
Jernej Čopič *

January 28, 2014

Abstract

Actor equilibrium is defined for static strategic environments, where decentralized and specialized actors do not know the distribution over uncertain parameters and form assessments as classical statisticians. In equilibrium, actors optimize and can justify observed distribution over own payoffs. The recovery problem due to incomplete observations permits adverse selection (private parameters) and moral hazard (private actions). The assessments supporting an equilibrium can (or sometimes must) differ for different actors. Equilibrium outcomes are given by a computationally-feasible set of linear inequalities and form a subset of outcomes satisfying minimal equilibrium requirements defined by common belief. An additional condition yields equivalence. The notion of actors is useful in applications to artificial intelligence. In a hawk-dove example and a partnership example equilibrium outcomes have negative welfare consequences, in contrast with Bayesian predictions that assume away the statistical problem. 

JEL codes: C10, C70, D80

*jcopic@econ.ucla.edu. I am especially grateful to Faruk Gul for helpful discussions, and to Luciano Pomatto for an example. I am also grateful to Pierpaolo Battigalli, Eddie Dekel, Goerge Georgiadis, Matt Jackson, Mihai Manea, Debrah Meloso, Joe Ostroy, Romans Panes, Hugo Sonnenschein, Juan Pablo Xandri and Bill Zame for comments and discussions, and to the seminar audiences at Caltech, Princeton and UCLA.
Keywords: equilibrium, actors, consistency, recovery, moral hazard, adverse selection, common belief, Hawk-Dove.

1 Introduction

In his pioneering work, Harsanyi (1967-68) defined Nash equilibrium for environments with uncertainty, where players are rational Bayesian decision makers. Recognizing the need to allow for heterogeneity in players’ beliefs regarding the uncertainty led Harsanyi to think of these players as holding Bayesian beliefs also over beliefs of others, then over these higher-order beliefs, and so on. These players are therefore required to have beliefs over very complex informational constructs – they are in a sense super rational. The question is also what, if anything, anchors such beliefs as that may affect the predictions of the model. It is therefore natural to instead think of these rational players as classical statisticians: assessments over uncertainty must be consistent (in the limit) with the data that each has had access to, while each is able to similarly justify the outcome as resulting from behavior of such rational players. On the theoretical level, it is additionally desirable that such equilibrium outcomes have a clear epistemic interpretation, and on the applied level, that equilibria can be determined from a computationally feasible set of conditions.

This paper proposes a notion of equilibrium in static strategic environments with

---

1See also Mertens and Zamir (1985) and Brandenburger and Dekel (1993). Under the additional desideratum that rational players can justify the others’ behavior, this can lead to a process akin to rationalizability, see e.g., Battigalli and Siniscalchi (2003). That process can in itself be complex.

2Several literatures have emerged, each addressing some of these issues. The literature on conjectural equilibrium fundamentally still views players as Bayesian, see e.g., Rubinstein and Wolinsky (1994), Battigalli and Guatoli (1997), and most recently Esponda (2013). The literature on Self-confirming equilibrium is closer to the present interpretation in terms of classical statistics. But there players either do not justify the others’ behavior, as in e.g., Fudenberg and Levine (1993a, 1993b) and Dekel et al. (2004), or such equilibrium outcomes are difficult to characterize and there isn’t a clear epistemic foundation – see Fudenberg and Kamada (2013) for such a study in a setting with no uncertainty. On a more applied note, Fershtman and Pakes (2012) propose an approach to dynamic games based on computational issues alone. Morris (1995) is an early proponent of notions of Bayes Nash equilibrium with no common prior, see also Battigalli et al. (2011) for a recent comprehensive study, but these notions are less related to the present approach.
informationally decentralized and specialized actors who are classical statisticians. A player is here thought of as a collection of all her actors, i.e., type-action combinations pertaining to that player – each actor is characterized by one private parameter, or type, and only one action with which they can act when that type of player is realized by a move of Nature. These actors are assumed to have no priors in a Bayesian sense. Instead, their probabilistic assessments of uncertainty (over both, type and action profiles) are interpreted as some consistent estimates of data they have obtained from the equilibrium interaction itself. For each actor this data is in the form of own payoffs obtained from actual play, so that if the information of all actors pertaining to a player were pooled, then such a player would have accumulated the data of own payoffs conditional on her types and actions that were taken. The actors’ assessments of uncertainty are required to be consistent with the actors’ infinite private datasets, that is, they are \textit{actor-minimally consistent} with the truth, and must also allow each actor to justify her data as having been generated from her interaction with other consistent and optimizing actors. Each actor may then be called upon to play when her assessment of uncertainty indicates that to be optimal. What is meant by equilibrium behavior is that it can be justified as a fixed point of such a procedure. Such \textit{actor equilibrium} satisfies the three above desiderata: rational actors’ assessments are anchored by the objective truth; equilibrium outcomes have an epistemic foundation; and these outcomes are characterized by linear inequalities.

Since actors have no explicit data regarding the others’ actions or types, the strategic interaction is susceptible to a combination of familiar informational prob-

\footnote{The starting point for that is a normal-form game with uncertainty viewed as an extensive-form game where Nature first draws players’ types, see Harsanyi (1967). This is then combined with the notion of the agent-normal form due to Selten (1975), and with the population interpretation of mixed strategies similar to that in Maynard-Smith and Price (1973). That is, an agent of a player (defined by a type) taking two different actions is interpreted as two different actors – that is the sense in which these actors are specialized. I use the term actors to not confuse that with the standard game-theoretic notion of agents, who are not specialized.

The actors therefore know the payoff structure, and those who have acted know that their payoffs arose as a result of a strategic interaction with the other actors, rather than from a random process.}
lems of moral hazard (private actions) and adverse selection (private types). Consequently, the actors might be unable to recover the true underlying probability distribution over the uncertain outcome. Consistent assessments might be different for actors pertaining to different players, and even for actors pertaining to the same player. Indeed, in some cases the actors can justify one-another’s optimal behavior only because the recovery problem allows for each actor to hold different actor-minimally-consistent assessments. In two examples, such equilibrium outcomes are shown to have positive and normative implications. The first example is a variation on the Hawk-Dove game – there is no exogenous uncertainty – and it illustrates moral hazard in an actor equilibrium. The second example is an example of partnership formation under uncertainty, and illustrates an equilibrium outcome exhibiting adverse selection. Bayesian equilibrium predictions, which assume away the statistical problem, would point only to equilibrium outcomes that are not problematic in terms of social welfare. In contrast, the allocations resulting from the newly-described equilibrium outcomes have negative welfare consequences.

Formally, an actor equilibrium outcome is defined as follows. There exist assessments for each actor such that: (i) given her assessment, the actor maximizes player’s utility; (ii) every actor’s assessment is actor-minimally-consistent with the truth; (iii) every actor’s assessment is actor-minimally consistent with every other actor’s assessment. The first condition means that if the actor is called upon to play, then her incentive constraints must hold vis-a-vis other actors with the same type of that player. The second condition is the consistency requirement explained above. The third condition guarantees that in such an outcome actors could justify the behavior of other actors in an infinite regress akin to rationalizability. This condition might seem unusual at first blush, but it is in fact intuitive: an actor’s assessment is in her view the truth, so that the other actor’s assessments ought to be

---

5 Throughout the paper, I use the word recover (as in Mass-Colell, Whinston and Green (1995)) rather than identify because the latter has a specific meaning in econometrics.
consistent with that truth. When the information of actors of each player is pooled, i.e., centralized, this yields the definition of a player equilibrium outcome. In either case, these conditions are expressed as linear equations that refer only to the actors’ assessments over first-order uncertainty and the payoff structure.

I then provide an exhaustive epistemic definition of what is meant by equilibrium behavior: in a minimal equilibrium outcome, actors can be rational and actor-minimally consistent, and there exists a common belief in rationality and actor-minimal consistency. The three conditions of actor equilibrium imply minimal equilibrium, but minimal equilibrium turns out to be slightly more general than actor equilibrium. This is illustrated by another partnership game exhibiting moral hazard in an environment with no exogenous uncertainty. Under an additional condition, with the interpretation that actors report their assessments over uncertainty to one another, the definitions of actor equilibrium and minimal equilibrium are equivalent.

A key distinction of these definitions is that equilibrium is specified in terms of an outcome that can be supported by some assessments, rather than an outcome along with specific actors’ assessments. First, an outcome might be supportable by a variety of different assessments satisfying these conditions, which is essentially the recovery problem. A subtler inspection of equilibrium conditions indicates that actors themselves are assumed to operate on that principle: such assessments are not required to be factually correct but merely to exist. Second, even while an outcome might be supported by assessments satisfying these conditions, it might well be supported by other assessments that do not satisfy any of these conditions – actors might in principle hold assessments whereby none of the actors is even rational. This distinction therefore crucially emphasizes the difference between what is known and

\[\text{If each players’ actors’ information were centralized and condition (iii) were dropped, that would imply that players thought their payoffs came about as a result of a random process, rather than a strategic interaction with other actors. That would in fact yield a version of self-confirming equilibrium defined by Fudenberg and Levine (1993). Note also that a common prior can be interpreted as a result of consistent estimation of full observations of the outcome. Under such a common prior, a Bayes Nash equilibrium outcome satisfies the three conditions of player equilibrium.}\]
what can be believed, but may never in fact be known.

As described above, rather than Bayesians, actors are here assumed to be classical statisticians. I do not specify any particular learning process that could lead to a subset of equilibrium points. Instead, I show that an equilibrium outcome can be thought of as a stable point of a process whereby in that outcome, actors have accumulated infinite datasets of private observations of their payoffs, then derived their private assessments as some consistent estimates, behaved optimally, and were able to justify the others’ behavior as a result of such a procedure, including the justification itself. Therefore, the equilibrium conditions are necessary conditions for limit points of any learning process of rational and sophisticated actors who can adjust their behavior and their statistical estimates of uncertainty. Since the information of actors pertaining to a player is decentralized this allows for more informational indeterminacy than what the player version of equilibrium allows for. These equilibrium conditions are therefore also necessary conditions for limit points of any learning process of rational and sophisticated players.\footnote{Equilibrium here assumes no correlation. Therefore, in such a learning process, actors or players should not be able to correlate their actions by way of the process itself, as in e.g., Hart and Mas-Colell (2000). However, since actors only observe own payoffs, such correlation might be difficult to achieve. For a survey of learning theories see Fudenberg and Levine (2009).}

This paper is built around several ideas, some of which are new. First, at its core is the interpretation of sophisticated decision makers as classical statisticians in a game theoretic setting: there are no prior beliefs and these decision makers are assumed to only make deductions from whatever data they have accumulated. Second, a player is reformulated as a collection of specialized and informationally decentralized actors. Third, it provides a minimal requirement of what data each such actor must observe in order for the resulting interaction to still allow, in the limit, for an interpretation as a one-shot game: this yields the notion of actor-minimal consistency. Fourth, the actor equilibrium and player equilibrium are formulated \textit{via} conditions, in the form of linear equations, which are derived from the primitives of the payoff structure.
alone, and can be verified in a computationally-feasible manner. Fifth, the first three
ideas are combined with the general epistemic definition of equilibrium, which yields
minimal equilibrium of actors who are rational and sophisticated. Sixth, there is a
cose relationship with the standard notions of moral hazard and adverse selection
and the examples, which are both new and original, are of some interest per-se.

