

Online Appendix to “Model Selection in Panel Data Models: A Generalization of the Vuong Test”

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January 31, 2026

Abstract

This supplement consists of four appendices. Appendix 1 collects the proofs of the auxiliary lemmas used to establish the main results in Section 2 of Hahn, Liao, Menzel, and Vuong (2026). Appendix 2 contains the proofs of the auxiliary lemmas used in deriving the main results in Section 3 of Hahn, Liao, Menzel, and Vuong (2026). Appendix 3 provides the proofs of the main results presented in Section C of Hahn, Liao, Menzel, and Vuong (2026). Finally, Appendix 4 presents several illustrative examples based on the linear panel model.

1 Proofs for Lemmas in Appendix A.1 of Hahn, Liao, Menzel, and Vuong (2026)

PROOF OF LEMMA 1. (a) First, by Assumptions 1 and 8(iii), we can write

$$\begin{aligned} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \psi_j(z_{i,t}; \phi_{j,i}^*) \right)^2 \right] &\leq KT^{-1} \sum_{0 \leq h \leq T} \sum_{t \leq T-h} |\text{Cov}(\psi_j(z_{i,t}; \phi_{j,i}^*), \psi_j(z_{i,t+h}; \phi_{j,i}^*))| \\ &\leq KT^{-1} \sum_{0 \leq h \leq T} \sum_{t \leq T-h} \max_{t \leq T} \|\psi_j(z_{i,t}; \phi_{j,i}^*)\|_{2+\delta}^2 \alpha_i^{\delta/(2+\delta)}(h) \\ &\leq \max_{t \leq T} \|\psi_j(z_{i,t}; \phi_{j,i}^*)\|_{2+\delta}^2 \sum_{0 \leq h \leq T} (1 - h/T) a^{\delta h/(2+\delta)} \leq K, (S1) \end{aligned}$$

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where the second inequality follows from the covariance inequality for strong mixing processes (see, e.g., Corollary 14.3 in Davidson (1994)), and the last inequality uses Assumption 1(iv). Therefore,

$$T^{1/2} \sum_{i \leq n} \mathbb{E}[\tilde{V}_{j,i}] = 2 \sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \psi_j(z_{i,t}; \phi_{j,i}^*) \right)^2 \right] \leq K G_j.$$

It follows that

$$\left| T^{1/2} \sum_{i \leq n} \mathbb{E}[\tilde{V}_i] \right| = \left| T^{1/2} \sum_{i \leq n} (\mathbb{E}[\tilde{V}_{1,i}] - \mathbb{E}[\tilde{V}_{2,i}]) \right| \leq K(G_1 + G_2). \quad (\text{S2})$$

Moreover, by Assumptions 8(iii) and 9, and (55) (which follows from Lemma 2 under Assumptions 1 and 2 without imposing Assumption 3), we obtain

$$R_{j,n,T}(\hat{\phi}_j) = \sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \sigma_{j,\gamma,i}^2 + O_p(1) = O_p(G_j). \quad (\text{S3})$$

Combining (S2) and (S3) with Theorem 1, we obtain

$$\begin{aligned} MQLR_{n,T} &= \overline{QLR}_{n,T} + (MQLR_{n,T}^* - \overline{QLR}_{n,T}) \\ &\quad - (R_{1,n,T}(\hat{\phi}_j) - R_{2,n,T}(\hat{\phi}_j)) + T^{1/2} \sum_{i \leq n} \mathbb{E}[\tilde{V}_i] \\ &= \overline{QLR}_{n,T} + O_p(\omega_{n,T} + G_1 + G_2) = \overline{QLR}_{n,T} + O_p(G_1 + G_2), \end{aligned} \quad (\text{S4})$$

where the last equality follows from Lemma 5(ii). From the definition of $\hat{\omega}_{n,T}^2$ and Lemma S20,

$$\hat{\omega}_{n,T}^2 \leq \max \{ \hat{\sigma}_{n,T}^2 + \hat{\sigma}_{U,n,T}^2, \hat{\sigma}_{U,n,T}^2 \} \leq K, \quad (\text{S5})$$

with probability approaching 1. It follows from (S4) and (S5) that

$$\mathbb{E}[\varphi_{n,T}^{2\text{-side}}(p)] = \mathbb{P}(|MQLR_{n,T}| \geq \hat{\omega}_{n,T} z_{1-p/2}) \geq \mathbb{P}(|\overline{QLR}_{n,T}| \geq K z_{1-p/2} + O_p(G_1 + G_2)) - o(1).$$

Since $|\overline{QLR}_{n,T}| \succ G_1 + G_2$, the right-hand side converges to 1. This establishes $\mathbb{E}[\varphi_{n,T}^{2\text{-side}}(p)] \rightarrow 1$, as $n, T \rightarrow \infty$.

(b) The proof for the one-sided test follows by analogous arguments and is therefore omitted. *Q.E.D.*

PROOF OF LEMMA 2. The result in (73) follows from Assumption 1(i) together with Lemma S2, Lemma S4 and Lemma S6. The claim in (74) is obtained by applying Assumption 1(i), Lemma S3, Lemma S5 and Lemma S6. The assertion in (75) is established in Lemma S7. Finally, the last claim of the lemma follows from Lemma S14. *Q.E.D.*

PROOF OF LEMMA 3. To prove the claim in (77), we first note that, by Lemma S12,

$$\begin{aligned}
& n^{-1} \sum_{i \leq n} (\widehat{\mathbb{E}}_T[\Delta\psi(z_{i,t}, \hat{\phi}_i)^2] - \mathbb{E}_T[\Delta\psi(z_{i,t})^2]) \\
&= 2(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \Delta\psi_\phi(z_{i,t}; \phi_i^*) (\hat{\phi}_i - \phi_i^*) \\
&\quad + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_i - \phi_i^*)^\top \Delta\psi_\phi(z_{i,t}; \phi_i^*)^\top \Delta\psi_\phi(z_{i,t}; \phi_i^*) (\hat{\phi}_i - \phi_i^*) \\
&\quad + O_p(T^{-3/2} + \omega_{n,T} T^{-1} + \omega_{n,T}^2 (nT)^{-1/2}).
\end{aligned} \tag{S6}$$

Together with Lemma S8, Lemma S17 and Assumption 1(i), this implies

$$\begin{aligned}
& n^{-1} \sum_{i \leq n} (\widehat{\mathbb{E}}_T[\Delta\psi(z_{i,t}, \hat{\phi}_i)^2] - \mathbb{E}_T[\Delta\psi(z_{i,t})^2]) \\
&= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_i - \phi_i^*)^\top \Delta\psi_\phi(z_{i,t}; \phi_i^*)^\top \Delta\psi_\phi(z_{i,t}; \phi_i^*) (\hat{\phi}_i - \phi_i^*) \\
&\quad + O_p(T^{-3/2} + \omega_{n,T} (nT)^{-1/2}).
\end{aligned} \tag{S7}$$

