Supplemental Appendix to
Macro-Finance Decoupling: Robust Evaluations of
Macro Asset Pricing Models

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May 24, 2020

Abstract

This supplemental appendix contains additional technical details of Cheng, Dou, and Liao (2020). Section SA provides the proofs of several lemmas on the asymptotic convergence of the random components in the test statistic $T$ and the conditional critical value $c_\alpha(\hat{d})$. Section SB verifies the bounded Lipschitz properties of the test statistic and the conditional critical value, which are used to show their weak convergence in large sample. Section SC includes some auxiliary lemmas. Section SD derives the Euler equations that serve as the asset pricing moments in the long-run risk model and the disaster risk model. Section SE considers the long-run risk model and shows that the Gaussian limit is an innocuous assumption.

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SA Proofs for Asymptotic Convergence

This section provides the proofs of Lemmas A2, A3, A7, A8 in Cheng, Dou, and Liao (2020) under their Assumptions 1, 2, 3 and 4. Throughout this appendix, we use $\lambda_{\text{min}}(A)$ to denote the minimal eigenvalue of a square matrix $A$, and $\| \cdot \|$ denotes the matrix Frobenius norm.

Proof of Lemma A2. (a) Define $C_{1,n} \equiv \sup_{\theta \in \Theta} | n^{-1} g(\theta)'(\hat{\Omega}(\theta))^{-1} g(\theta) - n^{-1} G(\theta)'(\Omega(\theta))^{-1} G(\theta) |$ where $G(\theta) \equiv E [g(\theta)]$. Then by Assumptions 1(i, iii), 2(i) and 3(iii), 

$$C_{1,n} = o_p(1), \quad (SA.1)$$

uniformly over $\mathbb{P} \in \mathcal{P}_0$. Consider any $\varepsilon > 0$. By the definition of $\hat{\theta}$ and $\theta_0$,

$$\sup_{\mathbb{P} \in \mathcal{P}_0} \mathbb{P} \left( \left\| \hat{\theta} - \theta_0 \right\| \geq \varepsilon \right) \leq \sup_{\mathbb{P} \in \mathcal{P}_0} \mathbb{P} \left( \min_{\theta \in B_{\varepsilon}(\theta_0)} n^{-1} g(\theta)'(\hat{\Omega}(\theta))^{-1} g(\theta) \leq n^{-1} g(\theta_0)'(\hat{\Omega}(\theta_0))^{-1} g(\theta_0) \right) \leq \sup_{\mathbb{P} \in \mathcal{P}_0} \mathbb{P} \left( \min_{\theta \in B_{\varepsilon}(\theta_0)} n^{-1} G(\theta)'(\Omega(\theta))^{-1} G(\theta) \leq 2C_{1,n} \right) \leq \sup_{\mathbb{P} \in \mathcal{P}_0} \mathbb{P} \left( c_{\delta^2} \leq 2C_{1,n} \right), \quad (SA.2)$$

where the second inequality is by the definition of $C_{1,n}$ and the third inequality is by Assumption 3. Combining the results in (SA.1) and (SA.2), we deduce that

$$\lim_{n \to \infty} \sup_{\mathbb{P} \in \mathcal{P}_0} \mathbb{P} \left( \left\| \hat{\theta} - \theta_0 \right\| \geq \varepsilon \right) = 0 \text{ for any } \varepsilon > 0. \quad (SA.3)$$

Let $C_{2,n} \equiv \sup_{\theta \in \Theta} \| q(\theta) - Q(\theta) \|$. Then by Assumption 1(ii),

$$C_{2,n} = o_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}. \quad (SA.4)$$

Applying the first order expansion, we get

$$g(\hat{\theta}) = g(\theta_0) + q(\tilde{\theta}) n^{1/2} (\hat{\theta} - \theta_0), \quad (SA.5)$$

where $\tilde{\theta}$ is the mean value between $\theta_0$ and $\hat{\theta}$ and it may vary across rows. By Assumption 1(ii), the consistency of $\hat{\theta}$ and (SA.4),

$$q(\hat{\theta}) = Q(\tilde{\theta}) + o_p(1) = Q + o_p(1) = O_p(1), \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. \quad (SA.6)$$

Similarly, we can show that

$$q(\hat{\theta}) = Q(\hat{\theta}) + o_p(1) = Q + o_p(1) = O_p(1), \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. \quad (SA.7)$$
By Assumption 2(i, ii) and the consistency of \( \hat{\theta} \),

\[
\hat{\Omega} \equiv \hat{\Omega}(\hat{\theta}) = \Omega + o_p(1) \tag{SA.8}
\]

uniformly over \( P \in \mathcal{P}_0 \). Applying the chain rule, we get the first order condition of \( \hat{\theta} \):

\[
0_{d_\theta \times 1} = 2q(\hat{\theta})'\hat{\Omega}^{-1}g(\hat{\theta}) - \left( n^{-1/2}g(\hat{\theta})'\hat{\Omega}^{-1}\frac{\partial \hat{\Omega}(\hat{\theta})}{\partial \theta_j}\hat{\Omega}^{-1}g(\hat{\theta}) \right)_{j=1,...,d_\theta}, \tag{SA.9}
\]

where \((a_j)_{j=1,...,d_\theta} \equiv (a_1, \ldots, a_{d_\theta})'\) for any real numbers \(a_1, \ldots, a_{d_\theta}\). By Assumptions 1(i, iii), 2(iv) and 3(iii), the consistency of \( \hat{\theta} \), (SA.5), (SA.6), (SA.7) and (SA.8),

\[
0_{d_\theta \times 1} = q(\hat{\theta})'\hat{\Omega}^{-1}g(\hat{\theta}) = q(\hat{\theta})'\hat{\Omega}^{-1}g(\theta_0) + q(\hat{\theta})'\hat{\Omega}^{-1}q(\hat{\theta})n^{1/2}(\hat{\theta} - \theta_0) \\
= Q'\Omega^{-1}g(\theta_0) + (Q'\Omega^{-1}Q + o_p(1)) n^{1/2}(\hat{\theta} - \theta_0) + o_p(1) \tag{SA.10}
\]

uniformly over \( P \in \mathcal{P}_0 \). Similarly, we can show that for \( j = 1, \ldots, d_\theta \),

\[
n^{-1/2}g(\hat{\theta})'\hat{\Omega}^{-1}\frac{\partial \hat{\Omega}(\hat{\theta})}{\partial \theta_j}\hat{\Omega}^{-1}g(\hat{\theta}) = n^{1/2}(\hat{\theta} - \theta_0)o_p(1) + o_p(1) \text{ uniformly over } P \in \mathcal{P}_0. \tag{SA.11}
\]

Combining the results in (SA.9), (SA.10), (SA.11), and applying Assumptions 1(iii) and 3(iii), we deduce that

\[
n^{1/2}(\hat{\theta} - \theta_0) = -(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}g(\theta_0) + o_p(1) = O_p(1) \tag{SA.12}
\]

uniformly over \( P \in \mathcal{P}_0 \), which proves the first claim of the lemma.

(b) This claim follows by Assumptions 1 and 3(iii), (SA.5), (SA.6) and (SA.12).

(c) This claim has been proved in (SA.7).

(d) By Assumptions 2(iv) and 3(iii), and (SA.8),

\[
\lim_{n \to \infty} \inf_{P \in \mathcal{P}_0} \mathbb{P} \left( K^{-1} \leq \lambda_{\text{min}}(\hat{\Omega}) \leq \lambda_{\text{max}}(\hat{\Omega}) \leq K \right) = 1. \tag{SA.13}
\]

By Assumptions 1(iii) and 3(iii), (SA.7) and (SA.8), we have

\[
q(\hat{\theta})'\hat{\Omega}^{-1}q(\hat{\theta}) = Q'\Omega^{-1}Q + o_p(1) \text{ uniformly over } P \in \mathcal{P}_0, \tag{SA.14}
\]

which together with Assumption 3(iii) implies that

\[
\lim_{n \to \infty} \inf_{P \in \mathcal{P}_0} \mathbb{P} \left( K^{-1} \leq \lambda_{\text{min}}(q(\hat{\theta})'(\hat{\Omega})^{-1}q(\hat{\theta})) \leq \lambda_{\text{max}}(q(\hat{\theta})'(\hat{\Omega})^{-1}q(\hat{\theta})) \leq K \right) = 1. \tag{SA.15}
\]
Let $\|\cdot\|_S$ denote the matrix operator norm. By Exercise 7.2.18 in Horn and Johnson (1990),
\[
\|\hat{\Omega}^{1/2} - \Omega^{1/2}\|_S \leq \|\hat{\Omega} - \Omega\|_S \|\Omega^{-1}\|_S,
\]  
(SA.16)
which together with Assumption 3(iii) and (SA.8) implies that
\[
\|\hat{\Omega}^{1/2} - \Omega^{1/2}\|_S = o_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. 
\]  
(SA.17)
By (SA.17) and the relation between the operator norm and the Frobenius norm,
\[
\|\hat{\Omega}^{1/2} - \Omega^{1/2}\| = o_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. 
\]  
(SA.18)
By Assumptions 1(iii) and 3(iii), (SA.7) and (SA.18),
\[
\hat{\Omega}^{-1/2} q(\hat{\theta}) = \Omega^{-1/2} Q + o_p(1) = O_p(1)
\]  
(SA.19)
uniformly over $\mathbb{P} \in \mathcal{P}_0$. The claim in the lemma follows by (SA.14), (SA.15) and (SA.19).