The idea that economic actors should be thought of as classical statisticians rather
than Bayesians is related to similar ideas in the literature. In general equilbirum,
Anderson and Somnenschein (1985) provide a definition of rational expectations equi-
librium where instead of being Bayesian or classical statisticians, economic agents
use linear econometric models. In game theory, similar ideas are present in notions
of conjectural equilibrium due to Hahn (1977), Rubinstein and Wolinsky (1994),
Battigali and Guaitoli (1997), and most recently Esponda (2013). Among these, the
most closely related is Esponda (2013), but the setting there is still Bayesian in that
there is one realization of the state of the world, and other presumably possible, or
probable, states are imagined but never realized. This ideas is present in the notions
of self-confirming equilibrium, due to Fudenberg and Levine (1993a, 1993b), Dekel et
al (2004), and Fudenberg and Kamada (2013). Finally, the ideas here are related to
the econometric literature on partial identification, see for example Heckman (1995)
or Manski (1993, 2004). The difference is that the setting here is strategic, the payoff
structure is assumed to be common knowledge, and no identifying assumptions are
made.\footnote{With respect to an explicit statistical interpretation in a setting with uncertainty, the most
closely related among these is Dekel et al (2004). There, players do not justify other players’
behavior. In Fudenberg and Kamada (2013) they do, but the focus there is mostly on extensive-
form games with complete information; information structures and epistemic assumptions are also
different, and they focus on refinements and possible correlation through the play of the game itself.
Related is also Čopič and Galeotti (2007).}

As mentioned above, the idea of reformulating a player as a collection of actors
\footnote{Less related are models with statistical mistakes, e.g., McKelvey and Palfrey (1995), or models
where some incorrect beliefs are allowed to persist in the steady state, e.g., Jackson and Kalai
(1997), Eyster and Rabin (2005), Esponda (2008), and Jehiel and Koessler (2009). See also the
survey by Kurz (2011) of such models in dynamic macroeconomic settings.}
combines the ideas of Harsanyi (1967-68), Maynard-Smith and Price (1973), and Selten (1975). The actors pertaining to the same side, i.e., a player, still coordinate which of them should act: if under a given realization of the private parameter the assessment of an actor were to indicate that, a different actor with that parameter would optimally be called upon. In that sense, a player is conceived as a team of such actors, one actor for each private parameter and each action, and these actors are coordinated by a common incentive to maximize the payoff to the team as a whole, that is, the player.\textsuperscript{10} The notion of actors is suitable for applications where actors on the same side may be completely decentralized, in particular, in artificial intelligence and computer science.

The issue of what data the actors observe is at the heart of the notions of conjectural equilibrium and Self-confirming equilibrium mentioned above. In that literature, this is modeled by an exogenous parameter, called the feedback, or the observation function.\textsuperscript{11} Such a feedback function then induces a partition over the space of uncertainty. The approach here is similar, but there are two differences. First, given the payoff structure, the observational criterion is here not an exogenous parameter. Instead, I consider the weakest observational criterion for the interaction to have an interpretation as a one-shot strategic interaction between von-Neumann-Morgenstern expected-utility maximizers. In a decision problem, such a decision maker must have the information regarding the payoffs from potentially probabilistic consequences of her different acts – here too, for each action and private parameter, the actor observes the data regarding her payoffs. One could consider an even weaker criterion, where the actor observes only her average payoffs from each action. But that would then correspond to a stable point of a different kind of game with “averaged payoffs”, rather than a one-shot game with uncertainty. Such more radical departure from

\textsuperscript{10}Inspiring this idea is also the classic book on teams by Radner and Marschak (1972).

\textsuperscript{11}In Fudenberg and Kamada (2013), that is modeled directly as a partition on the terminal nodes of the extensive-form game.
the standard model of a game with uncertainty seems interesting, but I limit myself here to the observational assumption that still allows for the standard one-shot interpretation. The second difference here is to consider actors who are informationally decentralized, and in that sense have less information than players in models of conjectural and Self-confirming equilibrium.

An important issue with equilibrium outcomes is their computability.\textsuperscript{12} When players may have different assessments over uncertainty, in order to justify one-another’s play they must also make assessments regarding each-other’s assessments and so on. But an equilibrium outcome supporting such assessments might be computationally very demanding to verify. This concern is addressed to some extent in the notion of minimal equilibrium: while an epistemic hierarchy of assessments might be relevant, any outcome of a minimal equilibrium can be supported by a hierarchy of assessments over payoff relevant parameters alone. That is in itself a large reduction in the dimensionality of the problem. In actor and player equilibrium outcomes the computational problem is further simplified: in order to verify whether an outcome is supportable in actor equilibrium, only the existence of supporting first-order assessments must be verified. To the extent that there might be one such assessment to verify for each actor, this is potentially more computationally demanding than say, Bayes Nash equilibrium. However, by focusing on the outcome, that actually reduces the complexity of the problem since the set of supporting assessments is given by linear incentive constraints, i.e., it is convex.

The epistemic definition of minimal equilibrium outcomes is that they are supported by actor-minimal consistency and optimality, and a common belief (in the epistemic sense) in actor-minimal consistency and optimality. This is motivated by similar ideas in the literature, in particular, Brandenburger and Dekel (1987), Battigalli and Siniscalchi (2003), and especially Esponda (2013).\textsuperscript{13} This epistemic

\textsuperscript{12}In a dynamic context, this concern is the main motivation for recent work by Fershtman and Pakes (2012).

\textsuperscript{13}As mentioned above, some main differences with Esponda (2013) are in the notion of consis-
definition then yields a simple characterization of actor equilibrium, where additionally, actors’ assessments must be common belief. In contrast, Nash equilibrium has traditionally been characterized in terms of knowledge rather than belief, see Aumann (1987) and in particular Aumann and Brandenburger (1995).\textsuperscript{14} In the present setting with private information, where actual behavior by others is not observed, but can only be intuited from observations, belief rather than knowledge is more appropriate: a player might never truly know the assessments of other players. Indeed, a player may never even truly know that another player’s strategy is optimal.\textsuperscript{15} That these properties are common belief rather than common knowledge therefore merely makes it possible for players to justify each other’s behavior, rather than requiring that any particular such justification should in fact be true.

One of the motivations here was to consider the statistical problem pertaining to traditional economic problems of information: adverse selection and moral hazard.\textsuperscript{16} As mentioned before, actor-minimal consistency can be thought of as the minimal informational requirement for one-shot games played by expected-utility maximizing actors. Alternatively, actor-minimal consistency can also be thought of as the informational criterion which is characterized by allowing for moral hazard and adverse selection: under moral hazard, an actor does not directly observe the others’

---

\textsuperscript{14}The difference between knowledge and belief can be heuristically described in the sense of classical statisticians considered here. If a decision maker knows something, then the event where that is false is not possible. If a decision maker believes something, then the event where that is true is possible, while the event where that is false may also be possible.

\textsuperscript{15}To illustrate this point, think of the following situation. Player 1 has two different types, $\theta_1$ and $\theta_1'$, and two different actions, $a_1$ and $a_1'$, where $a_1$ is strictly dominant for $\theta_1$, and $a_1'$ is strictly dominant for $\theta_1'$. Suppose player 2 can recover the likelihoods of $a_1$ and $a_1'$, say 0.5 and 0.5. Then, if player 2 believed player 1 to be optimizing, then player 2 could conclude that the likelihoods of $\theta_1$ and $\theta_1'$ were 0.5 and 0.5. However, even if player 2 knew that the likelihoods of $\theta_1$ and $\theta_1'$ were 0.5 and 0.5, player 2 still couldn’t know that player 1 was optimizing unless player 2 observed player 1’s actual choices of actions for $\theta_1$ and $\theta_1'$, that is, player 1’s actual strategy.

\textsuperscript{16}The literature on such informational problems is vast. Some classics are Akerlof (1970), Spence (1973), Prescott and Townsend (1984), but the literature dates even much further back.
actions, and under adverse selection, an actor does not directly observe the others’ types. However, in environments exhibiting such informational problems, it seems illusory that actors should by default have correct estimates of underlying uncertainty—these are precisely the environments where actors face recovery problems which then allow for a variety of consistent estimates. Such assessments can then support outcomes that could otherwise never be supported even if the actors’ assessments merely coincided (but did not necessarily coincide with the truth).

The examples here are of that sort. The first example is a game I call Hawk/Harpy-Dove. It illustrates the notion of actor equilibrium in a setting with no uncertainty, and the particular outcome concerns the case of pure moral hazard: because hawk actors cannot recover whether they are facing a hawk or a harpy their assessments allow for hawks to be optimal even in the absence of harpies, i.e., (hawk,hawk) is an actor equilibrium of that game. The outcome of the Hawk-Dove game thus changes by introducing an element of moral hazard into the game, which can have welfare consequences. The second example illustrates the case of pure adverse selection in partnership formation under uncertainty. The player equilibrium outcome there is slightly more intricate, and it shows how the potential partners might not engage in a partnership even when that is the unique Pareto efficient and robust outcome. The reason is that when no partnership forms, a player cannot recover the distribution over the other’s types. If players’ assessments put sufficient mass on less desirable types on the other side, none wants to unilaterally deviate from no partnership. The equilibrium outcome is driven purely by different consistent assessments, and has negative welfare consequences.\footnote{In a related paper, Čopić (2013), I investigate in the setting of pure adverse selection conditions under which equilibrium outcomes cannot be supported under any common assessment.} In both cases, these failures are due to a combination of behavior and the \textit{resulting} informational problem, which then generates the data supporting that behavior. In a sense described here, these outcomes are therefore driven purely by expectations, which are nonetheless consistent, and rational.
The rest of the paper is organized as follows. Section 2 defines the notions of player and actor equilibrium, and gives the relationship between these notions and Bayes Nash equilibrium under the common prior. Section 3 defines minimal equilibrium, and relate it to actor equilibrium. Section 4 shows how such equilibrium outcomes can be interpreted as steady states of recurrent play of the game by classical statisticians. Section 5 provides examples of a player equilibrium exhibiting adverse selection and an actor equilibrium exhibiting moral hazard. One might also find it useful to skip from Section 2 straight to Section 5, and then read the other two sections.

2 Actor equilibrium and player equilibrium

An \( n \)-player game with uncertainty \( \Gamma \) is given by a set of players \( N = \{1, \ldots, n\} \), a product of finite sets of actions \( A = A_1 \times \ldots \times A_n \), a product of finite sets of players’ payoff types, \( \Theta = \Theta_1 \times \ldots \times \Theta_n \), and a vector of (bounded) payoff functions \( u: \Theta \times A \rightarrow \mathbb{R}^n \). Thus, \( \Gamma \equiv (N, A, \Theta, u) \). The player indexed by \( n \) can be interpreted as the state of fundamentals of the economy, in which case \( A_n \equiv \{a_n\} \), \( u_n \equiv \text{const.} \), and the types of the other players are interpreted as their signals.

Interpreting the state of the fundamentals of the economy as a player allows working within the simpler normal-form representation with types alone, while preserving the generality of the model that includes the states of fundamentals apart from players’ types or signals. In such a description, “payoff types” of players \( 1, \ldots, n - 1 \) describe players’ signals, i.e., \( \theta_i \) is the signal to player \( i \) when the state of the world is \( \theta_n \), with the appropriate conditional payoff consequences to player \( i \). Note that under the usual Bayesian assumption of a common prior over the states of the world and the signal structure, such signaling model is of course reducible to the normal-form payoff-type representation in the standard way. That is, by computing for each player the conditional expected payoffs for each vector of signals to all players. However,
when there is no common prior, as is of course the case in the present setting, such standard procedure is no longer possible, as different assessments may induce different normal-form representations. When the “states of fundamentals” are included as a “player” in the game as described here, that allows for such normal-form representation in that case as well. The description here therefore conveniently embeds the presumably more general model with the states of fundamentals into a reduced-form framework without having to consider the states of fundamentals in any special way. The fundamentals of the economy are indexed by “player $n$” rather than $0$ because index $0$ will be used to describe the objective outcome.\footnote{To my knowledge, I am not aware of any previous argument in the literature that would show how models with the explicit states of fundamentals can be represented in the normal form even when players’ information may be different.}

Given $i \in N$, denote $\Omega_i = \{ (\theta_i, a_i) \mid \theta_i \in \Theta_i, a_i \in A_i \}$, and let $\Omega = \times_{i \in N} \Omega_i$. The interpretation is that $\Omega = \Theta \times A$, and $\Omega_i$ is the set of actors of player $i$. One can therefore imagine a player as a collection of highly specialized actors, each with some payoff characteristic (type), and each equipped with an action that they can take.\footnote{Under such interpretation, some distinction between types and actions describing the actors of a player is still maintained: the player has no control over the randomness by which types are chosen, but the player determines the probabilities with which actors of different types are chosen. Note that the collection of actors representing a player is partitioned into subsets along either the types, or the actions. Hence one could more generally imagine a situation where no such distinction is made, and the player is simply a collection of abstract actors, partitioned into subsets. However, it seems difficult to imagine what sort of “game” such abstract representation would render.}

Denote $\omega_i, \Theta_i = \theta_i, \omega_i, A_i = a_i$, and $\omega = (\omega_0, \omega_1, ..., \omega_n)$. Given a player $i$’s actor $\omega_i = \Theta_i, \omega_i, A_i = a_i$, denote by $u_{\omega_i}$ “$\omega_i$’s utility function,”\footnote{Index $-i$ denotes elements of $N$ other than $i$.}

\[ u_{\omega_i} : \Omega_{-i} \to \mathbb{R}, \]

where $u_{\omega_i}(\omega_{-i}) = u_i(\theta_i, \omega_{-i}, a_i, a_{-i})$.