Under Assumption 3, we have

$$\mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^2 \right] = T^{-1} \sum_{t \leq T} \mathbb{E}[\tilde{\psi}_{j,\gamma}^*(z_{i,t})^2] = \sigma_{j,\gamma,i}^2$$

and

$$T^{-1} \sum_{t, t' \leq T} \mathbb{E}[\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})] = T^{-1} \sum_{t \leq T} \mathbb{E}[\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{1,\gamma}^*(z_{i,t})] = \sigma_{12,\gamma,i},$$

where $\tilde{\psi}_{j,\gamma}^*(z_{i,t}) \equiv \psi_{j,\gamma}^*(z_{i,t}) / (-\Psi_{j,\gamma\gamma,g})^{1/2}$ ($j = 1, 2$), for any $i \in I_{2,g}$ and any $g \in \mathcal{G}_2$. Combining these results with Assumption 1(i), (S7) and Lemma S11 yields

$$\begin{aligned}
& n^{-1} \sum_{i \leq n} (\widehat{\mathbb{E}}_T[\Delta\psi(z_{i,t}, \hat{\phi}_i)^2] - \mathbb{E}_T[\Delta\psi(z_{i,t})^2]) \\
&= (nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(\sum_{i \in I_{2,g}} \sigma_{1,\gamma,i}^4 + n_{2,g}^{-2} \sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \sum_{i \in I_{j,g}} \sigma_{j,\gamma,i}^2 - 2n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{12,\gamma,i}^2 \right) \\
&\quad + O_p(T^{-3/2} + \omega_{n,T} (nT)^{-1/2}).
\end{aligned} \tag{S8}$$

Combining (S8) with the definition of $\sigma_{S,n,T}^2$ establishes the claim in (77). The result in (78) is proved in Lemma S16. *Q.E.D.*

PROOF OF LEMMA 4. The claims in (79), (80), (81) and (82) follow from Lemmas S13, S15, S18 and S19, respectively. *Q.E.D.*

PROOF OF LEMMA 5. The claims in part (i) and part (ii) follow directly from Lemma S1(ii) and Lemma S17, respectively. *Q.E.D.*

1.1 Auxiliary Lemmas

This section provides the lemmas used in the proof of Lemmas 2-4. We begin by recalling notation introduced in the main text and then introduce additional notation. For $j = 1, 2$, and any $g \in \mathcal{G}_j$,

$$\begin{aligned}\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) &\equiv n_{j,g}^{-1} \sum_{i \in I_{j,g}} \hat{\mathbb{E}}_T [\hat{\psi}_{j,\gamma\gamma}(z_{i,t})], \\ \Psi_{j,\gamma\gamma,g} &\equiv n_{j,g}^{-1} \sum_{i \in I_{j,g}} \mathbb{E}_T [\psi_{j,\gamma\gamma}(z_{i,t})], \\ \hat{\Psi}_{j,\gamma,g} &\equiv n_{j,g}^{-1} \sum_{i \in I_{j,g}} \hat{\mathbb{E}}_T [\psi_{j,\gamma}^*(z_{i,t})],\end{aligned}$$

where $\hat{\psi}_{j,\gamma\gamma}(z_{i,t}) \equiv \psi_{j,\gamma\gamma}(z_{i,t}; \hat{\phi}_{j,g})$, $\psi_{j,\gamma}^*(z_{i,t}) \equiv \psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]$, $\psi_{j,\gamma\gamma}(z_{i,t}) \equiv \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g}^*)$ and $\psi_{j,\gamma}(z_{i,t}) \equiv \psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*)$.

Lemma S1. *Suppose Assumptions 1 and 2 hold. For model j ($j = 1, 2$), we have:*

- (i) $\max_{g \in \mathcal{G}_j} |\hat{\gamma}_{j,g} - \gamma_{j,g}^*| = o_p(1)$;
- (ii) $\min_{g \in \mathcal{G}_j} (-\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})) \geq K^{-1}$ wpa1;
- (iii) $\hat{\theta}_j = \theta_j^* + O_p((nT)^{-1/2})$;
- (iv) $\sum_{g \in \mathcal{G}_j} (n_{j,g}T)^{1/2} |\hat{\gamma}_{j,g} - \gamma_{j,g}^*|^s = O_p(G_j)$ for $s \in \{1, \dots, 4\}$;
- (v) $\sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 = O_p(G_j T^{-1})$;
- (vi) $\sum_{g \in \mathcal{G}_j} n_{j,g} |\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}| = O_p((nT^{-1})^{1/2})$;
- (vii) $\sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g})^2 = O_p(T^{-1})$;
- (viii) $\max_{g \in \mathcal{G}_j} n_{j,g}^2 \mathbb{E}[\hat{\Psi}_{j,\gamma,g}^4] = O_p(T^{-2})$.

PROOF OF LEMMA S1. We begin the proof by noting that Assumptions 7, 8, 9 hold for both models 1 and 2 under Assumption 2(i). Additionally, Assumption 10 has been verified for both models under Assumption 1(i) in the proof of Theorem 1. Under these assumptions, the claims in parts (i), (ii), (iii), and (iv) of the lemma are established using Lemma S58, Lemma S60(ii), Lemma S62, and Lemma S65 from Section C, respectively.

To establish the claim in part (v) of the lemma, we begin by applying the Cauchy-Schwarz inequality to get

$$\sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 \leq 2 \sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \hat{\Psi}_{j,\gamma\gamma,g})^2 + 2 \sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\Psi}_{j,\gamma\gamma,g} - \Psi_{j,\gamma\gamma,g})^2, \quad (\text{S9})$$

where $\hat{\Psi}_{j,\gamma\gamma,g} \equiv \hat{\Psi}_{j,\gamma\gamma}(\phi_{j,g}^*)$ and $\hat{\Psi}_{j,\gamma\gamma}(\phi_{j,g}^*)$ is defined analogously to $\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})$ with $\hat{\phi}_{j,g}$ replaced by $\phi_{j,g}^*$. Under Assumptions 1 and 8(iii), we obtain:

$$\mathbb{E} \left[\sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\Psi}_{j,\gamma\gamma,g} - \Psi_{j,\gamma\gamma,g})^2 \right] = \sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \mathbb{E} \left[\left| T^{-1} \sum_{t \leq T} (\psi_{j,\gamma\gamma}(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma\gamma}(z_{i,t})]) \right|^2 \right] \leq K G_j T^{-1}.$$

By Markov's inequality, it follows that:

$$\sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\Psi}_{j,\gamma\gamma,g} - \Psi_{j,\gamma\gamma,g})^2 = O_p(G_j T^{-1}). \quad (\text{S10})$$

Next, using Assumption 8(i) and the Cauchy-Schwarz inequality, we have:

$$\sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \hat{\Psi}_{j,\gamma\gamma}(\phi_{j,g}^*))^2 \leq \left(\sum_{g \in \mathcal{G}_j} n_{j,g}^2 \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4 \right)^{1/2} \times \left(\sum_{g \in \mathcal{G}_j} (n_{j,g} T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} M(z_{i,t})^4 \right)^{1/2}. \quad (\text{S11})$$

The second term in the product satisfies:

$$\sum_{g \in \mathcal{G}_j} (n_{j,g} T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} M(z_{i,t})^4 = O_p(G_j) \quad (\text{S12})$$

by Assumption 8(ii) and Markov's inequality. From parts (i) and (iii) of the lemma, we know:

$$\sum_{g \in \mathcal{G}_j} n_{j,g}^2 \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4 = K \left(\sum_{g \in \mathcal{G}_j} n_{j,g}^2 |\hat{\gamma}_{j,g} - \gamma_{j,g}^*|^4 + n^2 \|\hat{\theta}_j - \theta_j^*\|^4 \right) = O_p(G_j T^{-2}), \quad (\text{S13})$$

which, together with (S11)-(S13), implies that

$$\sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\Psi}_{j,\gamma\gamma,g}(\hat{\phi}_{j,g}) - \hat{\Psi}_{j,\gamma\gamma,g})^2 = O_p(G_j T^{-1}). \quad (\text{S14})$$

The claim in part (v) of the lemma follows from (S9), (S10) and (S14).

