpretation is possible.

In this paper, we provide the first repeated-choice foundation for stochastic choice. Given an agent's stochastic choice, we obtain necessary and sufficient conditions under which the agent's observed stochastic choice can be represented as a limit frequency of his repeated choices over time; in the representation, the agent repeatedly chooses today's consumption and tomorrow's continuation menu, aware that future preferences will evolve according to a *subjective utility process*.

Applying our model, we show that whenever the agent has non-standard intertemporal preferences, such as a preference for early (or late) resolution of uncertainty, his stochastic choice would be highly sensitive to the frequency of the repetition. In particular, an agent who prefers early resolution of uncertainty would choose risky options more frequently in repeated than static choice. As a result, failure to take repetition into account would naturally lead an analyst (an outside observer such as an econometrician) to biased estimates of the agent's preferences. Even with the population interpretation, we can use our results to understand the systematic ways in which non-standard intertemporal preferences affect the estimation of any dynamic discrete choice model.

To present our model, we first describe the formal setup. Based on the works of Kreps and Porteus (1978), Epstein and Zin (1989), and Gul and Pesendorfer (2004), we develop

[1] Early work on models of stochastic choice include Thurstone (1927), Luce (1959), Block and Marschak (1960), and Falmagne (1978). The adoption of these models in economics to study unobserved heterogeneity naturally led to the population interpretation. For an overview of this history, see McFadden (2001).

an infinite-horizon framework to study the agent's problem. Every period, the agent faces a *menu* (i.e., a choice set) which consists of risky prospects over consumption today and a continuation menu tomorrow. We focus on menus such that regardless of what he chooses or which outcome is realized, the agent will always face the same menu again after some finite time.[2] We call such menus *repeated*. Repeated menus are important since, given the infinite time horizon, the agent could choose from the same menu infinitely many times, generating an infinite time series of choices. Thus, an agent's *stochastic choice* on repeated menus can be interpreted as the long-run frequency of choices from repeated menus.

Based on this interpretation, we introduce a new tractable model of stochastic choice. The agent's utility at time period t depends on some state variable s_t that evolves according to an ergodic Markov process. The Markov process is fixed and known to the agent but unknown to the analyst, which makes the agent's choice stochastic from the perspective of the analyst. For example, the state could be the agent's mood on a particular day, which affects how risk-averse and how impatient he is that day. Given the realization of state s_t at time t, the agent's utility of a pair (c, z) of today's consumption c and tomorrow's continuation menu z is recursively given by

$$u_t(c, z) = \phi_{s_t}\left(c, \mathbb{E}_{s_t}\left[\max_{p \in z} u_{t+1}(p)\right]\right). \tag{1}$$

There are two parts to this utility. First, the stochastic aggregator ϕ_{s_t} specifies the agent's intertemporal attitudes toward current consumption and future continuation value. Second, continuation values are evaluated by taking expectations with respect to the Markov process of the state. In other words, the agent is fully sophisticated; he knows the Markov process and takes expectations with the understanding that he will be choosing from the menu z tomorrow. The utility function (1) can be seen as a stochastic version of the model from Kreps and Porteus (1978) where continuation values are evaluated according to the additive linear representation of Dekel et al. (2001).

The *utility process* u_t defined in (1) is ergodic and describes the agent's stochastic intertemporal preferences at every time period t. In our representation theorem, for any menu z that repeats every t periods, the probability $\rho_z(p)$ that an option p is chosen from the

[2] It is straightforward to extend our domain to include menus that repeat with some positive probability as long as the probability does not depend on the agent's choice (otherwise, selection issues may complicate the identification exercise).

menu z is given by

$$\rho_z(p) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} 1\left\{u_{it+1}(p) \geq u_{it+1}(q) \text{ for all } q \in z\right\}, \tag{2}$$

where $1\{\cdot\}$ is the indicator function and $u_t(p) = \int u_t(c,z)\, dp$ with $u_t(c,z)$ described as in (1). In this case, we say ρ is *ergodic*. Here, the probability that p is chosen from a set z is exactly the long-run frequency of the event that p is the best element in z according to the utility process. This is exactly the individual interpretation of stochastic choice models. We thus provide a theoretical foundation for this interpretation.

The representation has two interesting features. First, despite the generality of the model and the fact that our domain is restricted to only repeated menus, we show that the analyst can fully identify the agent's utility process from stochastic choice over repeated binary menus. Second, the analyst can use the identified utility process to pin down the distribution of the entire time series of choices (once an initial state is determined), even though the observable data only consists of stochastic choice, i.e. the long-run choice frequencies.

We discuss two applications to illustrate the types of biases that can arise if the analyst ignores repetition and the agent's intertemporal preferences. In both applications, we consider a special case in which the stochastic aggregator ϕ takes on the well-known formula provided by Epstein and Zin (1989); we call this special case *stochastic Epstein-Zin*. Consider an analyst interested in eliciting an agent's risk aversion. Understanding that the agent's preferences may be stochastic, the analyst asks the agent to repeatedly choose between a safe option (e.g., \$5 for sure) and a risky option (e.g., \$10 or \$0 with equal probability) every day. If the agent is myopic and only considers current consumption, then the long-run frequency of choosing the safe option would correspond exactly to the probability that the agent is risk-averse, which is the standard individual interpretation of stochastic choice.

However, if the agent is sophisticated, then he would take into account the fact that he will be choosing again between the safe and risky options tomorrow. We show that if he has a preference for early resolution of uncertainty, then the probability of choosing the risky option increases when repetition becomes more frequent. In the Epstein-Zin model, a preference for early resolution of uncertainty corresponds to the agent's desire for consumption smoothing being lower than his relative risk aversion. For such an agent, the risky option feels "safer" under repeated choice; intuitively, even if today's outcome is bad, repeating the choice means that there is always a chance that tomorrow's outcome will be good. As a result, the risky option becomes more attractive as repetition becomes more frequent. This is a novel behavioral phenomenon absent in stochastic choice models that do not explicitly address repetition. Moreover, if the analyst misspecified the model and ignored repetition, then she will *underestimate* the agent's atemporal risk aversion. All this demonstrates the importance of modeling repetition when analyzing stochastic choice data.

In the second application, we consider a simple two-period example of a dynamic discrete choice model. Based on the same insight as in the first application, we illustrate the inherent inference issues that can arise if intertemporal preferences are not taken into account in applications of dynamic discrete choice estimation.

The two applications suggest that modeling repetition is crucial for inference when agents have non-standard intertemporal preferences. We also address the question of when an agent's preferences can be correctly inferred *without* modeling repetition explicitly. We define this formally using an axiom called *Independence of Continuation Menu (ICM)* and show that it is satisfied if and only if the utility process is *standard*, i.e., the stochastic aggregator takes the form of $\phi_s(c, v) = (1 - \beta_s) w_s(c) + \beta_s v$ where w_s is a random von Neumann–Morgenstern (vNM) utility function and β_s is a random discount factor. In the case of stochastic Epstein-Zin preferences, *Indifference to Timing of Resolution of Uncertainty (IRU)* would ensure that the utility process is standard. In general, however, this is not true; IRU characterizes a stochastic version of Uzawa-Epstein preferences in which discount factors also depend on consumption.[3] In this case, ICM would still be violated since continuation menus would still affect inference. We show that the gap between IRU and ICM is exactly a repeated version of the classic independence axiom, which we call *Repeated Independence (RI)*. We thus demonstrate the following three-way equivalence:

$$\text{ICM} \;\Leftrightarrow\; \text{IRU} + \text{RI} \;\Leftrightarrow\; \text{Standard Utility.}$$

The takeaway is that any generalization of standard utility will require the analyst to take into account repeated choice when conducting estimation or inference from stochastic choice.

Finally, we provide an axiomatic characterization of our model. While we focus only on the smaller domain of repeated menus, we show that the set of repeated menus is in fact dense in the set of all menus. In other words, for any generic menu z, we can construct a sequence of repeated menus that approximate z with arbitrary closeness. By considering a

[3] The model is originally proposed by Uzawa (1968) and later axiomatized by Epstein (1983) in an extended lottery setup.

continuous extension, we can therefore focus on a stochastic choice function ρ over all finite (but not necessarily repeated) menus.

For our representation, we construct a random expected utility model on an infinite-dimensional space where continuation menus are evaluated according to the representation in Dekel et al. (2001). This exercise faces two technical challenges. First, extending Dekel et al. (2001) to countably-additive probability measures on an infinite-dimensional space is difficult due to the lack of compactness in the infinite-dimensional setting (see Krishna and Sadowski (2014) for an outline of the technical issues). Second, the extension of Gul and Pesendorfer (2006) to an infinite-dimensional space with a countably-additive measure is also highly nontrivial.[4] We provide a unified methodology using the set of Lipschitz continuous utilities to address both challenges.

Our axioms combine the axioms of Gul and Pesendorfer (2006) with the linearity and continuity axioms of Dekel et al. (2001). We introduce three new axioms. The first two axioms (*Deterministic Stationarity* and *Average Stationarity*) are weaker analogs of the stationarity axiom of Koopmans (1960) for stochastic choice.[5] They allow us to construct a recursive and stationary Markov utility process. The last axiom (*D-continuity*) is a continuity condition stating that preference for flexibility is robust to small perturbations. It ensures ergodicity of the utility process. Finally, the representation is obtained by an application of the Birkhoff ergodic theorem. See the discussion after Theorem 4 for details.

The rest of the paper is organized as follows. Section 2 introduces our repeated menus setup and our model with ergodic utilities. Section 3 presents the two applications of estimation under stochastic Epstein-Zin and dynamic discrete choice. In Section 4 we introduce ICM and its relationship with intertemporal preferences. Finally, Section 5 contains the axiomatic characterization. All omitted proofs are contained in the appendices.

1.1 Related Literature

Our paper is mainly related to four strands of literature in the following areas: (i) random expected utility, (ii) menu preferences, (iii) intertemporal choice, and (iv) dynamic discrete choice. The first strand of literature is on stochastic choice models of random expected utility. Gul and Pesendorfer (2006), Ahn and Sarver (2013), Lu (2016), and Lu and Saito

[4] See Ma (2018) and Frick et al. (2018) for recent extensions.

[5] A similar axiom appears in Lu and Saito (2018).

(2018) study static models of stochastic choice, while Fudenberg and Strzalecki (2015) and Frick et al. (2018) study dynamic random choice.[6] Our paper is most closely related to the latter. The main differences are in motivation and the mathematical modeling. Given their motivation to study history dependency, Frick et al. (2018) study stochastic choice conditional on past menus, past choices, and consumption realizations, while our stochastic choice function is not conditional on these. Although they can interpret stochastic choice in their model as the result of a single agent, in contrast to our paper, they mainly focus on the population interpretation as it facilitates the interpretation of their primitive.[7] They consider any menus in a finite-horizon setup, while we consider repeated menus in an infinite-horizon setup.[8]

The second relevant strand consists of the modern literature on menu preferences, which began with Dekel et al. (2001) and Gul and Pesendorfer (2001). The former was extended to an objective state space by Dillenberger et al. (2014). Gul and Pesendorfer (2004) extends menu preferences to a dynamic setting by proposing an infinite-horizon consumption setup, which we have adopted in our paper. Other papers that make use of this framework include Higashi et al. (2009) and Krishna and Sadowski (2016). The first considers a random discounting model in which the agent anticipates the stochasticity of his future discount factor. The second extends the additive linear representation into an infinite-dimensional space. While their extension is finitely additive, our extension is countably additive while still preserving the uniqueness of the representation. More recently, Krishna and Sadowski (2014) and Dillenberger et al. (2017) augment the dynamic setup with an informational structure. See Dillenberger et al. (2017) for a review of this literature.

Thirdly, our paper is related to the classical literature on intertemporal choice. As mentioned, our model can be seen as a stochastic version of Kreps and Porteus (1978), including the popular special case of Epstein and Zin (1989) and Weil (1990). We also characterize a stochastic version of Uzawa-Epstein preferences, which was originally proposed by Uzawa

[6] A more recent paper is Duraj (2018), which extends Frick et al. (2018) to a setting with an objective state space. Ke (2018) also studies expected utility in a Luce model.

[7] As explained above, our motivation is to provide a theoretical repeated-choice foundation for the stochastic choice of a single agent. Although we can adopt the population interpretation in some cases (see Section 3.2), we mainly focus on the individual interpretation.

[8] On the technical side, they also provide an extension of Gul and Pesendorfer (2006) to an infinite-dimensional setting. While they use the finiteness condition of Ahn and Sarver (2013) to extend the representation to a finitely additive measure, we use Lipschitz continuity to extend the representation to a countably additive one.

(1968) and later axiomatized by Epstein (1983) in an extended setup with lotteries.[9] More recently, Bommier et al. (2017) also characterize standard utility via a monotonicity axiom.

Finally, our paper is related to the large literature on dynamic discrete choice. While the importance of considering non-standard intertemporal preferences (e.g., a preference for early resolution of uncertainty) is well-known, the literature has assumed standard intertemporal preferences for the sake of tractability.[10] As far as we know, we are the first to analyze the effects of non-standard intertemporal preferences on inference under dynamic discrete choice. In addition, our ergodic representation (2) features in estimation methods of dynamic discrete choice models. Expanding on the work of Rust (1987), Hotz and Miller (1993) introduced an estimation methodology that is computationally less demanding. Their method of calculating conditional choice probabilities (CCP) from a sequences of choices uses a formula similar to our ergodic representation (2). On the other hand, a typical model in dynamic discrete choice assumes both observable states as well as unobservable states. While our model only includes unobservable states, it would be possible to extend our model to allow for observable states as well.[11]

2 A Model of Ergodic Utility

In this section, we first formally define repeated menus and then introduce our stochastic choice primitive. We then define a utility process and present our general model, an ergodic representation of stochastic choice. Finally, we discuss identification and uniqueness.

2.1 Repeated Menus

This section describes the basic setup of the model. Let time $T = \{1, 2, \dots\}$ be discrete and $M = [0, m]$ denote a closed interval representing consumption (e.g., money). The agent is faced with an infinite-horizon consumption problem (IHCP), that is, a menu of choice options

[9] Recent papers that study the macroeconomic implications of stochastic intertemporal preferences include Alvarez and Atkeson (2017) and Barro et al. (2017).

[10] From Rust (1994), "expected-utility models imply that agents are indifferent about the timing of the resolution of uncertain events, whereas human decision-makers seem to have definite preferences over the time at which uncertainty is resolved. The justification for focusing on expected utility is that it remains the most tractable framework for modeling choice under uncertainty."

[11] Such an extension would study stochastic choices conditional on the observable state, which corresponds exactly to CCP.

in which each option corresponds to a lottery over consumption today and a continuation menu tomorrow. We will refer to IHCPs simply as *menu*s and denote them by $z \in Z$. From Gul and Pesendorfer (2004), we know that Z is homeomorphic to $\mathcal{K}(\Delta(M \times Z))$, where $\Delta(\cdot)$ denotes the set of probability measures and $\mathcal{K}(\cdot)$ denotes the set of nonempty compact subsets. Thus, we will associate Z with $\mathcal{K}(\Delta(M \times Z))$ without loss of generality. We also let $X = M \times Z$ denote the set of possible *outcomes*. For $x \in X$, we sometimes let $x \in \Delta X$ denote the degenerate lottery δ_x. For $p \in \Delta X$, we also use $p \in Z$ to denote the singleton menu $\{p\}$. We let $ap+(1-a)q \in \Delta X$ denote the usual mixture between any two probability measures $p, q \in \Delta X$ and $a \in [0,1]$.

The main focus of our study will be on menus that repeat themselves after a fixed number of periods. The following example illustrates what we mean by such *repeated* menus.

Example 1 (Safe vs. Risky Option)**.** Consider an analyst interested in eliciting an agent's risk aversion which may be stochastic every period. Every day, the agent is offered a choice between a safe option p and a risky option q from the menu $z = \{p, q\}$. The safe option $p \in \Delta X$ yields \$5 for sure today and the menu $z \in Z$ again for sure tomorrow. The risky option $q \in \Delta X$ yields either \$10 or \$0 with equal probability today and the menu $z \in Z$ again for sure tomorrow. Note that the agent is sophisticated and understands that regardless of what he chooses today and which outcome is realized, he will always be faced with the menu z again for sure tomorrow.

Example 1 illustrates a menu that is repeated every period. More generally, we consider menus such that, regardless of what the agent chooses and which outcome is realized, he will always face the menu again for sure after a fixed number of time periods. Formally, for $z \in Z$, let $R_0(z) = \{z\}$ and for $t \in T$, define

$$R_t(z) := \mathcal{K}(\Delta(M \times R_{t-1}(z))).$$

Thus, $R_t(z) \subset Z$ are the subset of menus that yield z for sure after t periods.

Definition. A menu z is *t-period* if $z \in R_t(z)$. The menu z is *repeated* if it is t-period for some $t > 0$.

The menu in Example 1 is 1-period since $z \in R_1(z)$. Let $Z^r \subset Z$ denote the set of repeated menus. In general, for a repeated menu, the agent will always face the *same* menu again after some fixed number of time periods. For example, if the menu is t-period, then the agent chooses from the menu at periods $1, 1+t, 1+2t$ and so forth. Since this is repeated ad infinitum, this can generate an infinite time series of choice data.

Repeated menus have three interesting properties. First, for repeated menus, repetition is independent of the agent's choices. As a result, the analyst need not worry about selection biases interfering with the data collection process.

Second, even though repeated menus are restrictive, they are rich enough in that they are in fact dense in the set of all menus. In other words, repeated menus can be used to approximate any menu. This is especially important for the analyst when performing identification which we will see in Section 2.5. Section 5.1 discuss this property in further detail.

Third, there is always some minimal t^* for which z is t^*-period. Note that every t-period menu is also trivially kt-period for any positive integer k. In fact, t^* is the greatest common divisor of all possible periods of the menu; this is simply the first time z appears after the initial period. See Section F.2 in the Appendix for details.

Finally, let us mention an extension of our setup which can incorporate more common consumption-savings problems. In many consumption-savings problems however, menus may *not* be repeated; the agent may not face the same menu again in finite periods, independent of his choices. To address this, it is possible to extend our domain to include menus that repeat with some positive probability. We can then approximate a consumption-saving problem by making the probability of repetition arbitrarily small. This extension is straightforward as long as the repetition probability does not depend on the agent's choice.

2.2 Stochastic Choice

In our model, the main observable data, or primitive, is *stochastic choice.* Given repeated menus, we can interpret stochastic choice as the long-run frequency of the time series of choices. This interpretation of stochastic choice is standard in the literature, although it has not been modeled explicitly. For instance, in the random expected utility model of Gul and Pesendorfer (2006), stochastic choice can be interpreted as the long-run frequency of the time series choices from 1-period menus. See Luce (1959) and Luce and Suppes (1965) for more detailed descriptions of the individual interpretation of stochastic choice.

We now provide a formal definition of stochastic choice. Let $Z^f \subset Z$ denote the set of finite menus and let $Z^* = Z^r \cap Z^f$ denote the set of finite repeated menus.

Definition. A *stochastic choice* is a mapping $\rho : Z^* \to \Delta(\Delta X)$ such that for every $z \in Z^*$, ρ_z is a probability distribution on z.

Given a repeated menu $z \in Z^*$ and an option $p \in z$, the stochastic choice $\rho_z(p)$ designates the probability of choosing p from z. We deal with ties following Lu (2016) and Lu and Saito (2018) in allowing for some probabilities to be unspecified. This is analogous to how under standard deterministic choice, indifference characterizes exactly when the model is silent about which option the agent will choose. This approach allows the analyst to be agnostic about data that is orthogonal to the parameters of interest. For example, if two options are tied, then the stochastic choice is silent about the choice frequency for each option. Formally, we model this as non-measurability and let ρ denote the corresponding outer measure without loss of generality.[12] To simplify notation going forward, we sometimes let $\rho(z, y) = \rho_{z\cup y}(z)$ for $z, y \in Z^*$.[13] Thus, $\rho(p, q)$ denotes the frequency with which p is chosen over q.

While we assume that the analyst observes stochastic choice (i.e. the long-run choice frequency), we do *not* assume she also observes the actual time series of choices. This is a common assumption in many applications of stochastic choice, especially those that adopt the population interpretation. In dynamic discrete choice for instance, the analyst collects choices across both time and agents who are observationally identical under a standard i.i.d. assumption.[14] Since agents are i.i.d. across time, keeping track of the actual time series choice data is unnecessary so most models assume only stochastic choice is observable.

For the individual interpretation, our paper is the first to connect stochastic choice with long-run choice frequencies; we represent stochastic choice as if it is generated from an infinite time series of choices. Our focus on stochastic choice as a primitive is motivated by the existing literature and the fact that stochastic choice in our model is sufficient for identifying all the relevant parameters (see Theorem 1). Studying models that adopt time series choice data as a primitive would be interesting avenues for future research.[15]

[12] Let $\mathcal{F}$ be a σ-algebra on ΔX. Given any $z \in Z^*$, let ρ_z be a measure on the σ-algebra generated by $\mathcal{F} \cup \{z\}$. We can let ρ denote the outer measure with respect to this σ-algebra without loss of generality. See Lu (2016) for details.

[13] Note that if z contains no ties, then $\rho(z, y) = \sum_{p\in z} \rho_{z\cup y}(p)$ as all choice probabilities are specified. Otherwise, $\rho_{z\cup y}(z)$ denotes the outer measure.

[14] That is, the distribution of states is i.i.d. across both time and agents.

[15] If we consider the time series of choices as a primitive, then the behavioral restrictions (on time series choice data) for representation would be more stringent. This is because there are different choice paths that generate the same long-run choice frequency. We thank Tomasz Strzalecki for discussions on this issue.

2.3 Utility Process

In our model, the agent's utility at every period is stochastic and depends on the realization of state variable $s \in S$ that is unobserved by the analyst. We could interpret S as a set of subjective states that influence the agent's utility. For example, the state could be the agent's mood on a particular day, which affects how risk-averse or how patient he is on that day. We could also interpret the state as the realization of some private news arriving every period which affects the agent's utility that period.

The state evolves according to a Markov process $(s_t)_{t\in T}$ with transition probabilities $P : S \to \Delta S$ and a stationary distribution $\pi \in \Delta S$. The Markov process is fixed and known to the agent but unknown to the analyst. We assume the Markov process satisfies the continuity condition that $P_s \geq \delta\pi$ for some $\delta > 0$. This ensures that the Markov process has full support with respect to its stationary distribution and guarantees ergodicity.[16] Going forward, we let $[P]$ denote such a Markov process on the subjective state space.