Define $X_0 \equiv \Delta(\Omega)$ as the objective outcome space of $\Gamma$; $x \in \Delta(\Omega)$ is an objective outcome of $\Gamma$. Define $x_\Theta$ as the objective payoff-type distribution, or objective distri-
bution of Nature’s moves; $x_{A|\Theta_i}$ is the objective strategy of player $i$. Given a player $i \in N$, $x_i \in X_0$ is called player $i$’s subjective assessment. Similarly, given an actor $\omega_i \in \Omega_i$, $x_{\omega_i} \in X_0$ is called actor $\omega_i$’s subjective assessment.

It is more common to think of equilibrium as a situation concerning the play of non-specialized players, rather than actors, so I first formulate the more familiar form of incentive constraints of a player. Denote by $\mathcal{E}$ the expectation operator. An assessment $x \in X_0$ satisfies incentive constraints for player $i$ (IC-$i$), if,

$$\mathcal{E}_{x_{\Omega_i}|\theta_i} u_i(\theta_i, a_i, \omega_i) \geq \mathcal{E}_{x_{\Omega_i}|\theta_i'} u_i(\theta_i, a_i', \omega_i), \forall a_i \in A_i, \text{ s.t., } x_{\Omega_i}(\theta_i, a_i) > 0, \forall \theta_i \in \Theta_i.$$ 

Under the assessment $x$, every action that each type of player $i$ plays with a positive probability must deliver a maximal expected payoff to that type of $i$. Thus, the incentive constraints can be rewritten in the actor form: each actor occurring with a positive probability must deliver a maximal expected payoff to that type of player $i$. An assessment $x \in X_0$ satisfies incentive constraints for actor $\omega_i$ (IC-$\omega_i$), if,

$$x_{\Omega_i}(\omega_i) > 0 \Rightarrow \mathcal{E}_{x_{\Omega_i}|\omega_i} u_{\omega_i}(\omega_i) \geq \mathcal{E}_{x_{\Omega_i}|\omega_i'} u_{\omega_i'}(\omega_i), \forall \omega_i \in \Omega_i, \text{ s.t., } \omega_i, \Theta_i = \omega_i', \Theta_i'.$$ (1)

Hence, an assessment $x$ satisfies IC-$i$, if and only if, it satisfies IC-$\omega_i$, $\forall \omega_i \in \Omega_i$.

**DEFINITION 1.** An assessment $x$ satisfies optimality for $\omega_i$ if it satisfies IC-$\omega_i$. An assessment $x$ satisfies optimality for $i$ if it satisfies IC-$i$.

Next, I formally describe how in a given outcome actors and players recover information from their payoffs that they observe under that outcome. Given a $\sigma$-algebra $\mathcal{F}$ on $\Omega$, and outcomes $x, x' \in \Delta(\Omega)$, $x'$ is $\mathcal{F}$-consistent with $x$, if,

$$x'(E) = x(E), \forall E \in \mathcal{F}.$$
Denote \( V_{\omega_i} = \text{Image}(u_{\omega_i}) \), and denote \( V_i = \bigcup_{\omega_i \in \Omega_i} V_{\omega_i} \). Given an \( x \in X_0 \), denote

\[
V^x_{\omega_i} = \{ u_{\omega_i}(\omega^{-i}) \mid x(\omega_i, \omega^{-i}) > 0 \}.
\]

Thus, \( V^x_{\omega_i} \) is the set of payoffs, which \( \omega_i \) observes with a positive probability.

Given an \( x \in X_0 \), and \( \omega_i \in \Omega_i \), denote by \( \mathcal{F}^x_{\omega_i} \) the \( \sigma \)-algebra over \( \Omega^{-i} \) generated by events \( \{ u_{\omega_i}^{-1}(v) \mid v \in V^x_{\omega_i} \} \). Adopt the convention that \( u_{\omega_i}^{-1}(\emptyset) = \emptyset \), so that if \( V^x_{\omega_i} = \emptyset \), then \( \mathcal{F}^x_{\omega_i} \equiv \{ \emptyset, \Omega_i \} \). Let \( \mathcal{F}^x_{\omega_i} \) be the \( \sigma \)-algebra over \( \Omega \) generated by events \( \{ u_{\omega_i}^{-1}(v) \times \{ \omega_i \} \mid v \in V^x_{\omega_i} \} \cup \{ \Omega_i \setminus \{ \omega_i \} \times \Omega^{-i} \} \). Thus, \( \mathcal{F}^x_{\omega_i} \) represents the information available to \( \omega_i \) when the outcome is \( x \), and \( \omega_i \) observes the different payoffs which he obtains with a positive probability under \( x \). Note that \( \mathcal{F}^x_{\omega_i} \) in principle contains no information regarding the events concerning \( \omega'_i \in \Omega_i, \omega'_i \neq \omega_i \), and it also contains no information regarding the likelihood with which \( \omega_i \) occurs. In turn, \( \mathcal{F}^x_{\omega_i} \), contains information about the likelihood with which \( \omega_i \) has occured.

Given \( i \in N \), and a collection of \( \sigma \)-algebras over \( \Omega \) for each actor of \( i \), \( \{ \mathcal{F}_{\omega_i} \mid \omega_i \in \Omega_i \} \), let \( \mathcal{F}_i \) be the \( \sigma \)-algebra describing the information available to player \( i \),

\[
\mathcal{F}_i = \bigwedge_{\omega_i \in \Omega_i} \mathcal{F}_{\omega_i}.
\]

In particular, \( \mathcal{F}^x_i = \bigwedge_{\omega_i \in \Omega_i} \mathcal{F}^x_{\omega_i} \) is the information of a (non-purified) player \( i \) when the outcome is \( x \). \( \mathcal{F}^x_i \) is called the minimal information of player \( i \).\(^{21}\)

Denote by \( x|_{\omega_i} \) the conditional probability distribution over \( \Omega^{-i} \), given an \( \omega_i \in \Omega_i \).

**DEFINITION 2.** Take an outcome \( x \in \Delta(\Omega) \). An assessment \( x' \in \Delta(\Omega) \) is \( \omega_i \)-minimally-consistent with \( x \), if, \( x_{\Omega_i}(\omega_i) > 0 \) implies that \( x'_{\Omega_i}(\omega_i) > 0 \) and \( x'|_{\omega_i} \) is \( \mathcal{F}^x_{\omega_i} \)-consistent with \( x|_{\omega_i} \). That is, if conditionally on occurring, the actor’s assessment is minimally-consistent with the objective outcome. Assessment \( x' \) is \( i \)-minimally-consistent with \( x \), if \( x' \) is \( \mathcal{F}^x_i \)-consistent

\(^{21}\)In the spirit of the agent-normal form due to Selten (1975), one could similarly describe the information available to player \( i \)’s agent-type \( \theta_i \) by \( \mathcal{F}_{\theta_i} = \bigwedge_{\omega_i \in \Omega_i, \omega_i = \theta_i} \mathcal{F}_{\omega_i} \).
with $x$.

Minimal-consistency is related to familiar concepts in economics of information. It describes informational indeterminacy due to a combination of moral hazard and adverse selection. Under moral hazard, a player does not directly observe other players' actions but may only infer those from own payoffs, and perhaps some signals regarding the state of the world. Under adverse selection, a player does not directly observe other players' types, but may only infer those from own payoffs, and perhaps some information regarding these other players' actions. Under minimal-consistency, a player does not directly observe either other players' actions, or their types, so that minimal-consistency is equivalent to moral hazard and adverse selection combined. If a player had an infinite dataset of observations of own types, actions, and payoffs, and would then come up with his assessment as some limit-consistent estimate of the objective outcome derived from that dataset, then such assessment would satisfy minimal consistency with the objective outcome. I will make this interpretation precise in Section 4.

There is one important distinction between the player and the actor versions of minimal-consistency. Player-minimal-consistency requires that the marginal probability with which each actor of that player occurs under the player's assessment equals the marginal probability with which each actor occurs under the objective outcome. Actor-minimal-consistency only requires that conditional on the actor occurring, that actor's assessment is actor-minimally-consistent with the objective outcome. Therefore, actor-minimal-consistency implies no relationship between the marginal probability of an actor occurring under that actor's assessment and under the objective outcome. Thus, under the steady-state interpretation, the actor does not know the relative frequency of his occurrences relative to the occurrences of other actors.
I now provide the definitions of player equilibrium and actor equilibrium.  

**Definition 3.** An outcome \( x_i^* \in X_0 \) is supportable in *player equilibrium*, or is a *player equilibrium outcome*, if there exists an assessment for each player, \( x_i^* \in X_0 \), \( i \in N \), such that,

1. \( x_i^* \) is optimal for \( i, i \in N \).
2. \( x_i^* \) is \( i \)-minimally-consistent with \( x_0^* \), \( \forall i \in N \).
3. \( x_j^* \) is \( j \)-minimally-consistent with \( x_i^* \), \( \forall i, j \in N \).

**Definition 4.** An outcome \( x_i^* \in X_0 \) is supportable in *actor equilibrium*, or is an *actor equilibrium outcome*, if for every \( i \in N \), and every \( \omega_i \in \Omega_i \), there exists an assessment \( x_{\omega_i}^* \in X_0 \), such that,

1. \( x_{\omega_i}^* \) is optimal for \( \omega_i \), that is, if \( x_{\omega_i,\Omega_i}^*(\omega_i) > 0 \), then,
   \[
   \mathcal{E}_{x_{\omega_i,\Omega_i}}(\omega_i) \geq \mathcal{E}_{x_{\omega_i,\Omega_i}}(\omega_{-i}), \forall \omega_i, \omega'_{-i} \in \Omega_i, \text{ s.t. }, \omega_i, \Theta_i = \omega_{-i}'\Theta_i, \forall i \in N.
   \]
2. \( x_{\omega_i}^* \) is \( \omega_i \)-minimally-consistent with \( x_0^* \), \( \forall \omega_i \in \Omega_i, \forall i \in N \), that is,
   if \( x_{0,\Omega_i}^*(\omega_i) > 0 \), then \( x_{\omega_i,\Omega_i}^*(\omega_i) > 0 \) and
   \[
   \frac{1}{x_{\omega_i,\Omega_i}^*(\omega_i)} \sum_{\omega_{-i} \in \mathcal{E}} x_{\omega_i}^*(\omega_i, \omega_{-i}) = \frac{1}{x_{\omega_i,\Omega_i}^*(\omega_i)} \sum_{\omega_{-i} \in \mathcal{E}} x_0^*(\omega_i, \omega_{-i}), \forall E \in \mathcal{F}_{\omega_i}^x.
   \]
3. \( x_{\omega_i}^* \) is \( \omega_i \)-minimally-consistent with \( x_{\omega_j}^* \), \( \forall \omega_i \in \Omega_i, \forall \omega_j \in \Omega_j, \forall i, j \in N \), that is,
   if \( x_{\omega_j,\Omega_j}^*(\omega_i) > 0 \), then \( x_{\omega_i,\Omega_i}^*(\omega_i) > 0 \) and
   \[
   \frac{1}{x_{\omega_i,\Omega_i}^*(\omega_i)} \sum_{\omega_{-i} \in \mathcal{E}} x_{\omega_i}^*(\omega_i, \omega_{-i}) = \frac{1}{x_{\omega_j,\Omega_j}^*(\omega_i)} \sum_{\omega_{-i} \in \mathcal{E}} x_j^*(\omega_i, \omega_{-i}), \forall E \in \mathcal{F}_{\omega_i}^x.
   \]

The equilibrium of Definition 3 renders the following interpretation (one can similarly interpret the actor equilibrium of Definition 4). Each player \( i \) recovers some distribution over uncertainty, given by his assessment \( x_i^* \), which is consistent with

---

\(^{22}\)In the definition of player equilibrium, note that if requirement 3 is removed, that would define a Self-confirming equilibrium for the case of incomplete information (Dekel et al (2004)), with the feedback function given by each player’s own types, actions, and payoffs. Therefore the notion of Self-confirming equilibrium is weaker than player equilibrium. I am not aware of any notion of equilibrium analogous to the actor equilibrium defined here.
The truth insofar as player \( i \) can tell. Under \( x^*_i \), player \( i \) is behaving optimally. When justifying the outcome in terms of other players’ behavior, say player \( j \), player \( i \) considers not \( x^*_i \), but some other distribution over the uncertainty \( x^*_j \) that \( j \) could recover if \( x^*_i \) were the truth. Similarly, when considering player \( j \), \( i \) can take \( j \)'s perspective and thus justify the outcome from \( j \)'s perspective, and so on, ad infinitum. Thus, the vector consisting of the objective outcome and players’ assessment \((x^*_0, x^*_1, ..., x^*_n)\) is such, that each player’s assessment is minimally consistent with the objective outcome, each player optimizes, and an equilibrium outcome admits a common belief in minimal consistency and optimality. This will be formally defined in Section 3.