To establish the claim in part (vi) of the lemma, observe that from (S623) of Lemma S61:

$$(\hat{\gamma}_{j,g} - \gamma_{j,g}^*) + \frac{\hat{\Psi}_{j,\gamma,g}}{\hat{\Psi}_{j,\gamma\gamma,g}} = -\frac{\hat{\Psi}_{j,\gamma\theta,g}}{\hat{\Psi}_{j,\gamma\gamma,g}} (\hat{\theta}_j - \theta_j^*) - \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \tilde{\Psi}_{j,\gamma\phi\phi,g} (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{2\hat{\Psi}_{j,\gamma\gamma,g}}, \quad (\text{S15})$$

where

$$\hat{\Psi}_{j,\gamma\theta,g} \equiv (n_{j,g} T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\partial^2 \psi_j(z_{i,t}; \phi_{j,g}^*)}{\partial \gamma_{j,g} \partial \theta_j^\top} \text{ and } \tilde{\Psi}_{j,\gamma\phi\phi,g} \equiv (n_{j,g} T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\partial^2 \psi_{j,\gamma}(z_{i,t}; \tilde{\phi}_{j,g})}{\partial \phi_{j,g} \partial \phi_{j,g}^\top}.$$

Here, $\tilde{\phi}_{j,g}$ lies between $\hat{\phi}_{j,g}$ and $\phi_{j,g}^*$. By Assumption 8(ii), the triangle inequality, and Markov's inequality, we have

$$n^{-1} \sum_{g \in \mathcal{G}_j} n_{j,g} \|\hat{\Psi}_{j,\gamma\theta,g}\| \leq (nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \left\| \frac{\partial^2 \psi_j(z_{i,t}; \phi_{j,g}^*)}{\partial \gamma_{j,g} \partial \theta_j^\top} \right\| = O_p(1). \quad (\text{S16})$$

This, combined with parts (ii) and (iii) of the lemma, shows:

$$\left| \sum_{g \in \mathcal{G}_j} n_{j,g} \frac{\hat{\Psi}_{j,\gamma\theta,g}}{\hat{\Psi}_{j,\gamma\gamma,g}} (\hat{\theta}_j - \theta_j^*) \right| \leq \frac{\|\hat{\theta}_j - \theta_j^*\| \sum_{g \in \mathcal{G}_j} n_{j,g} \|\hat{\Psi}_{j,\gamma\theta,g}\|}{\min_{g \in \mathcal{G}_j} |\hat{\Psi}_{j,\gamma\gamma,g}|} = O_p((nT^{-1})^{1/2}). \quad (\text{S17})$$

By the triangle inequality and the Cauchy-Schwarz inequality:

$$\left| \sum_{g \in \mathcal{G}_j} n_{j,g} \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \tilde{\Psi}_{j,\gamma\phi\phi,g} (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{\hat{\Psi}_{j,\gamma\gamma,g}} \right| \leq \frac{\max_{g \in \mathcal{G}_j} \|\tilde{\Psi}_{j,\gamma\phi\phi,g}\|}{\min_{g \in \mathcal{G}_j} |\hat{\Psi}_{j,\gamma\gamma,g}|} \sum_{g \in \mathcal{G}_j} n_{j,g} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2. \quad (\text{S18})$$

In (S628) of the Supplementary Appendix, we have shown that under Assumptions 1 and 7-10:

$$\max_{g \in \mathcal{G}_j} \|\tilde{\Psi}_{j,\gamma\phi\phi,g}\| = O_p(1). \quad (\text{S19})$$

Using parts (iii, iv) of the lemma, we have

$$\sum_{g \in \mathcal{G}_j} n_{j,g} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 \leq K \left(\sum_{g \in \mathcal{G}_j} n_{j,g} |\hat{\gamma}_{j,g} - \gamma_{j,g}^*|^2 + n \|\hat{\theta}_j - \theta_j^*\|^2 \right) = O_p(T^{-1}). \quad (\text{S20})$$

From part (ii) of the lemma and (S18)-(S20), it follows that:

$$\sum_{g \in \mathcal{G}_j} n_{j,g} \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \tilde{\Psi}_{j,\gamma\phi\phi,g} (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{\hat{\Psi}_{j,\gamma\gamma,g}} = O_p(T^{-1}). \quad (\text{S21})$$

Combining this result with (S15) and (S17) establishes the claim in this part.

To establish the claim in part (vii), similar arguments as in (S16) yield:

$$n^{-1} \sum_{g \in \mathcal{G}_j} n_{j,g} \|\hat{\Psi}_{j,\gamma\theta,g}\|^2 \leq (nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \left\| \frac{\partial^2 \psi_j(z_{i,t}; \phi_{j,g}^*)}{\partial \gamma_{j,g} \partial \theta_j^\top} \right\|^2 = O_p(1).$$

Combining this with parts (ii) and (iii) of the lemma, we obtain:

$$\sum_{g \in \mathcal{G}_j} n_{j,g} \left| \frac{\hat{\Psi}_{j,\gamma\theta,g}}{\hat{\Psi}_{j,\gamma\gamma,g}} (\hat{\theta}_j - \theta_j^*) \right|^2 \leq \frac{\|\hat{\theta}_j - \theta_j^*\|^2 \sum_{g \in \mathcal{G}_j} n_{j,g} \|\hat{\Psi}_{j,\gamma\theta,g}\|^2}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma,g}^2} = O_p(T^{-1}). \quad (\text{S22})$$

The Cauchy-Schwarz inequality, part (ii) of the lemma, and earlier bounds in (S13) and (S19) further imply:

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} n_{j,g} \left| \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \tilde{\Psi}_{j,\gamma\phi\phi,g} (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{\hat{\Psi}_{j,\gamma\gamma,g}} \right|^2 \\ & \leq \frac{\max_{g \in \mathcal{G}_j} \|\tilde{\Psi}_{j,\gamma\phi\phi,g}\|^2 \sum_{g \in \mathcal{G}_j} n_{j,g} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma,g}^2} = O_p(G_j T^{-2}). \end{aligned} \quad (\text{S23})$$

The claim in part (vii) of the lemma now follows from (S15), (S22) and (S23).