(e) By Assumption 2(i, ii) and the consistency of $\hat{\theta}$,
\[
\sup_{\theta \in \Theta} \|\hat{\Omega}(\theta, \hat{\theta}) - \Omega(\theta, \theta_0)\| = o_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. 
\]  
(SA.20)
By (SA.13) and (SA.20),
\[
\sup_{\theta \in \Theta} \left\| (\hat{\Omega}(\theta, \hat{\theta}) - \Omega(\theta, \theta_0))\hat{\Omega}^{-1} \right\| \leq (\lambda_{\min}(\hat{\Omega}))^{-1} \sup_{\theta \in \Theta} \left\| \hat{\Omega}(\theta, \hat{\theta}) - \Omega(\theta, \theta_0) \right\| = o_p(1) 
\]  
(SA.21)
uniformly over $\mathbb{P} \in \mathcal{P}_0$, where the inequality is by the Cauchy-Schwarz inequality. Similarly,
\[
\sup_{\theta \in \Theta} \left\| \Omega(\theta, \theta_0)(\hat{\Omega}^{-1} - \Omega^{-1}) \right\| = \sup_{\theta \in \Theta} \left\| \Omega(\theta, \theta_0)\hat{\Omega}^{-1}(\hat{\Omega}^{-1} - \Omega^{-1}) \right\|
\]  
\[
\leq (\lambda_{\min}(\hat{\Omega})\lambda_{\min}(\Omega))^{-1} \sup_{\theta, \hat{\theta} \in \Theta} \|\Omega(\theta, \hat{\theta})\| \|\hat{\Omega}^{-1} - \Omega^{-1}\| = o_p(1) 
\]  
(SA.22)
uniformly over $\mathbb{P} \in \mathcal{P}_0$, where the inequality is by the Cauchy-Schwarz inequality, and the last equality is by Assumptions 2(iv) and 3(iii), (SA.8) and (SA.13). Collecting the results in (SA.21) and (SA.22), we deduce that
\[
\sup_{\theta \in \Theta} \left\| \hat{\Omega}(\theta) - \Omega(\theta) \right\| \leq \sup_{\theta \in \Theta} \left\| (\hat{\Omega}(\theta, \hat{\theta}) - \Omega(\theta, \theta_0))\hat{\Omega}^{-1} \right\| + \sup_{\theta \in \Theta} \left\| \Omega(\theta, \theta_0)(\hat{\Omega}^{-1} - \Omega^{-1}) \right\| = o_p(1)
\]  
(SA.23)
Proof of Lemma A3. To link $\hat{\xi}$ and $\xi^*$, we first define

$$
\tilde{\xi} \equiv (\tilde{v}', \tilde{m}(\cdot)', \text{vec}(V(\cdot))', \text{vech}(\Omega)' , \text{vech}(\Omega_0(\cdot))', \text{vech}(M)')',
$$

(SA.25)

where $\tilde{v} \equiv \Omega^{1/2}M\Omega^{-1/2}g(\theta_0)$ and $\tilde{m}(\cdot) \equiv g_0(\cdot) - V(\cdot)\tilde{v}$. The difference between $\tilde{\xi}$ and $\xi^*$ lie in the first two elements, where $\tilde{v}$ and $\tilde{m}(\cdot)$ in $\tilde{\xi}$ involve the empirical process $g(\cdot)$, but $\Omega^{1/2}Mv^*$ and $m^*(\cdot)$ in $\xi^*$ involve the limiting Gaussian process $\psi(\cdot)$. Under Assumption 1(i), $g(\cdot) - E[g(\cdot)]$ weakly converges to $\psi(\cdot)$, which is equivalent to

$$
\lim_{n \to \infty} \sup_{P \in \mathcal{P}_0} \sup_{f \in BL_1} \||E[f(g - E[g])] - E[f(\psi)]|| = 0.
$$

(SA.26)

Furthermore, Assumptions 1(iii), 2(iv) and 3(iii) imply that $\tilde{v}$ and $\tilde{m}(\cdot)$ are bounded and Lipschitz in $g(\cdot)$, which together with (SA.26) implies that

$$
\lim_{n \to \infty} \sup_{P \in \mathcal{P}_0} \sup_{f \in BL_1} \||E[f(\tilde{\xi})] - E[f(\xi^*)]|| = 0.
$$

(SA.27)

Next, note that the difference between $\tilde{\xi}$ and $\tilde{\xi}$ is that $\hat{V}(\theta)$, $\hat{\Omega}$ and $\hat{M}$ in $\tilde{\xi}$ are replaced by their probability limits in Lemma A2(c, d, e) of Cheng, Dou, and Liao (2020). Thus, we have $\hat{\xi} = \tilde{\xi} + o_p(1)$ uniformly over $P \in \mathcal{P}_0$ following Lemma A2(c, d, e) in Cheng, Dou, and Liao (2020), which implies

$$
\lim_{n \to \infty} \sup_{P \in \mathcal{P}_0} \sup_{f \in BL_1} \||E[f(\hat{\xi})] - E[f(\tilde{\xi})]|| = 0.
$$

(SA.28)

The desirable result follows from (SA.27), (SA.28), and the triangle inequality. Q.E.D.

Proof of Lemma A7. Since $0 \leq L(v; d^*) \leq v'Mv$ for any $v \in \mathbb{R}^k$ and $ut_C(u) \leq C$ for any $u \geq 0$, we have

$$
|L_C(v; d^*) - \overline{L}_C(v; d^*)| = L(v; d^*)tc(v'Mv)I\{|v|^2 > C\} \leq CI\{|v|^2 > C\}
$$

(SA.29)

for any $v \in \mathbb{R}^k$, which implies that

$$
P\left(|L_C(v^*, d^*) - \overline{L}_C(v^*, d^*)| > \varepsilon\right) \leq P\left(I\{|v^*|^2 > C\} > \varepsilon/C\right) \leq P\left(|v^*|^2 > C\right).
$$

(SA.30)
Since \( \|v^*\|^2 \) follows the Chi-square distribution with degree of freedom \( k \), there exists a finite constant \( C_\delta \) such that \( P(\|v^*\|^2 > C_\delta) \leq \delta/4 \) which together with (SA.30) implies that for any \( C \geq C_\delta \)

\[
P \left( |L_C(v^*, d^*) - \mathcal{L}_C(v^*, d^*)| > \varepsilon \right) \leq \delta/4. \tag{SA.31}
\]

By the union bound of probability and (SA.31), we have for any \( C \geq C_\delta \),

\[
P (L_C(v^*, d^*) > c_{a,C}(d^*) + \varepsilon) \leq P (\mathcal{T}_C(v^*, d^*) + |L_C(v^*, d^*) - \mathcal{L}_C(v^*, d^*)| > c_{a,C}(d^*) + \varepsilon)
\leq P (\mathcal{T}_C(v^*, d^*) > c_{a,C}(d^*)) + \delta/4 \leq \alpha + \delta/4, \tag{SA.32}
\]

where the last inequality is by the definition of \( c_{a,C}(d^*) \). Since \( R_C(\xi^*) = L_C(v^*, d^*) \) by definition, the claim of the lemma follows from (SA.32).

**Proof of Lemma A8.** (a). Since \( v^* = O_p(1) \), by Assumptions 1(iii), 2(iv) and 3(iii), Lemma A2(b, d) in Cheng, Dou, and Liao (2020), and (SA.18), we have uniformly over \( \mathbb{P} \in \mathcal{P}_0 \),

\[
\hat{\Omega}^{1/2} \hat{M} v^* - g(\hat{\theta}) = \Omega^{1/2} M \Omega^{-1/2} (\Omega^{1/2} v^* - g(\theta_0)) + o_p(1) = O_p(1), \tag{SA.33}
\]

which together with Lemma A2(e) in Cheng, Dou, and Liao (2020) implies that

\[
\sup_{\theta \in \Theta} \left| \hat{V}(\theta) (\hat{\Omega}^{1/2} \hat{M} v^* - g(\hat{\theta})) \right| = O_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. \tag{SA.34}
\]

Therefore by the triangle inequality, Assumption 1 and (SA.34),

\[
\sup_{\theta \in \Theta} \left| n^{-1/2} (\hat{m}(\theta) + \hat{V}(\theta) \hat{\Omega}^{1/2} \hat{M} v^*) - \mathcal{E}[g_0(\theta)] \right| \\
\leq \sup_{\theta \in \Theta} \left| n^{-1/2} g_0(\theta) - \mathcal{E}[g_0(\theta)] \right| + \sup_{\theta \in \Theta} \left| n^{-1/2} \hat{V}(\theta) (\hat{\Omega}^{1/2} \hat{M} v^* - g(\hat{\theta})) \right| = o_p(1) \tag{SA.35}
\]

uniformly over \( \mathbb{P} \in \mathcal{P}_0 \). By Assumptions 2(i) and 4, and (SA.35), we can apply similar arguments in the proof of (SA.3) to deduce that

\[
\hat{\theta}^* = \theta_0 + o_p(1) \tag{SA.36}
\]

uniformly over \( \mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00} \). Applying the chain rule, we get the first order condition of \( \hat{\theta}^* \):

\[
0_{d_\theta \times 1} = 2 \left( \frac{\partial}{\partial \theta'} (\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*) v^*) \right)' \\
(\hat{\Omega}_0^*)^{-1} (\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*) v^*) \\
- \left( \hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*) v^* \right)' (\hat{\Omega}_0^*)^{-1} \frac{\partial \hat{\Omega}_0(\hat{\theta}^*)}{\partial \theta_j} (\hat{\Omega}_0^*)^{-1} (\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*) v^*) \right)_{j=1, \ldots, d_\theta}, \tag{SA.37}
\]
where \( \hat{\Omega}^*_0 \equiv \hat{\Omega}^*_0(\hat{\theta}^*) \) and \( \hat{V}_S(\hat{\theta}^*) \equiv \hat{V}(\hat{\theta}^*)\hat{\Omega}^{1/2}\hat{M} \). Using (SA.36) and similar arguments for showing (SA.5) and (SA.6), we obtain

\[
g_0(\hat{\theta}^*) = g_0(\theta_0) + n^{1/2}(\hat{\theta}^* - \theta_0)(Q_0 + o_p(1)) = O_p(1) + n^{1/2}(\hat{\theta}^* - \theta_0)O_p(1) \quad \text{(SA.38)}
\]

uniformly over \( \mathbb{P} \in \mathcal{P}_0 \). By (SA.34) and (SA.38),

\[
\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)v^* = g_0(\hat{\theta}^*) + \hat{V}(\hat{\theta}^*)(\hat{\Omega}^{1/2}\hat{M}v^* - g(\hat{\theta})) = O_p(1) + n^{1/2}(\hat{\theta}^* - \theta_0)O_p(1), \quad \text{(SA.39)}
\]

which together with Assumptions 2 and 3(iii), (SA.36) and the Cauchy-Schwarz inequality implies that

\[
n^{-1/2}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)v^*)(\hat{\Omega}^*_0)^{-1}\frac{\partial \hat{Q}_0(\hat{\theta}^*)}{\partial \theta_j}(\hat{\Omega}^*_0)^{-1}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)v^*)
\]

\[
= n^{1/2}(\hat{\theta}^* - \theta_0)o_p(1) + o_p(1), \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00}. \quad \text{(SA.40)}
\]

By Assumptions 2(iii, iv) and 3(iii),

\[
\max_{1 \leq j \leq d_\theta} \sup_{\theta \in \Theta} \left\| \hat{V}(\theta)/\partial \theta_j \right\| = O_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0, \quad \text{(SA.41)}
\]

which combined with (SA.33) implies that

\[
\frac{\partial}{\partial \theta^p} \hat{V}(\hat{\theta}^*)(\hat{\Omega}^{1/2}\hat{M}v^* - g(\hat{\theta})) = O_p(1) \quad \text{(SA.42)}
\]

uniformly over \( \mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00} \). By Assumption 1(ii) and (SA.36),

\[
n^{-1/2} \frac{\partial g_0(\hat{\theta}^*)}{\partial \theta^p} = Q_0 + o_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00}. \quad \text{(SA.43)}
\]