The following Proposition relates player equilibrium and actor equilibrium.

**Proposition 1.** An outcome \( x^*_0 \in X_0 \) is supportable in player equilibrium, if and only if, \( x^*_0 \) is supportable in actor-equilbrium, s.t., \( x^*_{\omega_i,\Omega_i}(\omega_i) = x^*_{0,\Omega_i}(\omega_i) \), and \( x^*_{\omega_i} = x^*_{\omega_i'}, \forall \omega_i, \omega_i' \in \Omega_i, \forall i \in N \).

**Proof.** If an outcome \( x^*_0 \in \Delta(\Omega) \) is supportable in player equilibrium, then let \( x^*_{\omega_i} = x^*_i, \forall \omega_i \in \Omega_i, i \in N \). Since \( x^*_i \) is \( i \)-min-consistent with \( x^*_0, x^*_{i,\Omega_i}(\omega_i) = x^*_{0,\Omega_i}(\omega_i) \). Optimality for each \( \omega_i \) is implied by optimality for \( i \), and minimal-consistency for each \( \omega_i \) is implied by the fact that \( F^x_i \) is a refinement of \( F^{x_{\omega_i}} \), for each \( \omega_i \in \Omega_i \). Thus, \( x^*_0 \) is supportable in actor equilibrium.

For the converse, suppose \( x^*_0 \) is supportable in actor equilibrium, and let \( x^*_{\omega_i} = x^*_i, \omega_i \in \Omega_i, i \in N \), be the supporting assessments, where \( x^*_{\omega_i} = x^*_{\omega_i'} \), \forall \omega_i, \omega_i' \in \Omega_i, \forall i \in N \). Let \( x^*_i = x^*_{\omega_i} \), for some \( \omega_i \). Since \( x^*_{\omega_i,\Omega_i}(\omega_i) = x^*_{0,\Omega_i}(\omega_i) \), \( x^*_i \) satisfies \( \omega_i \)-minimal-consistency for all \( \omega_i \in \Omega_i \), and \( F^{x_i} = \wedge_{\omega_i \in \Omega_i} F^{x_{\omega_i}} \), it follows that \( x^*_i \) satisfies \( i \)-minimal-consistency. Optimality for \( i \) of \( x^*_i \) is obvious.

One may ask why the equilibrium notions of definitions 3 and 4 are not stated as it is usually done, i.e., “an equilibrium is given by \( x^*_0 \) and assessments \( x^*_i, i \in N \), s.t.,
equilibrium conditions hold.” These definitions make explicit the distinction between what is objective and what is subjective: For example, in definition 3 the outcome $x_0^*$ is objective, and is at least in principle observable by an econometrician, while assessments $x_i^*, i \in N$, are subjective and there is no good reason to think they should be observable or recoverable. The outcome $x_0^*$ could also be supportable by a host of other subjective assessments, and except for some special cases, e.g., generic environments, there seems little hope for either an econometrician (or each player) to be able to recover the subjective assessments that players hold. In fact, in general a minimal notion of equilibrium is a much more complicated object, which is in principle described by an infinite regress of players’ assessment over a more general space of universal hierarchies of assessments. A formal definition is provided in Section 3, and the relationship between a minimal equilibrium and actor equilibrium is characterized in Theorem 4. Definitions 3 and 4 provide a transparent and econometrically feasible way to describe equilibria in terms of their outcomes.

The next Theorem 1 relates actor equilibrium to the familiar notion of Bayes-Nash equilibrium with a common prior over payoff types. Theorem 1 is the conceptual centerpiece of this article. It provides the crucial distinction between the present approach to equilibrium and the traditional view where players (or in the present case actors) are imputed with some prior beliefs, which are additionally further assumed to be known by the modeler. The crucial point of Theorem 1 is to view outcomes as realizations of actors’ play, and the definition of actor equilibrium then implies that these actors would in such an outcome generate that data that would sustain the outcome itself in some way. But neither are actors’ beliefs given as priors, nor can these beliefs necessarily be known. Thus, rather than Bayesian statisticians, actors are here interpreted as classical statisticians. This interpretation is the subject of Section 4.

**Theorem 1.** An outcome $x_0^*$ is a Bayes-Nash equilibrium outcome with some com-
mon prior, if and only if, $x^*_0$ is an actor equilibrium outcome, s.t., there exist supporting assessments $x^*_{\omega_i}, \omega_i \in \Omega_i, i \in N$, with $x^*_{\omega_i} = x^*_{\omega_j}, \forall \omega_i \in \Omega_i, \omega_j \in \Omega_j, i, j \in N$.

Proof. If $x^*_0$ is a Bayes-Nash equilibrium outcome with some common prior, then $x^*_0$ is a Bayes-Nash equilibrium outcome with the common prior equal to $x^*_{0,\Theta}$, and setting $x^*_{\omega_i} = x^*_0, \omega_i \in \Omega_i, i \in N$, shows that $x^*_0$ is an actor equilibrium outcome.

For the converse, take the supporting assessments that are equal for all actors, and set the common prior to equal $x^*_{\omega_i, \Theta}$, for some $\omega_i \in \Omega_i, i \in N$. Since all the assessments are equal, and $x^*_0$ is an actor equilibrium outcome, the incentive constraints for all actors are therefore satisfied, so that $x^*_0$ is a Bayes-Nash equilibrium outcome, with the common prior $x^*_{\omega_i, \Theta}$. Note that this common prior need not equal the true objective distribution over types. \hfill \qed

Combining Proposition 1 and Theorem 1 yields the next corollary.

**Corollary 1.** An outcome $x^*_0$ is a Bayes-Nash equilibrium outcome with some common prior, and in particular, with the true common prior, if and only if, $x^*_0$ is a player equilibrium outcome, s.t., there exist supporting assessments $x^*_i, i \in N$, with $x^*_i = x^*_j, \forall i, j \in N$.

Theorem 1 and especially its Corollary 1 can be viewed as reinterpretations of Bayes-Nash equilibrium with a common prior. The common prior is not assumed as a part of the exogenous description of the game. Instead, it just so happens that a particular equilibrium outcome admits a common assessment of uncertainty. However, this common assessment of uncertainty need not equal the objective distribution over types, and there may be a variety of such common assessments supporting a particular equilibrium outcome.

The idea of robustness of an outcome to the assessments that players could make yields the notion of ex-post Nash equilibrium.\textsuperscript{23} When an outcome is supportable by

\textsuperscript{23}Since I am not primarily concerned with robustness, I here treat just the case of players, as I
any common assessments, then that outcome is an *ex-post* Nash equilibrium outcome, which is in turn supportable by any consistent common assessments.

**Proposition 2.** An outcome \( x^* \) is an *ex-post* Nash equilibrium outcome, if and only if, \( \bar{x}^* \) is a player equilibrium outcome supportable by some common assessments, for every \( \bar{x}^* \), such that \( x^* \equiv x^*_{0,A_i|\Omega_i}, \forall i \in N. \)

**Proof.** By definition, an outcome \( x^* \) is an *ex-post* Nash equilibrium outcome, if and only if, \( x^* \) is a best reply to \( x^*_{-i} \), \( \forall i \in N, \forall \theta \in \Theta. \) Define \( x^* \) by setting its marginal over \( \Theta \) to be any probability distribution over \( \Delta(\Theta) \), \( x^*_{0,\Theta} = \text{Pr}(\Theta \in \Delta(\Theta)), \) and set \( x^*_{A|\Theta} = x^*_{A|\Theta} \). Now set \( \bar{x}^* \equiv x^* \), \( \forall i \in N. \)

For the converse, let \( \text{Pr}(\theta) = \text{Pr}(\theta \in \Delta(\Theta)), \) i.e., \( \text{Pr}(\theta) \) is a point-mass at \( \theta \), for some \( \theta \in \Theta \), then set \( x^*_{0,\Theta} = \text{Pr}(\Theta \in \Delta(\Theta)), \) and \( x^*_{A|\Theta} = x^*_{A|\Theta} \). By Theorem 1, \( x^* \) is a Bayes Nash equilibrium outcome under some common prior. Now observe that minimal consistency implies that \( i \)’s supporting assessment \( \bar{x}^* \) satisfies \( \bar{x}^* = 1_{\theta_i}. \) Since players’ assessments are common, \( \bar{x}^* \) is a Bayes Nash equilibrium outcome under the common prior \( \text{Pr}(\Theta). \) Therefore, \( x^*_{0,A_i|\theta_i} \) is a best reply to \( x^*_{-i} \), \( \forall i \in N, \forall \theta \in \Theta. \)

3 Minimal Equilibrium

In this section I define minimal equilibrium. Minimal equilibrium is defined epistemically, by minimal consistency and optimality, and a common belief in minimal consistency and optimality. In order to define a common belief in these notions I first define the appropriate space of universal hierarchies of assessments, or universal constructs.24

---

24 This construction is fundamentally the same as the construction of universal types in Mertens and Zamir (1987), and in particular Brandenburger and Dekel (1993), where the underlying state space is here given by \( \Omega \). In the sense that assessments are here over strategies as well, the
Take a product set $Y = \times_{i \in N} Y_i$, where each $Y_i$ is given by $Y_i = \bigcup_{\omega_i \in \Omega} Y_{i,\omega_i} = \{\theta_i\} \times \{a_i\} \times \bar{Y}_{\omega_i}$, for some set $\bar{Y}_{\omega_i}$. For example, setting $\bar{Y}_{\omega_i} \equiv \emptyset$ one obtains $Y \equiv \Omega$. A $y_i \in Y_i$ is an actor, and for $y_i = (\theta_i, a_i, \bar{y}_i)$, denote $y_i,\Theta = \theta_i$, $y_i,A = a_i$ and $y_i,\Omega = \omega_i = (\theta_i, a_i)$. Given such a set $Y$ and $y_i \in Y_i$, define the utility function of $y_i$, by $u_{y_i} : Y_i \to \mathbb{R}$, where

$$u_{y_i}(y_i) \equiv u_i(y_i,\Theta, y_i,\Theta, y_i,A, y_i,A).$$

An $x \in \Delta(Y)$ satisfies optimality for $y_i \in Y_i$, or IC-$y_i$ if,

$$y_i \in \text{support}(x_{Y_i}) > 0 \Rightarrow \mathcal{E}_{x_{Y_i}|y_i} u_{y_i} \geq \mathcal{E}_{x_{Y_i}|y_i} u_{y_i'}, \forall y_i, y_i' \in Y_i, \text{ s.t.}, y_i,\Theta = y_i,\Theta.$$