We now describe the agent's utility. Let U denote the set of all utilities $u : X \to [0, 1]$ normalized such that $u(\underline{x}) = 0$ and $u(\bar{x}) = 1$, where $\underline{x}$ and $\bar{x}$ correspond to consuming 0 and m forever, respectively. For any measurable $u \in U$, we let

$$u(p) := \int_X u(x)dp$$

denote the expected utility of $p \in \Delta X$.

Every period $t \in T$, a state $s_t \in S$ realizes and determines two things: (i) the agent's utility $u_{s_t} \in U$ at period t, and (ii) his expectation $\mathbb{E}_{s_t}$ about next period's state $s_{t+1} \in S$ according to the transition probability P_{s_t}. For example, the agent's mood determines his risk aversion and discount factor today and also informs his beliefs about his mood tomorrow. The agent is fully sophisticated and has correct beliefs; he anticipates what his mood will be tomorrow in order to determine his utility tomorrow as well as his beliefs about what his mood will be the day after, and so forth.

Following Kreps and Porteus (1978), we model utilities recursively as aggregator functions of current consumption and future continuation value. To accommodate changing utilities, we allow the aggregator function to be stochastic. A *stochastic aggregator* $\phi_s(c, v)$ specifies how the agent evaluates his current consumption c versus his future continuation value v

[16] The classic Doeblin's condition states that $P_s^n \geq \delta\lambda$ for some $n \geq 1$ and probability measure λ. Our condition obtains if we set $n = 1$ and $\lambda = \pi$.

given state $s \in S$. Formally, the stochastic aggregator $\phi_s : M \times [0,1] \to [0,1]$ is Lipschitz continuous (with some bound N) and strictly increasing in the second argument. Since the agent anticipates that he may be choosing again next period, future continuation values are evaluated via the additive linear representation of Dekel et al. (2001). We now define a utility process as follows.

Definition. A stochastic process $(u_t)_{t \in T}$ on U is a *utility process* if there exists a Markov process $[P]$ on S and a stochastic aggregator ϕ such that a.s.

$$u_t(c, z) = \phi_{s_t}\left(c, \mathbb{E}_{s_t}\left[\sup_{p \in z} u_{t+1}(p)\right]\right), \tag{3}$$

where the expectation $\mathbb{E}_{s_t}$ is taken with respect to P_{s_t}.

In this case, we say the utility process is *generated by* (P, ϕ). Given any Markov process $[P]$ and stochastic aggregator ϕ, we can always construct a utility process $(u_t)_{t \in T}$ generated by (P, ϕ). At a period $t \in T$, if $s_t = s$ for some $s \in S$, we sometimes write u_s or u_{s_t}, instead of u_t.

Every utility process is also an ergodic Markov process on the space of utilities. To see why it is a Markov process, note that if $u_s = u_{s'}$, then the agent's expectations $\mathbb{E}_s$ and $\mathbb{E}_{s'}$ are the same. Since the agent has correct beliefs, this means that the distribution of the next period's utility induced by P_s and $P_{s'}$ is also the same. Moreover, the following lemma shows that the utility process is ergodic as well.

Lemma 1. *A utility process is an ergodic Markov process.*

Proof. See Appendix A.1. □

2.4 Ergodic Representation of Stochastic Choice

We are now ready to define the main model. We say the utility process is *regular* if $u_s(p) = u_s(q)$ with π-probability of either zero or one for all $p, q \in \Delta X$. In other words, ties either never occur or occur for sure.

Definition. ρ is *ergodic* if there exists a regular utility process generated by (P, ϕ) such that for any t-period $z \in Z^*$, a.s.

$$\rho_z(p) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} 1\left\{u_{it+1}(p) \geq u_{it+1}(q) \text{ for all } q \in z\right\}.$$

If ρ is ergodic, then we say it is represented by (P, ϕ).

In our model, the stochastic choice of an option $p \in z$ corresponds exactly to the long-run frequency of choosing p in an infinite sequence of choices by the agent. At every period, p is chosen only if it is ranked the highest in z according to realization of the utility process u. Recall that the utility process has a rich intertemporal structure as discussed previously. Note that this is an as-if representation that corresponds exactly to the individual interpretation of stochastic choice in a repeated setup. Moreover, this features prominently in dynamic discrete choice estimation.[17] In Section 5, we provide the axiomatic characterization of the representation.

For a simple illustration, consider a well-known special case of our model.

Definition. A utility process is *standard* if there is a random vNM utility w_s and a random discount factor β_s such that a.s.

$$\phi_s(c, v) = (1 - \beta_s) w_s(c) + \beta_s v. \tag{4}$$

The standard utility process correspond to the random expected utility model of Gul and Pesendorfer (2006).

Example 2 (Random Expected Utility)**.** Let $[P]$ denote an i.i.d process and let the stochastic aggregator satisfy

$$\phi_s(c, v) = (1 - \beta_s) w_s(c) + \beta_s v,$$

where w_s is a random vNM utility and $\beta_s \in (0, 1)$ is a random discount factor. Thus,

$$u_t(c, z) = (1 - \beta_s) w_{s_t}(c) + \beta_{s_t} \mathbb{E}\left[\sup_{p \in z} u_{t+1}(p)\right].$$

Suppose ρ is represented by (P, ϕ). Consider a 1-period $z \in Z^*$. As a result, for any $p, q \in z$, we have $u_t(p) \geq u_t(q)$ if and only if $w_{s_t}(p) \geq w_{s_t}(q)$ by canceling out the continuation value of the menu z. From the ergodic representation, we thus have

$$\begin{aligned} \rho_z(p) &= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1\left\{w_{s_i}(p) \geq w_{s_i}(q) \text{ for all } q \in z\right\} \\ &= \pi\left\{s \in S \;:\; w_s(p) \geq w_s(q)\right\}, \end{aligned}$$

[17] For instance in Hotz and Miller (1993), similar formulas are used for the computation of conditional choice probabilities which are then used to estimate value functions for identifying parameters of interest. This methodological approach is now commonly used in the literature.

which corresponds to the random expected utility model of Gul and Pesendorfer (2006).

Example 2 illustrates the fact that when the aggregator is *standard* (i.e., additive and time-separable), the analyst does not need to model repetition explicitly. For instance, repetition can be delayed for an arbitrary number of periods without affecting stochastic choice. More generally, the agent's long-run choice frequency is the same regardless of how often choices are repeated, that is, stochastic choice is independent of future continuation menus. As we will see in Section 4, this is no longer true once we move away from standard utilities.

2.5 Identification and Uniqueness

Given an ergodic representation, Theorem 1 below shows that the analyst can completely identify the agent's utility process from stochastic choice. In other words, the analyst does not require the full time series of choices for identification. Moreover, this can be done by focusing only on repeated binary menus.

Theorem 1. *Let ρ and ρ' be represented by (P, ϕ) and (P', ϕ') respectively. Then the following are equivalent:*

(i) $\rho(p, q) = \rho'(p, q)$ *for all* $p, q \in z \in Z^*$.

(ii) (P, ϕ) *and* (P', ϕ') *generate the same utility process.*

Proof. See Appendix B. □

Note that Theorem 1 does not mean that the Markov process on S can be identified uniquely; nonuniqueness can be trivially obtained by relabeling or adding redundant states. Nevertheless, if we focus on a "minimal" state space such that no two states have the same utility, then unique identification holds.

Given that stochastic choice data consist of only long-run frequencies, one may wonder how it would be possible to identify the agent's utility process completely beyond its stationary distribution. To see how this is possible, consider two different utility processes where one is i.i.d. and the other exhibits persistence but both have the same stationary distribution. Since the agent's utility also encodes information about his expectation regarding tomorrow's utilities, the analyst can distinguish between the two processes via the agent's attitudes toward continuation menus. Intuitively, in the i.i.d. case, tomorrow's utilities are more dispersed than in the persistent case so the agent would exhibit a greater preference for larger menus in the i.i.d. case than in the persistent case.

3 Applications

We now demonstrate how the failure to take into account the agent's intertemporal preferences in stochastic choice models will lead to estimates and inferences that are biased. We present two applications. The first involves eliciting risk aversion under Epstein-Zin preferences. The second involves inferences in a simple two-period dynamic discrete choice example.

3.1 Stochastic Epstein-Zin

In this section, we apply our model to the widely used intertemporal preferences of Epstein and Zin (1989) and Weil (1990). We consider the case where Epstein-Zin preferences are stochastic at every period.

Definition. A utility process is *stochastic Epstein-Zin* if there are $RRA_s \neq 1$, $\psi_s < 1$, and $\beta_s \in (0,1)$ such that a.s.

$$\phi_s(c,v) = \left((1-\beta_s)\, c^{1-\psi_s} + \beta_s v^{\frac{1-\psi_s}{1-RRA_s}}\right)^{\frac{1-RRA_s}{1-\psi_s}}. \tag{5}$$

If ρ is ergodic with a stochastic Epstein-Zin utility process, then we say ρ is *stochastic Epstein-Zin.* In a stochastic Epstein-Zin utility process, each realized utility function is characterized by three stochastic parameters: (i) the relative risk aversion RRA, (ii) the elasticity of intertemporal substitution $EIS = \psi^{-1}$, and (iii) the discount rate β. Since EIS captures how the agent is willing to shift consumption across periods in response to changes in interest rates, its reciprocal $\psi = EIS^{-1}$ can be interpreted as the agent's preference for consumption smoothing.

A useful special case is when $\psi = RRA$, in which case the model reduces to random utility with constant relative risk aversion (CRRA).[18] Note that this is the only case when

[18] One simple case is when the subjective state space itself is $s = (RRA, EIS, \beta)$. Note that this is not without loss of generality, since utilities encode not only intertemporal preferences (in the form of the stochastic aggregator) but also the agent's expectations regarding tomorrow's state. The allowable subjective state space can thus be much richer than the three parameters (RRA, EIS, β).

the agent is indifferent to the timing of the resolution of uncertainty. The followings are extensions of the classic definitions of preference for early or late resolution of uncertainty in our repeated choice setup.

Definition. ρ satisfies *Preference for Early Resolution of Uncertainty (PEU)* if for all $\alpha \in [0,1]$,

$$\rho\left(\alpha\delta_{(c,z)} + (1-\alpha)\,\delta_{(c,y)}, \delta_{(c,\alpha z+(1-\alpha)y)}\right) = 1.$$

ρ satisfies *Preference for Late Resolution of Uncertainty (PLU)* if for all $\alpha \in [0,1]$,

$$\rho\left(\delta_{(c,\alpha z+(1-\alpha)y)}, \alpha\delta_{(c,z)} + (1-\alpha)\,\delta_{(c,y)}\right) = 1.$$

It is well known that PEU corresponds to $\psi \leq RRA$ and PLU corresponds to $\psi \geq RRA$ (see Epstein et al. (2014)). This naturally extends to our setup as well.

Corollary 1. *Suppose ρ is stochastic Epstein-Zin.*

- *Then ρ satisfies PEU if and only if a.s. $\psi_s \leq RRA_s$.*
- *Then ρ satisfies PLU if and only if a.s. $\psi_s \geq RRA_s$.*

Proof. The proof follows from Proposition 3 in Section 4. □

We now show how the proper modeling of repeated menus is important when the agent's utility process is stochastic Epstein-Zin. Consider Example 1, in which the analyst is eliciting the risk aversion of an agent by repeatedly offering him the choice between a safe option p that yields \$5 for sure and a risky option q that yields \$10 and \$5 with equal probability. In that example, repetition is modeled explicitly as occurring every period. On the other hand, in most models of stochastic choice (e.g., Gul and Pesendorfer (2006)), repetition is not modeled explicitly. In the following, we will show how ignoring repetition in stochastic choice models would lead to estimates and inferences that are biased.

3.1.1 Delayed Repetition

To demonstrate the importance of modeling repetition, suppose we elicited choice every two periods instead of one. Denote this delayed repeated menu as z^{+1}. Let $p^{+1} \in z^{+1}$ denote the delayed safe option which yields \$5 today, \$0 tomorrow, and the repeated menu z^{+1} on the day after. Let q^{+1} denote the delayed risky option which yields \$10 and \$5 with equal probability today, \$0 tomorrow, and the repeated menu z^{+1} on the day after. We call $z^{+1} = \{p^{+1}, q^{+1}\}$ "the menu z delayed by 1 period".

We can generalize this concept of delayed repetition to any finite number of time periods. Given a 1-period menu $z \in Z^*$ and $t \in T$, let z^{+t} denote the menu obtained by delaying repeated choice by t periods. Formally, for every $p \in z$, $p^{+t} \in z^{+t}$ is such that

$$p^{+t} = \left(p_1, \underbrace{\$0, \ldots, \$0}_{t} \; ; \; z^{+t} \right),$$

where $p_1 \in \Delta M$ is today's consumption distribution. Note that z^{+t} is $t+1$-period. The following result shows that when the agent's desire for consumption smoothing is less (more) than risk aversion, the probability that a safe option is chosen increases (resp., decreases) under delay.[19]

Proposition 1. *Suppose ρ is stochastic Epstein-Zin. For 1-period menu z and $p \in z$ such that $p = \delta_{(c,z)}$ for some $c \in M$,*

(i) *$\psi_s \leq RRA_s$ a.s. implies $\rho_z\left(\delta_{(c,z)}\right) \leq \rho_{z^{+t}}\left(\delta^{+t}_{(c,z)}\right)$.*

(ii) *$\psi_s \geq RRA_s$ a.s. implies $\rho_z\left(\delta_{(c,z)}\right) \geq \rho_{z^{+t}}\left(\delta^{+t}_{(c,z)}\right)$.*

Proof. First, suppose $\psi_s \leq RRA_s$ a.s. Let $r = (\$0, \ldots, \$0 \; ; \; z^{+t})$ and note that

$$v_{s_t}(z) = \mathbb{E}_{s_t}\left[\max_{q \in z} u_{s_{t+1}}(q)\right] \geq \mathbb{E}_{s_t}\left[u_{s_{t+1}}(r)\right] = v_{s_t}(r).$$

Let $v_2 = v_s(r)$ and $v_1 = v_s(z)$ so $v_2 \leq v_1$. Define

$$\sigma_s := \frac{1 - \psi_s}{1 - RRA_s}.$$

Since $\phi_s(c, v_2)^{\sigma_s} - \beta_s v_2^{\sigma_s} = (1 - \beta_s)\, c^{1-\psi_s}$, we have

$$\phi_s(c, v_1) = \left(\phi_s(c, v_2)^{\sigma_s} + \beta_s\left(v_1^{\sigma_s} - v_2^{\sigma_s}\right)\right)^{\sigma_s^{-1}}.$$

Now, if $RRA_s < 1$, then $\sigma_s \geq 1$ as $\psi_s \leq RRA_s$. On the other hand, if $RRA_s > 1$, then $\sigma_s < 0$ as $\psi_s < 1$. In either case, this means that $\phi_s(\cdot, v_1)$ is more convex than $\phi_s(\cdot, v_2)$ so

[19] For convenience, we present Proposition 1 in its weak form but it also holds with strictness. That is, if $\psi_s < RRA_s$ holds with some probability implies $\rho_z\left(\delta_{(c,z)}\right) < \rho_{z^{+t}}\left(\delta^{+t}_{(c,z)}\right)$.

$\phi_s(\cdot, v_2)$ is more risk-averse than $\phi_s(\cdot, v_1)$. This implies that for every $q \in z$, if

$$u_s\left(\delta_{(c,z)}\right) = \phi_s(c, v_1) \geq \int_M \phi_s(c', v_1)\, dq_1 = u_s(q)$$

then

$$u_s\left(\delta^{+t}_{(c,z)}\right) = \phi_s(c, v_2) \geq \int_M \phi_s(c', v_2)\, dq_1^{+t} = u_s\left(q^{+t}\right).$$

The conclusion follows. The case for $\psi_s \geq RRA_s$ a.s. is analogous. □

To understand the implication of Proposition 1, consider Example 1. In that example, the menu z contains only two options, the safe option and the risky option. Consider an agent whose desire for consumption smoothing is always smaller than his relative risk aversion (i.e., $\psi_s \leq RRA_s$ a.s.). Notice that Proposition 1 implies that the probability of choosing the risky option increases when repetition becomes more frequent (i.e., the delay $+t$ becomes smaller). This result can be understood intuitively as follows: under repeated choice the risky option feels “safer” because even if today’s outcome is bad, there is always a chance that tomorrow’s outcome will be good. Thus, the risky option becomes more attractive as repetition becomes more frequent whenever the agent’s preference for consumption smoothing is low compared to his risk aversion.[20]

This behavior may be natural in our daily life; for instance, a consumer may choose more “risky” brands if he knows he will visit the grocery store every day but stick to “safer” brands if he can visit the store only seldom. The reasoning for $\psi \geq RRA$ is symmetric. A static model of stochastic choice that ignores repetition would fail to capture such behavioral phenomena.

3.1.2 Biased Estimation

We now show how ignoring the repetition of choice would lead to systematic biases in estimation of the agent’s risk aversion. To illustrate, note that by making delay arbitrarily long, we can set the value of continuation menus arbitrarily small so that the agent behaves as if he ignores repetition. In this way, we can approximate the agent’s static stochastic choice. Applying Proposition 1 to Example 1, we get that when $\psi \leq RRA$,

$$\rho_z\left(\delta_{(\$5,z)}\right) \leq \lim_{t\to\infty} \rho_{z^{+t}}\left(\delta^{+t}_{(\$5,z)}\right) = \pi\left\{w_s(5) \geq \frac{1}{2}w_s(10) + \frac{1}{2}w_s(0)\right\}, \tag{6}$$

[20] Another example is when people would be willing to bet on a repeated lottery but not on a one-time lottery as in the well-known Law of Large Numbers fallacy of Samuelson (1963).

where $w_s(c) = c^{1-RRA_s}$.

The right-hand side is the stochastic choice of an agent with standard CRRA utility, which exactly coincides with the static distribution of risk aversion. The inequality in (6) implies that an analyst who incorrectly assumes standard intertemporal preferences would *underestimate* the agent's risk aversion if the agent in fact has a preference for early resolution of uncertainty. In other words, the analyst may incorrectly conclude that the agent is mostly risk-loving, while in reality, he is mostly risk-averse but chooses the risk-free option infrequently due to intertemporal preferences.

Note that in the special case in which $\psi = RRA$, intertemporal preferences are standard (see Example 2) and repetition does not matter. In this case, any inference from a static model that ignores repetition would be correct. For this reason, there is an implicit assumption in static models of stochastic choice that the agent's intertemporal preferences are standard. In general, whenever intertemporal preferences are non-standard, there will always be some biases in estimation. We formally show this in Section 4.

3.2 Dynamic Discrete Choice

In this subsection, we apply our model to a simple two-period dynamic discrete choice example to illustrate the effects of intertemporal preferences on inference. Following most applications in dynamic discrete choice, we adopt the population interpretation of stochastic choice in this subsection only. In other words, we consider a population of observationally identical agents facing the same choice problem. This is possible in our model under two assumptions. First, even though choices are not technically repeated (we consider only two periods), we can model this as the limit of delaying repetition for an arbitrarily number of periods (see Section 3.1.1). Second, we assume the state follows an i.i.d. process where the distribution of each agent's state tomorrow is exactly equal to the population distribution π.[21] Under these assumptions, the long-run choice frequency that corresponds to stochastic choice also reflects the population choice. We can thus reinterpret stochastic choice in our ergodic model as a result of unobserved heterogeneity in a population of agents. The latter assumption is a typical assumption when estimating conditional choice probabilities in the dynamic discrete choice literature (see Hotz and Miller (1993)).

[21] We can relax this assumption as long as the stationary distribution of the (possibly non-i.i.d.) state process is the same as the population distribution.

The setup is as follows. There is a population of agents who decide whether to purchase phone insurance (e.g., AppleCare) at the beginning of years 1 and 2. We are interested in modeling their choice of insurance. Let c_s be the annual consumption value of the phone for an agent at state $s \in S$. We assume s is i.i.d. with stationary distribution π, which is also the population distribution of s. The price of insurance is a. In year $t \in \{1, 2\}$, there is p_t probability that the phone breaks down, in which case an agent's estimated repair cost for fixing a broken phone is θ_s. The analyst knows a, p_1, and p_2 and would like to estimate the distribution of the repair cost θ_s. For simplicity, we assume that $c_s \geq a$ and $c_s \geq \theta_s$ so all agents have positive final consumption. Note that in contrast to the application in Section 3.1, utilities in this example appear stochastic to the analyst due to unobserved heterogeneity in the population (e.g., each agent's repair cost).

First, consider the case where all agents have risk-neutral standard preferences (i.e., stochastic Epstein-Zin from (5) with $RAA_s = \psi_s = 0$). We study whether agents choose to buy insurance in year 1. Let β_s be the discount rate and v denote an agent's continuation value.[22] An agent will choose insurance if the following holds:

$$(1-\beta_s)(c_s - a) + \beta_s v \geq p_1\left((1-\beta_s)(c_s - \theta_s) + \beta_s v\right) + (1-p_1)\left((1-\beta_s)c_s + \beta_s v\right),$$

or, equivalently, $\theta_s \geq a/p_1$. If we let p denote the "buy insurance" option, q denote the "not buy insurance" option and $z = \{p, q\}$ denote the menu, then the probability that insurance is purchased is given by

$$\rho_z^*(p) = \pi\left\{s \in S : \theta_s \geq a/p_1\right\}. \tag{7}$$

Naturally, lower values of θ_s correspond to fewer agents choosing insurance.

Next, we consider the case where all agents have non-standard preferences. For instance, suppose the utility of an agent in state $s \in S$ is given by stochastic Epstein-Zin with risk neutrality (i.e., $RAA_s = 0$):

$$\phi_s(x, v) = \left((1-\beta_s)x^{1-\psi_s} + \beta_s v^{1-\psi_s}\right)^{\frac{1}{1-\psi_s}}, \tag{8}$$

where ψ_s captures the agent's desire for consumption smoothing as in the previous subsection. Note that when the continuation value v is zero, this reduces to standard risk-neutral utility.

[22] This is the same for all agents since the distribution of next period's state is π for everyone.

Now, the probability that insurance is chosen is given by

$$\rho_z(p) = \pi\{s \in S : \phi_s(c_s - a, v) \geq p_1 \phi_s(c_s - \theta_s, v) + (1 - p_1)\phi_s(c_s, v)\}, \tag{9}$$

where

$$v := \int_S \max\{\phi_s(p'), \phi_s(q')\} \, d\pi$$

is the value of the continuation menu $z' = \{p', q'\}$, where p' and q' correspond to purchasing insurance or not respectively.