Collecting the results in (SA.42) and (SA.43), we have

\[
n^{-1/2} \frac{\partial}{\partial \theta^p}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)v^*)
\]

\[
= n^{-1/2} \frac{\partial}{\partial \theta^p}g_0(\hat{\theta}^*) + n^{-1/2} \frac{\partial}{\partial \theta^p}\hat{V}(\hat{\theta}^*)(\hat{\Omega}^{1/2}\hat{M}v^* - g(\hat{\theta})) = Q_0 + o_p(1) \quad \text{(SA.44)}
\]
uniformly over \( P \in \mathcal{P}_0 \cap \mathcal{P}_00 \). Applying the first order expansion to get

\[
\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)v^* \\
= g_0(\hat{\theta}^*) + \hat{V}(\hat{\theta}^*)(\hat{\Omega}^{1/2}Mv^* - g(\hat{\theta})) \\
= g_0(\theta_0) + \hat{V}(\theta_0)(\hat{\Omega}^{1/2}Mv^* - g(\hat{\theta})) \\
+ \frac{\partial g_0(\hat{\theta}^*)}{\partial \theta'}(\hat{\theta}^* - \theta_0) + \frac{\partial \hat{V}(\hat{\theta}^*)}{\partial \theta'}(\hat{\Omega}^{1/2}Mv^* - g(\hat{\theta}))(\hat{\theta}^* - \theta_0) \\
= g_0(\theta_0) + \hat{V}(\theta_0)(\hat{\Omega}^{1/2}Mv^* - g(\hat{\theta})) + (Q_0 + o_p(1))n^{1/2}(\hat{\theta}^* - \theta_0) + o_p(1) \quad (\text{SA.45})
\]

uniformly over \( P \in \mathcal{P}_0 \cap \mathcal{P}_00 \), where the third equality is by Assumption 1(ii), (SA.33), (SA.36) and (SA.41). By \( V(\theta_0) = S_0 \), Lemma A2(e) in Cheng, Dou, and Liao (2020) and (SA.33),

\[
\hat{V}(\theta_0)(\hat{\Omega}^{1/2}Mv^* - g(\hat{\theta})) = S_0\Omega^{1/2}M\Omega^{-1/2}(\hat{\Omega}^{1/2}v^* - g(\theta_0)) + o_p(1) \quad (\text{SA.46})
\]

uniformly over \( P \in \mathcal{P}_0 \). By (SA.45) and (SA.46), we have uniformly over \( P \in \mathcal{P}_0 \cap \mathcal{P}_00 \),

\[
\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)v^* = (Q_0 + o_p(1))n^{1/2}(\hat{\theta}^* - \theta_0) \\
+ Q_0(\Omega'\Omega^{-1}Q)^{-1}Q'(\Omega'\Omega^{-1}g(\theta_0)) + S_0\Omega^{1/2}Mv^* + o_p(1). \quad (\text{SA.47})
\]

By Assumptions 3(iii) and 4(ii), \( Q_0'\Omega_0^{-1}Q_0 \) is positive definite. Therefore collecting the results in (SA.37), (SA.40), (SA.44), (SA.47), and applying Assumption 2(i, ii) and (SA.36) to \( \hat{\Omega}_0^* \), we obtain

\[
n^{1/2}(\hat{\theta}^* - \theta_0) = -(Q'\Omega^{-1}Q)^{-1}Q'(\Omega'\Omega^{-1}g(\theta_0)) - (Q_0'\Omega_0^{-1}Q_0)^{-1}Q_0'\Omega_0^{-1}S_0\Omega^{1/2}Mv^* + o_p(1) \quad (\text{SA.48})
\]

uniformly over \( P \in \mathcal{P}_0 \cap \mathcal{P}_00 \), which proves part (a) of the lemma.

(b) By (SA.48), Assumptions 3(iii) and 4,

\[
n^{1/2}(\hat{\theta}^* - \theta_0) = O_p(1) \quad \text{uniformly over } P \in \mathcal{P}_0 \cap \mathcal{P}_00. \quad (\text{SA.49})
\]

By Assumptions 2(iv), 3(iii) and 4, (SA.47) and (SA.48), we have uniformly over \( P \in \mathcal{P}_0 \cap \mathcal{P}_00 \),

\[
\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)v^* = -Q_0(Q_0'\Omega_0^{-1}Q_0)^{-1}Q_0'\Omega_0^{-1}S_0\Omega^{1/2}Mv^* + S_0\Omega^{1/2}Mv^* + o_p(1) \\
= \Omega_0^{1/2}M_0\Omega_0^{-1/2}S_0\Omega^{1/2}Mv^* + o_p(1) \\
= \Omega_0^{1/2}M_0\Omega_0^{-1/2}S_0\Omega^{1/2}v^* + o_p(1) = O_p(1), \quad (\text{SA.50})
\]

where \( M_0 \equiv I_{k_0} - \Omega_0^{-1/2}Q_0(Q_0'\Omega_0^{-1}Q_0)^{-1}Q_0'\Omega_0^{-1/2} \) and the last equality is by \( M_0\Omega_0^{-1/2}Q_0 = 0_{k_0 \times 1} \).
By Assumptions 2(i, ii) and 3(iii), (SA.36) and (SA.50), we deduce that uniformly over \( P \in \mathcal{P}_0 \cap \mathcal{P}_00 \),

\[
(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)v^*)(\Omega_0^{-1/2}S_0\Omega_0^{1/2})^{-1}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)v^*) = v^*\tilde{M}_0v^* + o_p(1), \tag{SA.51}
\]

where \( \tilde{M}_0 \equiv (\Omega_0^{-1/2}S_0\Omega_0^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega_0^{1/2}) \). Since \( v^* = O_p(1) \), Lemma A2(e) implies that

\[
v^*\tilde{M}_0v^* = v^*Mv^* + o_p(1) \tag{SA.52}
\]

uniformly over \( P \in \mathcal{P}_0 \), which together with (SA.51) finishes the proof.

(c). Since \( M \equiv I_k - \Omega^{-1/2}Q(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1/2}, M^2 = M \). Moreover,

\[
\tilde{M}_0^2 = (\Omega_0^{-1/2}S_0\Omega_0^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega_0^{1/2})(\Omega_0^{-1/2}S_0\Omega_0^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega_0^{1/2}) = \tilde{M}_0 \tag{SA.53}
\]

and

\[
\tilde{M}_0M = \tilde{M}_0 - (\Omega_0^{-1/2}S_0\Omega_0^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega_0^{1/2})\Omega^{-1/2}Q(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1/2}
\]

\[
= (\Omega_0^{-1/2}S_0\Omega_0^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega_0^{1/2}) = \tilde{M}_0, \tag{SA.54}
\]

where the second equality is by \( M_0\Omega_0^{-1/2}Q_0 = 0_{k_0 \times 1} \). Similarly, \( MM_0 = \tilde{M}_0 \). Therefore, \((M - \tilde{M}_0)^2 = M^2 - M\tilde{M}_0 - \tilde{M}_0M + \tilde{M}_0^2 = M - \tilde{M}_0 \) which implies that \( M - \tilde{M}_0 \) is an idempotent matrix. The rank of \( M - \tilde{M}_0 \) equals the trace of \( M - \tilde{M}_0 \) since \( M - \tilde{M}_0 \) is idempotent. By the definition of \( M \) and \( \tilde{M}_0 \),

\[
tr(M) = k - tr(\Omega^{-1/2}Q(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1/2}) = k - d_\theta \tag{SA.55}
\]

and

\[
tr(\tilde{M}_0) = tr((\Omega_0^{-1/2}S_0\Omega_0^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega_0^{1/2}))
\]

\[
= tr(M_0(\Omega_0^{-1/2}S_0\Omega_0^{1/2})'S_0\Omega_0^{-1/2}S_0\Omega_0^{1/2}) = tr(M_0) = k_0 - d_\theta, \tag{SA.56}
\]

which implies that \( tr(M - \tilde{M}_0) = k - k_0 = k_1 \). Therefore, \( M - \tilde{M}_0 \) is an idempotent matrix with rank \( k_1 \) which together with \( v^* \sim N(0, I_k) \) proves the claim (c).

\[Q.E.D.\]

**SB  Proofs for Bounded Lipschitz Conditions**

This section contains the proofs of Lemmas A5 and A6 in Cheng, Dou, and Liao (2020) under their Assumptions 2 and 3, and some auxiliary results used to show them.

**Lemma SB1.** Consider \( \xi \equiv (x', d')' \) where \( d \) satisfies Assumptions 2 and 3. Then we have for any
given $C > 0$:

(i) $R(\xi)$ is bounded and Lipschitz in $\xi$ on the set $\{\xi : R(\xi) \geq 0 \text{ and } x'\Omega_d^{-1}x \leq C\}$;

(ii) $L(v; d)$ is bounded and Lipschitz in $d$ on the set $\{(v, d) : L(v; d) \geq 0 \text{ and } ||v|| \leq C\}$.

**Proof of Lemma SB1.** (i) Let $S_\xi = \{\xi : R(\xi) \geq 0 \text{ and } x'\Omega_d^{-1}x \leq C\}$. By the definition of $R(\xi)$, $R(\xi) \leq x'\Omega_d^{-1}x \leq C$ for any $\xi \in S_\xi$, which shows that $R(\xi)$ is bounded on $S_\xi$. Next, we want to show that for any $\xi_1, \xi_2 \in S_\xi$,

$$|R(\xi_1) - R(\xi_2)| \leq C_R ||\xi_1 - \xi_2||_s$$

(SB.57)

for some constant fixed $C_R$. By the triangle inequality,

$$|R(\xi_1) - R(\xi_2)| \leq R(\xi_1) + R(\xi_2) \leq x_2'\Omega_d^{-1}x_2 + x_1'\Omega_d^{-1}x_1 = 2C,$$  

(SB.58)

which implies that the claimed result holds with a Lipschitz constant $C_R = 2$ if $||\xi_1 - \xi_2||_s > C$. Thus, it is only necessary to consider the case that $||\xi_1 - \xi_2||_s \leq C$.

Define $\bar{A}_j(\theta) \equiv m_{d,j}(\theta) + V_{d,j}(\theta)x_j$ for $j = 1, 2$. Consider any $\xi_1, \xi_2 \in S_\xi$, by the triangle inequality,

$$|R(\xi_1) - R(\xi_2)| \leq \left|x_1'\Omega_d^{-1}x_1 - x_2'\Omega_d^{-1}x_2\right| + \min_{\theta \in \Theta} A_1(\theta)'(\Omega_{0,d,1}(\theta))^{-1}A_1(\theta) - \min_{\theta \in \Theta} A_2(\theta)'(\Omega_{0,d,2}(\theta))^{-1}A_2(\theta)$$ .