To establish the claim in part (viii), observe that:

$$n_{j,g}^2 \hat{\Psi}_{j,\gamma,g}^4 = T^{-2} \left((n_{j,g} T)^{-1/2} \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma}^*(z_{i,t}) \right)^4. \quad (\text{S24})$$

Using Rosenthal's inequality (see, e.g., (S553) in Lemma S47) and Assumptions 2(iii), we have

$$\mathbb{E} \left[\left((n_{j,g}T)^{-1/2} \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma}^*(z_{i,t}) \right)^4 \right] \leq K. \quad (\text{S25})$$

The claim follows from (S24) and (S25). $Q.E.D.$

Lemma S2. *Under Assumptions 1 and 2, we have*

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\widehat{\mathbb{E}}_T[\hat{\psi}_{j,\gamma}^2(z_{i,t})] - \widehat{\mathbb{E}}_T[\psi_{j,\gamma}^2(z_{i,t})]}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} = O_p \left((nT^{-1})^{1/2} + G_j T^{-1} \right). \quad (\text{S26})$$

PROOF OF LEMMA S2. Define:

$$\psi_{j,\gamma\phi}(z_{i,t}; \phi_{j,g}) \equiv \frac{\partial \psi_{j,\gamma}(z_{i,t}; \phi_{j,g})}{\partial \phi_{j,g}} \quad \text{and} \quad \psi_{j,\gamma\phi\phi}(z_{i,t}; \phi_{j,g}) \equiv \frac{\partial^2 \psi_{j,\gamma}(z_{i,t}; \phi_{j,g})}{\partial \phi_{j,g} \partial \phi_{j,g}^\top}. \quad (\text{S27})$$

Using the Taylor expansion, we obtain:

$$\begin{aligned} & \sum_{i \in I_{j,g}} \sum_{t \leq T} (\hat{\psi}_{j,\gamma}^2(z_{i,t}) - \psi_{j,\gamma}^2(z_{i,t})) \\ &= 2 \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\phi}^\top(z_{i,t}; \phi_{j,g}^*) (\hat{\phi}_{j,g} - \phi_{j,g}^*) \\ &+ (\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g}) \psi_{j,\gamma\phi}^\top(z_{i,t}; \tilde{\phi}_{j,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*) \\ &+ (\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}; \tilde{\phi}_{j,g}) \psi_{j,\gamma\phi\phi}(z_{i,t}; \tilde{\phi}_{j,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*), \end{aligned} \quad (\text{S28})$$

where $\tilde{\phi}_{j,g}$ lies between $\hat{\phi}_{j,g}$ and $\phi_{j,g}^*$ for any $g \in \mathcal{G}_j$. From this expression, it is evident that (S26) follows if the following conditions are satisfied:

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\phi}^\top(z_{i,t}; \phi_{j,g}^*) (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{(n_{j,g}T) \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} = O_p((nT^{-1})^{1/2} + G_j T^{-1}), \quad (\text{S29})$$

$$\sum_{g \in \mathcal{G}_j} \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g}) \psi_{j,\gamma\phi}^\top(z_{i,t}; \tilde{\phi}_{j,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{(n_{j,g}T) \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} = O_p(G_j T^{-1}), \quad (\text{S30})$$

$$\sum_{g \in \mathcal{G}_j} \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}; \tilde{\phi}_{j,g}) \psi_{j,\gamma\phi\phi}(z_{i,t}; \tilde{\phi}_{j,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{(n_{j,g}T) \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} = O_p(G_j T^{-1}). \quad (\text{S31})$$

We now proceed to verify (S29), (S30) and (S31).

Let $C_{1,j,g}(z_{i,t}) \equiv \psi_{j,\gamma}(z_{i,t}) \psi_{j,\gamma\phi}^\top(z_{i,t}, \phi_{j,g}^*)$ and $C_{1,j,g}^*(z_{i,t}) \equiv C_{1,j,g}(z_{i,t}) - \mathbb{E}[C_{1,j,g}(z_{i,t})]$. Applying Assumptions 1, 8, and the covariance inequality for strong mixing processes, we obtain:

$$\mathbb{E} \left[\left\| T^{-1/2} \sum_{t \leq T} C_{1,j,g}^*(z_{i,t}) \right\|^2 \right] \leq K. \quad (\text{S32})$$

Using the triangle inequality and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
& \left| \sum_{g \in \mathcal{G}_j} \frac{(n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{1,j,g}^*(z_{i,t})}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} (\hat{\phi}_{j,g} - \phi_{j,g}^*) \right| \\
& \leq \frac{\sum_{g \in \mathcal{G}_j} \left| (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{1,j,g}^*(z_{i,t}) (\hat{\phi}_{j,g} - \phi_{j,g}^*) \right|}{\min_{g \in \mathcal{G}_j} (|\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})|)} \\
& \leq \frac{\left(\sum_{g \in \mathcal{G}_j} \left\| (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{1,j,g}^*(z_{i,t}) \right\|^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 \right)^{1/2}}{\min_{g \in \mathcal{G}_j} (|\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})|)}. \tag{S33}
\end{aligned}$$

Under Assumption 1 and (S32), we have

$$\begin{aligned}
\mathbb{E} \left[\sum_{g \in \mathcal{G}_j} \left\| (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{1,j,g}^*(z_{i,t}) \right\|^2 \right] &= \sum_{g \in \mathcal{G}_j} (n_{j,g}T)^{-2} \sum_{i \in I_{j,g}} \mathbb{E} \left[\left\| \sum_{t \leq T} C_{1,j,g}^*(z_{i,t}) \right\|^2 \right] \\
&\leq \sum_{g \in \mathcal{G}_j} (n_{j,g}T)^{-2} \sum_{i \in I_{j,g}} KT \leq KG_j T^{-1},
\end{aligned}$$

which, combined with Markov's inequality, implies that

$$\sum_{g \in \mathcal{G}_j} \left\| (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{1,j,g}^*(z_{i,t}) \right\|^2 = O_p(G_j T^{-1}). \tag{S34}$$

Thus, by collecting the results in Lemma S1(ii), (S20), (S33) and (S34), we conclude:

$$\sum_{g \in \mathcal{G}_j} \left(\frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} C_{1,j,g}^*(z_{i,t})}{(n_{j,g}T) \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} \right) (\hat{\phi}_{j,g} - \phi_{j,g}^*) = O_p(G_j T^{-1}). \tag{S35}$$

By the triangle inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
& \left\| \sum_{g \in \mathcal{G}_j} \frac{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}}{(n_{j,g}T) \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{1,j,g}(z_{i,t})] (\hat{\phi}_{j,g} - \phi_{j,g}^*) \right\| \\
& \leq \frac{\left(\sum_{g \in \mathcal{G}_j} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 \right)^{1/2}}{T \min_{g \in \mathcal{G}_j} (|\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}|)} \\
& \quad \times \left(\sum_{g \in \mathcal{G}_j} \left\| (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{1,j,g}(z_{i,t})] \right\|^2 \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 \right)^{1/2}. \tag{S36}
\end{aligned}$$

Since $\|\mathbb{E}[C_{1,j,g}(z_{i,t})]\| \leq K$ under Assumption 8(iii), by (S20) and the Cauchy-Schwarz inequality, we have

$$\sum_{g \in \mathcal{G}_j} \left\| (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{1t,j,g}(z_{i,t})] \right\|^2 \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 \leq K \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 = O_p(G_j T^{-1}). \tag{S37}$$

Combining this with Assumption 9, which is upheld in Assumption 2(i), Lemma S1(ii, v) and (S36), it follows that

$$\sum_{g \in \mathcal{G}_j} \frac{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}}{(n_{j,g}T)\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})\Psi_{j,\gamma\gamma,g}} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{1,j,g}(z_{i,t})](\hat{\phi}_{j,g} - \phi_{j,g}^*) = O_p(G_j T^{-1}). \quad (\text{S38})$$