We now demonstrate how ignoring intertemporal preferences would lead to biased estimation of θ_s in this dynamic discrete choice problem. Suppose that agents' utilities are non-standard and given by equation (8) and, hence, the insurance adoption rate is given by $\rho_z(p)$ from equation (9). The analyst however misspecifies the model and assumes that utilities are standard. In this misspecified model, the insurance adoption rate is given by $\rho_z^*(p)$ from equation (7). The following proposition characterizes the comparison between $\rho_z^*(p)$ and $\rho_z(p)$ depending on the agents' intertemporal preferences.

Proposition 2. *Suppose that ρ^* and ρ are given as in equations (7) and (9), respectively.*

(i) $\psi_s \leq 0$ *(i.e. RRA_s) a.s. implies* $\rho_z(p) \leq \rho_z^*(p)$.

(ii) $\psi_s \geq 0$ *(i.e. RRA_s) a.s. implies* $\rho_z(p) \geq \rho_z^*(p)$.

Proof. Note that $\phi_s(\cdot, v)$ is convex if $\psi_s \leq 0$. Thus, $\phi_s(\cdot, v)$ is risk-loving so

$$\phi_s(c_s - a, v) \geq p_1 \phi_s(c_s - \theta_s, v) + (1 - p_1)\phi_s(c_s, v)$$

implies $c_s - a \geq p_1(c_s - \theta_s) + (1 - p_1)c_s$. This means that $\rho_z(p) \leq \rho_z^*(p)$ as desired. The case for $\psi_s \geq 0$ is symmetric. □

Proposition 2 implies that if almost all agents prefer early resolution of uncertainty (i.e., ψ_s is negative a.s.), then ignoring intertemporal preferences will result in *underestimation* of repair costs. To see this, note that the analyst misinterprets the observed adoption rate $\rho_z(p)$ as $\rho_z^*(p)$ and will estimate θ_s based on the misspecified model (7). Proposition 2 shows that $\rho_z(p) \leq \rho_z^*(p)$ when ψ_s is negative a.s. This means that if the analyst observes a low adoption rate, she would incorrectly infer that repair costs are low.[23] In reality however, agents are more willing to decline insurance due to their intertemporal preferences. The implication

[23] Recall that a lower adoption rate corresponds to lower values of θ_s from equation (7).

for when almost all agents prefer late resolution of uncertainty (i.e., ψ_s is positive a.s.) is symmetric.

For an intuitive understanding of why Proposition 2 holds, recall Proposition 1. Note that buying (not buying) insurance in Proposition 2 corresponds to choosing the safe option (resp., the risky option) in Proposition 1. This is because if agents purchase insurance, their payoffs are constant. Note also that assuming the standard model corresponds to delaying forever (i.e., $\rho_z^* = \rho_{z+\infty}$). Therefore, under the assumption of risk neutrality (i.e., $RAA = 0$), statements (i) and (ii) in Proposition 2 correspond respectively to statements (i) and (ii) in Proposition 1 with infinite delay (i.e., $t = \infty$). The reasoning for Proposition 2 then follows as in Proposition 1.

This example illustrates how our model can be readily applied to problems of discrete choice estimation that allow for more general temporal preferences. Although we assumed risk neutrality for simplicity, this example can be easily generalized to accommodate non-trivial risk attitudes. Our example is straightforward but it serves to illustrate the inherent inference issues that can arise if intertemporal preferences are not taken into account in many applications of dynamic discrete choice estimation. While ignoring intertemporal preferences would obviously affect inference, our main point is understanding the systematic way in which intertemporal preferences affect estimation as outlined in Proposition 2.

3.2.1 Relationship with Dynamic Discrete Choice

We end this section on intertemporal preferences with a discussion of our results in relation to models of dynamic discrete choice. In a typical model of dynamic discrete choice, the agent's utility satisfies

$$u_{s_t}(c,z) = (1-\beta)\left(w(c) + \epsilon_{s_t}(c,z)\right) + \beta \mathbb{E}_{s_t}\left[\sup_{p \in z} u_{s_{t+1}}(p)\right], \tag{10}$$

where the shocks ϵ are i.i.d. across consumption c and continuation menus z. If shocks do not depend on continuation menus or the continuation menu is the same regardless of what option the agent chooses, then this model (10) coincides with our standard utility process. This is the case in the example of Section 3.2, where shocks represent unobserved repair costs, as well as in many dynamic discrete choice problems such as Rust (1987) and Hotz and Miller (1993).

To see how the dynamic discrete choice model (10) coincides with our standard model, note that when shocks are independent of continuation menus, we can write the shock $\epsilon_s(c,z)$ simply as $\epsilon_s(c)$. We can then express the sum of current consumption utility w and the shock ϵ_s as a new current consumption utility w_s as

$$w_s(c) := w(c) + \epsilon_s(c)$$

As a result, model (10) coincides with our model with a standard utility process. In this way, we can consider extensions of typical models in dynamic discrete choice to allow for more general aggregators such as stochastic Epstein-Zin.[24]

With the above relationship with mind, it is easy to see how to extend the result from the previous section to a more general dynamic discrete choice setting with additive shocks. Recall that the stochastic Epstein-Zin aggregator with risk-neutral utility (i.e., $RRA_s = 0$) is given by

$$\phi_s(c,v) := \left((1-\beta_s)\,c^{1-\psi_s} + \beta_s v^{1-\psi_s}\right)^{\frac{1}{1-\psi_s}}$$

Now, the utility if the phone breaks is given by

$$u_s(c_s - \theta_s, z) = \phi_s(c_s - \theta_s + \epsilon_s(c_s - \theta_s), v_s(z))$$

where θ_s is the repair cost. The same argument for Proposition 2 then applies in this setting. If almost all agents prefer early resolution of uncertainty (i.e., ψ_s is negative a.s.), then ignoring intertemporal preferences will result in underestimation of repair costs. Vice-versa, if almost all agents prefer late resolution of uncertainty (i.e., ψ_s is positive a.s.), then ignoring intertemporal preferences will result in overestimation of repair costs.

4 Intertemporal Preferences

4.1 Independence of Continuation Menus

In Section 3, we demonstrated how the explicit modeling of repeated choice is paramount for an analyst interested in elicitation or inference when the agent has non-standard intertemporal preferences. In this section, we formalize when repeated choice needs to be taken into

[24] One difference is that utilities in our model are bounded and Lipschitz continuous which would not be technically satisfied if shocks are extreme-value distributed. However, if we consider only a finite subset of choice options which is the case in most applications, then our conditions can be satisfied without loss of generality.

account by the analyst versus when it is unnecessary to do so as in static random choice. In the case of the latter, we say the stochastic choice satisfies an axiom called *Independence of Continuation Menus.*

To illustrate, recall Example 1 where the menu consists of a risky option that yields \$10 and \$0 with equal probability and a safe option that yields \$5 for sure. Proposition 1 implies that the probability of choosing the risky option over the safe option depends on the timing of the next repetition; in other words, continuation menus matter unless the agent is indifferent to the timing of resolution of uncertainty. On the other hand, in Example 2 where we assume standard utility, the only thing that matters is the distribution of current consumption; in that case, choice is independent of continuation menus.

We now formalize these concepts. Fix a menu $z \in Z$ and for every $p \in z$, let $p_1^Z \in \Delta Z$ denote the distributions of next-period continuation menus. Given a menu z, suppose $p_1^Z = q_1^Z$ for all $p, q \in z$ so the distribution of the agent's next-period continuation menu is the same regardless of what the agent chooses. We call such a menu 1-*period invariant.*[25]

The following definition characterizes when choice is independent of next-period continuation menus. To introduce the definition, for any menu $z \in Z$ and for every $p \in z$, let $p_1^M \in \Delta M$ denote the distributions of current consumption and let

$$z_1^M := \left\{ p_1^M \in \Delta M \ : \ p \in z \right\}$$

denote the menu of consumption distributions.

Consider a menu z where $p_1^Z = r$ for all $p \in z$ so z is 1-period invariant. Now, construct another menu from z by switching the distribution of next-period menus from r to r' but leaving the distribution of current consumption the same. Call this new menu y. In other words, $z_1^M = y_1^M$ and $q_1^Z = r'$ for all $q \in y$. Note that both z and y are 1-period invariant. *1-Period Independence of Continuation Menus* states that choice probabilities in y and z are the same; in other words, switching the common distribution of next-period menus does not alter stochastic choice.

Definition. ρ satisfies *1-Period Independence of Continuation Menus (1-ICM)* if for all 1-period invariant $z, y \in Z^*$, $p \in z$ and $q \in y$,

$$p_1^M = q_1^M \text{ and } z_1^M = y_1^M \implies \rho_z(p) = \rho_y(q).$$

[25] Note that every 1-period menu is 1-period invariant. The converse is not true.

Under 1-ICM, the agent evaluates current consumption independent of next-period continuation menus. In fact, it implies the separability axiom of Frick et al. (2018) which is the stochastic analog of the standard separability axiom of Fishburn (1970). This follows from the fact when current consumption is evaluated independent of next-period continuation menus, the agent will naturally ignore correlations between current consumption and next-period menus.

1-ICM is applicable only to menus that are 1-period invariant. This is the case in Proposition 1 where $y = z^{+t}$ for some t so $z_1^M = y_1^M$. Note that for all $p \in z$, $p_1^Z = \delta_z$ while for all $q \in y$, $q_1^Z = \delta_{(0,\dots,y)}$ which corresponds to 0 consumption for t periods followed by y. Hence, both y and z are 1-period invariant. Proposition 1 implies that if $\psi_s = RRA_s$ a.s., then $\rho_z\left(\delta_{(c,z)}\right) = \rho_{z^t}\left(\delta_{(c,z)}^{+t}\right)$ which agrees exactly with 1-ICM. In general however, we may consider menus that are *not* 1-period invariant. Suppose the analyst is interested in eliciting the agent's discount factor. In order to do this, she would need to offer repeated menus of *at least* 2 periods. For instance, let p correspond to an early option of consuming \$10 today and q correspond to a later option of consuming \$15 tomorrow. Let $z = \{p, q\}$ where $p = (10, 0; z)$ and $q = (0, 15; z)$. In this case, $p_1^Z = \delta_{(0,z)} \neq \delta_{(15,z)} = q_1^Z$ so z is not 1-period invariant. As a result, 1-ICM no longer applies.

We now extend our notion of independence beyond the first period. For simplicity, we will focus on menus such that every continuation menu before time t is degenerate. We call such menus *t-simple*. For every option in a t-simple menu, we can consider its distributions over t-period consumptions and continuation menus. Formally, let $M_1 := M$, and recursively define $M_t := M \times \Delta M_{t-1}$. Let $p_t^M \in \Delta M_t$ denote the t-period distribution of consumption and let

$$z_t^M := \left\{p_t^M \in \Delta M_t \ : \ p \in z\right\}$$

denote the menu of t-period consumption distributions. Also let $p_t^Z \in \Delta\left(\Delta\left(\cdots \Delta Z\right)\right)$ denote the t-period distribution of continuation menus where the $\Delta\left(\cdot\right)$ operator is applied t times. Given a menu z, if $p_t^Z = q_t^Z$ for all $p, q \in z$, then the menu is *t-period invariant.*[26]

The next definition characterizes when choice is independent of all continuation menus. It extends 1-ICM from one period to t periods. Similar to the reasoning for 1-ICM, ICM implies that switching the common distribution of continuation menus does not alter stochastic

[26] As in 1-period menus, every simple t-period menu is t-period invariant but the converse is not true.

choice.

Definition. ρ satisfies *t-Period Independence of Continuation Menus (t-ICM)* if for all t-period invariant $z, y \in Z^*$, $p \in z$ and $q \in y$,

$$p_t^M = q_t^M \text{ and } z_t^M = y_t^M \implies \rho_z(p) = \rho_y(q).$$

Moreover, ρ satisfies *Independence of Continuation Menus (ICM)* if it satisfies t-ICM for all $t \in T$.

In the following, we characterize utility processes that satisfy ICM. First, consider the following class of separable utility processes.

Definition. A utility process is *separable* if there is a random vNM utility w_s, a random function φ_s and a random discount factor β_s such that a.s.

$$\phi_s(c, v) = (1 - \beta_s) w_s(c) + \beta_s \varphi_s(v).$$

Note that a separable utility process is standard if and only if $\varphi_s(v) = v$ a.s.

The main result of this section shows that 1-ICM exactly characterizes separable utility while ICM exactly characterizes standard utility. Standard utility has been widely assumed in dynamic discrete choice analysis (see Section 3.2.1 for details).[27]

Theorem 2. *Suppose ρ is ergodic. Then,*

- *it satisfies 1-ICM if and only if its utility process is separable.*
- *it satisfies ICM if and only if its utility process is standard.*
- *it satisfies ICM if and only if it satisfies 1-ICM and 2-ICM.*

Proof. See Appendix E.1. □

Theorem 2 shows that while separability is sufficient to ensure 1-ICM, it is insufficient to ensure ICM. In other words, when an agent has a separable utility process, the analyst can ignore repetition for 1-period menus but not 2-period ones. Satisfying both 1-ICM and 2-ICM is sufficient for full ICM. Moreover, only a standard utility process will ensure full ICM.

[27] It corresponds to an infinite-horizon Markovian version of the Bayesian Evolving Utility model of Frick et al. (2018).

Standard utility was exactly the case assumed in Example 2 where our ergodic model reduces to the static model of random expected utility. While it may not be surprising that standard utility ensures ICM, Theorem 2 interestingly shows that ICM implies standard utility. In other words, whenever the agent has non-standard intertemporal preferences (i.e., non-standard utility), there exists some repeated choice problem where continuation menus matter; ignoring repeated choice in such a problem would result in biased inference.

4.2 Resolution of Uncertainty and Repeated Independence

In this subsection, we relate ICM with other well-studied intertemporal preferences. This allows us to provide an alternate characterization of ICM and clarifies its relationship with other behavioral properties studied in the literature.

Consider the stochastic Epstein-Zin preferences of Section 3.1 and note that if the agent satisfies *Indifference to Timing of Resolution of Uncertainty (IRU)* (i.e., both PEU and PLU), then the utility process is standard (i.e., $\psi_s = RAA_s$ a.s.). Given Theorem 2, this means that under stochastic Epstein-Zin preferences, IRU ensures ICM is satisfied. For general utility processes however, IRU does not imply ICM; it implies a stochastic version of the classic Uzawa-Epstein preferences.

Definition. A utility process is *stochastic Uzawa-Epstein* if there are vNM utilities w_s and β_s such that a.s.

$$\phi_s(c, v) = (1 - \beta_s(c)) w_s(c) + \beta_s(c) v.$$

Proposition 3. *Suppose ρ is ergodic. Then,*

- *it satisfies PEU (PLU) if and only if $\phi_s(c, \cdot)$ is convex (resp., concave) a.s.*
- *it satisfies IRU if and only if its utility process is stochastic Uzawa-Epstein.*

Proof. Suppose ρ exhibits PEU. We thus have a.s.

$$\alpha\phi_s(c, v_s(z)) + (1 - \alpha)\phi_s(c, v_s(y)) \geq \phi_s(c, \alpha v_s(z) + (1 - \alpha) v_s(y)).$$

Since this is true for all z and y, the result follows. The case for PLU is symmetric. If $\phi(c, \cdot)$ is both concave and convex, then it is linear. Thus, $\phi_s(c, v) = (1 - \beta_s(c)) w_s(c) + \beta_s(c) v$ for $\beta_s(c) > 0$ for all $c \in M$. □

Proposition 3 is the stochastic analog of Theorem 1 of Epstein (1983). Since stochastic Uzawa-Epstein is strictly more general than the standard model, Proposition 3 implies that IRU is too weak to ensure ICM. In fact, since Uzawa-Epstein utilities are not even separable, IRU will not even ensure 1-ICM. It is easy to see this in the functional form of Uzawa-Epstein utility as the value of continuation menus has nontrivial effects on current consumption utility via the term $\beta_s(c)$.

Given that IRU does not ensure ICM but ICM implies IRU (since every standard utility satisfies IRU), it is natural to ask what additional property will bridge the gap between IRU and ICM. It turns out to be a repeated version of classic independence axiom. To illustrate, recall Example 1 where the 1-period menu z consists of a risky option that yields \$10 and \$0 with equal probability and a safe option that yields \$5 for sure. Suppose we wanted to test the independence axiom in this repeated setup by mixing both the risky and safe options with a third option r that yields \$3 for sure. Let y denote this new 50-50 mixture of z and r. Note that y is also a 1-period menu and consists of two options: one option that yields \$10 with probability 0.25, \$0 with probability 0.25, and \$3 with probability 0.50; the other option yields \$3 and \$0 with equal chance. Importantly, regardless of what happens, the agent will face y for sure next period so this mixture is repeated every period ad infinitum. We use the notation $y = 0.5z \otimes 0.5r$ to denote this 50-50 repeated mixture between z and r. This corresponds exactly to repeated testing of the classic independence axiom.

We now formalize this concept. First consider a 1-period menu $z \in Z^*$ in which every $p \in z$ can be expressed as $(p_1\ ;\ z)$. Consider repeatedly mixing z with some $r \in \Delta M$. This yields the new 1-period menu, denoted by $\alpha z \otimes (1-\alpha) r \in Z^*$, such than any element of the 1-period menu is of the form

$$(\alpha p_1 + (1-\alpha) r\ ;\ \alpha z \otimes (1-\alpha) r).$$

In other words, every option is mixed with r every period. We denote the element of $\alpha z \otimes (1-\alpha) r \in Z^*$ by $\alpha p \otimes (1-\alpha) r$. We can extend this to all t-period simple menus (see Appendix G) and define repeated independence as follows.

Definition. ρ satisfies *Repeated Independence (RI)* if for all t-simple $z \in Z^*$, $\alpha > 0$ and $r \in \Delta M$

$$\rho_z(p) = \rho_{\alpha z \otimes (1-\alpha) r}(\alpha p \otimes (1-\alpha) r).$$

RI is exactly the classic independence axiom in our repeated choice setup. In fact, it corresponds to the linearity axiom in the static random expected utility model of Gul and Pesendorfer (2006). The main result of this subsection shows that IRU in addition to RI exactly characterizes ICM. Moreover, under IRU, RI is equivalent to 1-ICM.

Theorem 3. *Suppose ρ is ergodic. Then the following statements are equivalent:*

(i) *it satisfies ICM.*

(ii) *it satisfies IRU and RI.*

(iii) *it satisfies IRU and 1-ICM.*

Proof. For the equivalence between (i) and (ii), see Appendix E.3. The equivalence between (i) and (iii) follows from Theorem 2, Proposition 3 and the fact that any separable Uzawa-Epstein utility must be standard. □

Theorem 3 suggests that with stochastic choice, intertemporal preferences complicate tests of the classic independence axiom. Even though the agent may satisfy the static independence axiom for a single time period, he may violate this repeated version of the independence axiom (i.e., RI). [28] Moreover, as we will show in the next section, any ergodic ρ satisfies the independence axiom over menus (i.e., Linearity (Axiom 1.2)). These facts show the importance of specifying the appropriate domain when we test the independence axiom with stochastic choice.

5 Characterization

This section provides an axiomatic characterization of our model. First, we show how repeated menus can be used to approximate any menu. This allows us to extend our primitive to the set of all (finite) menus.

5.1 Extending Repeated Menus

Given any menu $z \in Z$, consider replicating the menu z for the first t periods and ending with a menu $y \in Z$ for sure. We use the notation $r_{y,t}(z)$ to denote such a menu and construct

[28] In Appendix G, we study the relationship between non-standard intertemporal preferences and particular patterns of RI violations along with comparative statics.

it inductively as follows. First, for any $y \in Z$, let $r_{y,0}(z) = y$. Given $r_{y,t-1}$, for any $p \in \Delta X$, let $p_{y,t} \in \Delta X$ denote the lottery induced by $r_{y,t-1}$, that is, for all measurable $A \times B$,

$$p_{y,t}(A \times B) = p\left(A \times r_{y,t-1}^{-1}(B)\right).$$

Finally, for any $z \in Z$, define

$$r_{y,t}(z) := \{p_{y,t} \ : \ p \in z\}.$$

In other words, $r_{y,t}(z) \in Z$ is the menu that follows z for the first t periods ending with y for sure. Lemma 13 shows that this is well-defined.

Given any menu $z \in Z$, we can now define what it means to construct a repeated menu that approximates z up to t periods. We let z^t denote this t-period repeated version of z.

Definition. Given $z \in Z$, let z^t be t-period such that $z^t = r_{z^t,t}(z)$.

The following lemma shows this is well-defined. Moreover, given any menu $z \in Z$, we can use its t-period repeated version to approximate it as we increase the number of periods between each repetition.

Lemma 2. *For every $z \in Z$, z^t exists and $z^t \to z$ as $t \to \infty$.*

Proof. See Appendix F.1. □

Recall $Z^* = Z^r \cap Z^f$ where Z^f is the set of finite menus. We can now use finite repeated menus to approximate any finite menu.

Corollary 2. *Z^* is dense in Z^f.*

Proof. Fix some finite menu $z \in Z^f$ so from Lemma 2 above, we can find repeated menus z^t such that $z^t \to z$. Since $z^t = r_{z^t,t}(z)$ and z is finite, z^t is also finite by definition. Thus, $z^t \in Z^*$ as desired. □

5.2 Axiomatic Characterization

The results in the previous section allow us to extend the observed stochastic choice on repeated finite menus to all finite menus as follows. Consider a random choice $\bar{\rho}$ on all finite menus Z^f such that $\bar{\rho}_z = \rho_z$ for every $z \in Z^*$. In other words, $\bar{\rho}$ agrees with ρ on all repeated menus Z^*. From Corollary 2, we know that Z^* is dense in Z^f. Thus, for any $z \in Z^f$, we can find $z^t \in Z^*$ such that $z^t \to z$. If $\bar{\rho}$ is continuous, then ignoring ties,

$$\bar{\rho}_z = \lim_t \rho_{z^t}$$

Thus, we can think of $\bar{\rho}$ as the continuous extension of ρ from Z^* to Z^f. We model ties in the same way as ρ (see the discussion on ties in Section 2.1) and let $Z^\circ \subset Z^f$ denote the set of finite menus that contain no ties. To simplify notation going forward, we let ρ denote $\bar{\rho}$ without loss of generality.