Given an $x \in \Delta(Y)$, denote

$$V_{y_i}^x = \{u_{y_i}(y_i) | x(u_{y_i}(y_i)) > 0\}.$$

Thus, $V_{y_i}^x$ is the set of payoffs that actor $y_i$ observes with a positive probability under $x$. Given an $x \in \Delta(Y)$, define the information $\sigma$-algebra of $y_i$, $\mathcal{F}_{y_i}^x$, as the $\sigma$-algebra over $Y_i$ generated by events $\{u_{y_i}^{-1}(v) | v \in V_{y_i}^x\}$. Given $x, x' \in \Delta(Y)$, $x$ satisfies $y_i$-min-consistency with $x'$ if $x'|_{y_i}$ is $\mathcal{F}_{y_i}^x$-consistent with $x|_{y_i}$, i.e., if conditional on occuring, for actor $y_i$, assessment $x$ is minimally-consistent with $x'$, insofar as $y_i$ can tell.\footnote{As in Section 2, one can define the player version of minimal consistency.}

Inductively define the spaces of $k$-th order assessments, $Y_k$, $k \geq 0$,
\[Y_0 = \Omega \equiv \times_{i \in N} \Omega_i \equiv \cup_{\omega \in \Omega} Y_{0,\omega},\]
\[Y_1 = Y_0 \times \Delta(Y_0) \equiv \cup_{\omega \in \Omega} \times_{i \in N} (Y_{0,\omega} \times \Delta(Y_0)),\]
\[\vdots\]
\[Y_k = Y_{k-1} \times \Delta(Y_{k-1}) \equiv \cup_{\omega \in \Omega} \times_{i \in N} (\{\omega_i\} \times Y_{k-1,\omega} \times \Delta(Y_{k-1})),\]
\[\vdots\]

Observe that, \(Y_k = \times_{i \in N} Y_{k,i}\), where \(Y_{k,i} = \cup_{\omega \in \Omega_i} \{\omega_i\} \times Y_{k,\omega_i}\), e.g., \(Y_0 = \emptyset\). As in Brandenburger and Dekel (1993), a universal construct of actor \(\omega_i \in \Omega_i\) is a hierarchy of assessments,

\[t^{\omega_i} = (t^{\omega_i}_0, t^{\omega_i}_1, \ldots) \in \times_{k=0}^{\infty} \Delta(Y_k).\]

Let, \(T = \times_{k=0}^{\infty} \Delta(Y_k)\).

A \(t \in T\) satisfies belief in optimality at all orders \(k > 0\), if \(x_k \in \Delta(Y_k)\) satisfies IC-\(y_{k,i}\), \(\forall y_{k,i} \in Y_{k,i}, \forall i \in N, \forall k > 0\).

A profile of universal constructs \((t^{\omega_i})_{\omega_i \in \Omega_i, i \in N}\) satisfies a common belief in optimality if \(t^{\omega_i}\) satisfies a belief in optimality at all orders, \(\forall \omega_i \in \Omega_i, \forall i \in N\).

For a \(y \in Y_k\), \(y_i = (y'_i, x)\), where \(y'_i \in Y_{k-1,i}\), \(x \in \Delta(Y_{k-1})\), let \(y_i, Y_{k-1}\) denote the \(y'_i \in Y_{k-1}\), and let \(y_i, \Delta(Y_{k-1})\) denote the \(x\).

Let \(x \in \Delta(Y_{k-1})\) and \(x' \in \Delta(Y_k)\). Then, \(x'\) is min-consistent with \(x\), if \(y_{i, \Delta(Y_{k-1})}\) is \(y_{i, Y_{k-1}}\)-min-consistent with \(x\), \(\forall i \in N, \forall y \in \text{support}(x')\). A \(t \in \times_{k=0}^{\infty} \Delta(Y_k)\) satisfies belief in minimal consistency at all orders \(k > 0\), if \(t_k\) is min-consistent with \(t_{k-1}, \forall k > 0\). A profile of universal assessments \((t^{\omega_i})_{\omega_i \in \Omega_i, i \in N}\) satisfies a common belief in minimal consistency if \(t^{\omega_i}\) satisfies a belief in minimal consistency at all orders, \(\forall \omega_i \in \Omega_i, \forall i \in N\).

**Definition 5.** An outcome \(x^*_0 \in \Delta(\Omega)\) is supportable in minimal equilibrium, or is a minimal equilibrium outcome, if it admits minimal consistency and optimality, and
a common belief in minimal consistency and optimality. That is, if, there exists a profile of universal constructs \((t^\omega_i)_{\omega_i \in \Omega_i, i \in N}\), s.t.,

1. \(t^\omega_0\) satisfies optimality for \(\omega_i\), \(\forall \omega_i \in \Omega_i, \forall i \in N\).

2. \(t^\omega_0\) satisfies \(\omega_i\)-minimal consistency with \(x^*_0\), \(\forall \omega_i \in \Omega_i, \forall i \in N\).

3. \((t^\omega_i)_{\omega_i \in \Omega_i, i \in N}\) satisfies a common belief in optimality.

4. \((t^\omega_i)_{\omega_i \in \Omega_i, i \in N}\) satisfies a common belief in minimal consistency.

Much simpler than the space of universal constructs is the space of reduced constructs, \(T^R\), which I inductively define next.

Let \(Y^R_0 = Y_0\), and let

\[
Y^R_k = \{y \in Y_k \mid y_i, \Delta(y_{k-1}) \in \Delta(Y^R_{k-1}), y_i, \Delta(y_{k-1}) = y'_i, \Delta(y_{k-1}), \forall y_i, y'_i \in Y_{k,i}, \text{s.t.,} y_i, \Omega = y'_i, \Omega, \forall i \in N\}.
\]

Let \(T^R = \times_{k=0}^{\infty} \Delta(Y^R_k)\).

One way to intuitively describe the space \(T^R\), is that it is the space of universal constructs, such that, at every order, the assessments depend only on payoff-relevant information. That is, take two constructs, \(t, t' \in T^R\), and if at every order \(k\), \(t_k, t'_k \in Y^R_k\) have the same payoff relevant component \(\omega_i \in \Omega_i\), then \(t = t'\). For a \(k > 0\) let \(T^{R,k} = \times_{k=0}^{k} \Delta(Y^R_k)\).

An assessment in \(Y^R_k\) has a natural and intuitive representation: it describes what an actor “thinks” that some other actor “thinks” that some other actor “thinks,” at order \(k\) in a chain of length \(k\), where the last element in the chain describes the actor under consideration, and the assessment of that actor is simply an element in \(\Delta(\Omega)\). Hence, a reduced construct is simply a collection of all such assessments, and the space of reduced constructs can therefore be represented by the space of sequences of probability distributions over \(\Omega\). This representation is similar (but richer) to the representation of the systems of beliefs in the notion rationalizability, see Bernheim.

24
Take a $k > 0$, and let $L^k$ be the set of sequences of length $k$ of elements from $\Lambda = \cup_{i \in N} \Omega_i$, i.e., $L^k = \{ \ell \in \Lambda^k \}$. Let $L = \cup_{k=0}^{\infty} L^k$, and for a $\omega_i \in \Omega_i$, let $L^k_{\omega_i} \subset L$ be the set of all sequences of length $k$ starting with $\omega_i$. Adopt the convention that when $k = 0$, so that $L^0 = \{ \varnothing \}$, the $\varnothing$ when appearing as an index is interpreted as a 0.

**Proposition 3.** There is a homeomorphism $h : T^R \to \times_{k \in L} \Delta(\Omega)$. For each $k > 0$, the projection of $h$ on $T^R_k$ is the identity map.

**Proof.** Take a $t \in T^R$, $t = (t_k)_{k=0}^{\infty}$. For $x_0 \in \Delta(Y_0^R)$, let $h_0(x_0) = x^0$. Take a $k > 0$. For $x \in \Delta(Y^k)$, take an $i \in N$, and $\omega_i \in \Omega_i$. If $\exists y_i \in Y_{k,\omega_i}$, s.t., $y_i \in \text{support}(x_{Y_{k,\omega_i}})$, then let $h_{k,\omega_i}(x) \equiv y_i, \Delta(Y_{k-1})$. Since $t$ is reducible and by the inductive hypothesis, $h_{k,\omega_i}(x) \in \Delta(\Omega)$. If $\not\exists y_i \in Y_{k,\omega_i}$, s.t., $y_k, \omega_i \in \text{support}(x_{Y_{k,\omega_i}})$, then take any $y_i \in Y_{k,\omega_i}$ and let $h_{k,\omega_i}(x) \equiv y_i, \Delta(Y_{k-1}) \in \Delta(\Omega)$. Let $h_k(x) = \times_{i \in N} \times_{\omega_i \in \Omega_i} h_{k,\omega_i}(x)$, and let $h(t) = \times_{k=0}^{\infty} h_k(t_k)$. \( \square \)

A $t \in T^R$ can be thought of as a hierarchy of an actor’s assessments, when such a hierarchy is in the space of reduced constructs $T^R$. Proposition 3 states that each $t \in T^R$ can be represented as $t \equiv (x_\ell)_{\ell \in L}$. Moreover, for $t, t' \in T^R$, $t = t'$ if and only if all the elements of respective representations of $t$ and $t'$ are the same.

**Theorem 2.** If an outcome $x^*_0 \in \Delta(\Omega)$ is an actor equilibrium outcome, then $x^*_0$ is a minimal equilibrium outcome.

**Proof.** Take an actor equilibrium outcome $x^*_0$, and the supporting assessments $(x^*_{\omega_i})_{\omega_i \in \Omega_i}$. Take an $i \in N$, $\omega_i \in \Omega_i$, and $\ell \in L_{\omega_i}$. Define $x_\ell = x^*_{\omega_j}$, where $\omega_j = \text{last}(\ell)$ and $\text{last}(\ell)$ denotes the last element of $\ell$. Let $t^*_{\omega_i} = (x_\ell)_{\ell \in L_{\omega_i}}$, so that $t^*_{\omega_i} \in T^R$.

When $\ell = (\omega_i)$, $x_\ell = x^*_{\omega_i}$, $x^*_{\omega_i}$ is $\omega_i$-min-consistent with $x^*_0$ and $x^*_{\omega_i}$ is optimal for $\omega_i$. Since $x^*_{\omega_i}$ is optimal for $\omega_j$, it follows that $t^*_{\omega_i}$ satisfies a common belief in

\(^{26}\) Apart from other differences, e.g., incomplete information, it is here convenient to formally define actors’ assessments regarding themselves.

25
optimality. Finally, take $\ell, \ell' \in L_{\omega_i}$, such that, $last(\omega_i) = \omega_j$, and $\ell' = \ell \cup \{\omega_k\}$. Since $x^*_{\omega_k}$ is $\omega_k$-min-consistent with $x^*_{\omega_j}$, it follows that $t_{\omega_i}$ satisfies a common belief in minimal consistency. Therefore, $x_0^*$ is a minimal-equilibrium outcome.

The next result of this section is that, in general, if an outcome is supportable in a minimal equilibrium, then it is supportable in the space of reduced assessments. Thus, any minimal equilibrium is a much simpler object than any notion which critically depends on players' universal types. Note that the crucial property for this is minimal consistency – minimal consistency links the assessments with the payoff-relevant information.

**Theorem 3.** Take a minimal equilibrium outcome $x_0 \in \Delta(\Omega)$. Then there exists a profile of reduced assessments $(\bar{t}_{\omega_i})_{\omega_i \in \Omega_i, i \in N} \in [T^R]^{\Omega}$, s.t.,

1. $\bar{t}_{\omega_i}$ satisfies optimality for $\omega_i$, $\forall \omega_i \in \Omega_i, \forall i \in N$.
2. $\bar{t}_0$ satisfies $\omega_i$-minimal consistency with $x_0$, $\forall \omega_i \in \Omega_i, \forall i \in N$.
3. $(\bar{t}_{\omega_i})_{\omega_i \in \Omega_i, i \in N}$ satisfies a common belief in optimality.
4. $(\bar{t}_{\omega_i})_{\omega_i \in \Omega_i, i \in N}$ satisfies a common belief in minimal consistency.