By the triangle inequality, we have

$$\begin{aligned} & \left\| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{1,j,g}(z_{i,t})]}{(n_{j,g}T)\Psi_{j,\gamma\gamma,g}} (\hat{\phi}_{j,g} - \phi_{j,g}^*) \right\| \\ & \leq \left\| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\theta}^\top(z_{i,t}; \phi_{j,g}^*)]}{(n_{j,g}T)\Psi_{j,\gamma\gamma,g}} (\hat{\theta}_j - \theta_j^*) \right\| \\ & \quad + \left\| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g}^*)]}{(n_{j,g}T)\Psi_{j,\gamma\gamma,g}} (\hat{\gamma}_{j,g} - \gamma_{j,g}^*) \right\|, \end{aligned} \quad (\text{S39})$$

where $\psi_{j,\gamma a}(z_{i,t}; \phi_{j,g}) \equiv \partial \psi_{j,\gamma}(z_{i,t}; \phi_{j,g}) / \partial a$ for $a \in \{\theta_j, \gamma_{j,g}\}$. The first term in the inequality above can be bounded as follows:

$$\begin{aligned} & \left\| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\theta}^\top(z_{i,t}; \phi_{j,g}^*)]}{(n_{j,g}T)\Psi_{j,\gamma\gamma,g}} (\hat{\theta}_j - \theta_j^*) \right\| \\ & \leq \frac{\sum_{g \in \mathcal{G}_j} (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \left\| \mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\theta}^\top(z_{i,t}; \phi_{j,g}^*)] \right\|}{\min_{g \in \mathcal{G}_j} (-\Psi_{j,\gamma\gamma,g})} \|\hat{\theta}_j - \theta_j^*\| \\ & \leq KG_j \|\hat{\theta}_j - \theta_j^*\| = O_p((nT^{-1})^{1/2}). \end{aligned} \quad (\text{S40})$$

where the first inequality follows from the triangle and Cauchy-Schwarz inequalities, the second inequality follows from Assumptions 8(iii) and 9, and the equality holds due to Lemma S1(iii). For the second term after the inequality in (S39), we have:

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g}^*)]}{(n_{j,g}T)\Psi_{j,\gamma\gamma,g}} (\hat{\gamma}_{j,g} - \gamma_{j,g}^*) \right| \\ & \leq \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g}^*)]}{(n_{j,g}T)\Psi_{j,\gamma\gamma,g}} \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g} \right| \\ & \quad + \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g}^*)]}{(n_{j,g}T)\Psi_{j,\gamma\gamma,g}} (\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}) \right|. \end{aligned} \quad (\text{S41})$$

Using Assumptions 8(iii) and 9, and Lemma S1(vi), we get

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g}^*)]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}} (\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}) \right| \\ & \leq K \sum_{g \in \mathcal{G}_j} |\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}| = O_p((nT^{-1})^{1/2}). \end{aligned} \quad (\text{S42})$$

By Assumptions 1, 8 and 9, and Lemma S1(viii), it follows that

$$\mathbb{E} \left[\left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g}^*)]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}} \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g} \right|^2 \right] \leq K \sum_{g \in \mathcal{G}_j} \mathbb{E}[\hat{\Psi}_{j,\gamma,g}^2] \leq KnT^{-1}.$$

This, together with Markov's inequality, (S39)-(S42) shows that

$$\sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{1,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}} (\hat{\phi}_{j,g} - \phi_{j,g}^*) = O_p((nT^{-1})^{1/2}). \quad (\text{S43})$$

Collecting the results in (S35), (S38) and (S43), we obtain

$$T^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*) \psi_{j,\gamma\phi}^\top(z_{i,t}; \phi_{j,g}^*) (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} = O_p\left((nT^{-1})^{1/2} + G_j T^{-1}\right),$$

which establishes (S29).

We now proceed to verify (S30). Using the triangle inequality and the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_j} \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g}) \psi_{j,\gamma\phi}^\top(z_{i,t}; \tilde{\phi}_{j,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{(n_{j,g}T) \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} \right| \\ & \leq \frac{\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \|\psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g})\|^2}{\min_{g \in \mathcal{G}_j} (|\hat{\Psi}_{j,\gamma\gamma,g}(\hat{\phi}_{j,g})|)}. \end{aligned} \quad (\text{S44})$$

By Assumption 8 and the Cauchy-Schwarz inequality, we have:

$$\sum_{i \in I_{j,g}} \sum_{t \leq T} \|\psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g})\|^2 \leq K \sum_{i \in I_{j,g}} \sum_{t \leq T} \left(M(z_{i,t})^2 \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 + \|\psi_{j,\gamma\phi}(z_{i,t}; \phi_{j,g}^*)\|^2 \right).$$

This implies that

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \|\psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g})\|^2 \\ & \leq K \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} M(z_{i,t})^2 \\ & + K \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \|\psi_{j,\gamma\phi}(z_{i,t}; \phi_{j,g}^*)\|^2 \\ & \leq K \left(1 + \max_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 \right) \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \tilde{M}_1(z_{i,t})^2, \end{aligned} \quad (\text{S45})$$

where $\tilde{M}_1(z_{i,t}) \equiv M(z_{i,t}) + \|\psi_{j,\gamma\phi}(z_{i,t}; \phi_{j,g}^*)\|$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \tilde{M}_1(z_{i,t})^2 \\ & \leq \left(\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_j} (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \tilde{M}_1(z_{i,t})^4 \right)^{1/2}. \end{aligned} \quad (\text{S46})$$

From Assumptions 8(ii, iii) and Markov's inequality, it follows that

$$\sum_{g \in \mathcal{G}_j} (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[\tilde{M}_1(z_{i,t})^4] \leq O_p(G_j), \quad (\text{S47})$$

which together with (S13) and (S46) establishes that

$$\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \tilde{M}_1(z_{i,t})^2 = O_p(G_j T^{-1}).$$

This along with Lemma S1(i, ii) and (S45) shows that

$$\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \|\psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g})\|^2 = O_p(G_j T^{-1}). \quad (\text{S48})$$

Combining Lemma S1(ii), and results (S44) and (S48), we verify that:

$$\sum_{g \in \mathcal{G}_j} \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g}) \psi_{j,\gamma\phi}^\top(z_{i,t}; \tilde{\phi}_{j,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{(n_{j,g}T) \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} = O_p(G_j T^{-1}),$$

which shows that (S30) holds.