We are now ready to state our axioms on stochastic choice. The first set of axioms consists of conditions on random expected utility. Note that mixtures here are taken ex-ante at time 0 and we let $\text{ext}(z)$ denote the extreme options of some menu $z \in Z^f$. Also recall that $\bar{x}$ and $\underline{x}$ are the consumption streams that yield the best outcome (i.e., m) and the worst outcome (i.e., 0) respectively forever. Note that we sometimes let x denote the singleton menu that yields consumption $x \in X$ forever.

Axiom 1.1 (Monotonicity). *For any $z, y \in Z^f$ and $p \in z$,*

$$z \subset y \implies \rho_z(p) \geq \rho_y(p).$$

Axiom 1.2 (Linearity). *For any $z \in Z^f$, $\alpha > 0$, $p \in z$, and $q \in \Delta X$,*

$$\rho_z(p) = \rho_{\alpha z + (1-\alpha) q}(\alpha p + (1-\alpha) q).$$

Axiom 1.3 (Extremeness). *For any $z \in Z^f$, $\rho_z(\text{ext}(z)) = 1$.*

Axiom 1.4 (Continuity). *$\rho : Z^\circ \to \Delta(\Delta X)$ is continuous.*

Axiom 1.5 (Best-Worst). *$\rho(\underline{x}, \bar{x}) = 0$ and $\rho(\bar{x}, x) = \rho(x, \underline{x}) = 1$ for all $x \in X$.*

Axiom 1.6 (L-continuity). *There exists $N > 0$ such that for any $\alpha \in [0,1]$ and any $x, x' \in X$,*

$$|x - x'| \leq \frac{\alpha}{N} \implies \rho\left(\alpha\delta_{\bar{x}} + (1-\alpha)\delta_x, \alpha\delta_{\underline{x}} + (1-\alpha)\delta_{x'}\right) = 1.$$

Axioms 1.1-1.4 are direct from Gul and Pesendorfer (2006). Best-Worst (Axiom 1.5) ensures that $\bar{x}$ and $\underline{x}$ truly are the best and worst outcomes. Finally, L-continuity (Axiom 1.6) is the stochastic version of the Lipschitz continuity axiom from Dekel et al. (2007). It guarantees that utilities are sufficiently well-behaved in that they are Lipschitz continuous with respect to some common bound N. This is important for the representation and ensures that it is unique.[29] To understand L-continuity intuitively, note that when $N = \infty$, then $x_1 = x_2 = x$ for some x and the axiom reduces to

$$\rho\left(\alpha\delta_{\bar{x}} + (1-\alpha)\,\delta_x, \alpha\delta_{\underline{x}} + (1-\alpha)\,\delta_x\right) = 1,$$

which holds by Best-Worst and Linearity. L-continuity requires that this holds for large enough but finite N.[30] Taken together, Axiom 1 characterizes a random expected Lipschitz utility with best and worst outcomes. Continuation Linearity (Axiom 2) below ensures that agent's preference toward continuation menus satisfy linearity with respect to ex-post mixing. First, we define component-wise ex-post mixing. For $\lambda \in [0,1]$, $c, c' \in M$ and $z, z' \in Z$, define ex-post mixing as

$$\lambda\delta_{(c,z)} \oplus (1-\lambda)\,\delta_{(c',z')} := \delta_{(\lambda c+(1-\lambda)c', \lambda z+(1-\lambda)z')}.$$

Here, the first mixture $\lambda c + (1-\lambda)\,c'$ corresponds to the standard mixing of monetary consumptions (i.e., real numbers) while the second mixture $\lambda z + (1-\lambda)\,z'$ corresponds to Minkowski mixing of menus.[31] For any $c \in M$, let Z_c^f be the set of finite menus such that every option $p \in z$ is degenerate and yields consumption c for sure today (i.e., $p = \delta_{(c,w)}$ for some $w \in Z$). For any $z \in Z_c^f$, define

$$\lambda z \oplus (1-\lambda)\,\delta_{(c',z')} := \left\{\lambda p \oplus (1-\lambda)\,\delta_{(c',z')} \ : \ p \in z\right\},$$

which is the Minkowski version of ex-post mixing.

Consider a lottery p in a menu $z \in Z_c^f$. Lets mix p and z with a pair (c', z') ex post and call them q and y, respectively (i.e., $q = \lambda p \oplus (1-\lambda)\,\delta_{(c',z')}$ and $y = \lambda z \oplus (1-\lambda)\,\delta_{(c',z')}$). Then $y \in q$ and the independence axiom with respect to the ex-post mixing would state that

$$\rho_z(p) = \rho_y(q).$$

The axiom below strengthens this to independence even with respect to mixtures between z

[29] When the outcome space is infinite-dimensional, allowing for all possible vNM utilities would be too permissive and result in identification issues.

[30] Notice that if the condition is satisfied for N, then it must also be satisfied for all $N' \geq N$ so testing the axiom involves finding a large enough N such that the condition holds.

[31] One could only impose mixing in menus in cases where tomorrow's consumption is the same. The same characterization would then lead to a random utility model where the transition probabilities P_s could also depend on the consumption each period and they all share the same stationary distribution. This could accommodate consumption-dependent stochastic preferences such as habit formation or experimentation.

and y.

Axiom 2 (Continuation Linearity)**.** *If $p \in z \in Z_c^f$, $y = \lambda z \oplus (1-\lambda)\,\delta_{(c',z')}$ and $q = \lambda p \oplus (1-\lambda)\,\delta_{(c',z')}$ for $c, c' \in M$, $z' \in Z$ and $\lambda > 0$, then*

$$\rho_z(p) = \rho_{\alpha z + (1-\alpha) y}(\alpha p + (1-\alpha) q).$$

The next two axioms are conditions with respect to the classic stationarity axiom originally proposed by Koopmans (1960). In classic stationarity, an agent's choices remain unchanged if all consumptions are delayed by the same number of time periods. Given stochastic preferences, classic stationarity would obviously be violated. One way to extend stationarity to a stochastic setup is to require an agent's choice frequencies to remain unchanged if all consumptions are delayed by the same number of time periods.[32] Formally, for any $z, y \in Z^f$ and $c \in M$,

$$\rho(z, y) = \rho\left(\delta_{(c,z)}, \delta_{(c,y)}\right).$$

Classic stationarity is normatively appealing and necessary if the agent is a standard exponential discounter. Stochastic stationarity retains much of the flavor of classic stationarity but allows for stochastic choice due to stochastic utilities.

However, stochastic stationarity would be violated in our model of ergodic utility. For example, consider the standard utility process, in which the state follows an i.i.d. process (i.e., $P_s = \pi$ for all $s \in S$). Let p correspond to the option of consuming c_1 today and 0 tomorrow and q correspond to the option of consuming 0 today and c_2 tomorrow. Thus,

$$\rho(p, q) = \pi\left\{w(c_1) \geq \beta_{s_1} w(c_2)\right\},$$

which depends on the distribution of the stochastic discount rate β_{s_1}. Here, the choice between the original options depends on the realization of the agent's stochastic discount rate. On the other hand, if all consumption is delayed by one period, then

$$\rho\left(\delta_{(c,p)}, \delta_{(c,q)}\right) = \pi\left\{\cancel{\beta_{s_1}} w(c_1) \geq \cancel{\beta_{s_1}} \delta w(c_2)\right\} = \pi\left\{w(c_1) \geq \delta w(c_2)\right\},$$

which is not stochastic as $\delta = \mathbb{E}[\beta_{s_2}]$ is deterministic. Notice here that, the choice between the delayed options depends on the agent's *expectation* of the discount rate, which is deterministic in this i.i.d. example. In general, when realizations and expectations are different,

[32] See Lu and Saito (2018) for a stochastic version of the stationarity axiom in a different setup.

stochastic stationarity will be violated.[33]

Given the example above, we consider two relaxations of Stochastic Stationarity. The first condition, Deterministic Stationarity (Axiom 3) is exactly the classic deterministic stationarity axiom of Koopmans (1960) extended to menus.[34] It states that choices should satisfy stationarity whenever they are deterministic.

Axiom 3 (Deterministic Stationarity)**.** *For any $z, y \in Z^f$ and $c \in M$,*

$$\rho(z, y) = 1 \;\;\Rightarrow\;\; \rho\left(\delta_{(c,z)}, \delta_{(c,y)}\right) = 1.$$

The second condition, Average Stationarity (Axiom 4), states that choice frequencies should satisfy stationarity on "average". The axiom can be interpreted as the stationarity on the *surplus* of menus. To see the interpretation, recall from McFadden (1978, 1981) that the surplus of a menu z is given by

$$\int_S \max_{p\in z} u_s(p)\, d\pi. \tag{11}$$

Following this definition, the surplus of a menu z delayed by one period is given by

$$\int_S \left(\int_S \max_{p\in z} u_{s'}(p)\, dP_s\right) d\pi. \tag{12}$$

If the Markov process is stationary (i.e., $\pi = \int_S P_s d\pi$), then these two surpluses must be the same. This is exactly the implication of the axiom.

We now show how surpluses are calculated using "average" stochastic choice. For $\alpha \in [0,1]$, let p_α denote the lottery that yields the best outcome with probability α and the worst outcome with probability $1-\alpha$, that is

$$p_\alpha := \alpha\delta_{\bar{x}} + (1-\alpha)\,\delta_{\underline{x}}.$$

Thus, p_α is the worst option when $\alpha = 0$ and the best option when $\alpha = 1$. Since $\rho(z, p_\alpha)$ is

[33] In the i.i.d. example, the discount factor for consumption at period t is given by

$$\beta_{s_1}\mathbb{E}\left[\beta_{s_2}\mathbb{E}\left[\cdots\beta_{s_{t-1}}\mathbb{E}\left[\beta_{s_t}\right]\right]\right] = \beta_{s_1}\mathbb{E}\left[\beta\right]^{t-1} = \beta_{s_1}\delta^{t-1},$$

where $\delta := \mathbb{E}[\beta]$. Interestingly, this particular example corresponds to a model of random quasi-hyperbolic discounting where present bias occurs if $\beta_{s_1} < \delta$ and future bias occurs if $\beta_{s_1} > \delta$.

[34] It is very similar to the menu stationarity axiom of Higashi et al. (2009) except we only require implication in one direction.

the demand for z given the outside option p_α, we can interpret

$$\bar{z} := \int_0^1 \rho\left(z, p_\alpha\right) d\alpha \tag{13}$$

as the "average" demand for z. It is straightforward to show that $\bar{z}$ is exactly the surplus of the menu and coincides with (11) via standard integration by parts.[35] Similarly, we can define $\int_0^1 \rho\left(\delta_{(c,z)}, \delta_{(c,p_\alpha)}\right) d\alpha$ as the "average" demand for z delayed by one period and show that it coincides with (12). The following axiom states that average stochastic choice is the same if consumption is delayed by one period.

Axiom 4 (Average Stationarity)**.** *For any $z \in Z^f$ and $c \in M$,*

$$\int_0^1 \rho\left(z, p_\alpha\right) d\alpha = \int_0^1 \rho\left(\delta_{(c,z)}, \delta_{(c,p_\alpha)}\right) d\alpha.$$

While Average Stationarity ensures stationarity of the utility process, it does not guarantee ergodicity of the utility process which is crucial for our representation. This is obtained by a final axiom called D-continuity (Axiom 5). First, note that by Monotonicity, if $z \supset y$, then clearly $\rho\left(z, y\right) = 1$. By Deterministic Stationarity, this implies that

$$\rho\left(\delta_{(c,z)}, \delta_{(c,y)}\right) = 1,$$

which demonstrates classic preference for flexibility. We now require preference for flexibility to be "robust" in the following sense. For any menu $z \in Z$, let $p_{\bar{z}} := \bar{z}\delta_{\bar{x}} + \left(1 - \bar{z}\right)\delta_{\underline{x}}$ denote its *probability-equivalent* where $\bar{z}$ is its average demand from equation (13). Since average demand is equivalent to the surplus of the menu, the agent is ex-ante indifferent between the menu and its probability-equivalent. The last axiom states that preference for flexibility is robust even if we perturb the menus z and y slightly by mixing them with the probability-equivalents $p_{\bar{y}}$ and $p_{\bar{z}}$ respectively.

Axiom 5 (D-continuity)**.** *There exists $\varepsilon > 0$ such that for any $z, y \in Z$ and $c \in M$,*

$$z \supset y \implies \rho\left(\delta_{(c,(1-\varepsilon)z+\varepsilon p_{\bar{y}})}, \delta_{(c,(1-\varepsilon)y+\varepsilon p_{\bar{z}})}\right) = 1.$$

D-continuity implies that the utility process satisfies Doeblin's condition and is thus ergodic. We are now ready to state our main representation theorem.

[35] To see this, note that $\bar{z} = \int_0^1 \pi\left\{s : \max_{p\in z} u_s\left(p\right) \geq \alpha\right\} d\alpha = \int_S \max_{p\in z} u_s\left(p\right) d\pi$. This is similar to the use of test functions in Lu (2016)

Theorem 4. *ρ satisfies Axioms 1-5 if and only if it is ergodic.*

Proof. See Appendix D. □

We now provide an outline for the proof of Theorem 4. The first step is the construction of a random expected utility representation where the probability measure is countably additive and continuation menus are evaluated according to the additive linear utility function of Dekel et al. (2001). This exercise faces two technical challenges. First, we need to extend the random expected utility representation of Gul and Pesendorfer (2006) to an infinite-dimensional space while keeping the countable additivity (Theorem 5 in the Appendix). Next, we need to extend the representation of Dekel et al. (2001) to countably-additive probability measures in an infinite-dimensional setting (Theorem 6 in the Appendix). Both extensions are known challenges in the literature as the set of utilities over an infinite-dimensional space (without any restrictions) can be no longer compact.[36] We employ a unified methodology that achieves both. The main technical innovation is focusing on the set of Lipschitz continuous utilities with common bound; this forms a nice compact set according to the Arzela-Ascoli theorem (see Appendix A). This is obtained using the L-continuity (Axiom 1.6) which is the stochastic version of the Lipschitz continuity axiom from Dekel et al. (2007). Note that this is not only important for the representation but also crucial for identification in both settings (Theorem 1). In fact, without such a restriction on the set of utilities, identification would not be possible.

Once we have a random expected utility representation where continuation menus are evaluated according to the additive linear functional form, the next step is to show that the random utilities are derived from the stationary distribution of an ergodic utility process. This is where the last three axioms come into play. First, by using Deterministic and Average Stationarity, we show that the random utility is recursive. This allows us to construct a Markov utility process with a stationary distribution that coincides exactly with the distribution of the random utility from the representation. Next, D-continuity ensures that this Markov utility process is ergodic. Finally, the representation is obtained by an application of the Birkhoff ergodic theorem.

[36] For instance, the unit ball is compact in finite-dimensional space but not in infinite-dimensional space. See the discussion after Theorem 3 in Krishna and Sadowski (2014) for more details.

Appendices

A Lipschitz Continuous Utilities

Remember that $X = M \times Z$. Since M and Z are compact metric spaces, X is a compact metric space. Let $C(X)$ denote the set of continuous functions defined on X, $L(X)$ denote the set of Lipschitz continuous functions defined on X, and $L_N(X)$ the set of Lipschitz functions defined on X with Lipschitz bound N. We endow $C(X)$ with the topology of uniform convergence. Fix $\overline{x}, \underline{x} \in X$ and define

$$U_N := \{u \in L_N(X) \ : \ 0 = u(\underline{x}) \leq u(x) \leq u(\overline{x}) = 1 \text{ for all } x \in X\}. \tag{14}$$

For each $u \in C(X)$ and $p \in \Delta X$, let

$$u(p) = \int_X u \, dp$$

denote its expectation. The following result shows that the set of utilities we consider is compact. It is crucial for both characterization and identification, and highlights the role of Lipschitz functions.

Lemma 3. *U_N is compact in $C(X)$.*

Proof. We will show this using the Arzela-Ascoli Theorem (Theorem 4.43 of Folland (2013)). First, we show that $L_N(X)$ is equicontinuous. Fix $x \in X$ and $\varepsilon > 0$ and consider $y \in X$ such that $|x - y| < \frac{1}{N}\varepsilon$. Thus, for all $u \in L_N(X)$

$$|u(x) - u(y)| \leq N|x - y| < \varepsilon.$$

Since this holds for all $x \in X$, U_N is equicontinuous. Since $0 \leq |u| \leq 1$ for all $u \in U_N$, U_N is pointwise bounded.

Next, we show that U_N is closed. Consider $u_k \in U_N$ such that $u_k \to u$. We will show that $u \in U_N$. Since u_k is bounded, we have

$$u(x) - u(y) = \lim_k (u_k(x) - u_k(y)) \leq \lim_k N|x - y| = N|x - y|$$

for all $x, y \in X$. Thus, $u \in L_N(X)$. Next, note that for all k,

$$0 = u_k(\underline{x}) \leq u_k(x) \leq u_k(\bar{x}) = 1$$

so $0 = u(\underline{x}) \leq u(x) \leq u(\bar{x}) = 1$. This shows $u \in U_N$, hence U_N is closed. By the Arzela-Ascoli (Theorem 4.43 of Folland (2013)) , U_N is compact in $C(X)$. ■

A.1 Proof of Lemma 1

We first show the following lemma which characterizes distributions on a compact subset of $C(X)$.

Lemma 4. *Let $\mu, \nu \in \Delta U$ where U is a compact subset of $C(X)$. If for all $r \geq 0$ and $p \in \Delta X$,*

$$\int_U e^{ru(p)} d\mu = \int_U e^{ru(p)} d\nu$$

then $\mu = \nu$.

Proof. Let Φ denote the set of continuous functions ϕ defined on U such that

$$\phi(u) = \sum_{i=1}^{n} a_i e^{r_i u(p_i)}$$

for some n, $a_i \in \mathbb{R}$, $r_i \geq 0$ and $p_i \in \Delta X$ for each $i \in \{1, \ldots, n\}$. Thus, for all $\phi \in \Phi$,

$$\int_U \phi(u)\, d\mu = \int_U \sum_{i=1}^{n} a_i e^{r_i u(p_i)} d\mu = \int_U \sum_{i=1}^{n} a_i e^{r_i u(p_i)} d\nu = \int_U \phi(u)\, d\nu$$

We will show that Φ is uniformly dense in $C(U)$ by the Stone-Weierstrass Theorem (Theorem 9.13 of Aliprantis and Border (2006) (henceforth, AB)). First note that Φ is a vector space that includes constants since $e^{0u(p)} = 1 \in \Phi$.

To show that Φ is closed under multiplication. Consider $a_1 e^{r_1 u(p_1)}, a_2 e^{r_1 u(p_2)} \in \Phi$. If $r_1 + r_2 > 0$, then

$$a_1 e^{r_1 u(p_1)} a_2 e^{r_2 u(p_2)} = a_1 a_2 e^{(r_1+r_2) u\left(\frac{r_1}{r_1+r_2} p_1 + \left(1 - \frac{r_1}{r_1+r_2}\right) p_2\right)} \in \Phi$$

On the other hand, if $r_1 + r_2 = 0$, then $r_1 = r_2 = 0$ and

$$a_1 e^{r_1 u(p_1)} a_2 e^{r_2 u(p_2)} = a_1 a_2 \in \Phi$$

This means that Φ is closed under multiplication.

Next, we show that Φ separates points in U. Suppose $u, v \in U$ such that $u \neq v$. Thus, there is some $x \in X$ such that $u(x) > v(x)$ without loss of generality. If we let $p = \delta_x$, then $u(p) = u(x) > v(x) = v(p)$ so $e^{u(p)} > e^{v(p)}$. This establishes that Φ separates points in U.

Since U is compact, Φ is a subalgebra, contains the constant function and separates points in U, Φ is uniformly dense in $C(U)$ by the Stone-Weierstrass Theorem. This means that for any $\phi \in C(U)$, we can find $\phi_k \in \Phi$ such that $\phi_k \to \phi$ uniformly. Hence, if we fix some $\varepsilon > 0$, then there exists some n such that $|\phi_k - \phi| \leq \varepsilon$ for all $k > n$. This implies that for all $u \in U$,

$$\phi_k(u) \leq |\phi_k(u) - \phi(u)| + |\phi(u)| \leq |\phi(u)| + \varepsilon.$$

Thus, ϕ_k are all dominated by a integrable function, so by dominated convergence,

$$\int_U \phi(u)\, d\mu = \lim_k \int_U \phi_k(u)\, d\mu = \lim_k \int_U \phi_k(u)\, d\nu = \int_U \phi(u)\, d\nu.$$

By AB Theorem 15.1, $\mu = \nu$. ■

We now prove Lemma 1. Define the mapping $\xi : S \to U$ as in equation (3), or

$$\xi_s(c, z) = \phi_s\left(c, \int_S \sup_{p \in z} u_{\tilde{s}}(p)\, dP_s\right).$$

Consider two states $s, s' \in S$ such that $\xi_s = \xi_{s'}$. We will show that this means that $P_s \circ \xi^{-1} = P_{s'} \circ \xi^{-1}$. Let $\nu = P_s \circ \xi^{-1}$, $\nu' = P_{s'} \circ \xi^{-1}$ and $z = \left\{p, \alpha\delta_{\bar{x}} + (1-\alpha)\delta_{\underline{x}}\right\}$. Since $\xi_s = \xi_{s'}$ and $\phi_.$ is strictly increasing in the second argument, we have

$$\int_U \max\{u(p), \alpha\}\, d\nu = \int_S \sup_{p \in z} u_{\tilde{s}}(p)\, dP_s = \int_S \sup_{p \in z} u_{\tilde{s}}(p)\, dP_{s'} = \int_U \max\{u(p), \alpha\}\, d\nu'$$

for any $\alpha \in [0, 1]$. By Theorem 1.57 of Müller and Stoyan (2002), for any increasing convex function φ,

$$\int_U \varphi(u(p))\, d\nu = \int_U \varphi(u(p))\, d\nu'.$$

Thus by Lemma 4, $\nu = \nu'$ because ν and ν' are probability measures on U_N, which is compact by Lemma 3.