(SB.59)

By the triangle inequality, the Cauchy-Schwarz inequality, $x_1'\Omega_d^{-1}x_1 \leq C$ and $x_2'\Omega_d^{-1}x_2 \leq C$,

$$\left|x_1'\Omega_d^{-1}x_1 - x_2'\Omega_d^{-1}x_2\right| \leq \left|x_1 - x_2\right|'(\Omega_d^{-1}x_1) + \left|x_2'\Omega_d^{-1}(\Omega_{d,1} - \Omega_{d,2})\Omega_d^{-1}x_1\right| + \left|x_2'\Omega_d^{-1}(x_1 - x_2)\right|$$

$$\leq \left[\frac{(x_1'\Omega_d^{-1}x_1)^{1/2}}{(\lambda_{\min}(\Omega_{d,1}))^{1/2}} + \frac{(x_2'\Omega_d^{-1}x_2)^{1/2}}{(\lambda_{\min}(\Omega_{d,2}))^{1/2}}\right]||x_1 - x_2||$$

$$+ \frac{(x_1'\Omega_d^{-1}x_1)^{1/2}(x_2'\Omega_d^{-1}x_2)^{1/2}}{(\lambda_{\min}(\Omega_{d,1}))^{1/2}(\lambda_{\min}(\Omega_{d,2}))^{1/2}} ||\Omega_{d,1} - \Omega_{d,2}||$$

$$\leq 2c^1_{\lambda}C^{1/2}||x_1 - x_2|| + c^1_{\lambda}C ||\Omega_{d,1} - \Omega_{d,2}|| \leq C_1 ||\xi_1 - \xi_2||_s$$  

(SB.60)

for some fixed constant $C_1$.

Let $\theta_j$ denote the minimizer of $A_j(\theta)'(\Omega_{0,d,j}(\theta))^{-1}A_j(\theta)$ for $j = 1, 2$. By the triangle inequality,
we have
\[
\begin{align*}
&\left| \min_{\theta \in \Theta} A_1(\theta)'(\Omega_{0,d,1}(\theta))^{-1}A_1(\theta) - \min_{\theta \in \Theta} A_2(\theta)'(\Omega_{0,d,2}(\theta))^{-1}A_2(\theta) \right| \\
&\leq \max_{\{\theta_1, \theta_2\}} \left| A_1(\theta)'(\Omega_{0,d,1}(\theta))^{-1}A_1(\theta) - A_2(\theta)'(\Omega_{0,d,2}(\theta))^{-1}A_2(\theta) \right| \\
&\leq \max_{\{\theta_1, \theta_2\}} \left| (A_1(\theta) - A_2(\theta))'(\Omega_{0,d,1}(\theta))^{-1}A_1(\theta) \right| \\
&+ \max_{\{\theta_1, \theta_2\}} \left| A_2(\theta)'(\Omega_{0,d,1}(\theta))^{-1}(A_1(\theta) - A_2(\theta)) \right| \\
&+ \max_{\{\theta_1, \theta_2\}} \left| A_2(\theta)'((\Omega_{0,d,1}(\theta))^{-1} - (\Omega_{0,d,2}(\theta))^{-1})A_2(\theta) \right| . \quad \text{(SB.61)}
\end{align*}
\]

We next investigate the three terms after the second inequality of (SB.61) one by one. By the triangle inequality and the Cauchy-Schwarz inequality,
\[
\begin{align*}
&\max_{\{\theta_1, \theta_2\}} \|A_1(\theta) - A_2(\theta)\| \\
&\leq \sup_{\theta \in \Theta} \|m_{d,1}(\theta) - m_{d,2}(\theta)\| + \|x_1\| \sup_{\theta \in \Theta} \|V_{d,1}(\theta) - V_{d,2}(\theta)\| + \sup_{\theta \in \Theta} \|V_{d,2}(\theta)\| \|x_1 - x_2\| \\
&\leq \|\xi_1 - \xi_2\|_s + \lambda_{\max}(\Omega_{d,1})(x_1'\Omega_{d,1}^{-1}x_1)^{1/2} \|\xi_1 - \xi_2\|_s + C_V \|\xi_1 - \xi_2\|_s , \quad \text{(SB.62)}
\end{align*}
\]
where \(C_V \equiv c_\lambda^{-1}C_\Omega\) and the second inequality is by the definition of \(\|\xi_1 - \xi_2\|_s\) and \(\sup_{\theta \in \Theta} \|V_{d,2}(\theta)\| \leq c_\lambda^{-1}C_\Omega\) (which is proved in Lemma A2 of Cheng, Dou, and Liao (2020)). Therefore, by Assumption 2(iv) and (SB.62),
\[
\begin{align*}
&\max_{\{\theta_1, \theta_2\}} \|A_1(\theta) - A_2(\theta)\| \leq (1 + (CC_\Omega)^{1/2} + C_V) \|\xi_1 - \xi_2\|_s , \quad \text{(SB.63)}
\end{align*}
\]
which together with the triangle inequality, Assumption 2(iv) and \(\|\xi_1 - \xi_2\|_s \leq C\) implies that
\[
\begin{align*}
&\max_{\{\theta_1, \theta_2\}} \|A_1(\theta)\| \leq \|A_1(\theta_1)\| + \|A_1(\theta_2)\| \\
&\leq \|A_1(\theta_1)\| + \|A_2(\theta_2)\| + \|A_1(\theta_2) - A_2(\theta_2)\| \\
&\leq C_\Omega^{1/2} \left((A_1(\theta)'(\Omega_{d,1,0}(\theta))^{-1}A_1(\theta))^{1/2} + (A_2(\theta)'(\Omega_{d,2,0}(\theta))^{-1}A_2(\theta))^{1/2}\right) \\
&+ (1 + (CC_\Omega)^{1/2} + C_V) \|\xi_1 - \xi_2\|_s \\
&\leq 2(C_\Omega C)^{1/2} + (1 + (CC_\Omega)^{1/2} + C_V)C . \quad \text{(SB.64)}
\end{align*}
\]
By the same arguments, the inequality in (SB.64) applies to \(\max_{\{\theta_1, \theta_2\}} \|A_2(\theta)\|\). By the Cauchy-
Schwarz inequality, Assumption 3(iii), \((\text{SB.63})\) and \((\text{SB.64})\),
\[
\max_{\{\theta_1, \theta_2\}} \left| (A_1(\theta) - A_2(\theta))' (\Omega_{d,1,0}(\theta))^{-1} A_1(\theta) \right|
\leq (\lambda_{\min}(\Omega_{0,d,1}(\theta)))^{-1} \max_{\{\theta_1, \theta_2\}} \| A_1(\theta) - A_2(\theta) \| \max_{\{\theta_1, \theta_2\}} \| A_1(\theta) \| \leq C_2 \| \xi_1 - \xi_2 \|_s \tag{\text{SB.65}}
\]
for some constant fixed \(C_2\). Similarly, we can show that
\[
\max_{\{\theta_1, \theta_2\}} \left| A_2(\theta)'(\Omega_{0,d,1}(\theta))^{-1} (A_1(\theta) - A_2(\theta)) \right| \leq C_2 \| \xi_1 - \xi_2 \|_s, \tag{\text{SB.66}}
\]
By the Cauchy-Schwarz inequality, Assumption 3(iii) and \((\text{SB.64})\),
\[
\max_{\{\theta_1, \theta_2\}} \left| A_2(\theta)'((\Omega_{0,d,1}(\theta))^{-1} - (\Omega_{0,d,2}(\theta))^{-1}) A_2(\theta) \right|
\leq \frac{\max_{\{\theta_1, \theta_2\}} \| A_2(\theta) \|^2 \| \Omega_{0,d,1}(\theta) - \Omega_{0,d,2}(\theta) \|}{(\inf_{\theta \in \Theta} \lambda_{\min}(\Omega_{0,d,1}(\theta))) (\inf_{\theta \in \Theta} \lambda_{\min}(\Omega_{0,d,2}(\theta)))^{-1}} \leq C_3 \sup_{\theta \in \Theta} \| \Omega_{0,d,1}(\theta) - \Omega_{0,d,2}(\theta) \| \leq (2C_2 + C_3) \| \xi_1 - \xi_2 \|_s, \tag{\text{SB.67}}
\]
for some constant \(C_3\). Collecting the results in \((\text{SB.61})\), \((\text{SB.65})\), \((\text{SB.66})\) and \((\text{SB.67})\), we get
\[
\left| \min_{\theta \in \Theta} A_1(\theta)'(\Omega_{0,d,1}(\theta))^{-1} A_1(\theta) - \min_{\theta \in \Theta} A_2(\theta)'(\Omega_{0,d,2}(\theta))^{-1} A_2(\theta) \right| \leq (2C_2 + C_3) \| \xi_1 - \xi_2 \|_s, \tag{\text{SB.68}}
\]
which together with \((\text{SB.59})\) and \((\text{SB.60})\) implies that the lipschitz constant is \(C_R = C_1 + 2C_2 + C_3\).

(ii) Note that under the condition \(L(v; d) \geq 0\) and \(\| v \| \leq C\), we have \(0 \leq L(v; d) \leq v'M_d v \leq C^2\). To show \(L(v; d)\) is Lipschitz in \(d\), we write
\[
L(v; d) \equiv v'M_d v - \min_{\theta \in \Theta} (m_d(\theta) + V_{d,S}(\theta)v)' (\Omega_{d,0}(\theta))^{-1} (m_d(\theta) + V_{d,S}(\theta)v), \tag{\text{SB.69}}
\]
where \(V_{d,S}(\cdot) \equiv V_{d}(\cdot) \Omega_{d}^{1/2} M_d\). This functional form is analogous to \(R(\xi)\), with \(\Omega_d\) and \(V_d(\cdot)\) in \(R(\xi)\) replaced by \(M_d\) and \(V_{d,S}(\cdot)\), respectively. Given that \(V_{d,S}(\cdot)\) is Lipschitz in \(d\) (established in Lemma \text{SC4} below) and \(\sup_{\theta \in \Theta} \| V_{d,S}(\theta) \| \leq C_{d}^{1/2} C_V\) (by Assumptions 2(iv) and 3(iii)), showing \(L(v; d)\) is Lipschitz in \(d\) is analogous to showing \(R(\xi)\) is Lipschitz in \(\xi\). The only difference is that \(M_d\) is not a full rank matrix, unlike \(\Omega_d\), which is the reason that we have to bound \(\| v \|\) directly instead of bounding \(v'M_d v\). Because \((\text{SB.60})\) in the proof of part (i) uses the full rank condition of \(\Omega_d\), we replace \((\text{SB.60})\) with the following argument to show \(v'M_d v\) is Lipschitz in \(M_d\). Given \(\| v \| \leq C\), we have
\[
|v'M_{d,1} v - v'M_{d,2} v| = |v'(M_{d,1} - M_{d,2}) v| \leq \| v \|^2 \| M_{d,1} - M_{d,2} \| \leq C^2 \| M_{d,1} - M_{d,2} \|, \tag{\text{SB.70}}
\]
because \(M_{d,1}\) and \(M_{d,2}\) are both idempotent. The rest of the proof is analogous to those in the
Proof of Lemma SB1(i) and hence is omitted. \textit{Q.E.D.}

**Proof of Lemma A5.** The truncation function \( t_C(u) \) satisfies the following properties: (i) for any \( u \in \mathbb{R}^+ \), \( 0 \leq t_C(u) \leq 1 \) and \( ut_C(u) \leq u \); (ii) for any \( u_1, u_2 \in \mathbb{R} \), \( |t_C(u_1) - t_C(u_2)| \leq C^{-1} |u_1 - u_2| \), which implies that \( t_C(u) \) is Lipschitz. Therefore,

\[
0 \leq R_C(\xi) \leq (x'\Omega_d^{-1}x) t_C(x'\Omega_d^{-1}x) \leq C \quad \text{(SB.71)}
\]

which means that \( R_C(\xi) \) is bounded.