To verify (S31), we first note that, by the triangle inequality and the Cauchy-Schwarz inequality:

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_j} \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}; \tilde{\phi}_{j,g}) \psi_{j,\gamma\phi\phi}(z_{i,t}; \tilde{\phi}_{j,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{(n_{j,g}T) \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} \right| \\ & \leq \frac{\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \|\psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g})\| \|\psi_{j,\gamma\phi\phi}(z_{i,t}; \tilde{\phi}_{j,g})\|}{\min_{g \in \mathcal{G}_j} (|\hat{\Psi}_{j,\gamma\gamma,g}(\hat{\phi}_{j,g})|)}. \end{aligned} \quad (\text{S49})$$

Using Assumption 8 and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} & \sum_{i \in I_{j,g}} \sum_{t \leq T} \|\psi_{j,\gamma\phi}(z_{i,t}; \tilde{\phi}_{j,g})\| \|\psi_{j,\gamma\phi\phi}(z_{i,t}; \tilde{\phi}_{j,g})\| \\ & \leq \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 \sum_{i \in I_{j,g}} \sum_{t \leq T} M(z_{i,t})^2 + \sum_{i \in I_{j,g}} \sum_{t \leq T} \|\psi_{j,\gamma\phi}(z_{i,t}; \phi_{j,g}^*)\| \|\psi_{j,\gamma\phi\phi}(z_{i,t}; \phi_{j,g}^*)\| \\ & \quad + \|\hat{\phi}_{j,g} - \phi_{j,g}^*\| \sum_{i \in I_{j,g}} \sum_{t \leq T} M(z_{i,t}) (\|\psi_{j,\gamma\phi}(z_{i,t}; \phi_{j,g}^*)\| + \|\psi_{j,\gamma\phi\phi}(z_{i,t}; \phi_{j,g}^*)\|) \\ & \leq K(1 + \max_{g \in \mathcal{G}_j} (\|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 + \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|)) \sum_{i \in I_{j,g}} \sum_{t \leq T} \tilde{M}_2(z_{i,t})^2, \end{aligned}$$

where $\tilde{M}_2(z_{i,t}) \equiv \tilde{M}_1(z_{i,t}) + \|\psi_{j,\gamma\phi\phi}(z_{i,t}; \phi_{j,g}^*)\|$. Therefore,

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_j} \frac{(\hat{\phi}_{j,g} - \phi_{j,g}^*)^\top \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}; \tilde{\phi}_{j,g}) \psi_{j,\gamma\phi\phi}(z_{i,t}; \tilde{\phi}_{j,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*)}{(n_{j,g}T) \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} \right| \\ & \leq \frac{K(1 + \max_{g \in \mathcal{G}_j} (\|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 + \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|))}{\min_{g \in \mathcal{G}_j} |\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})|} \\ & \quad \times \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \tilde{M}_2(z_{i,t})^2. \end{aligned} \quad (\text{S50})$$

Following similar arguments to those used to establish (S48), we can show that

$$\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \tilde{M}_2(z_{i,t})^2 = O_p(G_j T^{-1}),$$

which, combined with (S50) and Lemma S1(i, ii, iii), verifies (S31). Q.E.D.

Lemma S3. *Under Assumptions 1 and 2, we have*

$$\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(\hat{\mathbb{E}}_T[\hat{\psi}_{2,\gamma}(z_{i,t})])^2 - (\hat{\mathbb{E}}_T[\psi_{2,\gamma}(z_{i,t})])^2}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} = O_p((nT^{-1})^{1/2} + G_2 T^{-1}). \quad (\text{S51})$$

PROOF OF LEMMA S3. First, we notice that the left-hand side (LHS) of (S51) can be written as

$$\begin{aligned} & \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(\sum_{t \leq T} \hat{\psi}_{2,\gamma}(z_{i,t}))^2 - (\sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}))^2}{n_{2,g} T^2 \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} \\ & = \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(\sum_{t \leq T} (\hat{\psi}_{2,\gamma}(z_{i,t}) - \psi_{2,\gamma}(z_{i,t})))^2}{n_{2,g} T^2 \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} \\ & \quad + 2 \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{\sum_{t \leq T} (\hat{\psi}_{2,\gamma}(z_{i,t}) - \psi_{2,\gamma}(z_{i,t}))}{n_{2,g} T^2 \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}). \end{aligned} \quad (\text{S52})$$

By Assumption 8(i) and the Cauchy-Schwarz inequality, the first term on the right-hand side (RHS) of the above equation can be bounded as:

$$\begin{aligned} & \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(\sum_{t \leq T} (\hat{\psi}_{2,\gamma}(z_{i,t}) - \psi_{2,\gamma}(z_{i,t})))^2}{n_{2,g} T^2 |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|} \\ & \leq \frac{\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (T^{-1} \sum_{t \leq T} M(z_{i,t}) \|\hat{\phi}_{2,g} - \phi_{2,g}^*\|)^2}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|} \\ & \leq \frac{\left(\sum_{g \in \mathcal{G}_2} \|\hat{\phi}_{2,g} - \phi_{2,g}^*\|^4 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} (n_{2,g}T)^{-1} \sum_{i \in I_{2,g}} \sum_{t \leq T} M(z_{i,t})^4 \right)^{1/2}}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|}. \end{aligned} \quad (\text{S53})$$

From Assumption 8 and Markov's inequality, it follows that

$$\sum_{g \in \mathcal{G}_2} (n_{2,g}T)^{-1} \sum_{i \in I_{2,g}} \sum_{t \leq T} M(z_{i,t})^4 = O_p(G_2), \quad (\text{S54})$$

which along with (S13) and (S53) implies that

$$\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(\sum_{t \leq T} (\hat{\psi}_{2,\gamma}(z_{i,t}) - \psi_{2,\gamma}(z_{i,t})))^2}{n_{2,g}T^2 \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} = O_p(G_2T^{-1}). \quad (\text{S55})$$

To investigate the second term in the RHS of (S52), we first apply the Taylor expansion to obtain:

$$\begin{aligned} & \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} (\hat{\psi}_{2,\gamma}(z_{i,t}) - \psi_{2,\gamma}(z_{i,t}))}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \\ &= \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} \psi_{2,\gamma\phi}(z_{i,t}, \tilde{\phi}_{2,g})(\tilde{\phi}_{2,g} - \phi_{2,g}^*)}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}), \end{aligned} \quad (\text{S56})$$

where $\tilde{\phi}_{j,g}$ lies between $\hat{\phi}_{j,g}$ and $\phi_{j,g}^*$. By Assumption 8 and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} (\psi_{2,\gamma\phi}(z_{i,t}, \tilde{\phi}_{2,g}) - \psi_{2,\gamma\phi}(z_{i,t}, \phi_{2,g}^*))(\tilde{\phi}_{2,g} - \phi_{2,g}^*)}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \right| \\ & \leq \max_{i \leq n} \left| T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \right| \frac{\sum_{g \in \mathcal{G}_2} \|\tilde{\phi}_{2,g} - \phi_{2,g}^*\|^2 (n_{2,g}T)^{-1} \sum_{i \in I_{2,g}} \sum_{t \leq T} M(z_{i,t})}{\min_{g \in \mathcal{G}_2} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})}. \end{aligned} \quad (\text{S57})$$

Note that by a maximal inequality under the L_p -norm,

$$\left\| \max_{i \leq n} \left| T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \right| \right\|_p \leq K n^{1/p} \max_{i \leq n} \left\| T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \right\|_p. \quad (\text{S58})$$

By Assumption 8, we have

$$\mathbb{E}[|\psi_{2,\gamma}(z_{i,t})|^p] \leq K, \quad (\text{S59})$$

which implies that $\|\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]\|_p \leq K$. Thus, applying Rosenthal's inequality, we get

$$\mathbb{E} \left[\left| \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \right|^p \right] \leq K T^{p/2}$$

which along with (S58) shows that

$$\left\| \max_{i \leq n} \left| T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \right| \right\|_p \leq K n^{1/p} T^{-1/2}.$$

Combining this result with Markov's inequality and Assumption 1(i), we establish that

$$\max_{i \leq n} \left| T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \right| = O_p(T^{-1/2+1/p}). \quad (\text{S60})$$