We can now define a transition kernel ν_v on U such that $\nu_v := P_s \circ \xi^{-1}$ where $v = u_s$. If we let $\mu = \pi \circ \xi^{-1}$, then

$$\int_U \nu_v(B)\, d\mu = \int_S \nu_{u_s}(B)\, d\pi = \int_S P_s\left(\xi^{-1}(B)\right) d\pi = \pi\left(\xi^{-1}(B)\right) = \mu(B),$$

where the first and the last equality hold by the definition of μ, the second equality holds by definition of ν_v, and the third equality holds because π is a stationary distribution of P. Thus, the utility process is a stationary Markov process. Moreover, for any measurable B, we have μ-a.s.

$$\nu_v(B) = P_s\left(\xi^{-1}(B)\right) \geq \delta\pi\left(\xi^{-1}(B)\right) = \delta\mu(B)$$

so the Markov process satisfies Doeblin's condition and is thus ergodic.

B Proof of Theorem 1 (Uniqueness)

From Lemma 1, the utility process is ergodic so let μ and μ' denote the stationary utility distributions for ρ and ρ' respectively. For every $z = \{p, q\} \in Z^*$, we have

$$\rho_z(p) = \lim_{n\to\infty} \frac{1}{n} \sum_{k\geq 0} 1_{B(p,z)}(s_{tk+1}) = \mu\{u \in U \ : \ u(p) \geq u(q)\}$$

and likewise for ρ' and μ', where the first equality is by the ergodic representation and the second equality is by the Birkoff ergodic theorem.

Choose any binary menu $z = \{p, q\} \in Z$. For each $t \in T$, define

$$p^t = p_{z^t,t}, \quad q^t = q_{z^t,t}.$$

Then $p^t \to p$ and $q^t \to q$ as $t \to \infty$. By definition $z^t = \{p^t, q^t\} \in Z^*$ and $z^t \to z$ by Lemma 2.

Step 1: If $u(p) = u(q)$ with μ-measure zero, then $\lim_{t\to\infty} \rho(p^t, q^t) = \mu\{u(p) \geq u(q)\}$.

Proof. First, note that μ-a.s.

$$\lim_t 1_{u(p^t)\geq u(q^t)} = 1_{u(p)\geq u(q)}$$

To see why, first suppose $u(p) \geq u(q)$, but $\liminf_t 1_{u(p^t)\geq u(q^t)} = 0$. Thus, we can find a subsequence p^k, q^k such that $u\left(p^k\right) < u\left(q^k\right)$ so $u(p) \leq u(q)$ yielding a contradiction as $u(p) \neq u(q)$ μ-a.s.. On the other hand, if $u(p) < u(q)$, then clearly $\limsup_t 1_{u(p^t)\geq u(q^t)} = 0$. By the dominated convergence theorem, we thus have

$$\lim_t \rho\left(p^t, q^t\right) = \lim_t \int_U 1_{u(p^t)\geq u(q^t)} d\mu = \int_U 1_{u(p)\geq u(q)} d\mu = \mu\{u(p) \geq u(q)\}$$

as desired. □

Step 2: If $u(p) = u(q)$ with μ'-a.s., then $u(p) = u(q)$ with μ-a.s.

Proof. Let $q = p_\alpha := \alpha\delta_{\bar{x}} + (1-\alpha)\delta_{\underline{x}}$ and suppose that $u(p) = u(q) = \alpha$ μ'-a.s. We will show that this implies $u(p) = \alpha$ μ-a.s. Fix a positive number ε. Consider $p_{\alpha+\varepsilon}$ and $p_{\alpha-\varepsilon}$

and note that $u(p_{\alpha+\varepsilon}) > u(p) > u(p_{\alpha-\varepsilon})$ μ'-a.s. for all $\varepsilon > 0$. By regularity, without loss of generality, we can choose ε such that $u(p) = u(p_{\alpha+\varepsilon})$ and $u(p) = u(p_{\alpha-\varepsilon})$ with μ-measure zero. Thus,

$$\mu\{u(p) \geq u(p_{\alpha-\varepsilon})\} = \lim_t \rho\left(p^t, p^t_{\alpha-\varepsilon}\right) = \lim_t \rho'\left(p^t, p^t_{\alpha-\varepsilon}\right) = \mu'\{u(p) \geq u(p_{\alpha-\varepsilon})\} = 1,$$

where the first and the third equality hold by Step 1, the second equality holds by the supposition of Theorem 1 that ρ and ρ' coincide on binary sets, and the last equality holds by the supposition that $u(p) = \alpha$ μ'-a.s.. By the symmetric argument for p and $p_{\alpha+\varepsilon}$,

$$\mu\{u(p_{\alpha+\varepsilon}) \geq u(p)\} = \lim_t \rho\left(p^t_{\alpha+\varepsilon}, p^t\right) = \lim_t \rho'\left(p^t_{\alpha+\varepsilon}, p^t\right) = \mu'\{u(p_{\alpha+\varepsilon}) \geq u(p)\} = 1.$$

Thus, $u(p) \in [\alpha-\varepsilon, \alpha+\varepsilon]$ μ-a.s. Since ε is an arbitrary positive number, $u(p) = \alpha$ μ-a.s. as desired. □

Step 3: For any $p \in \Delta(X)$, $u(p)$ has the same distribution under μ and under μ'.

Proof. Fix any $p \in \Delta(X)$ and $\alpha \in \mathbb{R}$ to show

$$\mu\{u(p) \geq \alpha\} = \mu'\{u(p) \geq \alpha\}.$$

By the regularity of μ, it suffices to consider the following two cases.

Case 1: The case when $\mu\{u(p) = \alpha\} = 0$. Let $q = p_\alpha := \alpha\delta_{\bar{x}} + (1-\alpha)\delta_{\underline{x}}$. By Step 1

$$\mu\{u(p) \geq \alpha\} = \lim_{t\to\infty} \rho(\rho^t, q) = \lim_{t\to\infty} \rho'(\rho^t, q) = \mu'\{u(p) \geq \alpha\}.$$

Case 2: The case when $\mu\{u(p) = \alpha\} = 1$. By Step 2, $\mu'\{u(p) = \alpha\} = 1 = \mu\{u(p) = \alpha\}$. □

Now, by Step 3,

$$\int_U e^{ru(p)} d\mu = \int_U e^{ru(p)} d\mu'$$

for all $r \geq 0$ and $p \in \Delta X$. Since μ and μ' are probability measure on U_N, which is compact by Lemma 3. Thus, $\mu = \mu'$ by Lemma 4. Since each $u \in U$ determines the transition kernel on U, this means that the Markov utility process induced by μ and μ' are the same. The converse is trivial.

C Extension Theorems

In this section, we employ a unified methodology to extend both Gul and Pesendorfer (2006) (henceforth GP) and Dekel et al. (2001) (henceforth DLR)[37] to countably-additive probability measures in infinite-dimensional settings. In both cases, we achieve this by focusing on the set of Lipschitz continuous utilities with a common bound. Note that this is a compact set by the same argument as in Lemma 3 which ensures our representations are unique. We first focus on finite-dimensional settings and then apply Kolmogorov's extension theorem followed by Tietze extension theorem (Theorem 4.16 of Folland (2013)). On an abstract level, this is analogous to the extension to uniformly continuous paths for the construction of Brownian motion.[38]

Throughout this section, we will let X be a compact metric space and U_N be the set of Lipschitz continuous utilities with common bound N defined by (14). We will assume that X contains two elements $\bar{x}$ and $\underline{x}$.

The following preliminary lemma modified from Dekel et al. (2007) characterizes Lipschitz continuous functions on a dense subset.

Lemma 5. *Let X^* be a dense subset of X and suppose $v : X^* \to \mathbb{R}$ is such that $v(\bar{x}) = 1$ and $v(\underline{x}) = 0$. Then the following statements are equivalent:*

(i) *There exist $N > 0$ such that if $|x_1 - x_2| \leq \frac{\alpha}{N}$ for $x_1, x_2 \in X^*$ and $\alpha \in [0,1]$, then*

$$\alpha v(\bar{x}) + (1-\alpha) v(x_1) \geq \alpha v(\underline{x}) + (1-\alpha) v(x_2).$$

(ii) *v is Lipschitz continuous with bound N.*

Proof. Suppose (i) is true. Fix some $\bar{\alpha} < 1$ and consider $x_1, x_2 \in X^*$. First suppose $|x_1 - x_2| N = \alpha \leq \bar{\alpha} < 1$. We thus have $\alpha v(\underline{x}) + (1-\alpha) v(x_2) \leq \alpha v(\bar{x}) + (1-\alpha) v(x_1)$. Hence

$$v(x_2) - v(x_1) \leq \frac{\alpha}{1-\alpha} = \frac{N}{1-\alpha} |x_1 - x_2| \leq \frac{N}{1-\bar{\alpha}} |x_1 - x_2|.$$

[37] See also Dekel et al. (2007).

[38] Other papers that also employ Kolmogorov's extension in this manner include Lu and Saito (2018), who do not address the continuity of utilities, and Frick et al. (2018), who obtain a measure with finite support (ignoring ties).

Now suppose $|x_1 - x_2| N = \alpha > \bar{\alpha}$. Since X is a convex metric space, we can find $y_i := \left(1 - \frac{i}{n}\right) x_1 + \frac{i}{n} x_2 \in X$ for $i \in \{0, 1, \ldots, n\}$ such that

$$|y_{i+1} - y_i| = \frac{1}{n} |x_1 - x_2| < \frac{\bar{\alpha}}{N}$$

Since X^* is dense in X and the metric mapping is continuous, we can choose n large enough such that for each $\varepsilon > 0$, we can find $y_i^* \in X^*$ such that $|y_i - y_i^*| \leq \varepsilon$ and $\left|y_{i+1}^* - y_i^*\right| < \frac{\bar{\alpha}}{N}$ for all i. From the argument above, we have

$$\begin{aligned} v\left(y_{i+1}^*\right) - v\left(y_i^*\right) &\leq \frac{N}{1 - \bar{\alpha}} \left|y_{i+1}^* - y_i^*\right| \\ &\leq \frac{N}{1 - \bar{\alpha}} \left(|y_{i+1} - y_i| + \left|y_{i+1}^* - y_{i+1}\right| + |y_i^* - y_i|\right) \\ &\leq \frac{N}{1 - \bar{\alpha}} \left(|y_{i+1} - y_i| + 2\varepsilon\right) = \frac{N}{1 - \bar{\alpha}} \left(\frac{1}{n} |x_1 - x_2| + 2\varepsilon\right) \end{aligned}$$

Since we can let $y_0^* = y_0 = x_1$ and $y_n^* = y_n = x_2$, this implies that

$$v(x_2) - v(x_1) \leq \sum_{1 \leq i \leq n} \left|v\left(y_i^*\right) - v\left(y_{i-1}^*\right)\right| \leq \frac{N}{1 - \bar{\alpha}} \left(|x_1 - x_2| + 2n\varepsilon\right)$$

Taking $\varepsilon \to 0$ yields

$$v(x_2) - v(x_1) \leq \frac{N}{1 - \bar{\alpha}} |x_1 - x_2|$$

Since $\frac{N}{1-\bar{\alpha}} \to N$ as $\bar{\alpha} \to 0$, this means that $|v(x_2) - v(x_1)| \leq N |x_1 - x_2|$ for all $x_1, x_2 \in X^*$. Thus, v is Lipschitz continuous with bound N as desired.

Now, suppose (ii) is satisfied. Note that if $\alpha = 1$, then the result is trivial so assume $\alpha < 1$. Suppose that $|x_1 - x_2| \leq \frac{\alpha}{N}$ and since $v \in L_N(X^*)$,

$$v(x_2) - v(x_1) \leq N |x_1 - x_2| \leq \frac{N}{1 - \alpha} |x_1 - x_2| \leq \frac{\alpha}{1 - \alpha}$$

Rearranging yields

$$\alpha v(\underline{x}) + (1 - \alpha) v(x_2) \leq \alpha v(\bar{x}) + (1 - \alpha) v(x_1)$$

as desired. ■

C.1 Extension of Gul and Pesendorfer (2006)

In this section, we extend the main theorem of GP. Let $Z = \mathcal{K}(\Delta X)$ denote the set of non-empty compact subsets of ΔX. We consider a stochastic choice function ρ on Z^f, the finite menus in Z. That is for every $z \in Z^f$, ρ_z is a Borel probability measure over z. We model ties as in Lu (2016) and let $Z^\circ \subset Z^f$ denote the set of finite menus that contain no ties.

Condition 1.1 (Monotonicity). $z \subset y$ implies $\rho_z(p) \geq \rho_y(p)$

Condition 1.2 (Linearity). $\rho_z(p) = \rho_{\alpha z + (1-\alpha) q}(\alpha p + (1-\alpha) q)$

Condition 1.3 (Extremeness). $\rho_z(\text{ext}(z)) = 1$

Condition 1.4 (Continuity). $\rho : Z^\circ \to \Delta(\Delta X)$ is continuous

Condition 1.5 (Best-Worst). $\rho(\underline{x}, \bar{x}) = 0$ *and* $\rho(\bar{x}, x) = \rho(x, \underline{x}) = 1$ *for all* $x \in X$.

Condition 1.6 (L-continuity). There exists $N > 0$ such that for $\alpha \in [0,1]$, $|x_1 - x_2| \leq \frac{\alpha}{N}$ implies $\rho\left(\alpha\delta_{\bar{x}} + (1-\alpha)\delta_{x_1}, \alpha\delta_{\underline{x}} + (1-\alpha)\delta_{x_2}\right) = 1$.

We will now prove the following extension of GP to an infinite-dimensional setting. We say a probability measure on U_N is *regular* if $u(p) = u(q)$ occurs with probability zero or one for all $p, q \in \Delta X$

Theorem 5 (GP extension). *ρ satisfies C1 if and only if there exists a regular probability measure μ on U_N such that for any $z \in Z^f$,*

$$\rho_z(p) = \mu\{u \in U_N \;:\; u(p) \geq u(q) \;\; \textit{for all } q \in z\}$$

The necessity of the axioms is straightforward. C1.1-C1.3 follow from the same arguments as in GP while C1.4 follows from the same argument as in Lu (2016). It is easy to see C1.5 from the representation while C1.6 follows from Lemma 5 above.

We now show sufficiency and suppose ρ satisfies C1. Since X is separable, let $X^* \subset X$ be a countable dense subset of X and without loss of generality, assume $\underline{x}, \bar{x} \in X^*$.

Lemma 6. *There exists a probability measure μ on the Borel σ-algebra corresponding to uniform convergence on U_N such that for all finite $W \subset X^*$ and finite $z \subset \Delta W$,*

$$\rho_z(p) = \mu\{u \in U_N \;:\; u(p) \geq u(q) \;\; \textit{for all } q \in z\}.$$

Proof. We prove this in a series of steps.

Step 1: There exists a probability measure π on the Borel σ-algebra corresponding to pointwise convergence on $\mathbb{R}^{X^*}$ such that for all finite $W \subset X^*$ and finite $z \subset \Delta W$,

$$\rho_z(p) = \pi \left\{ u \in \mathbb{R}^{X^*} \ : \ u(p) \geq u(q) \text{ for all } q \in z \right\}.$$

Proof. From Gul and Pesendorfer (2006) and Lu (2016), C1.1-C1.4 imply that for each finite $W \subset X^*$ where $\underline{x}, \bar{x} \in W$, there exists a probability measure π_W on $\mathbb{R}^W$ such that for any finite $z \subset \Delta W$,

$$\rho_z(p) = \pi_W \left\{ u \in \mathbb{R}^W \ : \ u(p) \geq u(q) \text{ for all } q \in z \right\}$$

Moreover, C1.5 implies that we can assume μ-a.s. $0 = u(\underline{x}) \leq u(x) \leq u(\bar{x}) = 1$ for all $x \in X^*$ without loss of generality. By the uniqueness result of GP, all these π_W are consistent.[39] Thus, by Kolmogorov's extension, there exists a measure π on $\mathbb{R}^{X^*}$ such that for all finite $W \subset X^*$ and finite $z \subset \Delta W$,

$$\rho_z(p) = \pi \left\{ u \in \mathbb{R}^{X^*} \ : \ u(p) \geq u(q) \text{ for all } q \in z \right\}$$

Moreover, we can assume that π is a measure on the Borel σ-algebra corresponding to pointwise convergence on $\mathbb{R}^{X^*}$ (i.e., the product topology, see exercise I.6.35 of Çınlar (2011)). □

Step 2: There exists $N > 0$ such that π-a.s. for all $\alpha \in [0,1]$ and $x_1, x_2 \in X^*$,

$$|x_1 - x_2| \leq \frac{\alpha}{N} \implies \alpha + (1-\alpha) u(x_1) \geq (1-\alpha) u(x_2).$$

Proof. For $\alpha \in [0,1]$ and $x_1, x_2 \in X^*$, define

$$U_\alpha^{x_1,x_2} := \left\{ u \in \mathbb{R}^{X^*} \ : |x_1 - x_2| \leq \frac{\alpha}{N} \implies \alpha + (1-\alpha) u(x_1) \geq (1-\alpha) u(x_2) \right\}.$$

By C1.6, there exists $N > 0$ such that $\pi(U_\alpha^{x_1,x_2}) = 1$ for all $\alpha \in [0,1]$ and $x_1, x_2 \in X^*$. Let $U_\alpha := \bigcap_{x_1,x_2 \in X^*} U_\alpha^{x_1,x_2}$ so by the countable additivity of π and the fact that X^* is a countable dense subset of X, $\pi(U_\alpha) = 1$ for any $\alpha \in [0,1]$. Let I^* be the rationals in $[0,1]$ so by the same argument, $\pi\left(\bigcap_{\alpha \in I^*} U_\alpha\right) = 1$.

We will show that $\pi\left(\bigcap_{\alpha \in [0,1)} U_\alpha\right) = 1$. It suffices to show that $\bigcap_{\alpha \in I^*} U_\alpha \subset \bigcap_{\alpha \in [0,1)} U_\alpha$.

[39] Note that this requires normalized utilities.

We will show that for any $u \in \bigcap_{\alpha \in I^*} U_\alpha$ and $\alpha \in [0,1)$, $u \in U_\alpha$. Choose any $x_1, x_2 \in X^*$ such that $|x_2 - x_1| \leq \alpha/N$ and consider a sequence α_k of I^* such that $\alpha_k \to \alpha$ and $\alpha_k \geq \alpha$. Since $|x_2 - x_1| \leq \alpha_k/N$ and $u \in \bigcap_{\alpha \in I^*} U_\alpha$, we have $u(x_2) - u(x_1) \leq \alpha_k/(1-\alpha_k)$ for each k. Since $\alpha_k \to \alpha$, we have $u(x_2) - u(x_1) \leq \alpha/(1-\alpha)$ so $u \in U_\alpha$. Thus, $\pi\left(\bigcap_{\alpha \in [0,1]} U_\alpha\right) = 1$ as desired. □

By Step 2, Lemma 5 yields $\pi(L_N(X^*)) = 1$. By the Lipschitz version of the Tietze extension theorem (see McShane (1934)), we can extend π on $L_N(X^*)$ to a probability measure μ on $L_N(X)$.

Step 3: $\mu(U_N) = 1$.

Proof. For each $x \in X$, define

$$U_x := \left\{u \in L_N(X) \ : \ 0 = u(\underline{x}) \leq u(x) \leq u(\bar{x}) = 1\right\}.$$

Then $\pi(U_x) = 1$ for any $x \in X$. By the countable additivity of π, we have $\pi\left(\bigcap_{x \in X^*} U_x\right) = 1$. We will show that $\pi\left(\bigcap_{x \in X} U_x\right) = 1$. It suffices to prove that $\cap_{x \in X^*} U_x \subset \cap_{x \in X} U_x$. Suppose that $u \in \cap_{x \in X^*} U_x$ and consider $x \in X$. If $x \in X^*$, then the result holds trivially so suppose $x \notin X^*$. Since X^* is dense in X, there exists a sequence x_k of X^* such that $x_k \to x$. Since $0 \leq u(x_k) \leq 1$ for each k, we have $0 \leq u(x) \leq 1$ by the continuity of u. □

Finally, since X is compact, pointwise convergence is equivalent to uniform convergence on U_N. Thus, μ is a measure on the Borel σ-algebra corresponding to uniform convergence. ■

Define

$$B(p,z) := \{u \in U_N \ : \ u(p) \geq u(q) \text{ for all } q \in z\}$$

so $B(p,z)$ is μ-measurable. Also define $B(p,q) := B(p,\{p,q\})$ to simplify notation.

We will show that $\rho_z(p) = \mu(B(p,z))$. First, we prove a series of lemmas. The following is straightforward but will be useful for latter analysis.

Lemma 7. *For every $p \in \Delta X$, there exists a sequence $p_n \to p$ such that each p_n has a finite support in X^*.*

Proof. Since X^* is dense and Dirac measures are extreme points in ΔX, the result follows from the Krein-Milman theorem (AB Theorem 15.10). ■

The next two lemmas deals with ties in the stochastic choice.

Lemma 8. *Suppose $z \in Z^\circ$ and $p_n \to p$ for every $p \in z$ where each p_n has finite support in X^*. If $z_n := \{p_n \ : \ p \in z\} \in Z^\circ$, then $\rho_z(p) \leq \mu(B(p,z))$.*

Proof. First, note that since $p_n \to p$ for every $p \in z$, $z_n \to z$. Since $z_n, z \in Z^\circ$, Continuity (C1.4) implies that

$$\rho_z(p) = \lim_n \rho_{z_n}(p_n) = \lim_n \mu(B(p_n, z_n))$$

where the last equality follows from the representation as each p_n has finite support in X^*. Note that

$$\limsup_n 1_{B(p_n,z_n)} \leq 1_{B(p,z)}$$

To see why, note that if $\limsup_n 1_{B(p_n,z_n)}(u) = 1$, then there exists a subsequence $\{(p_k, z_k)\}$ such that $u(p_k) \geq u(q_k)$ for all $q_k \in z_k$ so $u(p) \geq u(q)$. Thus, we have

$$\rho_z(p) = \lim_n \int_{U_N} 1_{B(p_n,z_n)} d\mu \leq \int_{U_N} \limsup_n 1_{B(p_n,z_n)} d\mu \leq \int_{U_N} 1_{B(p,z)} d\mu = \mu(B(p,z)),$$

where the first inequality follows from Fatou's Lemma. ■

Lemma 9. *The following statements hold:*

(i) *If p and q are tied, then $u(p) = u(q)$ a.s.*

(ii) *If p and q are not tied, then $u(p) \neq u(q)$ a.s.*

Proof. First, we show that if p is not tied with $\underline{x}$, then $\rho(\underline{x}, p) = 0$. By Lemma 7, there exists $p_n \to p$ where p_n has finite support in X^*. Let $\tilde{p}_n := \left(1 - \frac{1}{n}\right) p_n + \frac{1}{n}\delta_{\bar{x}}$ and note that $\tilde{p}_n$ cannot be tied with $\underline{x}$ since a.s.

$$u(\tilde{p}_n) = \left(1 - \frac{1}{n}\right) u(p_n) + \frac{1}{n} > 0$$

Note that $\tilde{p}_n \to p$ and each $\tilde{p}_n$ also has finite support in X^*. Since $\{\underline{x}, \tilde{p}_n\} \in Z^\circ$ and $\{\underline{x}, \tilde{p}_n\} \to \{\underline{x}, p\} \in Z^\circ$, Continuity (C1.4) yields

$$\rho(\underline{x}, p) = \lim_n \rho(\underline{x}, \tilde{p}_n) = \lim_n \mu\{0 \geq u(\tilde{p}_n)\} = 0$$

as desired. We now prove the lemma via two steps.