**Proof.** Take a $\omega_i \in \Omega_i$ and let $t_{\omega_i} \in T$ be some universal construct satisfying requirements 1-4 from Definition 5, $t_{\omega_i} = (t^0_{\omega_i}, t^1_{\omega_i}, \ldots)$. Inductively define a mapping $\xi : T \rightarrow \times_{t \in \mathcal{L}} \Delta(\Omega)$ as follows. For $k = 0$ let $\xi_0(x_0) = x_0$. Take $k > 0$, $x \in \Delta(Y_k)$, $\omega \in \Omega$. If $\exists y_\omega \in support(x)$, then let $\xi_k(x_{Y_k,\omega}) = 1_{\{y_\omega\}}$, and,

$$
\xi_k(x)(y_\omega) = \int_{y'_\omega \in Y_k,\omega} dx(y'_\omega).
$$

Otherwise, take any $y_\omega \in Y_k,\omega$, let $\xi(x_{Y_k,\omega}) = 1_{\{y_\omega\}}$, and

$$
\xi_k(x)(y_\omega) = \int_{y'_\omega \in Y_k,\omega} dx(y'_\omega).
$$

26
Let $\xi(t) = \times_{k=0}^{\infty} \xi_k(t_k)$. It is evident that $\xi(t) \in T^R$, and that $\xi_k$ preserves belief in optimality and minimal consistency at $k$, for all $k > 0$. Letting $\tilde{\xi}_\omega = \xi(\bar{t}_\omega)$ completes the proof.

By Theorem 2, any actor equilibrium is a minimal equilibrium. An actor equilibrium is computationally a relatively simple object as it only requires first-order supporting assessments, which satisfy the equilibrium conditions. By Theorem 3, any minimal equilibrium outcome is supportable by reduced constructs. While much simpler than universal constructs, reduced constructs are still much more complicated than the assessments from the definition of actor equilibrium. One might therefore hope that any minimal equilibrium outcome is supportable in actor equilibrium. Unfortunately, that is in general not the case, which is shown by the following example.

The 3-player game in this example is called *The third wheel*, and the name shall be justified in a moment. The two payoff matrices below depict a complete-information game between three players, the row player, player 1, the column player, player 2, and the matrix player, player 3.\(^{27}\)

<table>
<thead>
<tr>
<th>$In$</th>
<th>$a_2$</th>
<th>$a'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>3, 1, 0</td>
<td>2, 0, 3</td>
</tr>
<tr>
<td>$a'_1$</td>
<td>3, 2, 4</td>
<td>2, 2, 0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$Out$</th>
<th>$a_2$</th>
<th>$a'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0, 0, 4</td>
<td>1, 2, 4</td>
</tr>
<tr>
<td>$a'_1$</td>
<td>0, 1, 4</td>
<td>−1, 4, −1</td>
</tr>
</tbody>
</table>

This game is a variation on the battle of the sexes, with the addition of moral hazard and another player, hence the name. Think of players 1 and 2 as a somewhat mismatched partnership, and player 3 as a potential replacement to player 2, or a potential additional (and better) partner to player 1. If player 3 chooses to participate that uniformly raises the payoffs to player 1, and in fact, for players 1 and 2 it is in a minimal equilibrium only possible to be in a partnership (match their actions), if

\(^{27}\)I am grateful and indebted to Luciano Pomatto, who sent me a slightly simpler example, which motivated the example below.
3 plays \( In \). Player 2 is better off playing action \( a'_2 \), except when that is mismatched with player 1 and player 3 participates – in that sense \( a'_2 \) bears some risk for player 2 when 3 participates and 1 mismatches with 2 at \( (a_1, a'_2, In) \). If player 3 plays \( In \), then as long as players 1 and 2 match their actions that is in general disadvantageous to player 3; Except if players 1 and 2 were to match their actions on player 2’s favorite, in which case, had 3 not participated, the payoffs to 1 and 3 would both be negative (an interpretation is that player 3 empathizes with player 1 to the extent that when player 1’s payoff is negative, player 3’s payoff is negative as well). Otherwise, player 3 is equally well off participating if player 1 breaks off the engagement with player 2, or not participating, the interpretation being that in that case player 3 can find some other equally-good partner.

Consider the outcome \( x_0 = (a_1, a_2, In) \). That outcome is a minimal equilibrium and is not an actor equilibrium. First, observe that, \( \{x_0\} \in F^{x_0}_{a_1} \), \( \{x_0, (a'_1, a_2, In)\} \in F^{x_0}_{a_2} \), and \( \{x_0, (a'_1, a'_2, In)\} \in F^{x_0}_{In} \); and these events are atomic in their respective \( \sigma \)-algebras. Next, \( x_0 \) satisfies optimality for \( a_1 \) and \( a_2 \), \( (a'_1, a'_2, In) \) satisfies optimality for \( In \), while \( (a'_1, a_2, In) \) does not satisfy optimality for \( a_2 \), and \( x_0 \) does not satisfy optimality for \( In \). Therefore, the only way to support \( x_0 \) in a minimal equilibrium is by setting \( x_\ell = x_0 \) for all \( \ell \), s.t., \( In \notin \ell \), and \( x_\ell = (a'_1, a'_2, In) \), otherwise. In other words, actors \( a_1 \) and \( a_2 \) assess the true outcome, while \( In \)’s assessment is that the outcome is \( (a'_1, a'_2, In) \). Evidently, \( x_{a_1} \) is not \( a_1 \)-consistent with \( x_{In} \), and \( x_{a_2} \) is not \( a_2 \)-consistent with \( x_{In} \), so that \( x_0 \) is not an actor equilibrium. Note that \( (a'_1, a_2, Out) \) is also a minimal-equilibrium outcome, which is slightly more complicated than \( x_0 \). Similarly to \( x_0 \) above, it is not supportable in an actor equilibrium.\(^{28}\)

Apart from the Nash equilibria of this game, \( (a'_1, a'_2, In) \) and \( (a_1, a'_2, Out) \), all other minimal-equilibrium outcomes involve some degree of moral hazard, in the

\(^{28}\)E.g., at the first order, it must be supported by \( x_{a_1} = (a'_1, a_2, Out), x_{Out} \in \{(a'_1, a_2, Out), (a_1, a'_2, Out)\}, x_{a_2} = (a_1, a_2, In) \); at the second order, \( x_{a_2} \) can be supported as \( x_0 \) above, and I leave it to the reader to construct other higher-order assessments.
sense that at least one player must make an assessment regarding the others’ behavior that is different from what the other players actually do. The example therefore illustrates how minimal equilibrium may represent quite intricate interactions of rational and sophisticated actors (or players) involving moral hazard. In a certain sense most of the moral hazard here comes from player 1, as player 1 in any minimal equilibrium correctly identifies the outcome. Still, for the present purpose, the main point of the example is that minimal equilibria in general need not be actor equilibria. The precise relationship between minimal equilibria and actor equilibria is the subject of the last theorem of this section.

**Theorem 4.** A minimal equilibrium outcome $x^*_0$ is supportable in actor equilibrium, if and only if, there exist supporting hierarchies of actors’ assessments, such that actors’ assessments are common belief.

**Proof.** If $x^*_0$ is supportable in actor equilibrium, then the existence of assessments that are common belief follows from the definition of actor equilibrium. For the converse, let the outcome $x^*_0$ be a minimal-equilibrium outcome, such that there exist supporting hierarchies of actors’ assessments that are common belief, denoted by $(x^*_i)_{i \in \mathcal{L}}$. Let $x^*_\lambda = x^*_\lambda(\lambda)$, i.e., $x^*_\lambda$ is the first-order supporting assessment of the minimal-equilibrium outcome $x^*_0$. By assumption $x^*_\lambda(\lambda') = x^*_\lambda(\lambda)$, $\forall \lambda' \in \lambda$, and since $x^*_\lambda(\lambda')$ is $\lambda$-min-consistent with $x^*_\lambda(\lambda)$, it follows that $x^*_0$ is an actor equilibrium. ☐

The following corollary is immediate.

**Corollary 2.** A minimal equilibrium outcome $x^*_0$ is supportable in player equilibrium, if and only if, there exist supporting hierarchies of players’ assessments, such that players’ assessments are common belief.

An interpretation of Theorem 4 is that the outcomes of minimal equilibrium and actor equilibrium are equivalent when actors can in an infinite recurrent play rely on truthfully communicating their respective supporting assessments to each
other, without revealing their private datasets. The interpretation of equilibrium as steady-state of infinite recurrent play of the game is the subject of the next section.

4 Equilibrium as a steady state of recurrent play

Minimal equilibrium of Definition 5 has an interpretation as a steady state when actors engage in a stationary infinite recurrent play of the game. This interpretation is quite straightforward. Rather than constructing specific learning processes, I show how minimal equilibrium satisfies a necessary condition for a steady state. Once all the behavior has settled, and actors accumulate ideal private datasets in the steady state, actors then consistently estimate the uncertainty from their private datasets, and make additional deductions given that these datasets have been generated from play by rational and sophisticated actors.

Let $Pr_{\Theta}$ be the objective distribution over Nature’s moves, let $s_i : \Theta_i \rightarrow A_i$ be the strategy of player $i$ in $\Gamma$, and let,

$$x(\theta, a) = Pr_{\Theta}(\theta) \times_{i \in N} s_i[\theta_i](a_i).$$

Thus, $x$ defines an objective outcome in $\Gamma$. Let for each $i$, $\bar{\Omega}_i$ be the set of $i$’s actors that occur with a positive probability, $\bar{\Omega}_i = \{\omega_i \in \Omega_i \mid x_{\Omega_i}(\omega_i) > 0\}$. 

Now suppose that $\Gamma$ is played recurrently, with no repeated-game effects, over infinitely-many periods, with $Pr_{\Theta}$ and $s = (s_i)_{i \in N}$ fixed, so that $x$ is also fixed and does not change over time, i.e., $x$ is stationary. Denote by $\tilde{x}^t$ the realization of $x$ in period $t$, and let $D^\tau = \{\tilde{x}_t\}_{t=1}^\tau$ be the sequence of realizations of $x$ in $\tau \geq 1$ periods. Let $D_{\omega_i}^\tau$ be the private dataset of observations of actor $\omega_i$ until period $\tau \geq 1$,

$$D_{\omega_i}^\tau = \{v_{\omega_i}^t \mid \omega_i = \tilde{x}_t^t, \text{ and } v_{\omega_i}^t = u_{\omega_i}(\tilde{x}_t^t)\}_{t=1}^\tau.$$
Thus, $D^\tau_{\omega_i}$ is a dataset of payoffs that actor $\omega_i$ received in all instances when $\omega_i$ was realized.

Denote by $\bar{\Omega}^\tau_i$ the set of all probability distributions over $\Omega$, satisfying IC-$\omega_i$ until period $\tau$,

$$
\bar{\Omega}^\tau_i = \{ \omega_i \in \Omega_i \mid \exists x \in D^\tau, \text{s.t.}, \omega_i = x_i \}.
$$

Note that whenever $\omega_i \notin \bar{\Omega}^\tau_i$, then $D^\tau_{\omega_i} = \emptyset$. Let $D^\infty = \lim_{\tau \to \infty} D^\tau$, $\bar{\Omega}^\infty_i = \cup_{\tau \geq 1} \bar{\Omega}^\tau_i$, and $D^\infty_{\omega_i} = \lim_{\tau \to \infty} D^\tau_{\omega_i}$, where $D^\infty_{\omega_i} = \emptyset$, $\forall \omega_i \notin \bar{\Omega}^\infty_i$.

Given $D^\tau$, and $\omega_i \in \bar{\Omega}^\tau_i$, let $\bar{x}^\tau_{\omega_i,V}$ be the empirical probability distribution over $V_{\omega_i}$ after $\tau$ periods. For a $\bar{v}_i \in V_{\omega_i}$, denote $\Omega_{-i|\bar{v}_i} = \{ \omega_{-i} \in \Omega_{-i} \mid u_{\omega_i}(\omega_{-i}) = \bar{v} \}$. Let $\bar{x}^\infty_{\omega_i,V} = \lim_{\tau \to \infty} \bar{x}^\tau_{\omega_i,V}$, where this limit is a point-wise limit. Finally, for every $\omega_i \in \bar{\Omega}^\infty_i$, let $X_{\omega_i}[x]$ be the set of all probability distributions $\hat{x}$ over $\Omega$, such that,

$$
\frac{\hat{x}_{\omega_i}(\Omega_{-i,\bar{v}})}{\hat{x}_{\omega_i}(\Omega_{-i})} = \bar{x}^\infty_{\omega_i,V}(\bar{v}), \forall \bar{v} \in V_{\omega_i}.
$$

(2)

For $\omega_i \notin \bar{\Omega}^\infty_i$, let $X_{\omega_i}[x] = \Delta(\Omega)$.