Therefore, from (S59) and (S60), it follows that

$$\begin{aligned} \max_{i \leq n} \left| T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \right| &\leq \max_{i \leq n} \left| T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \right| \\ &\quad + \max_{i \leq n} \left| T^{-1} \sum_{t \leq T} \mathbb{E}[\psi_{2,\gamma}(z_{i,t})] \right| = O_p(1). \end{aligned} \quad (\text{S61})$$

By the Cauchy-Schwarz inequality, Assumption 8(ii) and (S13), we have

$$\begin{aligned} &\frac{\sum_{g \in \mathcal{G}_2} \|\hat{\phi}_{2,g} - \phi_{2,g}^*\|^2 (n_{2,g}T)^{-1} \sum_{i \in I_{2,g}} \sum_{t \leq T} M(z_{i,t})}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|} \\ &\leq \frac{\left(\sum_{g \in \mathcal{G}_2} \|\hat{\phi}_{2,g} - \phi_{2,g}^*\|^4 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} (n_{2,g}T)^{-1} \sum_{i \in I_{2,g}} \sum_{t \leq T} M(z_{i,t})^2 \right)^{1/2}}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|} \\ &= O_p(G_2 T^{-1}). \end{aligned} \quad (\text{S62})$$

From (S56), (S57), (S61) and (S62), we establish:

$$\begin{aligned} &\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} (\hat{\psi}_{2,\gamma}(z_{i,t}) - \psi_{2,\gamma}(z_{i,t}))}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \\ &= \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} \psi_{2,\gamma\theta}(z_{i,t})(\hat{\theta}_2 - \theta_2^*)}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \\ &\quad + \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} \psi_{2,\gamma\gamma}(z_{i,t})}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) + O_p(G_2 T^{-1}). \end{aligned} \quad (\text{S63})$$

The first term on the RHS of (S63) can be bounded as

$$\begin{aligned} &\left| \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} \psi_{2,\gamma\theta}(z_{i,t})(\hat{\theta}_2 - \theta_2^*)}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \right| \\ &\leq \frac{\|\hat{\theta}_2 - \theta_2^*\| \max_{i \leq n} \left| T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \right|}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|} \sum_{g \in \mathcal{G}_2} (n_{2,g}T)^{-1} \sum_{i \in I_{2,g}} \sum_{t \leq T} \|\psi_{2,\gamma\theta}(z_{i,t})\|. \end{aligned} \quad (\text{S64})$$

By Assumption 8 and Markov's inequality, it follows that

$$\sum_{g \in \mathcal{G}_2} (n_{2,g}T)^{-1} \sum_{i \in I_{2,g}} \sum_{t \leq T} \|\psi_{2,\gamma\theta}(z_{i,t})\| = O_p(G_2),$$

which together with Lemma S1(ii, iii) and (S61) implies that

$$\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} \psi_{2,\gamma\theta}(z_{i,t})(\hat{\theta}_2 - \theta_2^*)}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) = O_p(G_2(nT)^{-1/2}). \quad (\text{S65})$$

To study the second term on the RHS of (S63), we first write it as

$$\begin{aligned} & \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} \psi_{2,\gamma\gamma}(z_{i,t})}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \\ &= \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} (\psi_{2,\gamma\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})])}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \\ &+ \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} \frac{\mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})]}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \\ &- \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} \frac{\mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g}) \Psi_{2,\gamma\gamma,g}} (\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g}) - \Psi_{2,\gamma\gamma,g}) \\ &+ \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}}. \end{aligned} \quad (\text{S66})$$

By the Cauchy-Schwarz inequality, we can bound the first term on the RHS of (S66) as

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{j,g} - \gamma_{j,g}^*) \sum_{i \in I_{2,g}} \frac{\sum_{t \leq T} (\psi_{2,\gamma\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})])}{(n_{2,g} T) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{j,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \right| \\ & \leq \frac{\max_{i \leq n} \left| T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}) \right|}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{j,g})|} \left(\sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{j,g} - \gamma_{j,g}^*)^2 \right)^{1/2} \\ & \times \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \left(T^{-1} \sum_{t \leq T} (\psi_{2,\gamma\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})]) \right)^2 \right)^{1/2}. \end{aligned} \quad (\text{S67})$$

Under Assumptions 1 and 8, we can apply the covariance inequality for mixing processes to obtain

$$\mathbb{E} \left[\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \left(T^{-1} \sum_{t \leq T} (\psi_{2,\gamma\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})]) \right)^2 \right] \leq K G_2 T^{-1},$$

which together with Markov's inequality, Lemma S1(ii, iv), (S61) and (S67) implies that we can bound the first term on the RHS of (S66) by

$$\sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} \frac{\sum_{t \leq T} (\psi_{2,\gamma\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})])}{(n_{2,g} T) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) = O_p(G_2 T^{-1}). \quad (\text{S68})$$

As for the second term on the RHS of (S66), we have by the Cauchy-Schwarz inequality and Assumption 8

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} \frac{\mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})]}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \right| \\ & \leq \frac{K \left(\sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*)^2 \right)^{1/2}}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \left(T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \right)^2 \right)^{1/2}. \end{aligned} \quad (\text{S69})$$

Under Assumptions 3 and 8, it follows that

$$\mathbb{E} \left[\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \left(T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) \right)^2 \right] \leq K G_2 T^{-1},$$

which together with Markov's inequality, Lemma S1(ii, iv) and (S69) implies that we can bound the second term on the RHS of (S66) by

$$\sum_{g \in \mathcal{G}_2} \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} \frac{\mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})]}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]) = O_p(G_2 T^{-1}). \quad (\text{S70})$$

As for the third term on the RHS of (S66), we have by the Cauchy-Schwarz inequality

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} \frac{\mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g}) \Psi_{2,\gamma\gamma,g}} (\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g}) - \Psi_{2,\gamma\gamma,g}) \right| \\ & \leq \frac{K \left(\sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*)^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} (\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g}) - \Psi_{2,\gamma\gamma,g})^2 \right)^{1/2}}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g}) \Psi_{2,\gamma\gamma,g}|} = O_p(G_2 T^{-1}) \end{aligned} \quad (\text{S71})$$

where the equality follows from Assumption 9 and Lemma S1(iv, v).

Combining the results from (S63), (S65), (S66), (S68), (S70) and (S71), we obtain

$$\begin{aligned} & \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{T^{-1} \sum_{t \leq T} (\hat{\psi}_{2,\gamma}(z_{i,t}) - \psi_{2,\gamma}(z_{i,t}))}{n_{2,g} \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})} T^{-1} \sum_{t \leq T} \psi_{2,\gamma}(z_{i,t}) \\ & = \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} + O_p(G_2 T^{-1}). \end{aligned} \quad (\text{S72})$$

In view of the results established in (S52), (S55) and (S72), it is evident that the claim of the lemma follows if

$$\sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} = O_p(G_2^{1/2} T^{-1/2}). \quad (\text{S73})$$

We next turn to verify (S73). Using the expression in (S15), we can write

$$\begin{aligned}
& \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} \\
&= - \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\Psi_{2,\gamma\gamma,g}} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} \\
&\quad - \sum_{g \in \mathcal{G}_2} (\hat{\Psi}_{2,\gamma\gamma,g}^{-1} - \Psi_{2,\gamma\gamma,g}^{-1}) \hat{\Psi}_{2,\gamma,g} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} \\
&\quad - \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma\theta,g} (\hat{\theta}_2 - \theta_2^*)}{\hat{\Psi}_{2,\gamma\gamma,g}} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} \\
&\quad - \sum_{g \in \mathcal{G}_2} \frac{(\hat{\phi}_{2,g} - \phi_{2,g}^*)^\top \tilde{\Psi}_{2,\gamma\phi\phi,g} (\hat{\phi}_{2,g} - \phi_{2,g}^*)}{2 \hat{\Psi}_{2,\gamma\gamma,g}} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}}. \tag{S74}
\end{aligned}$$