Step 1: If p and q are tied, then $u(p) = u(q)$ a.s.

Proof. First, suppose p is not tied with $\underline{x}$ so $\rho(\underline{x}, p) = 0$ from above. Let $p^\varepsilon := (1-\varepsilon) p + \varepsilon \delta_{\underline{x}}$ so $\rho(p^\varepsilon, p) = 0$ by Linearity (C1.2). Since p and q are tied, $\rho(p^\varepsilon, q) = 0$ by Lemma A.2 of

Lu (2016). Consider $z_n^\varepsilon = \{p_n^\varepsilon, q_n\}$ where $p_n^\varepsilon \to p^\varepsilon$, $q_n \to q$ and p_n^ε and q_n both have finite support in X^* as from Lemma 7. If p_n^ε is tied with q_n, let

$$\tilde{p}_n^\varepsilon := \left(1 - \frac{1}{n}\right) p_n^\varepsilon + \frac{1}{n}\delta_{\underline{x}}$$
$$\tilde{q}_n := \left(1 - \frac{1}{n}\right) q_n + \frac{1}{n}\delta_{\bar{x}}$$

so $\{\tilde{p}_n^\varepsilon, \tilde{q}_n\} \in Z^\circ$. Since $\{\tilde{p}_n^\varepsilon, \tilde{q}_n\} \to \{p^\varepsilon, q\} \in Z^\circ$, by Lemma 8,

$$1 = \rho\left(q, p^\varepsilon\right) \leq \mu\left(B\left(q, p^\varepsilon\right)\right) = \mu\left\{u\left(q\right) \geq \left(1 - \varepsilon\right) u\left(p\right)\right\}$$

Thus, a.s.

$$u\left(p\right) - u\left(q\right) \geq -\varepsilon u\left(p\right) \geq -\varepsilon$$

for all $\varepsilon > 0$ so $u\left(q\right) \geq u\left(p\right)$ a.s. By the symmetric reasoning, we have $u\left(p\right) \geq u\left(q\right)$ a.s. Hence $u\left(p\right) = u\left(q\right)$ a.s.

Finally, note that if p is tied with $\delta_{\underline{x}}$, then $\frac{1}{2}p + \frac{1}{2}\delta_{\bar{x}}$ is tied with $\frac{1}{2}\delta_{\underline{x}} + \frac{1}{2}\delta_{\bar{x}}$ where the latter is not tied with $\delta_{\underline{x}}$. Applying the above argument yields $\frac{1}{2}u\left(p\right) + \frac{1}{2} = \frac{1}{2}$ a.s. or $u\left(p\right) = 0$ a.s. as desired. □

Step 2: If p and q are not tied, then $u\left(p\right) \neq u\left(q\right)$ a.s.

Proof. Let p and q be not tied. Consider $p^\varepsilon := \left(1 - \varepsilon\right) p + \varepsilon\delta_{\underline{x}}$ and $q^\varepsilon := \left(1 - \varepsilon\right) q + \varepsilon\delta_{\bar{x}}$ for $\varepsilon > 0$. Note that if p^ε and q^ε are tied, then from (i), we have a.s.

$$u\left(p\right) = u\left(q\right) + \frac{\varepsilon}{1 - \varepsilon}$$

Thus, we can choose $\varepsilon \to 0$ such that p^ε and q^ε are not tied. Consider $z_n^\varepsilon = \{p_n^\varepsilon, q_n^\varepsilon\}$ where $p_n^\varepsilon \to p^\varepsilon$, $q_n^\varepsilon \to q^\varepsilon$ and p_n^ε and q_n both have finite support in X^* as above. Again, let

$$\tilde{p}_n^\varepsilon := \left(1 - \frac{1}{n}\right) p_n^\varepsilon + \frac{1}{n}\delta_{\underline{x}}$$
$$\tilde{q}_n^\varepsilon := \left(1 - \frac{1}{n}\right) q_n^\varepsilon + \frac{1}{n}\delta_{\bar{x}}$$

so $\{\tilde{p}_n^\varepsilon, \tilde{q}_n^\varepsilon\} \in Z^\circ$. Since $\{\tilde{p}_n^\varepsilon, \tilde{q}_n^\varepsilon\} \to \{p^\varepsilon, q^\varepsilon\} \in Z^\circ$, by Lemma 8,

$$\rho\left(p^\varepsilon, q^\varepsilon\right) \leq \mu\left(B\left(p^\varepsilon, q^\varepsilon\right)\right) = \mu\left\{u\left(p\right) - u\left(q\right) \geq \frac{\varepsilon}{1 - \varepsilon}\right\}$$

As $\varepsilon \searrow 0$, $\{p^\varepsilon, q^\varepsilon\} \to \{p, q\} \in Z^\circ$ so by Continuity (C1.4),

$$\rho(p,q) = \lim_{\varepsilon \searrow 0} \rho(p^\varepsilon, q^\varepsilon) \leq \lim_{\varepsilon \searrow 0} \mu \left\{ u(p) - u(q) \geq \frac{\varepsilon}{1-\varepsilon} \right\} = \mu \{u(p) > u(q)\}$$

By symmetric reasoning, we have $\rho(q,p) \leq \mu\{u(p) > u(q)\}$ so

$$1 = \rho(p,q) + \rho(q,p) \leq \mu\{u(p) > u(q)\} + \mu\{u(p) > u(q)\}$$

Thus, $u(p) = u(q)$ has μ-measure zero. □

■

We now complete the proof of Theorem 5. Let $z \in Z^\circ$ and $p_n \to p$ for every $p \in z$ where each p_n has finite support in X^*. Note that $z_n := \{p_n \ : \ p \in z\} \to z$. Suppose there exists an infinite subsequence such that $z_n \notin Z^\circ$. Thus, there must be a subsequence $p_n, q_n \in z_n$ that are tied for each n. By Lemma 9, $u(q_n) = u(p_n)$ a.s. so $u(q) = u(p)$ a.s. By Lemma 9 again, this means p and q are tied, contradicting $z \in Z^\circ$. Thus, we can assume that $z_n \in Z^\circ$ so by Lemma 8, we have $\rho_z(p) \leq \mu(B(p,z))$.

Finally, let $z_0 \subset z$ be such that $z_0 \in Z^\circ$ so $\rho_{z_0}(p) \leq \mu(B(p,z_0))$. Suppose $\rho_{z_0}(p) < \mu(B(p,z_0))$ for some $p \in z_0$. Thus,

$$1 = \sum_{p \in z_0} \rho_{z_0}(p) < \sum_{p \in z_0} \mu(B(p,z_0)) \leq 1$$

where the last inequality follows from Lemma 9 and the fact that z_0 has no ties. Since this yields a contradiction, it must be that $\rho_{z_0}(p) = \mu(B(p,z_0))$ for all $p \in z_0$. Now, for any $p \in z$, we can find some $p_0 \in z_0$ tied with p. By Lemma A.2 from Lu (2016), we have

$$\rho_z(p) = \rho_{z_0}(p_0) = \mu(B(p_0,z)) = \mu(B(p,z))$$

as desired.

C.2 Extension of Dekel et al. (2001)

In this section, we extend the main theorem of DLR. We consider a binary relation $\succeq$ on $Z = \mathcal{K}(\Delta X)$.[40] The methodology by which we extend DLR parallels the way in which we extended GP. The one technical difference is that there is no need to deal with ties, which

[40] While DLR formally considers all non-empty subsets of ΔX, it is without loss to focus on those that are compact.

simplifies the DLR extension.

Condition 2.1. $\succeq$ is a preference relation

Condition 2.2 (Flexibility)**.** $z \subset y$ implies $z \preceq y$

Condition 2.3 (Independence)**.** $z \succeq y$ implies $\alpha z + (1-\alpha)\, w \succeq \alpha y + (1-\alpha)\, w$

Condition 2.4 (Continuity)**.** $\succeq$ *is continuous*

Condition 2.5 (Best-Worst)**.** $\bar{x} \succeq \{x, \bar{x}\}$ *and* $x \succeq \{x, \underline{x}\}$ *for all* $x \in X$

Condition 2.6 (L-continuity)**.** There exists $N > 0$ such that for $\alpha \in [0,1]$, $|x_1 - x_2| \leq \frac{\alpha}{N}$ implies

$$(1-\alpha)\, \delta_{x_1} + \alpha \delta_{\bar{x}} \succeq \left\{ (1-\alpha)\, \delta_{x_1} + \alpha \delta_{\bar{x}},\ \ (1-\alpha)\, \delta_{x_2} + \alpha \delta_{\underline{x}} \right\}$$

We will now prove the following extension of DLR to an infinite-dimensional setting.

Theorem 6 (DLR extension)**.** $\succeq$ *satisfies C2 if and only if there exists a probability measure* ν *on* U_N *such that* $\succeq$ *is represented by the function* $v : Z \to \mathbb{R}$ *where*

$$v(z) = \int_{U_N} \sup_{p \in z} u(p)\, d\nu$$

The necessity of the axioms is straightforward. C2.1-C2.4 follow from the same arguments as in DLR. It is easy to see C2.5 from the representation while C2.6 follows from Lemma 5 above.

We now show sufficiency and suppose $\succeq$ satisfies C2. Since X is separable, let $X^* \subset X$ be a countable dense subset of X and without loss of generality, assume $\underline{x}, \bar{x} \in X^*$.

Lemma 10. *There exists a probability measure* ν *on* U_N *such that for all finite* $W \subset X^*$, *the function* $v : Z \to \mathbb{R}$ *where*

$$v(z) = \int_{U_N} \sup_{p \in z} u(p)\, d\nu$$

represents $\succeq$ *on* $\mathcal{K}(\Delta W)$.

Proof. From DLR, C2.1-C2.4 imply that for each finite $W \subset X^*$ where $\underline{x}, \bar{x} \in W$, there exists a probability measure μ_W on $\mathbb{R}^W$ such that

$$\int_{\mathbb{R}^W} \sup_{p \in z} u(p)\, d\mu_W$$

represents $\succeq$ on $\mathcal{K}(\Delta W)$. By C2.5, we have $u(\underline{x}) \leq u(x) \leq u(\bar{x})$ μ_W-a.s. for all $x \in X$. Thus, we can assume that $u(\bar{x}) = 1$ and $u(\underline{x}) = 0$ without loss of generality. With this normalization of utilities, the DLR representation is unique so all these μ_W are consistent. By Kolmogorov's extension, there exists a measure μ on $\mathbb{R}^{X^*}$ such that

$$\int_{\mathbb{R}^{X^*}} \sup_{p \in z} u(p)\, d\mu$$

represents $\succeq$ on $\mathcal{K}(\Delta W)$ for all finite $W \subset X^*$.

By C2.6, there exists $N > 0$ such that for all $\alpha \in [0,1]$ and $x_1, x_2 \in X^*$, μ-a.s. $|x_1 - x_2| \leq \alpha/N$ implies $\alpha + (1-\alpha) u(x_1) \geq (1-\alpha) u(x_2)$. Since X^* is countable dense subset of X, $[0,1]$ is separable, and μ is countably-additive, by the same argument as in Step 2 of Lemma 6, there exists $N > 0$ such that μ-a.s. for all $\alpha \in [0,1]$ and $x_1, x_2 \in X^*$, $|x_1 - x_2| \leq \alpha/N$ implies $\alpha + (1-\alpha) u(x_1) \geq (1-\alpha) u(x_2)$. Applying Lemma 5 yields μ-a.s. u is Lipschitz continuous with bound N.

By the Lipschitz version of the Tietze extension theorem, we can extend μ on $\mathbb{R}^{X^*}$ on to a probability measure ν on $L_N(X)$. Moreover, $0 = u(\underline{x}) \leq u(x) \leq u(\bar{x}) = 1$ ν-a.s. for all $x \in X^*$. Since X^* is countable dense in X and μ is countably additive, by the same argument as in Step 3 of Lemma 6, this means that ν-a.s. $0 = u(\underline{x}) \leq u(x) \leq u(\bar{x}) = 1$ for all $x \in X$ so $\nu(U_N) = 1$. Thus,

$$v(z) := \int_{U_N} \sup_{p \in z} u(p)\, d\nu$$

represents $\succeq$ on $\mathcal{K}(\Delta W)$ for all finite $W \subset X^*$. ■

We now complete the proof of Theorem 6. First we show that v is continuous. Note that $z_n \to z$ implies $\sup_{p \in z_n} u(p) \to \sup_{p \in z} u(p)$ for all $u \in U_N$. By dominated convergence, $v(z_n) \to v(z)$ so v is continuous.

Now, consider a generic $z \in Z$. Notice that $z \sim \alpha \delta_{\bar{x}} + (1-\alpha) \delta_{\underline{x}}$ where $\alpha = v(z)$. For any $p \in \Delta X$, by Lemma 7, we can find p_n with finite support in X^* such that $p_n \to p$. Let $z_n := \{p_n : p \in z\}$ so $z_n \to z$ and $z_n \in \mathcal{K}(\Delta W_n)$ for some finite $W_n \subset X^*$. Define $\alpha_n := v(z_n) \in [0,1]$ and without loss of generality, assume $\alpha_n \to \alpha^*$. Since v is continuous, $\alpha = v(z) = \alpha^*$. Note that by C2.4, $\underline{x} \preceq z_n \preceq \bar{x}$ for all z_n implies $\underline{x} \preceq z \preceq \bar{x}$. Now, suppose $z \succ \alpha \delta_{\bar{x}} + (1-\alpha) \delta_{\underline{x}}$ so we can find some $\beta > \alpha$ such that $z \succ \beta \delta_{\bar{x}} + (1-\beta) \delta_{\underline{x}}$. Since

$\alpha_n \to \alpha < \beta$, this means that for large enough n,

$$\beta\delta_{\bar{x}} + (1-\beta)\,\delta_{\underline{x}} \succ \alpha_n\delta_{\bar{x}} + (1-\alpha_n)\,\delta_{\underline{x}} \sim z_n$$

where the indifference follows from the representation. By C2.4, we have $\beta\delta_{\bar{x}}+(1-\beta)\,\delta_{\underline{x}} \succeq z$ yielding a contradiction. The case $z \prec \alpha\delta_{\bar{x}}+(1-\alpha)\,\delta_{\underline{x}}$ is symmetric so $z \sim \alpha\delta_{\bar{x}}+(1-\alpha)\,\delta_{\underline{x}}$. Finally, to complete the proof, note that $z \succeq y$ if and only if

$$v(z)\,\delta_{\bar{x}} + (1-v(z))\,\delta_{\underline{x}} \succeq v(y)\,\delta_{\bar{x}} + (1-v(y))\,\delta_{\underline{x}}$$

if and only if $v(z) \geq v(y)$. Thus, v represents $\succeq$ on Z.

Notice that the arguments in Lemma 10 corresponds exactly to those of Lemma 6 in the previous section. The remaining arguments are significantly simpler than those in Lemma 7–9 as there is no need deal with ties. Other than this technical difference, the methodology for extending DLR is identical to that for extending GP.

D Proof of Theorem 4 (Representation)

D.1 Sufficiency of Axioms

We first prove the sufficiency of the axioms. Note that Axiom 1 corresponds exactly to C1 so by Theorem 5, there exists a regular probability measure μ on U_N such that for any finite $z \in Z$,

$$\rho_z(p) = \mu\{u \in U_N \;:\; u(p) \geq u(q) \text{ for all } q \in z\}$$

Choose any $z_1, z_2 \in Z$. Let $z = \{p_1, p_2\}$ and $y = \{q_1, q_2\}$ where $p_i = \delta_{(c,z_i)}$ and $q_i = \frac{1}{2}p_i \oplus \frac{1}{2}\delta_{(d,w)}$ for $i \in \{1,2\}$. Applying Axiom 2 for $\alpha = \frac{1}{2}$, we have

$$\mu\{u(p_1) \geq u(p_2)\} = \rho_z(p_1) = \rho_{\frac{1}{2}z+\frac{1}{2}y}\left(\frac{1}{2}p_1 + \frac{1}{2}q_1\right) = \mu\{u(p_1) \geq u(p_2) \text{ and } u(q_1) \geq u(q_2)\}$$

Applying Axiom 2 for $\alpha = 0$ and $\alpha = \frac{1}{2}$, we have

$$\mu\{u(q_1) \geq u(q_2)\} = \rho_y(q_1) = \rho_{\frac{1}{2}z+\frac{1}{2}y}\left(\frac{1}{2}p_1 + \frac{1}{2}q_1\right) = \mu\{u(p_1) \geq u(p_2) \text{ and } u(q_1) \geq u(q_2)\}.$$

Thus, we have

$$0 = \mu\{u(p_1) \geq u(p_2) \text{ and } u(q_1) < u(q_2)\} = \mu\{u(p_1) < u(p_2) \text{ and } u(q_1) \geq u(q_2)\}$$

so $u(p_1) \geq u(p_2)$ if and only if $u(q_1) \geq u(q_2)$ μ-a.s.

For all $c, d \in M$, $z_1, z_2, w \in Z$, and $\lambda \in [0,1]$, we thus have μ-a.s.

$$\begin{aligned} & u(c, z_1) \geq u(c, z_2) \\ \Leftrightarrow \quad & u(\lambda c + (1-\lambda) d, \lambda z_1 + (1-\lambda) w) \geq u(\lambda c + (1-\lambda) d, \lambda z_2 + (1-\lambda) w) \end{aligned} \tag{15}$$

Since Z, M and $[0,1]$ are all separable and any $u \in U_n$ is continuous, by the countable additivity of μ, we have that the above holds μ-a.s. for all $c, d \in M$ and $z_1, z_2, w \in Z$ and $\lambda \in [0,1]$. In particular, this holds for the special case when $d = c$. Moreover, we also have that μ-a.s. that for all $c, c' \in M$ and $z_1, z_2, w \in Z$,

$$\begin{aligned} & u(c, z_1) \geq u(c, z_2) \\ \Leftrightarrow \quad & u\left(\frac{1}{2}c + \frac{1}{2}c', \frac{1}{2}z_1 + \frac{1}{2}w\right) \geq u\left(\frac{1}{2}c + \frac{1}{2}c', \frac{1}{2}z_2 + \frac{1}{2}w\right) \\ \Leftrightarrow \quad & u(c', z_1) \geq u(c', z_2) \end{aligned} \tag{16}$$

We can now define a preference relation $\succeq_u$ on Z for each $u \in U_N$ such that $z \succeq_u y$ if and only if $u(c, z) \geq u(c, y)$. Note that this is well-defined as it does not depend on $c \in M$ by (16) above.

We now show that $\succeq_u$ satisfies C2 μ-a.s. Note that C2.1 is trivial and C2.3 follows from (15) above. To see C2.2, note that from Axiom 3, for any z, y, if $z \supset y$, then

$$1 = \rho(z, y) = \rho\left(\delta_{(c,z)}, \delta_{(c,y)}\right) = \mu\{u(c, z) \geq u(c, y)\}.$$

Since μ is countably additive, $u \in U_N$ is continuous and Z is separable, C2.2 follows. Note that C2.4 follows from the continuity of $u \in U_N$. Finally, by applying Axiom 3 to Axioms 1.5 and 1.6, we obtain C2.5 and C2.6 respectively by the same argument as before.

Applying Theorem 6, this means that $\succeq_u$ is represented by

$$v_u(z) := \int_{U_N} \sup_{p \in z} \tilde{u}(p) \, d\nu_u$$

where ν_u is a probability measure on U_N. Since for every $c \in M$, $u(c, \cdot)$ and v_u represent the same preference, we can write

$$u(c, z) = \phi_u(c, v_u(z))$$

where $\phi_u : M \times [0,1] \to [0,1]$ is strictly increasing in the second argument. Note that this is well-defined as it does not depend on $c \in M$ by (16).

The following result shows that μ is the invariant measure of the transition kernel ν_u.

Lemma 11. *For any measurable set $B \subset U_N$,*

$$\mu(B) = \int_{U_N} \nu_u(B)\, d\mu$$

Proof. Define the measure μ^* on U_N such that for every measurable $B \subset U_N$, $\mu^*(B) = \int_{U_N} \nu_u(B)\, d\mu$. We will show that $\mu^* = \mu$. Consider finite $z \in Z$ and note that $\rho(z, p_\alpha) = \mu\left\{\sup_{p\in z} u(p) \geq \alpha\right\}$. Thus,

$$\int_{[0,1]} \rho(z, p_\alpha)\, d\alpha = \int_{U_N} \sup_{p\in z} u(p)\, d\mu \tag{17}$$

On the other hand, $\rho((c, z), (c, p_\alpha)) = \mu\{\phi_u(c, v_u(z)) \geq \phi_u(c, v_u(p_\alpha))\} = \mu\{v_u(z) \geq \alpha\}$, so

$$\int_{[0,1]} \rho\left(\delta_{(c,z)}, \delta_{(c,p_\alpha)}\right) d\alpha = \int_{U_N} v_u(z)\, d\mu. \tag{18}$$

Applying Axiom 4 to the left-hand sides of (17) and (18), we thus have

$$\int_{U_N} \sup_{p\in z} u(p)\, d\mu = \int_{U_N} v_u(z)\, d\mu = \int_{U_N} \left(\int_{U_N} \sup_{p\in z} \tilde{u}(p)\, d\nu_u\right) d\mu = \int_{U_N} \sup_{p\in z} u(p)\, d\mu^*.$$

Letting $z = \{p, p_\alpha\}$, we have $\int_{U_N} \max\{u(p), \alpha\}\, d\mu = \int_{U_N} \max\{u(p), \alpha\}\, d\mu^*$. By Theorem 1.57 of Müller and Stoyan (2002), for any increasing convex function ϕ,

$$\int_{U_N} \phi(u(p))\, d\mu = \int_{U_N} \phi(u(p))\, d\mu^*.$$

Since U_N is compact by Lemma 3, $\mu = \mu^*$ by Lemma 4. ■

Let U^1 be the set of $u \in U_N$ such that there exists ϕ_u and ν_u where

$$u(c, z) = \phi_u\left(c, \int_{U_N} \sup_{p\in z} \tilde{u}(p)\, d\nu_u\right)$$

so $\mu(U^1) = 1$. Recursively define $U^{n+1} := \{u \in U^n \;:\; \nu_u(U^n) = 1\}$ and let $U^* := \bigcap_{n=1}^{\infty} U^n$.