Next, we show that \( R_C(\xi) \) is Lipschitz in \( \xi \). That is for any \( \xi_j \) with \( R(\xi_j) \geq 0 \) \( (j = 1, 2) \),

\[
|R_C(\xi_1) - R_C(\xi_2)| \leq C_R \|\xi_1 - \xi_2\|, \quad \text{(SB.72)}
\]

where \( C_R \) is a finite constant. Without loss of generality, we assume that \( x_2'\Omega_d^{-1}x_2 \leq x_1'\Omega_d^{-1}x_1 \).

By the triangle inequality,

\[
|R_C(\xi_1) - R_C(\xi_2)| \leq \left| (R(\xi_1) - R(\xi_2))t_C(x_1'\Omega_d^{-1}x_1) \right| + \left| t_C(x_1'\Omega_d^{-1}x_1) - t_C(x_2'\Omega_d^{-1}x_2) \right| R(\xi_2). \quad \text{(SB.73)}
\]

We have (SB.72) holds with \( C_R = C_{R_1} + C_{R_2} \) if we can show that

\[
\left| (R(\xi_1) - R(\xi_2))t_C(x_1'\Omega_d^{-1}x_1) \right| \leq C_{R_1} \|\xi_1 - \xi_2\| \quad \text{(SB.74)}
\]

and

\[
\left| t_C(x_1'\Omega_d^{-1}x_1) - t_C(x_2'\Omega_d^{-1}x_2) \right| \leq C_{R_2} \|\xi_1 - \xi_2\| \quad \text{(SB.75)}
\]

for some finite constants \( C_{R_1} \) and \( C_{R_2} \).

We first consider (SB.74). First, note that it holds trivially if \( x_1'\Omega_d^{-1}x_1 > 2C \) because, in this case, \( t_C(x_1'\Omega_d^{-1}x_1) = 0 \). Second, given \( x_2'\Omega_d^{-1}x_2 \leq x_1'\Omega_d^{-1}x_1 \leq 2C \), we deduce that

\[
\left| (R(\xi_1) - R(\xi_2))t_C(x_1'\Omega_d^{-1}x_1) \right| \leq |R(\xi_1) - R(\xi_2)| \leq C_{R_1} \|\xi_1 - \xi_2\|, \quad \text{(SB.76)}
\]

where the first inequality is by property (i) of \( t_C(u) \), and the second inequality is by Lemma SB1(i).

Next, we show (SB.75). First, note that it holds trivially if \( 2C < x_2'\Omega_d^{-1}x_2 \). In this case,

\[
t_C(x_1'\Omega_d^{-1}x_1) = t_C(x_2'\Omega_d^{-1}x_2) = 0 \quad \text{(SB.77)}
\]

following the definition of \( t_C(u) \). Second, given \( x_2'\Omega_d^{-1}x_2 \leq 2C \), we have

\[
\left| t_C(x_1'\Omega_d^{-1}x_1) - t_C(x_2'\Omega_d^{-1}x_2) \right| R(\xi_2) \leq |R(\xi_2)| \leq x_2'\Omega_d^{-1}x_2 \leq 2C. \quad \text{(SB.78)}
\]

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Thus, (SB.75) holds with $C_{R_2} = 1$ if $\|\xi_1 - \xi_2\|_s > 2C$. Third, it remains to consider the case where $x_2 \Omega_{d,2}^{-1} x_2 \leq 2C$ and $\|\xi_1 - \xi_2\|_s \leq 2C$. In this case, Lemma SC3 in the Section SC implies that $x_1 \Omega_{d,1}^{-1} x_1 \leq C^*$ for some finite constant $C^*$. In this case,

$$\left| t_C(x_1 \Omega_{d,1}^{-1} x_1) - t_C(x_2 \Omega_{d,2}^{-1} x_2) \right| R(\xi_2) \leq t_C(x_1 \Omega_{d,1}^{-1} x_1) - t_C(x_2 \Omega_{d,2}^{-1} x_2) = 2C \right| x_1 \Omega_{d,1}^{-1} x_1 - x_2 \Omega_{d,2}^{-1} x_2 \right|$$

(SB.79)

using $0 \leq R(\xi_2) \leq x_2 \Omega_{d,2}^{-1} x_2 \leq 2C$ and $|t_C(x_1) - t_C(x_2)| \leq C^{-1}|x_1 - x_2|$ which follows from property (ii) of $t_C(x)$. Then we can show $|x_1 \Omega_{d,1}^{-1} x_1 - x_2 \Omega_{d,2}^{-1} x_2| \leq C_{R_2} \|\xi_1 - \xi_2\|_s$ for some constant $C_{R_2}$ by the same arguments that show (SB.60) but with $x_1 \Omega_{d,1}^{-1} x_1 \leq 2C$ replaced by $x_1 \Omega_{d,1}^{-1} x_1 \leq C^*$. Q.E.D.

**Lemma SB2.** Given $L(v; d) \geq 0$, $\mathcal{T}_C(v; d)$ is bounded and Lipschitz in $d$.

**PROOF OF LEMMA SB2.** The proof is analogous to that of Lemma A5 with the truncation function $t_C(x_1 \Omega_{d,1}^{-1} x)$ replaced by $t_C(v M_d v) I\{|v|^2 \leq C\}$ and Lemma SB1(i) replaced by Lemma SB1(ii), and hence is omitted. Q.E.D.

**PROOF OF LEMMA A6.** Since $0 \leq L(v; d) \leq v M_d v$ for any $v \in \mathbb{R}^k$ and $u t_C(u) \leq C$ for any $u \geq 0$, we have

$$\mathcal{T}_C(v; d) = L(v; d) t_C(v M_d v) I\{|v|^2 \leq C\} \leq L(v; d) t_C(v M_d v) \leq C,$$

(SB.80)

which implies that $c_{\alpha, C}(d)$ is bounded. For any $v$, any $d_1$ and $d_2$, by Lemma SB2 there exists a finite constant $C_L$ such that

$$\left| \mathcal{T}_C(v; d_1) - \mathcal{T}_C(v; d_2) \right| \leq C_L \|d_1 - d_2\|_s \cdot$$

(SB.81)

Since $\mathcal{T}_C(v; d_1) \geq \mathcal{T}_C(v; d_2) - C_L \|d_1 - d_2\|_s$ for any $v$ and $P^* (\mathcal{T}_C(v^*; d_1) > c_{\alpha, C}(d_1)) \leq \alpha$, we have $P (\mathcal{T}_C(v^*; d_2) > c_{\alpha, C}(d_1) + C_L \|d_1 - d_2\|_s) \leq \alpha$, which implies that

$$c_{\alpha, C}(d_2) \leq c_{\alpha, C}(d_1) + C_L \|d_1 - d_2\|_s \cdot$$

(SB.82)

Similarly, we also have

$$c_{\alpha, C}(d_1) \leq c_{\alpha, C}(d_2) + C_L \|d_1 - d_2\|_s \cdot$$

(SB.83)

Combining (SB.82) and (SB.83), we get

$$|c_{\alpha, C}(d_1) - c_{\alpha, C}(d_2)| \leq C_L \|d_1 - d_2\|_s \cdot$$

(SB.84)
which shows the claim of the lemma. \( Q.E.D. \)

**SC Additional Auxiliary Lemmas**

This section contains the proof of Lemma A4 in Cheng, Dou, and Liao (2020) and some other auxiliary results.

**Proof of Lemma A4.** For any square matrices \( A_1 \) and \( A_2 \), let \( \text{diag}(A_1, A_2) \) denote the block diagonal matrix created by aligning the input matrices \( A_1 \) and \( A_2 \) along the diagonal. Since \( \hat{\theta} \in \Theta \) is the minimizer of \( g(\theta)'(\hat{\Omega}(\theta))^{-1}g(\theta) \),

\[
\mathcal{T} \geq g(\hat{\theta})'\hat{\Omega}^{-1}g(\hat{\theta}) - g_0(\hat{\theta})'\hat{\Omega}_0^{-1}g_0(\hat{\theta}) = g(\hat{\theta})'\left(\hat{\Omega}^{-1} - \text{diag}\left(\hat{\Omega}_0^{-1}, 0_{k_1 \times k_1}\right)\right)g(\hat{\theta}) \quad (\text{SC.85})
\]

Let \( \hat{\Omega}_{0,1} \) denote the upper-right \( k_0 \times k_1 \) submatrix of \( \hat{\Omega} \) and \( \hat{\Omega}_{1,0} \equiv \hat{\Omega}_{0,1}^\top \). Since \( \hat{\Omega} \) is positive definite,

\[
\hat{\Omega}(\hat{\Omega}^{-1} - \text{diag}(\hat{\Omega}_0^{-1}, 0_{k_1 \times k_1}))\hat{\Omega} = \text{diag}(0_{k_0 \times k_0}, \hat{\Omega}_1 - \hat{\Omega}_{1,0}\hat{\Omega}_0^{-1}\hat{\Omega}_{0,1}) \quad (\text{SC.86})
\]

is a positive semi-definite matrix, which together with (SC.85) proves the first claim of the lemma.