**Proposition 4.** $x' \in X_{\omega_i}[x]$, if and only if, $x'$ is $\omega_i$-minimally consistent with $x$, $\forall \omega_i \in \bar{\Omega}^\tau_i, \forall i \in N$.

**Proof.** By the Kolmogorov’s strong law of large numbers, we have that $D^\infty_{\omega_i} = \emptyset$, if and only if, $\omega_i \notin \bar{\Omega}^\infty_i$. Hence, $\bar{\Omega}^\infty_i \equiv \bar{\Omega}_i[x]$ and the proposition holds for any $\omega_i \notin \bar{\Omega}_i[x]$.

Take an $\omega_i \in \bar{\Omega}_i$. Applying the strong law of large numbers again, we have that $\bar{x}^\infty_{\omega_i}(v_i) = x(u_{\omega_i}^{-1}(v_i))$, so that $x'$ satisfies (2), if and only if, $x'$ is $\omega_i$-minimally consistent with $x$.

Let for every $\omega_i \in \bar{\Omega}_i[x]$, $X^{(\omega_i)}_i[x] = \{ x' \in X_{\omega_i}[x] \mid x' \text{ IC - } \omega_i \}$, so that $X^{(\omega_i)}_i[x]$ is the set of all probability distributions over $\Omega$, satisfying IC-$\omega_i$, and $\omega_i$-minimal consistency with $x$. Now let $X^{(\omega_i,\omega_j)}_i[x] = \{ x' \in X^{(\omega_i)}_i[x] \mid \omega_j \in \bar{\Omega}_j[x'] \Rightarrow X^{(\omega_j)}_j[x'] \neq \emptyset \}$, and let for $\ell \in L_{\omega_i}$, $X^{(\ell,\omega_j)}_i[x] = \{ x' \in X^{(\ell)}_i[x] \mid \omega_j \in \bar{\Omega}_j[x'] \Rightarrow X^{(\omega_j)}_j[x'] \neq \emptyset \}$. Finally,
let

\[ X_{\omega_i}^{\infty}[x] = \cap_{\ell \in L_{\omega_i}} X_{\omega_i}^{(\ell)}[x]. \]

**Proposition 5.** An outcome \( x \in \Delta(\Omega) \) is supportable in a minimal equilibrium, if and only if, \( X_{\omega_i}^{\infty}[x] \neq \emptyset \), for all \( \omega_i \in \bar{\Omega}_i \), \( \forall i \in N \).

**Proof.** In one direction, suppose \( x \) is supportable in a minimal equilibrium. Then there exist a hierarchy of supporting assessments, for each \( \omega_i \in \Omega_i \), \( \forall i \in N \). It follows that \( X_{\omega_i}^{\infty}[x] \neq \emptyset \), \( \forall \omega_i \in \bar{\Omega}_i[x] \), since the first-order assessment of actor \( \omega_i \) is in \( X_{\omega_i}^{\infty}[x] \). In the other direction, observe that \( X_{\omega_i}^{\infty}[x] \) is the set of all probability distributions over \( \Omega \) satisfying \( \omega_i \)-IC, \( \omega_i \)-min-consistency with \( x \), and such that an infinite hierarchy of assessments satisfying a common belief in minimal consistency and optimality exists.

Proposition 5 yields the interpretation of minimal equilibrium as a steady state. Minimal equilibrium is a combination of *eductive* estimation of subjective data as well *deductive* interpretation of that data based on a purely strategic argument. In terms of generality of this steady-state interpretation, the equilibrium conditions are *necessary* conditions for limit points of *any learning process of rational and sophisticated actors* with the property that in the limit such a learning process satisfies limit consistency with the objective outcome.\(^{29}\) Of course, for a specific learning process, the set of outcomes of its limit points might be smaller than the set of equilibrium outcomes described here. However, if *any convergent learning process* is considered, then the equilibrium conditions here are also sufficient: for example, the “trivial, outcome-specific learning process” where at some point in time actors for any reason

---

\(^{29}\)A learning process will satisfy such limit consistency, for example, if there is no, or, depending on the environment, not too much informational decay between rounds of play. If there is informational decay, e.g., if players have a finite memory, or if players are in some other way boundedly rational, then the set of limit points of such learning processes might be larger. Note also that if by way of the learning process itself, actions can be correlated, e.g., as in Hart and Mas-Colell (2000), then that may result in a bigger set of limit points that allow for correlation. However, in the present setting where actors observe own payoffs alone such correlation would presumably be very difficult to achieve.
whatsoever act according to a particular equilibrium outcome, and then from then on their play is fixed, will lead to “learning” that equilibrium outcome, precisely as described here. The union of all such outcome-specific learning processes will yield all equilibrium outcomes described in Definition 5.\footnote{This learning interpretation also provides one way to see the difference between minimal equilibrium and the notion of rationalizable conjectural equilibrium (RCE) due to Esponda (2013). In RCE, the state of the world is drawn once and for all, players may be uncertain about the state, and then play their actions. If such a game is played recurrently, then in a steady state, each player is facing some sort of subjective game of complete information, given by the most likely state of the world given her dataset and her belief regarding other possible states. That is, RCE represents a steady state of some averaged complete-information game, where each player’s perception of the average game can be subjective but players are Bayesian regarding the draw of the state itself.}

To obtain a similar steady-state interpretation for actor equilibrium, one can appeal to Theorem 4. This is formally stated in the following Proposition 6. Its proof follows from Theorem 4 and Proposition 5.

**Proposition 6.** An outcome $x \in \Delta(\Omega)$ is supportable in actor equilibrium, if and only if, $X^\infty_{\omega}(x) \neq \emptyset$, for all $\omega_i \in \Omega_i$, $\forall i \in N$ and actors’ assessments are common belief.

The question then becomes under what circumstances is it plausible to think that the actors’ assessments would be common belief. One possibility might be to survey the actors’ assessments and make them public, without requiring the actors to reveal their private datasets. This would then narrow the set of minimal equilibria to actor-equilibria, provided that the actors in fact reported their assessments truthfully. However, this seems rather naive in that the actors or players might in fact have incentives not to truthfully report their assessments. For an example, consider the outcome $(a_1, a_2, In)$ in the third wheel, the three-player game of Section 3. There, player 1 would not want to truthfully communicate her assessment to player 3, since player 3 would then have a profitable deviation, which would be utility-diminishing to player 1.\footnote{Note that the payoffs in that game can easily modified so that 1’s payoff is in fact strictly greater in the outcome $(a_1, a_2, In)$ than in any other outcome. Thus, even after some process of readjustment, whereby the outcome would presumably settle on a different steady state, player 1 could never hope to obtain an equally high payoff.} Hence, one sufficient condition for actor equilibrium is that all actors
have incentives to truthfully communicate their assessments to each other. Of course, if the actors were willing to reveal their private datasets to each other, and thus pool their information, then by Theorem 1 and Corollary 1 the resulting outcome would in fact be a Bayes Nash equilibrium outcome under the common prior.

## 5 Examples

Both examples in this section illustrate equilibrium outcomes which are not Nash equilibrium outcomes, or Bayes-Nash equilibrium outcomes for any common prior. Objective outcomes that are supportable in Bayes-Nash equilibrium are in some sense less exciting, as such outcomes are by the results of Section 2 already known to be equilibrium outcomes. Thus, the equilibrium outcomes constructed here are such that would have previously been classified as non-equilibrium outcomes. The argument as to why the outcomes presented here are not Nash equilibrium outcomes is carefully explained in both examples. In order for this argument to work, the equilibrium outcome in the first example exhibits moral hazard in an actor equilibrium, and in the second example, the outcome exhibits adverse selection in a player equilibrium. Along the lines of the two examples provided here, one can easily construct examples that simultaneously exhibit both, adverse selection and moral hazard.

### 5.1 Actor equilibrium exhibiting pure moral hazard.

The first example describes an actor equilibrium outcome exhibiting moral hazard under no uncertainty. This is to my knowledge the first example of an actor equilibrium outcome, so let me give it some further interpretation. It is sensible to think of the game as an interaction of two teams of actors. Thus, imagine a player as a collection of actors where each actor can be thought of as a specialized member of a team,
a computer code, or a robot carrying out a specific task.\footnote{I limited the present example to no uncertainty to make the simplest possible illustration via a well-known game. In an example with uncertainty, an actor would further be type contingent, i.e., an actor can only be active when its type has been drawn. Upon request, I can provide such an example; in that example the assessments of different actors of the same players must be different.} If an actor is called upon to play, then it carries out the task, observes the distribution over realized payoffs, and comes up with an assessment over uncertainty consistent with that distribution (as described in Section 4). The actor can only be active if her assessment indicates that to be optimal – otherwise a different, utility improving actor should be active for that private parameter.

An alternative interpretation is in the spirit of Maynard-Smith and Price (1973). Under that interpretation, imagine a large population (a continuum) of actors, who interact with one other – these actors are randomly matched and the game is played. Each actor is programmed to only perform one action, and form assessments based on the likelihoods of its different payoffs. Each actor in the population, for whom there exist appropriate supporting assessments in the sense of actor equilibrium of Definition 4 can perform its one action. In equilibrium, these actors in the population appear with the right proportions. Out of equilibrium, the presumption is that some of the actors would otherwise be reprogrammed with a utility-improving action. A more specific narrative in the spirit of artificial intelligence is to imagine these actors as clients connected over the internet, who then engage in pairs to play the game at hand.\footnote{Note also that I do not consider here any refinement, such as, e.g., evolutionary stability. The question of such evolutionary stability is a separate one, and would have to be defined appropriately for the present case where supporting assessments play a crucial role.}

The game played by the actors is a variant of the Hawk-Dove game, with the twist that there are two kinds of hawks. The first kind of hawks is the usual kind, and these are called simply hawks: as usually, a hawk obliterates a dove, while a hawk against a hawk has undesirable consequences for both hawks. The second kind of hawk is an unusually cruel kind, called a harpy: vis-a-vis a hawk a harpy performs
exactly like a hawk, and *vis-a-vis* a dove, a harpy yields a large negative payoff to a dove. Any combination of hawks and harpies yields the same utility to both sides, and the payoffs to two doves maximize the aggregate welfare. The payoffs in the game Hawk/Harpy-Dove are summarized by the table below,

<table>
<thead>
<tr>
<th></th>
<th>hawk_2</th>
<th>harpy_2</th>
<th>dove_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>hawk_1</td>
<td>$-\frac{1}{3},-\frac{1}{3}$</td>
<td>$-\frac{2}{3},0$</td>
<td></td>
</tr>
<tr>
<td>harpy_1</td>
<td>$-\frac{1}{3},-\frac{1}{3}$</td>
<td>$-\frac{1}{3},-\frac{1}{3}$</td>
<td>$\frac{1}{2},-\frac{1}{2}$</td>
</tr>
<tr>
<td>dove_1</td>
<td>$0,\frac{2}{3}$</td>
<td>$-\frac{1}{2},\frac{1}{4}$</td>
<td>$\frac{1}{2},\frac{1}{2}$</td>
</tr>
</tbody>
</table>

As benchmarks, consider first the Nash equilibria of this game. Observe that *harpy_i* is weakly dominated by *hawk_i*, and strictly dominated when *dove_j* occurs with a positive probability. Therefore, whenever *dove_j* occurs with a positive probability, *harpy* cannot occur with a positive probability in any mixed-strategy Nash equilibrium. Observe, however that (*harpy_1, harpy_2*) is a pure-strategy Nash equilibrium, along with the usual (*hawk_1, dove_2*) and (*hawk_2, dove_1*). There is of course also the usual mixed-strategy Nash equilibrium, given by,

$$\left(\frac{1}{3} \times \text{hawk}_1 + \frac{2}{3} \times \text{dove}_1, \frac{1}{3} \times \text{hawk}_2 + \frac{2}{3} \times \text{dove}_2\right).$$

Finally, there are also mixed-strategy Nash equilibria of the form,

$$(p_1 \times \text{hawk}_1 + (1 - p_1) \times \text{dove}_1, p_2 \times \text{hawk}_2 + (1 - p_2) \times \text{dove}_2),$$

where $p_1, p_2 \in (0, \frac{1}{3}]$.