We show that $\mu(U^*) = 1$. First, we show that $\mu(U^n) = 1$ for all n by induction. Suppose $\mu(U^n) = 1$ so by Lemma 11,

$$1 = \mu(U^n) = \int_{U_N} \nu_u(U^n)\, d\mu.$$

Thus, $\nu_u(U^n) = 1$ μ-a.s. so $\mu(U^{n+1}) = 1$. Since $\mu(U^1) = 1$, this means that $\mu(U^n) = 1$ for all n. Since $U^{n+1} \subset U^n$, by Proposition 3.6 of Çınlar (2011), $\mu(U^*) = \lim_n \mu(U^n) = 1$.

By Lemma 11 again, we have

$$1 = \mu(U^*) = \int_{U_N} \nu_u(U^*)\, d\mu$$

so $\nu_u(U^*) = 1$ μ-a.s. This means that μ-a.s. that

$$u(c,z) = \phi_u\left(c, \int_{U^*} \sup_{p\in z} \tilde{u}(p)\, d\nu_u\right)$$

and $\rho_z(p) = \mu(B(p,z))$ for any finite $z \in Z$ where $B(p,z) := \{u \in U^* \ : \ u(p) \geq u(q) \text{ for all } q \in z\}$. We can now define a Markov process $[P]$ on $S := U^*$ with invariant distribution μ and transition kernel $P_s := \nu_u$ for all $s = u \in U^*$.

We now prove that the Markov process $[P]$ satisfies Doeblin continuity (i.e., there exists some $\delta > 0$ such that μ-a.s. $\nu_u(A) \geq \delta\mu(A)$ for all measurable A). For this purpose, we will show a lemma on the density of the set of support functions For any $z \in Z$, define the support function $\sigma_z : U_N \to \mathbb{R}$ by $\sigma_z(u) := \sup_{p\in z} u(p)$. Define the sets $\Sigma := \{r(\sigma_z - \sigma_y) \ : \ r > 0 \text{ and } z, y \in Z\}$ and $\Sigma^f := \left\{r(\sigma_z - \sigma_y) \ : \ r > 0 \text{ and } z, y \in Z^f\right\}$, where σ_z is the support function of $z \in Z$.

Lemma 12. Σ^f *is dense in* $C(U_N)$.

Proof. Note that for any $z \in Z$, we can find $z_k \in Z_f$ such that $z_k \to z$ (see Lemma 0 of Gul and Pesendorfer (2001)). Thus, $\sigma_{z_k} \to \sigma_z$ by Theorem 7.52 of AB. So Σ^f is dense in Σ. To show the lemma, therefore, it suffices to show that Σ is dense in $C(U_N)$.

First, we show that Σ is a linear subspace of $C(U_N)$. Consider the singleton menu $z = \delta_{\underline{x}}$ and note that by definition, $\sigma_z(u) = u(\underline{x}) = 0$ for all $u \in U_N$. Thus, $0 \in \Sigma$. Next, note that if $r(\sigma_z - \sigma_y) \in \Sigma$, then clearly $\lambda r(\sigma_z - \sigma_y) \in \Sigma$ for all $\lambda \in \mathbb{R}$. Finally, suppose $r_1(\sigma_{z_1} - \sigma_{y_1}), r_2(\sigma_{z_2} - \sigma_{y_2}) \in \Sigma$. Since $r_1, r_2 > 0$, define $\lambda := \frac{r_1}{r_1+r_2}$ so we have

$$r_1(\sigma_{z_1} - \sigma_{y_1}) + r_2(\sigma_{z_2} - \sigma_{y_2}) = (r_1 + r_2)\left[(\lambda\sigma_{z_1} + (1-\lambda)\sigma_{z_2}) - (\lambda\sigma_{y_1} + (1-\lambda)\sigma_{y_2})\right]$$

Since $\lambda\sigma_{z_1} + (1-\lambda)\sigma_{z_2} = \sigma_{\lambda z_1 + (1-\lambda)z_2} \in \Sigma$ (see Lemma 7.54 of AB), we have $r_1(\sigma_{z_1} - \sigma_{y_1}) + r_2(\sigma_{z_2} - \sigma_{y_2}) \in \Sigma$. This shows that Σ is a linear subspace of $C(U_N)$.

We now prove that Σ is dense in $C(U_N)$ using the Stone-Weierstrass Theorem. Note that for $z = \delta_{\overline{x}}$, $\sigma_z(u) = u(\overline{x}) = 1$ for all $u \in U_N$ so Σ includes constants. That Σ is a vector lattice follows from the same arguments as in Lemma 11 of DLR. Finally, we show that Σ separates $C(U_N)$. Choose any $u, v \in U_N$ such that $u \neq v$. Thus, there exists $x \in X$ such that $u(x) \neq v(x)$. If we let $z = \delta_x$, then

$$\sigma_z(u) = u(x) \neq v(x) = \sigma_z(v)$$

Thus, Σ separates $C(U_N)$. Since U_N is compact by Lemma 3, the Stone-Weierstrass Theorem (AB Theorem 9.12) shows that Σ is dense in $C(U_N)$. ■

Consider any $h \in C(U_N)$ such that $h \geq 0$. By Lemma 12, we can find $h_k \in Z_f$ such that $h_k \to h$. Define $g_k = \max\{h_k, 0\}$ and note that $g_k \to h$ as $h \geq 0$. Moreover, for $h_k = r(\sigma_z - \sigma_y)$ where $z, y \in Z^f$, we have

$$g_k = r \max\{\sigma_z - \sigma_y, 0\} = r(\sigma_{z\cup y} - \sigma_y) \in \Sigma^f$$

By Axiom 5, there exists some $\varepsilon > 0$ such that μ-a.s.

$$\begin{aligned} & \int_{U^*} \sigma_{(1-\varepsilon)(z\cup y)+\varepsilon p_{\varepsilon\bar{y}}} d\nu_u \geq \int_{U^*} \sigma_{(1-\varepsilon)y+\varepsilon p_{\varepsilon\overline{(z\cup y)}}} d\nu_u \\ \Leftrightarrow \quad & (1-\varepsilon)\int_{U^*} \sigma_{z\cup y} d\nu_u + \varepsilon\bar{y} \geq (1-\varepsilon)\int_{U^*} \sigma_y d\nu_u + \varepsilon\overline{(z\cup y)} \\ \Leftrightarrow \quad & \int_{U^*} (\sigma_{z\cup y} - \sigma_y) d\nu_u \geq \frac{\varepsilon}{1-\varepsilon}\left(\overline{(z\cup y)} - \bar{y}\right) = \delta \int_{U^*} (\sigma_{z\cup y} - \sigma_y) d\mu, \end{aligned}$$

where $\delta := \frac{\varepsilon}{1-\varepsilon}$. Thus, μ-a.s.$\int_{U^*} g_k d\nu_u \geq \delta \int_{U^*} g_k d\mu$. Since $g_k \to h$, this implies that μ-a.s. $\int_{U^*} h d\nu_u \geq \delta \int_{U^*} h d\mu$ by the dominated convergence theorem.

Since U_N is compact, $C(U_N)$ is separable by Lemma 3.99 of AB. Thus, by the countably additivity of μ, μ-a.s.

$$\int_{U^*} h d\nu_u \geq \delta \int_{U^*} h d\mu \tag{19}$$

for all nonnegative $h \in C(U_N)$. Now, by the regularity of ν_u and Urysohn's lemma (Theorem 4.15 of Folland (2013)), for any measurable $A \subset U^*$, there are nonnegative $h_k \in C(U_N)$ such that $h_k \to 1_A$ ν_u-a.s. Thus, by the dominated convergence theorem, μ-.a.s.

$$P_s(A) = \nu_u(A) = \int_{U^*} \lim_k h_k d\nu_u = \lim_k \int_{U^*} h_k d\nu_u \geq \lim_k \delta \int_{U^*} h_k d\mu = \delta \int_{U^*} \lim_k h_k d\mu = \delta\mu(A),$$

where the inequality is by (19).

Since this implies Doeblin's condition, the Markov process $[P]$ is uniformly ergodic (see

Theorem 16.2.3 of Meyn and Tweedie (2012)). By the Ergodic theorem, μ-a.s.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq 0}^{n}1_{B(p,z)}\left(s_k\right)=\mu\left(B\left(p,z\right)\right)=\rho\left(p,z\right)$$

for all $z\in Z^f$ as desired. This concludes the sufficiency proof.

D.2 Necessity of Axioms

We now show necessity of the axioms. Note that by Lemma 1, we can consider the ergodic utility process $u_t=u_{s_t}$ with stationary distribution μ. For any $z\in Z^f$, define

$$B\left(p,z\right):=\left\{u\in U_N\ :\ u\left(p\right)\geq u\left(q\right)\text{ for all }q\in z\right\}$$

By the Ergodic theorem, we have for every $z\in Z^f$,

$$\rho\left(p,z\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq 0}^{n}1_{B(p,z)}\left(u_k\right)=\mu\left(B\left(p,z\right)\right).$$

Axiom 1 then follows immediately from Theorem 5.

For Axiom 2, let $p\in z\in Z_c^f$, $y=\lambda z\oplus\left(1-\lambda\right)\delta_{(c',z')}$ and $q=\lambda p\oplus\left(1-\lambda\right)\delta_{(c',z')}\in y$ where $c,c'\in M$, $z'\in Z$ and $\lambda>0$. Note that for $p=\delta_{(c,w)}$, $u\left(p\right)\geq u\left(p'\right)$ for all $p'=\delta_{(c,w')}\in z$ if and only if $v_u\left(w\right)\geq v_u\left(w'\right)$ for all w' where

$$v_u\left(w\right):=\int_{U_N}\sup_{p\in w}\tilde{u}\left(p\right)d\nu_u$$

and ν_u is the transition kernel corresponding to the ergodic utility process. On the other hand, for all $p'\in z$ and all $q'=\lambda p'\oplus\left(1-\lambda\right)\delta_{(c',z')}\in y$,

$$\begin{aligned}u\left(q\right)\geq u\left(q'\right)&\Leftrightarrow u\left(\lambda c+\left(1-\lambda\right)c',\lambda w+\left(1-\lambda\right)z'\right)\geq u\left(\lambda c+\left(1-\lambda\right)c',\lambda w'+\left(1-\lambda\right)z'\right)\\&\Leftrightarrow v_u\left(\lambda w+\left(1-\lambda\right)z'\right)\geq v_u\left(\lambda w'+\left(1-\lambda\right)z'\right)\\&\Leftrightarrow v_u\left(w\right)\geq v_u\left(w'\right)\end{aligned}$$

for all w' as $\lambda>0$. Thus, $u\left(p\right)\geq u\left(p'\right)$ for all $p'\in z$ iff $u\left(q\right)\geq u\left(q'\right)$ for all $q'\in y$. This means that

$$\begin{aligned}\rho_z(p) &= \mu\{u(p) \geq u(p') \text{ for all } p' \in z\} \\ &= \mu\{\alpha u(p) + (1-\alpha)u(q) \geq \alpha u(p') + (1-\alpha)u(q') \text{ for all } p' \in z, q' \in y\} \\ &= \mu\{u(\alpha p + (1-\alpha)q) \geq u(\alpha p' + (1-\alpha)q') \text{ for all } p' \in z, q' \in y\} \\ &= \rho_{\alpha z + (1-\alpha)y}(\alpha p + (1-\alpha)q)\end{aligned}$$

as desired.

For Axiom 3, suppose $\rho(z,y) = 1$. Let

$$B := \left\{u \in U_N \ : \ \max_{p \in z} u(p) \geq \max_{q \in y} u(q)\right\}$$

so $\mu(B) = 1$. Since μ is the stationary distribution, $1 = \int_{U_N} \nu_u(B)\, d\mu$ so $\nu_u(B) = 1$ μ-a.s. This implies that $v_u(z) \geq v_u(y)$ μ-a.s. so $\rho\left(\delta_{(c,z)}, \delta_{(c,y)}\right) = 1$ as desired.

For Axiom 4, note that by the same arguments as in Lemma 11,

$$\begin{aligned}\int_{[0,1]} \rho(z, p_\alpha)\, d\alpha &= \int_{U_N} \sup_{p \in z} u(p)\, d\mu, \\ \int_{[0,1]} \rho\left(\delta_{(c,z)}, \delta_{(c,p_\alpha)}\right) d\alpha &= \int_{U_N} v_u(z)\, d\mu = \int_{U_N} \left(\int_{U_N} \sup_{p \in z} \tilde{u}(p)\, d\nu_u\right) d\mu.\end{aligned}$$

The result follows from the fact that μ is the stationary distribution.

Finally, for Axiom 5, suppose $y \subset z$ so

$$\bar{y} = \int_{U_N} \sup_{p \in y} u(p)\, d\mu \leq \int_{U_N} \sup_{p \in z} u(p)\, d\mu = \bar{z}$$

From Lemma 1, we know there exists some δ such that $\nu_u(B) \geq \delta\mu(B)$ for all measurable B so $\int_{U_N} \varphi d\nu_u \geq \int_{U_N} \varphi d\mu$ for all positive measurable functions φ. Let $\varepsilon := \frac{\delta}{1+\delta}$ so $\delta = \frac{\varepsilon}{1-\varepsilon}$. We thus have

$$\begin{aligned}v_u((1-\varepsilon)z + \varepsilon p_{\bar{y}}) - v_u((1-\varepsilon)y + \varepsilon p_{\bar{z}}) &= (1-\varepsilon)(v_u(z) - v_u(y)) + \varepsilon(v_u(p_{\bar{y}}) - v_u(p_{\bar{z}})) \\ &= (1-\varepsilon)\int_{U_N} \left(\sup_{p \in z} \tilde{u}(p) - \sup_{p \in y} \tilde{u}(p)\right) d\nu_u + \varepsilon(\bar{y} - \bar{z}) \\ &\geq (1-\varepsilon)\frac{\varepsilon}{1-\varepsilon}(\bar{z} - \bar{y}) - \varepsilon(\bar{z} - \bar{y}) = 0\end{aligned}$$

Thus, $\rho\left(\delta_{(c,(1-\varepsilon)z+\varepsilon p_{\bar{y}})}, \delta_{(c,(1-\varepsilon)y+\varepsilon p_{\bar{z}})}\right) = 1$ as desired. This concludes the proof.

E Proofs for Section 4

E.1 Proof of Theorem 2

Let μ denote the stationarity distribution for the utility process so by the Ergodic theorem, we have $\rho(p,q) = \mu\{u \in U \ : \ u(p) \geq u(q)\}$ for all $\{p,r\} \in Z^*$. That separability implies 1-ICM and standard implies both 1-ICM and 2-ICM are straightforward. We now show that 1-ICM implies the utility process is separable. Fix $z, y \in Z$. Consider $\{p,r\} \in Z^*$ such that

$$p = \frac{1}{4}\delta_{(0,z)} + \frac{1}{4}\delta_{(0,y)} + \frac{1}{4}\delta_{(c,z)} + \frac{1}{4}\delta_{(c,y)}, r = \frac{1}{2}\delta_{(0,z)} + \frac{1}{2}\delta_{(c,y)}$$

Note that $p^M = r^M = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_c$ and $p^Z = r^Z = \frac{1}{2}\delta_z + \frac{1}{2}\delta_y$. Let $\{q\} \in Z^*$ denote the singleton 1-period menu such that $q^M = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_c = p^M$ and $q^Z = \frac{1}{2}\delta_z + \frac{1}{2}\delta_y$. By 1-ICM, we thus have $1 = \rho_{\{q\}}(q) = \rho(p,r) = \rho(r,p)$. Thus a.s. $\frac{1}{4}u(0,z) + \frac{1}{4}u(0,y) + \frac{1}{4}u(c,z) + \frac{1}{4}u(c,y) = \frac{1}{2}u(0,z) + \frac{1}{2}u(c,y)$. That is, $u(0,y) + u(c,z) = u(0,z) + u(c,y)$.

Let $y = \underline{x}^t \to \underline{x}$ so $u(0,\underline{x}^t) \to u(\underline{x}) = 0$. If we let

$$v_s(z) := \mathbb{E}_s\left[\sup_{p \in z} u_{s'}(p)\right]$$

then $v_s(\underline{x}^t) \to 0$. Thus, we have a.s. $\phi_s(c, v_s(z)) = \phi_s(0, v_s(z)) + \phi_s(c,0)$. Letting $w_s(c) := \phi_s(c,0)$ and $\beta_s(v) = \phi_s(0,v)$, we have a.s. $\phi_s(c,v) = w_s(c) + \beta_s(v)$ as desired.

We now show that imposing 2-ICM in addition to 1-ICM implies the utility process must be standard. By 1-ICM, we have a.s. $\phi_s(c,v) = w_s(c) + \beta_s(v)$ from above. Consider $\{p,r\} \in Z^*$ such that

$$p = \frac{1}{4}\delta_{(0,p_0)} + \frac{1}{4}\delta_{\left(0,\delta_{(0,y)}\right)} + \frac{1}{4}\delta_{(0,q_0)} + \frac{1}{4}\delta_{(0,r_0)}, r = \frac{1}{2}\delta_{(0,p_0)} + \frac{1}{2}\delta_{(0,r_0)}$$

where $p_0 = b\delta_{(0,z)} + (1-b)\delta_{(0,y)}$, $q_0 = ab\delta_{(m,z)} + a(1-b)\delta_{(m,y)} + (1-a)b\delta_{(0,z)} + (1-a)(1-b)\delta_{(0,y)}$, and $r_0 = a\delta_{(m,y)} + (1-a)\delta_{(0,y)}$ for $a, b \in [0,1]$. Note that the distribution of 2-period consumptions of p and r are $\frac{1}{2}\delta_{(0,\delta_0)} + \frac{1}{2}\delta_{(0,a\delta_m+(1-a)\delta_0)}$ while their menu distributions are $\frac{1}{2}\delta_{(b\delta_z+(1-b)\delta_y)} + \frac{1}{2}\delta_{\delta_y}$. Thus, by 2-ICM, $1 = \rho(p,r) = \rho(r,p)$. Hence, we have a.s. $\frac{1}{4}u(0,p_0) + \frac{1}{4}u\left(0,\delta_{(0,y)}\right) + \frac{1}{4}u(0,q_0) + \frac{1}{4}u(0,r_0) = \frac{1}{2}u(0,p_0) + \frac{1}{2}u(0,r_0)$. That is, $u\left(0,\delta_{(0,y)}\right) + u(0,q_0) = u(0,p_0) + u(0,r_0)$. Thus, we have a.s.

$$\beta_s(\mathbb{E}_s[u_{s'}(0,y)]) + \beta_s(\mathbb{E}_s[u_{s'}(q_0)]) = \beta_s(\mathbb{E}_s[u_{s'}(p_0)]) + \beta_s(\mathbb{E}_s[u_{s'}(r_0)])$$

Let $y = \{\underline{x}^t\} \to \{\underline{x}\}$ and $z = \bar{x}^t \to \bar{x}$ so $u_s(0, \underline{x}^t) \to u_s(\underline{x}) = 0$ and $v_s(\bar{x}^t) \to 1$. By definition, $\beta_s(0) = \phi_s(0,0) = 0$ and $w_s(0) = \phi_s(0,0) = 0$. Thus we have a.s. $\beta_s(\mathbb{E}_s[aw_{s'}(m) + b\beta_{s'}(1)]) = \beta_s(\mathbb{E}_s[aw_{s'}(m)]) + \beta_s(\mathbb{E}_s[b\beta_{s'}(1)])$, or

$$\beta_s(a\mathbb{E}_s[w_{s'}(m)] + b\mathbb{E}_s[\beta_{s'}(1)]) = \beta_s(a\mathbb{E}_s[w_{s'}(m)]) + \beta_s(b\mathbb{E}_s[\beta_{s'}(1)]) \tag{20}$$

for all $a, b \in [0,1]$.

Let $\xi_s := \min\{\mathbb{E}_s[w_{s'}(m)], \mathbb{E}_s[\beta_{s'}(1)]\}$. Since

$$\mathbb{E}_s[w_{s'}(m)] + \mathbb{E}_s[\beta_{s'}(1)] = \mathbb{E}_s[w_{s'}(m) + \beta_{s'}(1)] = \mathbb{E}_s[\phi_s(m,1)] = 1,$$

$\xi_s > 0$. From equation (20), we have $\beta_s(x+y) = \beta_s(x) + \beta_s(y)$ for all $x, y \in [0, \xi_s]$. This is a Cauchy functional equation with bounded domain, and since β_s is continuous, we have a.s. $\beta_s(x) = \beta_s x$ for all $x \in [0, \xi_s]$ where β_s is a constant (see pg. 45 of Aczel (1966)). Now, for $v \in [0, 2\xi_s]$,

$$\beta_s(v) = \beta_s\left(\frac{v}{2} + \frac{v}{2}\right) = 2\beta_s\left(\frac{v}{2}\right) = \beta_s v.$$

By iteration, we have $\beta_s(v) = \beta_s v$ for all $v \in [0,1]$ as desired.