To prove the second claim, we first notice that \( \hat{m}(\hat{\theta}) = g_0(\hat{\theta}) - \hat{V}(\hat{\theta})g(\hat{\theta}) = 0_{k_0 \times 1} \) which implies that for any \( v \in \mathbb{R}^k \)

\[
L(v; \hat{\theta}) \geq v'\hat{M}v - v'\hat{V}_S(\hat{\theta})\hat{\Omega}_0^{-1}\hat{V}_S(\hat{\theta})v, \quad (\text{SC.87})
\]

where \( \hat{V}_S(\hat{\theta}) \equiv S_0\hat{\Omega}^{1/2}\hat{M} \). Therefore

\[
L(x; \hat{\theta}) \geq v'\hat{M}\hat{\Omega}_1^{1/2}\left(\hat{\Omega}^{-1} - \text{diag}(\hat{\Omega}_0^{-1}, 0_{k_1 \times k_1})\right)\hat{\Omega}_1^{1/2}\hat{M}v \geq 0, \quad (\text{SC.88})
\]

where the second inequality holds since the matrix in (SC.86) is positive semi-definite. \( Q.E.D. \)

**Lemma SC3.** For any \( \xi_1 \) and \( \xi_2 \) with \( x_2'\Omega_{d,2}^{-1}x_2 \leq C \) and \( \|\xi_1 - \xi_2\|_s \leq C \), where \( C \) is a finite constant, we have \( x_1\Omega_{d,1}^{-1}x_1 \leq C^* \) for some finite constant \( C^* \), which depends the constant \( C \) of the lemma, \( C_{\Omega} \) and \( c_\lambda \) in Assumptions 2(iv) and 3(iii) respectively.

**Proof of Lemma SC3.** Since \( \Omega_{d,1}^{-1} \) is symmetric and positive definite under Assumption 3(iii),

\[
x_1\Omega_{d,1}^{-1}x_1 \leq 2x_2'\Omega_{d,2}^{-1}x_2 + 2(x_1 - x_2)'\Omega_{d,1}^{-1}(x_1 - x_2)
\]

\[
= 2x_2'\Omega_{d,2}^{-1}x_2 + x_2'(\Omega_{d,1}^{-1} - \Omega_{d,2}^{-1})x_2 + 2(x_1 - x_2)'\Omega_{d,1}^{-1}(x_1 - x_2)
\]

\[
\leq 2C + 2x_2'(\Omega_{d,1}^{-1} - \Omega_{d,2}^{-1})x_2 + 2(x_1 - x_2)'\Omega_{d,1}^{-1}(x_1 - x_2), \quad (\text{SC.89})
\]
where the second inequality is by \( x_2' \Omega_{d,2}^{-1} x_2 \leq C \) as assumed in the lemma. By Assumption 3(iii) and \( \|\xi_1 - \xi_2\|_s \leq C \),

\[
(x_1 - x_2)' \Omega_{d,1}^{-1} (x_1 - x_2) \leq (\lambda_{\min}(\Omega_{d,1}))^{-1} \|x_1 - x_2\|^2 \leq C^2 c_\lambda^{-1}. \tag{SC.90}
\]

Similarly, by Assumption 3(iii) and \( \|\xi_1 - \xi_2\|_s \leq C \),

\[
\left| x_2'(\Omega_{d,1}^{-1} - \Omega_{d,2}^{-1}) x_2 \right|^2 = \left| x_2' \Omega_{d,1}^{-1} (\Omega_{d,1} - \Omega_{d,2}) \Omega_{d,2}^{-1} x_2 \right|^2 \leq \left( x_2' \Omega_{d,1}^{-1} x_2 \right) \left( x_2' \Omega_{d,2}^{-1} x_2 \right) \|\Omega_{d,1} - \Omega_{d,2}\|^2 \leq \frac{\lambda_{\max}(\Omega_{d,1})}{\lambda_{\min}(\Omega_{d,2})} \left( \lambda_{\min}(\Omega_{d,1}) \right)^2 \|\Omega_{d,1} - \Omega_{d,2}\|^2 \leq C^4 c_\lambda^{-3} C_\Omega, \tag{SC.91}
\]

where the last inequality is by \( x_2' \Omega_{d,2}^{-1} x_2 \leq C \), Assumption 3(iii) and \( \|\xi_1 - \xi_2\|_s \leq C \). The claim of the lemma follows from (SC.89)-(SC.91). \( Q.E.D. \)

**Lemma SC4.** For any \( \xi_1 \) and \( \xi_2 \), define \( V_{d,j}(\theta) \equiv V_{d,j}(\theta) \Omega_{d,j}^{1/2} M_{d,j} \) for \( j = 1, 2 \). Then we have

\[
\sup_{\theta \in \Theta} \|V_{d,1}(\theta) - V_{d,2}(\theta)\| \leq C_\Omega^{1/2} \left( 1 + c_\lambda^{-1} C_\Omega^{1/2} + c_\lambda^{-1} C_\Omega \right) \|\xi_1 - \xi_2\|_s, \tag{SC.92}
\]

where \( C_\Omega \) and \( c_\lambda \) are in Assumptions 2(iv) and 3(iii) respectively.

**Proof of Lemma SC4.** By definition,

\[
V_{d,1}(\theta) - V_{d,2}(\theta) = [V_{d,1}(\theta) - V_{d,2}(\theta)] \Omega_{d,1}^{1/2} M_{d,1} + V_{d,2}(\theta)(\Omega_{d,1}^{1/2} - \Omega_{d,2}^{1/2}) M_{d,1} + V_{d,2}(\theta) \Omega_{d,2}^{1/2} (M_{d,1} - M_{d,2}). \tag{SC.93}
\]

By the properties of \( \Omega_{d,j} \) and \( M_{d,j} \),

\[
\sup_{\theta \in \Theta} \left\| [V_{d,1}(\theta) - V_{d,2}(\theta)] \Omega_{d,1}^{1/2} M_{d,1} \right\| \leq C_\Omega^{1/2} \sup_{\theta \in \Theta} \|V_{d,1}(\theta) - V_{d,2}(\theta)\|. \tag{SC.94}
\]

By the properties of \( V_{d,j}(\theta) \) and \( M_{d,j} \), and Exercise 7.2.18 in Horn and Johnson (1990),

\[
\sup_{\theta \in \Theta} \left\| V_{d,2}(\theta)(\Omega_{d,1}^{1/2} - \Omega_{d,2}^{1/2}) M_{d,1} \right\| \leq \sup_{\theta \in \Theta} \|V_{d,2}(\theta)\| \left\| \Omega_{d,1}^{1/2} - \Omega_{d,2}^{1/2} \right\|_S \leq c_\lambda^{-1} C_\Omega \left\| \Omega_{d,1}^{1/2} - \Omega_{d,2}^{1/2} \right\| \leq C_\Omega^{3/2} \|\Omega_{d,1} - \Omega_{d,2}\|. \tag{SC.95}
\]

Similarly,

\[
\sup_{\theta \in \Theta} \left\| V_{d,2}(\theta) \Omega_{d,2}^{1/2} (M_{d,1} - M_{d,2}) \right\| \leq C_\Omega^{1/2} c_\lambda^{-1} C_\Omega^{3/2} \|M_{d,1} - M_{d,2}\|. \tag{SC.96}
\]

The the desirable result follows by (SC.94)-(SC.96) and the triangle inequality. \( Q.E.D. \)
SD Derivation of Asset Pricing Moments in Examples

SD.1 Equilibrium Solution of the Long-Run Risk Model

The stochastic discount factor (SDF) can be expressed as follows:

\[ M_{t+1} = \delta^{\vartheta} \left( \frac{C_{t+1}}{C_t} \right)^{-\vartheta / \psi} R_{c,t+1}, \quad \text{with} \quad \vartheta \equiv \frac{1}{1 - 1/\psi}, \]  

(SD.1)

where \( R_{c,t+1} \) is the return on the consumption claim. The log SDF can be written as

\[ m_{t+1} = \vartheta \log \delta - \frac{\vartheta}{\psi} \Delta c_{t+1} + (\vartheta - 1)r_{t+1}. \]  

(SD.2)

The state variable in the simplest long-run risk model is \( x_t \). To turn the system into an affine model, we first exploit the Campbell-Shiller log-linearization approximation:

\[ r_{t+1} = \kappa_0 + \kappa_1 z_{t+1} + \Delta c_{t+1} - z_t, \]  

(SD.3)

where \( z_t = \log(W_t/C_t) \) is the log wealth-consumption ratio and wealth is the “price” of consumption claims. The log-linearization constants are determined by long-run steady state:

\[ \kappa_0 = \log(1 + e^{\overline{z}}) - \kappa_1 \overline{z} \quad \text{and} \quad \kappa_1 = \frac{e^{\overline{z}}}{1 + e^{\overline{z}}}, \]  

(SD.4)

where \( \overline{z} \) is the mean of the log price-consumption ratio. Given the log-linearization approximation (SD.3) – (SD.4), we can search the equilibrium characterized by

\[ z_t = A_0 + A_1 x_t, \]  

(SD.5)

where the constants \( A_0 \) and \( A_1 \) are to be determined by the equilibrium conditions.

Thus, the log return on consumption claim can be written as

\[ r_{t+1} = \kappa_0 + \kappa_1 (A_0 + A_1 x_{t+1}) + \Delta c_{t+1} - (A_0 + A_1 x_t). \]  

(SD.6)

Therefore, the log SDF can be re-written in terms of state variables and exogenous shocks

\[ m_{t+1} = \Gamma_0 + \Gamma_1 x_t - \lambda_x \sigma_x \varepsilon_{c,t+1} - \lambda_x \psi \varepsilon_{x,t+1}, \]  

(SD.7)

where predictive coefficients are

\[ \Gamma_0 = \log \delta \quad \text{and} \quad \Gamma_1 = -\psi^{-1}, \]  

(SD.8)
and the market price of risk coefficients are

$$\lambda_c = \gamma \quad \text{and} \quad \lambda_x = (\gamma - \psi^{-1}) \frac{\kappa_1 \phi}{1 - \kappa_1 \rho}, \quad \text{(SD.9)}$$

The coefficients $A_j$’s are determined by the equilibrium condition (i.e., the Euler equation for the price of consumption claim) as follows:

$$1 = \mathbb{E}_t [M_{t+1} R_{c,t+1}] = \mathbb{E}_t \left[ e^{m_{t+1} + r_{t+1}} \right], \quad \text{(SD.10)}$$

where $\mathbb{E}_t[\cdot]$ denote the expectation given the information at time $t$. It leads to the equilibrium conditions:

$$A_0 = \frac{1}{1 - \kappa_1} (\log \delta + \kappa_0) \quad \text{and} \quad A_1 = \frac{1 - \psi^{-1}}{1 - \kappa_1 \rho} \phi. \quad \text{(SD.11)}$$

The long-run mean $\overline{z}$ is also determined endogenously in the equilibrium. More precisely, given all parameters fixed, we have $A_j = A_j(\overline{z})$ in equation (SD.11) because $\kappa_0$ and $\kappa_1$ are functions of $\overline{z}$. In the long-run steady state, we have

$$\overline{z} = A_0(\overline{z}) + A_2(\overline{z}) \sigma^2. \quad \text{(SD.12)}$$

Thus, in the equilibrium, the long-run mean $\overline{z}$ is a function of all parameters in the model, according to (SD.12) and the Implicit Function Theorem,

$$\overline{z} = \overline{z}(\rho, \phi, \sigma_c, \gamma, \psi, \delta). \quad \text{(SD.13)}$$

And hence, we can also solve $\kappa_0$ and $\kappa_1$ (whose explicit forms are usually not available) based on (SD.13) as follows,

$$\kappa_0 = \kappa_0(\rho, \phi, \sigma_c, \gamma, \psi, \delta) \quad \text{and} \quad \kappa_1 = \kappa_1(\rho, \phi, \sigma_c, \gamma, \psi, \delta). \quad \text{(SD.14)}$$

The gradients $\kappa_0$ and $\kappa_1$ with respect to the parameters, such as $\rho$ and $\nu$, can be calculated using the Implicit Function Theorem in (SD.12).