Now I show that $x_0^* = (\text{hawk}_1, \text{hawk}_2)$ is in fact an actor equilibrium. Let $\alpha_1, \alpha_2 \in (0, 1)$, and $p_1, p_2 \in [0, \frac{1}{3}]$, and let $x_{h_1}^*$ and $x_{h_1}^*$ be given by,
Set $x_{h_1}^* = x_{h_{2,\Omega_1}}^* = x_{dove_1}^* = x_{h_1}^*$ and $x_{h_2}^* = x_{h_{2,\Omega_1}}^* = x_{dove_2}^* = x_{h_2}^*$. These assessments then satisfy actor equilibrium conditions for $x_0^*$.

In these assessments, $p_j$ must be positive, and $\alpha_i \in (0, 1)$. The reason is that otherwise, condition (iii) in Definition 4 fails: e.g., if $\alpha_1 = 1$, then $x_{h_{2,\Omega_1}}^*(\text{harp}_{y_1}) > 0$, but $x_{h_{1,\Omega_1}}^*(\text{harp}_{y_1}) = 0$. Therefore, $x_{h_{1,\Omega_1}}^*(\text{hawk}_i) \neq x_{0,\Omega_1}^*(\text{hawk}_i)$. This implies that $x_0^*$ isn’t supportable in a player equilibrium. Note that as in Definition 4, assessments must be specified for all actors, even doves whose assessments turn out not to matter. The reason is that, as explained here for harpies, by condition (iii) in Definition 4 the assessments of actors who are thought to appear with a positive probability but in fact appear with probability zero actually matter, and must be specified. Therefore, it is not a priori possible to only specify assessments for actors who under the truth appear with a positive probability, and it is not a priori clear what other actors’ assessments will play a role. If this actor equilibrium were interpreted in the population sense, where there is only one large population, then in each above assessment, $\alpha_i = p_j$. That is, each actor’s assessment would then have to additionally correspond to the equilibrium being symmetric. To avoid that, one could interpret the actor equilibrium as an interaction of two large populations, one for each player.

An interpretation of this actor equilibrium is close to the literal distinction be-
tween hawks and harpies. That is, harpies are purely mythological creatures, and are therefore never called upon to act. Nevertheless, in this actor-equilibrium outcome harpies are thought to exist, and that creates moral hazard (or in the literal sense, an illusion thereof). This in turn implies the recovery problem due to which \((\text{hawk}_1,\text{hawk}_2)\) is supportable in an actor equilibrium.

One could also imagine that the game played might in fact be the usual Hawk-Dove – the example would then illustrate an extreme case of paranoia. If the actors were essentially programmed with the wrong game where harpies were possible, then hawk-hawk would be an actor equilibrium. But such rather philosophical interpretation is inconsistent with the assumption that the payoff structure is common knowledge – the correct interpretation here is the one given above in terms of the recovery problem due to moral hazard endogenous to the equilibrium behavior.

5.2 Player-equilibrium exhibiting pure adverse selection.

In the next example uncertainty is not of a purely strategic sort.\(^{34}\) Since this paper is also about exogenous uncertainty, such an example seems appropriate. In this example, I construct an actor equilibrium outcome, which exhibits pure adverse selection. That outcome is not supportable in a Bayes Nash equilibrium under any common prior. Whereas in the previous example, the actor equilibrium outcome \((\text{hawk}_1,\text{hawk}_2)\) yielded the same payoffs as some Nash equilibrium outcomes, e.g., \((\text{harpy}_1,\text{harpy}_2)\), here that will not be the case. The actor equilibrium outcome constructed here will in fact have payoff consequences – in the present example this outcome will have negative consequences for the aggregate welfare (under the objective truth) relative to a robust \((\text{ex-post})\) Nash equilibrium outcome. It will also turn out that this actor equilibrium outcome is supportable in a player equilibrium. I therefore narrate this example referring to players (not actors).

\(^{34}\)This example is similar to an example in Čopić (2013).
The payoff structure \( \hat{\Gamma} \) is given as follows. There are two players, \( N = \{1, 2\} \), each player has two types, \( \Theta_i = \{\theta_i, \theta'_i\}, \ i \in N \), two actions, \( A_i = \{a_i, a'_i\}, \ i \in N \), and the payoffs for each draw of types are specified by the following tables. Assume that the objective distribution over the moves by Nature is uniform over \( \Theta \). The interpretation of this payoff structure is that 1 and 2 can participate in a partnership, where actions \( a_i \) are interpreted as participating the partnership, while actions \( a'_i \) are interpreted as not participating in the partnership. If players play \( a_i \) regardless of the realization of their types, then “matched states,” \( (\theta_1, \theta_2) \) and \( (\theta_1, \theta_2) \) are beneficial to player 1 and detrimental to player 2, while “mismatched states,” \( (\theta'_1, \theta'_2) \) and \( (\theta'_1, \theta'_2) \), are beneficial to player 2 and detrimental to player 1. If both players play \( a'_i \) regardless of the realization of their types, then both obtain a zero payoff in every state. If only one plays \( a_i \), that still has payoff consequences for both players, which are different across the states. Note that when players are engaged in the partnership, they maximize their aggregate payoffs in every state.

<table>
<thead>
<tr>
<th>((\theta_1, \theta_2))</th>
<th>(a_2)</th>
<th>(a'_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>6,4</td>
<td>-1,3</td>
</tr>
<tr>
<td>(a'_1)</td>
<td>5,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>((\theta'_1, \theta_2))</th>
<th>(a_2)</th>
<th>(a'_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>4,6</td>
<td>2,5</td>
</tr>
<tr>
<td>(a'_1)</td>
<td>3,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>((\theta_1, \theta'_2))</th>
<th>(a_2)</th>
<th>(a'_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>6,4</td>
<td>-1,3</td>
</tr>
<tr>
<td>(a'_1)</td>
<td>5,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>((\theta'_1, \theta'_2))</th>
<th>(a_2)</th>
<th>(a'_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>4,6</td>
<td>-1,3</td>
</tr>
<tr>
<td>(a'_1)</td>
<td>5,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

As mentioned above, the objective probability distribution of Nature’s moves \( x_{0,\Theta} \) is given by,

<table>
<thead>
<tr>
<th>(x_{0,\Theta})</th>
<th>(\theta_2)</th>
<th>(\theta'_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta_1)</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>(\theta'_1)</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{4})</td>
</tr>
</tbody>
</table>
Forming a partnership is an *ex-post* Nash equilibrium of this game, i.e., each player plays action $a_i$ for every realization of her types.\textsuperscript{35} Denote this objective outcome of not forming the partnership along with the objective probability distribution over Nature’s moves by $\bar{x}$, i.e., $\bar{x}_\Theta = x_{0, \Theta}$ and $\bar{x}_{A_i|\Theta} = 1_{\{a_i\}}$. Since not forming the partnership is supportable in *ex-post* Nash equilibrium, it is supportable in a Bayes-Nash equilibrium under the common prior. Moreover, the outcome $\bar{x}$ is a fully-revealing player-equilibrium: to be minimally consistent with $\bar{x}$, a player must correctly assess the objective distribution over $\Omega$, i.e., $\mathcal{F}_i^\bar{x}$ is the discrete $\sigma$-algebra for each $i$. Since forming the partnership maximizes players’ aggregate payoffs in every state, this outcome is therefore also *ex-post* efficient; since it Pareto-dominates any other outcome it is therefore the unique *ex-post* efficient outcome. In other words, in this game, forming a partnership has desirable normative properties, and it is feasible in equilibrium, which is robust to players’ beliefs. If one assumed a common prior, and applied the notion of Bayes Nash equilibrium without regard to the statistical problem, one would therefore conclude that there was no issue in this example.

Now consider the objective outcome $x_0$, where each player doesn’t participate in the partnership for every realization of her types, along with the above objective probability distribution of Nature’s moves. Thus, the objective strategy of each player is given by $x_{0, A_i|\Theta} \equiv 1_{\{a_i\}}$, i.e., each player $i$ plays $a_i'$ regardless of her type $x_0$, and $x_0$ is the objective outcome where each of the four agent profiles $(\theta; a')$, $\theta \in \Theta$, $a' = (a_1', a_2')$ occur with equal probability $\frac{1}{4}$. First, $x_0$ is not supportable in Bayes-Nash equilibrium, *for any common prior*. To see that, denote such a common prior over $\Theta$ by $P(\ldots)$ and write down the incentive constraint of each type of each

\textsuperscript{35}A strategy profile is an *ex-post* Nash equilibrium if, given the other player’s strategy, a player has no incentives to deviate even if she knows the realization of the other player’s type. See also e.g., Bergemann and Morris (2007) and Bergemann and Morris (2013).
player under the strategy profile \((x_{0, A_1|\theta_1}, x_{0, A_2|\theta_2})\),

\[
0 \geq 2P(\theta_1, \theta_2') - P(\theta_1, \theta_2), \quad (3)
\]

\[
0 \geq 2P(\theta_1', \theta_2) - P(\theta_1', \theta_2'), \quad (4)
\]

\[
0 \geq 2P(\theta_1, \theta_2) - P(\theta_1', \theta_2), \quad (5)
\]

\[
0 \geq 2P(\theta_1', \theta_2') - P(\theta_1, \theta_2'). \quad (6)
\]

By summing up (3)-(6), we obtain

\[
0 \geq P(\theta_1, \theta_2) + P(\theta_1, \theta_2') + P(\theta_1', \theta_2) + P(\theta_1', \theta_2'),
\]

which is a contradiction since \(P\) is a probability distribution.

Observe that under the outcome \(x_0\), in order to satisfy \(i\)-min-consistency with \(x_0\), the assessment of each player \(i\) must satisfy \(x_{i, A_j|\Theta_j} = x_{0, A_j|\Theta_j}\). The reason is that, for example, when player 1 is of type \(\theta_1\), and plays \(a_1'\), the only way that she can obtain a payoff of 0 for sure is when player 2 plays \(a_2'\) for each of his types. Thus, in order to completely specify each player \(i\)’s assessment, as long as that is \(i\)-min-consistent with \(x_0\), it is enough to specify \(x_{i, \Theta}\). In this example, we can therefore imagine each player having a different min-consistent assessment over Nature’s moves, while their assessments of the objective strategy profile are correct. To see that \(x_0\) is in fact a player equilibrium, let the players’ assessment over Nature’s moves be given for example by,

\[
\begin{array}{ccc|ccc}
 x_{1, \Theta} & \theta_2 & \theta_2' & x_{2, \Theta} & \theta_2 & \theta_2' \\
 \theta_1 & \frac{2}{6} & \frac{1}{6} & \theta_1 & \frac{1}{6} & \frac{2}{6} \\
 \theta'_1 & \frac{1}{6} & \frac{2}{6} & \theta'_1 & \frac{2}{6} & \frac{1}{5}
\end{array}
\]

It is now immediate to verify that \(x_{1, \Theta}\) satisfies the incentive constraints of player 1, while \(x_{2, \Theta}\) satisfies the incentive constraints of player 2. In fact these incentive constraints are with such assessments satisfied with equalities. Observe also that \(i\)-min-consistency with \(x_0\) requires only that the marginal \(x_{i, \Theta_i}(\theta_i) = \frac{1}{2}, \theta_i \in \Theta_i\), which is satisfied. For the same reason, \(x_i\) also satisfies \(i\)-min-consistency with \(x_j\). Hence,
$x_0$ is a player-equilibrium outcome.\footnote{There are other assessments supporting $x_0$, such that players’ incentive constraints are satisfied with strict inequalities, e.g., for player 1,}

Under $x_0$, each player must correctly assess the other player’s strategy, but must have an incorrect assessment over the conditional distribution of the other player’s types, i.e., player knows what the opponent does, but doesn’t know the distribution over the opponent’s types. Therefore, this outcome exhibits pure adverse selection. Note that $x_0$ is of course an agent-equilibrium outcome, since it is a player-equilibrium outcome.

Thus, here the adverse-selection outcome of not forming a partnership is socially undesirable. It is only possible by players having diverse assessments over uncertainty. These diverse assessments are facilitated by the adverse-selection problem inherent to the outcome itself, whereby players cannot fully recover the objective uncertainty.

References


45