E.2 Definition of Repeated Independence (RI)

In the main part of the paper, we explained how to mix 1-period menus with a lottery $r \in \Delta M$. In this subsection, we formally define how to mix simple t-period menus. Fix some t-period menu $z \in Z^*$ that is also t-simple. For any lottery p that yields z in $t' \leq t$ periods, we will define $r_{t'}(p)$ as the t'-times repeated mixture between p and $r \in \Delta M$. This is constructed as follows. First, define $r_1(\cdot)$ exactly as in the 1-period case where for every p,

$$r_1(p) = (\alpha p_1 + (1-\alpha) r \ ; \ \alpha z \otimes (1-\alpha) r)$$

Now for $1 < t' \leq t$, we will recursively define $r_{t'}(\cdot)$. First, given $r_{t'-1}(\cdot)$ and some p, define two lotteries $\hat{p}$ and $\hat{r}$ such that

$$\begin{aligned}\hat{p}(A \times B) &:= p\left(A \times r_{t'-1}^{-1}(B)\right)\\ \hat{r}(A \times B) &:= r(A)\, p\left(\Delta M \times r_{t'-1}^{-1}(B)\right)\end{aligned}$$

for all measurable A and B. Note that $\hat{p}$ and $\hat{r}$ are the continuation lotteries where all future lotteries are also mixed with r. Next, define

$$r_{t'}(p) = \alpha\hat{p} + (1-\alpha)\hat{r}$$

Finally, set $\alpha p \otimes (1-\alpha) r = r_t(p)$ and

$$\alpha z \otimes (1-\alpha) r := \{\alpha p \otimes (1-\alpha) r \; : \; p \in z\}$$

E.3 Proof of Theorem 3

Note that by Theorem 2, all we need to show is that the utility process is standard if and only if ρ satisfies IRU and RI. Since a standard utility process trivially satisfies IRU and RI, we will show the converse. By IRU, we have

$$\phi_s(c,v) = w_s(c) + \beta_s(c)\, v$$

Note that $0 = \phi_s(0,0) = w_s(0)$ and $1 = \phi_s(m,1) = w_s(m) + \beta_s(m)$. Consider a 2-period $z = \{p_0, q_0\} \in Z^*$ where $p_0 = \frac{1}{2}\delta_{(c_1,p)} + \frac{1}{2}\delta_{\left(c_2,\delta_{(c_2,z)}\right)}$,$q_0 = \frac{1}{2}\delta_{(c_1,q)} + \frac{1}{2}\delta_{(c_2,q)}$, $p = \lambda_1\delta_{(m,z)} + (1-\lambda_1)\,\delta_{(c_2,z)}$ and $q = \lambda_2\delta_{(m,z)} + (1-\lambda_2)\,\delta_{(c_2,z)}$ for $c_1, c_2 \in (0,m)$. Note that

$$\begin{aligned} u_s(p_0) &= w_s\left(\frac{1}{2}\delta_{c_1} + \frac{1}{2}\delta_{c_2}\right) + \frac{1}{2}\beta_s(c_1)\,\mathbb{E}_s[u_{s'}(p)] + \frac{1}{2}\beta_s(c_2)\,\mathbb{E}_s[u_{s'}(c_2,z)] \\ u_s(q_0) &= w_s\left(\frac{1}{2}\delta_{c_1} + \frac{1}{2}\delta_{c_2}\right) + \left(\frac{1}{2}\beta_s(c_1) + \frac{1}{2}\beta_s(c_2)\right)\mathbb{E}_s[u_{s'}(q)] \end{aligned}$$

To simplify notation, let $\beta_i := \beta_s(c_i)$ and $\tilde{u}_s := \mathbb{E}_s[u_{s'}]$. Now, $u_s(p_0) \geq u_s(q_0)$ if and only if

$$\begin{aligned} u_s(p_0) \geq u_s(q_0) &\Leftrightarrow \beta_1\tilde{u}_s(p) + \beta_2\tilde{u}_s(c_2,z) \geq (\beta_1+\beta_2)\,\tilde{u}_s(q) \\ &\Leftrightarrow \beta_1\left(\tilde{u}_s(p) - \tilde{u}_s(q)\right) \geq \beta_2\left(\tilde{u}_s(q) - \tilde{u}_s(c_2,z)\right) \\ &\Leftrightarrow \beta_1(\lambda_1-\lambda_2)\left(\tilde{u}_s(m,z) - \tilde{u}_s(c_2,z)\right) \geq \beta_2\lambda_2\left(\tilde{u}_s(m,z) - \tilde{u}_s(c_2,z)\right) \\ &\Leftrightarrow \beta_1\lambda_1 \geq (\beta_1+\beta_2)\,\lambda_2, \end{aligned}$$

where the last inequality follows from the fact that $\tilde{u}_s(m,z) \geq \tilde{u}_s(c_2,z)$ a.s. as $m \geq c_2$.

Let $r = \delta_{c_2}$ and consider the 2-period $z' = az \otimes (1-a)\,r \in Z^*$. Note that $z' =$

$\{ap_0 \otimes (1-a)\,r, aq_0 \otimes (1-a)\,r\}$ where

$$ap_0 \otimes (1-a)\,r = \frac{1}{2}\left(a\delta_{(c_1,p')} + (1-a)\,\delta_{(c_2,p')}\right) + \frac{1}{2}\delta_{\left(c_2,\delta_{(c_2,z')}\right)},$$
$$aq_0 \otimes (1-a)\,r = \frac{1}{2}\left(a\delta_{(c_1,q')} + (1-a)\,\delta_{(c_2,q')}\right) + \frac{1}{2}\delta_{(c_2,q')}$$

and $p' = a\lambda_1\delta_{(m,z')} + (1-a\lambda_1)\,\delta_{(c_2,z')}$ and $q' = a\lambda_2\delta_{(m,z')} + (1-a\lambda_2)\,\delta_{(c_2,z')}$. Note that

$$u_s\left(ap_0 \otimes (1-a)\,r\right) = w_s\left(\frac{a}{2}\delta_{c_1} + \left(1-\frac{a}{2}\right)\delta_{c_2}\right) + \frac{1}{2}\left(a\beta_1 + (1-a)\,\beta_2\right)\tilde{u}_s\left(p'\right) + \frac{1}{2}\beta_2\tilde{u}_s\left(c_2, z'\right)$$
$$u_s\left(aq_0 \otimes (1-a)\,r\right) = w_s\left(\frac{a}{2}\delta_{c_1} + \left(1-\frac{a}{2}\right)\delta_{c_2}\right) + \left(\frac{a}{2}\beta_1 + \left(1-\frac{a}{2}\right)\beta_2\right)\tilde{u}_s\left(q'\right)$$

To simplify notation, let $\beta_a := a\beta_1 + (1-a)\,\beta_2$ and recall $\tilde{u}_s = \mathbb{E}_s\left[u_{s'}\right]$. Now we have

$$\begin{aligned}
& u_s\left(ap_0 \otimes (1-a)\,r\right) \geq u_s\left(aq_0 \otimes (1-a)\,r\right) \\
\Leftrightarrow\ & \beta_a\tilde{u}_s\left(p'\right) + \beta_2\tilde{u}_s\left(c_2, z'\right) \geq \left(\beta_a + \beta_2\right)\tilde{u}_s\left(q'\right) \\
\Leftrightarrow\ & \beta_a\left(\tilde{u}_s\left(p'\right) - \tilde{u}_s\left(q'\right)\right) \geq \beta_2\left(\tilde{u}_s\left(q'\right) - \tilde{u}_s\left(c_2, z'\right)\right) \\
\Leftrightarrow\ & \beta_a a\left(\lambda_1 - \lambda_2\right)\left(\tilde{u}_s\left(m, z'\right) - \tilde{u}_s\left(c_2, z'\right)\right) \geq \beta_2 a\lambda_2\left(\left(\tilde{u}_s\left(m, z'\right) - \tilde{u}_s\left(c_2, z'\right)\right)\right) \\
\Leftrightarrow\ & \beta_a\lambda_1 \geq \left(\beta_a + \beta_2\right)\lambda_2
\end{aligned}$$

where the last inequality again follows from the fact that $\tilde{u}_s(m, z) \geq \tilde{u}_s(c_2, z)$ a.s.

By RI, we have

$$\mu\left\{\frac{\beta_a}{\beta_a + \beta_2} \geq \frac{\lambda_2}{\lambda_1}\right\} = \rho\left(ap_0 \otimes (1-a)\,r, aq_0 \otimes (1-a)\,r\right) = \rho\left(p_0, q_0\right) = \mu\left\{\frac{\beta_1}{\beta_1 + \beta_2} \geq \frac{\lambda_2}{\lambda_1}\right\}.$$

Since this is true for all $\lambda_1, \lambda_2 \in (0, 1)$, it must be that $\frac{\beta_1}{\beta_1+\beta_2}$ and $\frac{\beta_a}{\beta_a+\beta_2}$ have the same distribution for all $a > 0$. If we let $\xi := \frac{\beta_1}{\beta_2}$, then ξ has the same distribution as

$$\frac{\beta_a}{\beta_2} = a\xi + (1-a)$$

Equivalently, this implies that $\xi - 1$ has the same distribution as $a(\xi - 1)$ for all $a > 0$. Let κ be the infimum of the support of $\xi - 1$. Since $\frac{\beta_1}{\beta_2} \geq 0$, $\kappa \geq -1$. Since $\xi - 1$ and $a(\xi - 1)$ have the same distribution, it must be that $\kappa = 0$. Thus, a.s.

$$0 \leq \xi - 1 = \frac{\beta_1}{\beta_2} - 1$$

or $\beta_s(c_1) = \beta_1 \geq \beta_2 = \beta_s(c_2)$ a.s. Since this was for arbitrary $c_1, c_2 \in (0, m)$, it must be that

$\beta(c_1) = \beta(c_2)$ for all $c_1, c_2 \in (0, m)$. Continuity of β then yields β must be constant on M.

F Repeated Menus

F.1 Proof of Lemma 2

In this section, we formally define z^t and prove Lemma 2. In order to do so, we first formally define the space of menus following Gul and Pesendorfer (2004). First, define $Z_0 := \{0\}$ and

$$Z_{t+1} := \mathcal{K}(\Delta(M \times Z_t))$$

Also, let $X_{t+1} = \Delta(M \times Z_t)$. Recall that $r_{y,t}(z)$ is the menu that follows $z \in Z$ for t periods and then ends with $y \in Z$ for sure. First, we show that this is well-defined.

Lemma 13. *For any $y \in Z$, $r_{y,t} : Z \to Z$ is well-defined.*

Proof. We will show by induction that $r_{y,t} : Z \to Z$ is continuous. Clearly this is true for $r_{y,0} = y$. Now, suppose that $r_{y,t-1}$ is continuous so $p_{y,t} \in \Delta X$ is well-defined. We show that $p_{y,t}$ is continuous in $p \in \Delta X$. Consider $p^n \to p$ and let $u : X \to \mathbb{R}$ be continuous and bounded. Note that since $r_{y,t-1}$ is continuous,

$$\int_X u(c,z)\, dp^n_{y,t} = \int_X u(c, r_{y,t-1}(z))\, dp^n \to \int_X u(c, r_{y,t-1}(z))\, dp = \int_X u(c,z)\, dp_{y,t}$$

so $p^n_{y,t} \to p_{y,t}$ as desired. Lemma 1(i) from Gul and Pesendorfer (2004) ensures that $r_{y,t}$ is continuous. Thus, by induction, $r_{y,t}$ is well-defined. ■

We now extend this notation to menus that end in finite periods, i.e. menus in Z_t. In other words, we will inductively construct the menu $r_{y,t}(z)$ that replicates $z \in Z_i$ for $t \leq i$ periods and ends with $y \in Z_j$ for sure for some j. First, for any $y \in Z_j$, let $r_{y,0}(z) = y$ for any $z \in Z_i$. Given $r_{y,t-1}$, for any $p \in \Delta X_i$ and $t \leq i$, let $p_{y,t} \in \Delta X_{t+j}$ denote the lottery induced by $r_{y,t-1}$, that is, for all measurable $A \times B$,

$$p_{y,t}(A \times B) = p\left(A \times r^{-1}_{y,t-1}(B)\right)$$

Thus, $p_{y,t}$ is the lottery that follows p for $t \leq i$ periods and then yields y for sure. Finally, for any $z \in Z_i$, define

$$r_{y,t}(z) := \{p_{y,t} \ : \ p \in z\}$$

In other words, $r_{y,t}(z) \in Z_{t+j}$ is the menu that follows $z \in Z_i$ for $t \leq i$ periods and then ends with y for sure. Note that by the same argument as in Lemma 13, $r_{y,t}$ is also well-defined.

In the following, we define z^t and show that z^t is t-period. If we let $y = 0 \in Z_0$, then $r_{0,t}(z) \in Z_t$ is the t-truncated version of $z \in Z_i$ for $t \leq i$. Following Gul and Pesendorfer (2004), we can now define the space of menus as

$$Z := \left\{ z \in \prod_{t \in T} Z_t \;\middle|\; z_t = r_{0,t}(z_{t+1}) \right\},$$

where z_t denote t-th argument of z for any $t \in T$. We endow Z with the product topology. Theorem A1 of Gul and Pesendorfer (2004) shows that Z is homeomorphic to $\mathcal{K}(\Delta(M \times Z))$.

Given z, we now formally define z^t by constructing a menu $\tilde{z} \in Z$ as follows. First, for any $i \leq t$, let $\tilde{z}_i = z_i$. For $i > t$, set $\tilde{z}_i = r_{\tilde{z}_{i-t},t}(z_t)$ iteratively. Thus, $\tilde{z}$ follows z for $i \leq t$ periods and then replicates itself going forward. Thus

$$\tilde{z} := (z_1, z_2, \ldots, z_t, r_{\tilde{z}_1,t}(z_t), r_{\tilde{z}_2,t}(z_t), \ldots) = r_{\tilde{z},t}(z)$$

We abuse the notation here and the following; the second equation means $\tilde{z} \in Z$ corresponds to $r_{\tilde{z},t}(z) \in \mathcal{K}(\Delta(M \times Z))$ by the homeomorphism between Z and $\mathcal{K}(\Delta(M \times Z))$. Define

$$z^t = \tilde{z}.$$

We now show that z^t is t-period. We show that $r_{y,t}(Z) \subset R_t(y)$ by induction. First, note that for all $z \in Z$,

$$r_{y,1}(z) = \{p_{y,1} \;:\; p \in z\} \in \mathcal{K}(\Delta(M \times \{y\})) = R_1(y)$$

so $r_{y,1}(Z) \subset R_1(y)$. Assume the induction step that $r_{y,t-1}(Z) \subset R_{t-1}(y)$. Thus, for any $p \in \Delta(M \times Z)$,

$$p_{y,t}(M \times R_{t-1}(y)) \geq p_{y,t}(M \times r_{y,t-1}(Z)) = p(M \times Z) = 1$$

Thus, we have

$$r_{y,t}(z) = \{p_{y,t} \;:\; p \in z\} \in R_t(y)$$

so $r_{y,t}(Z) \subset R_t(y)$. This shows that

$$z^t = r_{z^t,t}(z) \in R_t\left(z^t\right)$$

so z^t is t-period, where the equality means the correspondence based on the homeomorphism. Finally, since $z_i^t = z_i$ for all $i \leq t$, $z^t \to z$ as $i \to \infty$ in the product topology. This concludes the proof.

F.2 Property of Repeated Menus

As mentioned in Section 2.1, there is always some minimal t^* for which z is t^*-period and, in fact, t^* is simply the first time z appears after the initial period. The following lemma implies the results.

Suppose that z is a repeated menu (i.e., $z \in R_t(z)$ for some t) and the menu z appears at some period t' before period t (i.e.$z \in R_{t-t'}(z)$). The following lemma shows that the menu is also t'-period.

Lemma 14. *If z is a repeated menu (i.e., $z \in R_t(z)$ for some t) and $z \in R_{t-t'}(z)$ for some $t' < t$, then z is t'-period.*

Proof. We first show that $R_t(z) \cap R_\tau(z) \neq \varnothing$ implies $R_{t-1}(z) \cap R_{\tau-1}(z) \neq \varnothing$. Suppose $y \in R_t(z) \cap R_\tau(z)$ and choose some $p \in y$. By definition, $p(M \times R_{t-1}(z)) = p(M \times R_{\tau-1}(z)) = 1$ so

$$p(M \times (R_{t-1}(z) \cap R_{\tau-1}(z))) = 1$$

Thus, $R_{t-1}(z) \cap R_{\tau-1}(z) \neq \varnothing$.

We now prove the lemma. Suppose z is t-period and $z \in R_{t-t'}(z)$. Thus, $z \in R_t(z) \cap R_{t-t'}(z)$. Applying the above argument repeatedly yields $R_{t'}(z) \cap R_0(z) \neq \varnothing$. Since $R_0(z) = \{z\}$, we have $z \in R_{t'}(z)$ as desired. □

G Stochastic Epstein-Zin and RI

Under stochastic Epstein-Zin, non-standard intertemporal preferences manifest themselves in spurious violations of the classic independence axiom. Recall from Theorem 3 that RI along with IRU characterize ICM. For an Epstein-Zin agent, PEU (i.e., $\psi_s \leq RRA_s$) or PLU (i.e., $\psi_s \geq RRA_s$) can be detected by how RI is violated. Let $\geq_{FOSD}$ denote first-order stochastic dominance.

Proposition 4. *Suppose ρ is stochastic Epstein-Zin. For* 1*-period $z \in Z^*$ and $p_1 \geq_{FOSD} r$ for all $p \in z$,*

- $\psi_s \leq RRA_s$ *a.s. implies* $\rho_z\left(\delta_{(c,z)}\right) \leq \rho_{az\otimes(1-a)r}\left(a\delta_{(c,z)} \otimes (1-a)\,r\right)$
- $\psi_s \geq RRA_s$ *a.s. implies* $\rho_z\left(\delta_{(c,z)}\right) \geq \rho_{az\otimes(1-a)r}\left(a\delta_{(c,z)} \otimes (1-a)\,r\right)$

Proof. First, suppose $\psi_s \leq RRA_s$ a.s. Let $y = az \otimes (1-a)\,r$. Since $p_1 \geq_{FOSD} r$ for all $p \in z$,

$$v_{s_t}(z) = \mathbb{E}_{s_t}\left[\sup_{q\in z} u_{s_{t+1}}(q)\right] \geq \mathbb{E}_{s_t}\left[\sup_{q\in y} u_{s_{t+1}}(q)\right] = v_{s_t}(y)$$

Let $v_2 = v_s(z)$ and $v_1 = v_s(z)$ so $v_2 \leq v_1$. Now, for any $p \in z$,

$$u_s\left(\delta_{(c,z)}\right) \geq u_s(p) \Leftrightarrow \phi_s(c, v_1) \geq \int_M \phi_s(d, v_1)\,dp_1.$$

On the other hand,

$$\begin{aligned}
&u_s\left(a\delta_{(c,z)} \otimes (1-a)\,r\right) \geq u_s(ap \otimes (1-a)\,r) \\
\Leftrightarrow\; &au_s(c,y) + (1-a)\int_M u_s(c',y)\,dr \geq a\int_M u_s(c',y)\,dp_1 + (1-a)\int_M u_s(c',y)\,dr \\
\Leftrightarrow\; &\phi_s(c, v_2) \geq \int_M \phi_s(c', v_2)\,dp_1
\end{aligned}$$

Since $\psi_s \leq RRA_s$, $\phi_s(\cdot, v_1)$ is more convex than $\phi_s(\cdot, v_2)$ as in the proof of Proposition 1. Thus, for every $p \in z$, $u_s\left(\delta_{(c,z)}\right) \geq u_s(p)$ implies $u_s\left(a\delta_{(c,z)} \otimes (1-a)\,r\right) \geq u_s(ap \otimes (1-a)\,r)$ so the conclusion follows. The case for $\psi_s \geq RRA_s$ a.s. is symmetric. ■

Proposition 4 illustrates the type of permissible violation of the classic independence axiom in the repeated choice setup. For example, under strict PEU, if z consists of a risky and a safe option, then the probability of choosing the safe option will strictly increase if we mix all options with the worst consumption. Note that the act of mixing changes the agent's continuation value; when intertemporal preferences are non-standard as in Epstein-Zin, this generates violations of repeated independence. We can interpret this as a spurious violation of the independence axiom due to ignoring the intertemporal structure of the problem.

Note that this does not permit *any* violation of independence; for example, the agent will never strictly prefer mixtures. This is because the agent is still an expected utility maximizer on the larger outcome space of pairs of consumption and continuation menus. For example, given a repeated menu $z = \{(p_1, z), (q_1, z)\}$, the agent will never choose the mixture $\left(\frac{1}{2}p_1 + \frac{1}{2}q_1, y\right)$ in the repeated menu

$$y = \left\{(p_1, y), \left(\frac{1}{2}p_1 + \frac{1}{2}q_1, y\right), (q_1, y)\right\}$$

Even though there may be consumption smoothing due to intertemporal preferences, a stochastic Epstein-Zin agent will never exhibit a strict preference for ex-ante hedging; in other words, our model satisfies the stochastic version of betweenness from Dekel (1986) and Chew (1989).

Let $r = \delta_0$ and note that for any 1-period $z \in Z$,

$$a\delta_{(c,z)} \otimes (1-a)\,\delta_0 \to \delta_{\underline{x}}$$

as $a \to 0$. This suggests the following comparative statics result.

Proposition 5. *Suppose ρ and ρ' are both stochastic Epstein-Zin with respective risk aversion distributions π_{RRA} and π'_{RRA}. Then $\pi_{RRA} \geq_{FOSD} \pi'_{RRA}$ iff for all 1-period $z \in Z^*$,*

$$\lim_{a\to 0} \rho_{az\otimes(1-a)\delta_0}\left(a\delta_{(c,z)} \otimes (1-a)\,\delta_0\right) \leq \lim_{a\to 0} \rho'_{az\otimes(1-a)\delta_0}\left(a\delta_{(c,z)} \otimes (1-a)\,\delta_0\right)$$

Proof. Let $z = \left\{\delta_{(c,z)}, p\right\}$ and $y_a = az \otimes (1-a)\,\delta_0$. Note that $y_a \to \delta_{\underline{x}}$ as $a \to 0$ so

$$\lim_{a\to 0} v_{s_t}(y_a) = v_{s_t}(\underline{x}) = 0$$

Let $w_s(c) = c^{1-RRA_s}$ denote CRRA utility. We thus have

$$\begin{aligned}\lim_{a\to 0} \rho_{az\otimes(1-a)r}\left(a\delta_{(c,z)} \otimes (1-a)\,\delta_0\right) &= \lim_{a\to 0} \pi\left\{\phi_s(c, v(y_a)) \geq \int_M \phi_s(c', v(y_a))\,dp_1\right\} \\ &= \pi\left\{\phi_s(c, 0) \geq \int_M \phi_s(c', 0)\,dp_1\right\} \\ &= \pi_{RRA}\left\{w_s(c) \geq w_s(p_1)\right\}\end{aligned}$$

The conclusion follows from the fact that $\pi_{RRA} \geq_{FOSD} \pi'_{RRA}$ iff

$$\pi_{RRA}\left\{w_s(c) \geq w_s(p_1)\right\} \leq \pi'_{RRA}\left\{w_s(c) \geq w_s(p_1)\right\}$$

for all $c \in M$ and $p_1 \in \Delta M$. ■

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