From (SD.6) and (SD.11), it follows that

$$r_{t+1} - \mathbb{E}_t[r_{t+1}] = \beta_c \epsilon_{c,t+1} + \beta_x \epsilon_{x,t+1}, \quad \text{(SD.15)}$$

where the betas are

$$\beta_c = 1 \quad \text{and} \quad \beta_x = \kappa_1 A_1. \quad \text{(SD.16)}$$

The Euler equation for the log market return, denoted by $r_{t+1}$, and the risk free rate, denoted
by \( r_{f,t} \), can be written in one equation

\[
\mathbb{E}_t \left[ e^{m_{t+1}} \right] = \mathbb{E}_t \left[ e^{m_{t+1}+r_{t+1}^e} \right],
\]  
(SD.17)

where \( r_{t+1}^e \equiv r_{t+1} - r_{f,t} \) is the excess log return.

The Euler equation (SD.17) leads to

\[
\mathbb{E}_t \left[ r_{t+1}^e \right] = \lambda_c \beta_c \sigma_c^2 + \lambda_x \beta_x - \frac{1}{2} \left( \beta_c^2 \sigma_c^2 + \beta_x^2 \phi^2 \right) = \gamma \sigma_c^2 - \frac{1}{2} \sigma_c^2 + \frac{1}{2} (2 \gamma - \psi^{-1} - 1)(1 - \psi)(1 - \psi^{-1}) \frac{\kappa_1^2 \phi^2}{(1 - \kappa_1 \rho)^2}.
\]  
(SD.18)

Because \( \sigma_c^2 \approx 0 \) and \( \kappa_1 \approx 1 \), the expected log excess return is approximately equal to

\[
\mathbb{E}_t \left[ r_{t+1}^e \right] \approx \frac{1}{2} (2 \gamma - \psi^{-1} - 1)(1 - \psi)(1 - \psi^{-1}) \frac{\phi^2}{(1 - \rho)^2}.
\]  
(SD.19)

The excess log return \( r_{t+1}^e \) has the same conditional exposure to the shocks as the log return \( r_{t+1} \) since the log risk free rate \( r_{f,t} \) has zero conditional exposure to the shocks \( \varepsilon_{c,t+1} \) and \( \varepsilon_{x,t+1} \). Therefore, combining (SD.11), (SD.16), and (SD.19), we can obtain that the excess log return in the equilibrium can be represented as follows:

\[
r_{t+1}^e \approx \frac{1}{2} (2 \gamma - \psi^{-1} - 1)(1 - \psi)(1 - \psi^{-1}) \frac{\phi^2}{(1 - \rho)^2} + \varepsilon_{r,t+1}^e,
\]  
(SD.20)

where \( \varepsilon_{r,t+1}^e \equiv \sigma_c \varepsilon_{c,t+1} + (1 - \psi^{-1})(1 - \rho)^{-1} \phi \varepsilon_{x,t+1} + \sigma_r \varepsilon_{r,t+1} \). The Gaussian shock of the contemporaneous consumption growth is \( \varepsilon_{c,t+1} \); the Gaussian shock of the low-frequency component is \( \varepsilon_{x,t+1} \); and the measurement error \( \varepsilon_{r,t+1} \) is i.i.d. a standard normal variable and is independent of other shocks.

**SD.2 Equilibrium Solution of the Disaster Risk Model**

The dividend-price ratio is constant in the equilibrium, denoted by \( C/P \). The stock return is

\[
R_{t+1} = (C_{t+1} + P_{t+1})/P_t = (C/P + 1)C_{t+1}/C_t = (C/P + 1)e^{\sigma \varepsilon_{t+1} - \xi_{t+1}}.
\]  
(SD.21)

Thus, the expected log stock return is

\[
\mathbb{E}_t \left[ \ln R_{t+1} \right] = \ln(C/P + 1) - p (\psi + 1/\alpha).
\]  
(SD.22)
The equilibrium dividend-price ratio is
\[
P/C = \mathbb{E}_t \left[ e^{-\delta} (C_{t+1}/C_t)^{-\gamma} (P_{t+1}/C_{t+1} + 1) C_{t+1}/C_t \right]
\]
\[
= (P/C + 1) e^{-\delta} \mathbb{E}_t \left[ (C_{t+1}/C_t)^{-1-\gamma} \right]
\]
\[
= (P/C + 1) e^{-\delta + \frac{1}{2}(1-\gamma)^2 \sigma^2} \left[ 1 - p + pe^{-(1-\gamma)\mathbb{E}_t} \frac{\alpha}{\alpha - \gamma + 1} \right]. \quad \text{(SD.23)}
\]
Thus, the term \( \ln(C/P + 1) \) equals to
\[
\ln(C/P + 1) = \delta - \frac{1}{2}(1-\gamma)^2 \sigma^2 - \ln \left[ 1 - p + pe^{-(1-\gamma)\mathbb{E}_t} \frac{\alpha}{\alpha - \gamma + 1} \right]
\]
\[
\approx \delta - \frac{1}{2}(1-\gamma)^2 \sigma^2 + p - pe^{-(1-\gamma)\mathbb{E}_t} \frac{\alpha}{\alpha - \gamma + 1} \quad \text{(SD.24)}
\]
which together with \( r_{t+1} \equiv \ln R_{t+1} \) and (SD.22) implies that the expected log return is
\[
\mathbb{E}_t [r_{t+1}] \approx \delta - \frac{1}{2}(1-\gamma)^2 \sigma^2 - p(\mathbb{E}_t + 1/\alpha) + p - pe^{(\gamma-1)\mathbb{E}_t} \frac{\alpha}{\alpha - \gamma + 1}. \quad \text{(SD.25)}
\]
The equilibrium risk-free rate satisfies that
\[
R_f = \mathbb{E}_t \left[ e^{-\delta} (C_{t+1}/C_t)^{-\gamma} \right]^{-1} = e^{\delta - \frac{1}{2}\gamma^2 \sigma^2} \left[ 1 - p + pe^{\gamma\mathbb{E}_t} \frac{\alpha}{\alpha - \gamma} \right]^{-1}. \quad \text{(SD.26)}
\]
Thus, the log risk-free rate satisfies
\[
r_f \equiv \ln R_f = \delta - \frac{1}{2}\gamma^2 \sigma^2 - \ln \left[ 1 - p + pe^{\gamma\mathbb{E}_t} \frac{\alpha}{\alpha - \gamma} \right] \approx \delta - \frac{1}{2}\gamma^2 \sigma^2 + p - pe^{\gamma\mathbb{E}_t} \frac{\alpha}{\alpha - \gamma}. \quad \text{(SD.27)}
\]
Since the log excess return is defined as \( r_{t+1}^e \equiv r_{t+1} - r_f \), the expected log excess return is
\[
\mathbb{E}_t [r_{t+1}^e] \approx \gamma \sigma^2 - \frac{1}{2} \sigma^2 - p(\mathbb{E}_t + 1/\alpha) + p \alpha \left[ \frac{e^{\gamma\mathbb{E}_t}}{\alpha - \gamma} - \frac{e^{(\gamma-1)\mathbb{E}_t}}{\alpha - \gamma + 1} \right]. \quad \text{(SD.28)}
\]
Further, because \( \sigma^2 \approx 0 \), the key implication of the rare disaster risk model is
\[
\mathbb{E}_t [r_{t+1}^e] \approx -p(\mathbb{E}_t + 1/\alpha) + p \alpha \left[ \frac{e^{\gamma\mathbb{E}_t}}{\alpha - \gamma} - \frac{e^{(\gamma-1)\mathbb{E}_t}}{\alpha - \gamma + 1} \right]. \quad \text{(SD.29)}
\]
The excess log return \( r_{t+1}^e \) has the same conditional exposure to the shocks as the log return \( r_{t+1} \) since the log risk free rate \( r_f \) has zero exposure to the shocks. Therefore, the excess log return in
the equilibrium can be represented as follows:

$$r_{t+1}^e \approx -p(\gamma + 1/\alpha) + p\alpha \left[ \frac{e^{\gamma \rho}}{\alpha - \gamma} - \frac{e^{(\gamma-1)\rho}}{\alpha - \gamma + 1} \right] + \varepsilon_{r,t+1},$$

(SE.30)

where $\varepsilon_{r,t+1} \equiv \sigma \varepsilon_{t+1} - [x_{t+1}(\gamma + J_{t+1}) - p(\gamma + 1/\alpha)] + \sigma \varepsilon_{r,t+1}$. The Gaussian shock of the consumption growth is $\varepsilon_{t+1}$; the jump shock of the consumption growth is $- [x_{t+1}(\gamma + J_{t+1}) - p(\gamma + 1/\alpha)]$; and the measurement error $\varepsilon_{r,t+1}$ is a i.i.d. standard normal variable and is independent of other shocks.

**SE Gaussian Limit in the Long-Run Risk Model**

For the long-run risk model, we show that the Gaussian limit for the moment conditions is an innocuous assumption even if $\rho_n \to 1$ and $\phi_n \to 0$ at any rate, as long as $\theta_n = (1 - \rho_n) / \phi_n$ is bounded from above and below by some finite positive constants, where the subscript $n$ denotes the sample size $n$.

We start with the first moment condition. The observed process $\Delta c_t$ satisfies

$$\Delta c_{t+1} - \rho \Delta c_t = \phi_n (\rho_n x_{t-1} + \varepsilon_{x,t}) + \sigma_c \varepsilon_{c,t+1} - \rho (\phi_n x_{t-1} + \sigma_c \varepsilon_{c,t})$$

$$= \phi_n (\rho_n - \rho) x_{t-1} + \phi_n \varepsilon_{x,t} + \sigma_c \varepsilon_{c,t+1} - \rho \sigma_c \varepsilon_{c,t}. \quad (SE.1)$$

Multiply the first difference $\Delta c_{t+1} - \rho \Delta c_t$ in (SE.1) by $\Delta c_{t-1} = \phi_n x_{t-2} + \sigma_c \varepsilon_{c,t-1}$. We obtain

$$M_{1t} \equiv \Delta c_{t-1} (\Delta c_{t+1} - \rho \Delta c_t) = M_{1a,t} + M_{1b,t} + M_{1c,t} + M_{1d,t} + \mu_1, \quad (SE.2)$$

where

$$\mu_1 \equiv \mathbb{E} [\Delta c_{t-1} (\Delta c_{t+1} - \rho \Delta c_t)] = \phi_n^2 (\rho_n - \rho) \rho_n \mathbb{E}[x_{t-2}^2],$$

$$M_{1a,t} \equiv \phi_n^2 (\rho_n - \rho) \rho_n \left( x_{t-2}^2 - \mathbb{E}[x_{t-2}^2] \right),$$

$$M_{1b,t} \equiv \phi_n (\rho_n - \rho) \left( \sigma_c x_{t-1} \varepsilon_{c,t-1} + \phi_n x_{t-2} \varepsilon_{x,t-1} \right),$$

$$M_{1c,t} \equiv \phi_n x_{t-2} (\phi_n \varepsilon_{x,t} + \sigma_c \varepsilon_{c,t+1} - \rho \sigma_c \varepsilon_{c,t}),$$

$$M_{1d,t} \equiv \sigma_c \varepsilon_{c,t-1} (\phi_n \varepsilon_{x,t} + \sigma_c \varepsilon_{c,t+1} - \rho \sigma_c \varepsilon_{c,t}). \quad (SE.3)$$

The moment condition satisfies

$$M_1 \equiv n^{-1/2} \sum_{t=1}^n (M_{1t} - \mathbb{E}[M_{1t}]) = M_{1a} + M_{1b} + M_{1c} + M_{1d}, \quad (SE.4)$$

where $M_{1j} \equiv n^{-1/2} \sum_{t=1}^n M_{1j,t}$ for $j = a, b, c, d$. 

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Below we consider three separate cases: (i) \( \rho_n \) is bounded away from 1; (ii) \( \rho_n \) converges to 1 at the rate slower than \( n^{-1} \), i.e., \((1 - \rho_n)n \to \infty\); (iii) \( \rho_n \) converges to 1 at the rate \( n^{-1} \) or faster, i.e., \((1 - \rho_n)n \to c \in [0, \infty)\). In all three cases, \( \theta_n = (1 - \rho_n)/\phi_n \) is bounded below from 0 and above from infinity. Therefore, \( \phi_n \) always converges to 0 at the same rate at which \( \rho_n \) converges to 1.

In case (i), we can apply the central limit theorem (CLT) for weakly dependent triangular arrays and \( M_1 \) has a Gaussian limit.

Now we consider case (ii). We assume the initial condition satisfies \( \mathbb{E}[x_0^2] = o(n^{1/2}) \) as in GP. Following Lemma 1 of Giraitis and Phillips (2006) (hereafter GP),

\[
\frac{(1 - \rho_n^2)^{1/2}}{n^{-1/2}} \sum_{t=1}^{n} x_{t-1} \epsilon_{x,t} \to_d N(0, 1)
\]

( SE.5 )

using \( \text{Var}(\epsilon_{x,t}) = 1 \). We assume the initial condition satisfies \( \mathbb{E}[x_0^2] = o(n^{1/2}) \) as in GP. Following the proof for Lemma 1 of GP we also have

\[
\frac{(1 - \rho_n^2)^{1/2}}{n^{-1/2}} \sum_{t=1}^{n} x_{t-1} \epsilon_{x,t+1} \to_d N(0, 1),
\]

\[
\frac{(1 - \rho_n^2)^{1/2}}{n^{-1/2}} \sum_{t=1}^{n} x_{t-1} \epsilon_{c,t+j} \to_d N(0, \sigma_c^2),
\]

( SE.6 )

for any \( j \) following the independence of \( \epsilon_{x,t} \) and \( \epsilon_{c,t'} \) for any \( t \neq t' \). Following equation (20) in the proof of Lemma 2 of GP,

\[
n^{-1/2} \sum_{t=1}^{n} (x_{t-1}^2 - \mathbb{E}[x_{t-1}^2]) = (1 - \rho_n^2)^{-3/2} 2\rho_n Z_1 + (1 - \rho_n^2)^{-1} Z_2 + Z_3,
\]

( SE.7 )

where

\[
Z_1 \equiv \frac{(1 - \rho_n^2)^{1/2}}{n^{1/2}} \sum_{t=1}^{n} x_{t-1} \epsilon_{x,t} \to_d N(0, 1),
\]

\[
Z_2 \equiv n^{-1/2} \sum_{t=1}^{n} (\epsilon_{x,t}^2 - 1) \to_d N(0, V_{x^2}),
\]

\[
Z_3 \equiv \frac{x_0^2 - \mathbb{E}[x_0^2]}{n^{1/2}(1 - \rho_n^2)} - \frac{x_n^2 - \mathbb{E}[x_n^2]}{n^{1/2}(1 - \rho_n^2)},
\]

( SE.8 )

where the convergence for \( Z_1 \) holds by (SE.5), the convergence for \( Z_2 \) follows from the CLT with
\( V_x^2 \) being the variance of \( \epsilon_{x,t}^2 \). Then, we have \( M_{1a} = o_p(1) \) because

\[
\phi_n^2 \left( 1 - \rho_n^2 \right)^{-3/2} 2 \rho_n Z_1 = o_p(1),
\]
\[
\phi_n^2 \left( 1 - \rho_n^2 \right)^{-1} Z_2 = o_p(1),
\]
\[
\phi_n^2 Z_3 = o_p(1),
\]
(SE.9)

which in turn holds because \( \theta_n = (1 - \rho_n) / \phi_n \) is bounded, \( x_0^2 = o_p(n^{1/2}) \) by the initial condition, and \( \phi_n^2 \left( 1 - \rho_n^2 \right)^{-1} n^{-1/2} x_n^2 = o_p(1) \) by Lemma 3 of GP and the Markov inequality.

The other terms satisfy

\[
M_{1b} = \phi_n O_p((1 - \rho_n^2)^{-1/2}) = o_p(1),
\]
\[
M_{1c} = \phi_n O_p((1 - \rho_n^2)^{-1/2}) = o_p(1),
\]
(SE.10)

by \( \phi_n = O((1 - \rho_n)) \), (SE.5) and (SE.6). Then we can apply the CLT for triangular array of i.i.d. random variables to \( M_{1d} \) and the Gaussian limit holds for \( M_1 \).

Finally, we consider the case (iii), where \( n(1 - \rho_n) \to c \) for \( c \in [0, \infty) \). Following Phillips (1987),

\[
n^{-2} \sum_{t=1}^{n} x_{t-1}^2 = O_p(1), \quad n^{-1} \sum_{t=1}^{n} x_{t-1} \epsilon_{x, t} = O_p(1).
\]
(SE.11)

We also have

\[
n^{-1} \sum_{t=1}^{n} x_{t-1} \epsilon_{x, t+1} = O_p(1) \quad \text{and} \quad n^{-1} \sum_{t=1}^{n} x_{t-1} \epsilon_{c, t+j} = O_p(1)
\]
(SE.12)

for any \( j \), because \( \epsilon_{x, t} \) is i.i.d. with mean zero and it is independent of \( \epsilon_{c, t+j} \), following results for a vector process, see Park and Phillips (1988). Therefore,

\[
M_{1a} = \phi_n^2 \left[ O_p(n^{3/2}) - O(n^{1/2}(1 - \rho_n^2)^{-1}) \right] = o_p(1),
\]
\[
M_{1b} = \phi_n O_p(n^{1/2}) = o_p(1) \quad \text{and} \quad M_{1c} = \phi_n O_p(n^{1/2}) = o_p(1),
\]
(SE.13)

because \( \phi_n = O(n^{-1}) \). As in case (ii), we can apply the CLT to \( M_{1d} \) to obtain the Gaussian limit for \( M_1 \).

Next, we consider multiplying the first difference \( \Delta_c t+1 - \rho \Delta c_t \) in (SE.1) by \( \Delta c_t = \phi_n x_{t-1} + \sigma_c \epsilon_{c, t} \). We obtain

\[
M_{2,t} \equiv \Delta_c t (\Delta c_{t+1} - \rho \Delta c_t) = M_{2a,t} + M_{2b,t} + M_{2c,t} + M_{2d,t} + \mu_2,
\]
(SE.14)
where
\[
\begin{align*}
\mu_2 &\equiv \mathbb{E} [\Delta c_t (\Delta c_{t+1} - \rho \Delta c_t)] = \phi_n^2 (\rho_n - \rho) \mathbb{E}[x_{t-1}^2] - \rho \sigma_c^2, \\
M_{2a,t} &\equiv \phi_n^2 (\rho_n - \rho) (x_{t-1}^2 - \mathbb{E}[x_{t-1}^2]), \\
M_{2b,t} &\equiv \phi_n (\rho_n - \rho) \sigma_c x_{t-1} \epsilon_{c,t}, \\
M_{2c,t} &\equiv \phi_n x_{t-1} \left( \phi_n \epsilon_{x,t} + \sigma_c \epsilon_{c,t+1} - \rho \sigma_c \epsilon_{c,t} \right), \\
M_{2d,t} &\equiv \sigma_c \epsilon_{c,t} \left( \phi_n \epsilon_{x,t} + \sigma_c \epsilon_{c,t+1} - \rho \sigma_c \epsilon_{c,t} \right).
\end{align*}
\] (SE.15)

As above, we consider three cases. In case (i) where \(\rho\) is bounded away from 1, we can apply the CLT directly. In case (ii),
\[
\begin{align*}
M_{2a} &= n^{-1/2} \sum_{t=1}^{n} M_{2a,t} = \phi_n^2 (\rho_n - \rho) n^{-1/2} \sum_{t=1}^{n} (x_{t-1}^2 - \mathbb{E}[x_{t-1}^2]) = o_p(1) \tag{SE.16}
\end{align*}
\]
following the same arguments for \(M_{1a}\). Similarly, we can show that \(M_{2b}\) and \(M_{2c}\), defined similarly to \(M_{1b}\) and \(M_{1c}\) respectively, are both \(o_p(1)\) following the arguments for \(M_{1b}\) and \(M_{1c}\). Finally, the CLT always applies to \(M_{2d}\), the counterpart of \(M_{1d}\). In case (iii), same arguments for \(M_{1a}\) give
\[
M_{2a} = \phi_n^2 \left[ O_p(n^{3/2}) - O(n^{1/2}) (1 - \rho_n^2)^{-1} \right] = o_p(1). \tag{SE.17}
\]

Arguments for \(M_{1b}, M_{1c}\) and \(M_{1d}\) can be used to show the same results hold for their counterparts \(M_{2b}, M_{2c}\) and \(M_{2d}\) respectively.

Finally, the asset pricing moment condition always has a Gaussian limit because it is a location model with an i.i.d. error.

References