By Assumptions 1, 2(iii) and 8, and Lemma S1(viii)

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\Psi_{2,\gamma\gamma,g}} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} \right)^2 \right] \\
&= \sum_{g \in \mathcal{G}_2} \mathbb{E}[\hat{\Psi}_{2,\gamma,g}^2] \left(\frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}^2} \right)^2 \leq K \sum_{g \in \mathcal{G}_2} \mathbb{E}[\hat{\Psi}_{2,\gamma,g}^2] \leq K G_2 T^{-1},
\end{aligned}$$

where the second inequality follows from (S25). Thus, by Markov's inequality,

$$\sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\Psi_{2,\gamma\gamma,g}} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} = O_p(G_2^{1/2} T^{-1/2}). \tag{S75}$$

By the Cauchy-Schwarz inequality, Assumptions 8 and 9, we have

$$\begin{aligned}
& \left| \sum_{g \in \mathcal{G}_2} (\hat{\Psi}_{2,\gamma\gamma,g}^{-1} - \Psi_{2,\gamma\gamma,g}^{-1}) \hat{\Psi}_{2,\gamma,g} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} \right| \\
&\leq \frac{K \left(\sum_{g \in \mathcal{G}_2} (\hat{\Psi}_{2,\gamma\gamma,g} - \Psi_{2,\gamma\gamma,g})^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} \hat{\Psi}_{2,\gamma,g}^2 \right)^{1/2}}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|},
\end{aligned}$$

which along with Lemma S1(ii, v), (S25) and Markov's inequality shows that

$$\sum_{g \in \mathcal{G}_2} (\hat{\Psi}_{2,\gamma\gamma,g}^{-1} - \Psi_{2,\gamma\gamma,g}^{-1}) \hat{\Psi}_{2,\gamma,g} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} = O_p(G_2 T^{-1}). \tag{S76}$$

By the Cauchy-Schwarz inequality and Assumption 9, we have

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma\theta,g}(\hat{\theta}_2 - \theta_2^*)}{\hat{\Psi}_{2,\gamma\gamma,g}} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} \right| \\ & \leq \frac{K \|\hat{\theta}_2 - \theta_2^*\|}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|} \sum_{g \in \mathcal{G}_2} (\|\hat{\Psi}_{2,\gamma\theta,g} - \mathbb{E}[\hat{\Psi}_{2,\gamma\theta,g}]\| + \|\mathbb{E}[\hat{\Psi}_{2,\gamma\theta,g}]\|). \end{aligned} \quad (\text{S77})$$

Under Assumptions 1 and 8, we can apply the covariance inequality for mixing processes to obtain

$$\mathbb{E} \left[\|\hat{\Psi}_{2,\gamma\theta,g} - \mathbb{E}[\hat{\Psi}_{2,\gamma\theta,g}]\|^2 \right] \leq K(n_{2,g}T)^{-1},$$

which together with Markov's inequality, Assumption 8, Lemma S1(ii, iii) and (S77) implies that

$$\sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma\theta,g}(\hat{\theta}_2 - \theta_2^*)}{\hat{\Psi}_{2,\gamma\gamma,g}} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} = O_p(G_2 T^{-1}). \quad (\text{S78})$$

By the Cauchy-Schwarz inequality, Assumptions 8 and 9, we have

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_2} \frac{(\hat{\phi}_{2,g} - \phi_{2,g}^*)^\top \tilde{\Psi}_{2,\gamma\phi\phi,g}(\hat{\phi}_{2,g} - \phi_{2,g}^*)}{2\hat{\Psi}_{2,\gamma\gamma,g}} \frac{\sum_{i \in I_{2,g}} \mathbb{E}[\psi_{2,\gamma\gamma}(z_{i,t})] \mathbb{E}[\psi_{2,\gamma}(z_{i,t})]}{n_{2,g} \Psi_{2,\gamma\gamma,g}} \right| \\ & \leq \frac{K \max_{g \in \mathcal{G}_2} \|\tilde{\Psi}_{2,\gamma\phi\phi,g}\|}{\min_{g \in \mathcal{G}_2} |\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{j,g})|} \sum_{g \in \mathcal{G}_2} \|\hat{\phi}_{2,g} - \phi_{2,g}^*\|^2 = O_p(G_j T^{-1}) \end{aligned} \quad (\text{S79})$$

where the equality follows from Lemma S1(ii), (S19) and (S20). The claim in (S73) follows directly from (S74), (S75), (S76), (S78) and (S79). Q.E.D.

Lemma S4. *Under Assumptions 1 and 2, we have*

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{\hat{\mathbb{E}}_T[\psi_{j,\gamma}^2(z_{i,t})]}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{\hat{\mathbb{E}}_T[\psi_{j,\gamma}^2(z_{i,t})]}{n_{j,g} \Psi_{j,\gamma\gamma,g}} \right) = O_p((nT^{-1})^{1/2} + G_j T^{-1}).$$

PROOF OF LEMMA S4. Let $C_{2,j,g}(z_{i,t}) \equiv \psi_{j,\gamma}^2(z_{i,t})$ and $C_{2,j,g}^*(z_{i,t}) \equiv C_{2,j,g}(z_{i,t}) - \mathbb{E}[C_{2,j,g}(z_{i,t})]$. Using these notations, the following expression is obtained:

$$\begin{aligned} & T^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \left(\frac{\psi_{j,\gamma}^2(z_{i,t})}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{\psi_{j,\gamma}^2(z_{i,t})}{n_{j,g} \Psi_{j,\gamma\gamma,g}} \right) \\ & = \sum_{g \in \mathcal{G}_j} \left(\frac{1}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{1}{\Psi_{j,\gamma\gamma,g}} \right) (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})] \\ & + \sum_{g \in \mathcal{G}_j} \left(\frac{1}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{1}{\Psi_{j,\gamma\gamma,g}} \right) (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{2,j,g}^*(z_{i,t}). \end{aligned} \quad (\text{S80})$$

Applying the triangle inequality and the Cauchy-Schwarz inequality, we can bound the second term on the right-hand side of (S80) as follows:

$$\begin{aligned}
& \left| \sum_{g \in \mathcal{G}_j} \left(\frac{1}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{1}{\Psi_{j,\gamma\gamma,g}} \right) (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{2,j,g}^*(z_{i,t}) \right| \\
& \leq \frac{\sum_{g \in \mathcal{G}_j} |(\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})(n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{2,j,g}^*(z_{i,t})|}{\min_{g \in \mathcal{G}_j} (|\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}|)} \\
& \leq \frac{\left(\sum_{g \in \mathcal{G}_j} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_j} ((n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{2,j,g}^*(z_{i,t}))^2 \right)^{1/2}}{T \min_{g \in \mathcal{G}_j} (|\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}|)}. \quad (\text{S81})
\end{aligned}$$

Using similar arguments to those leading up to (S34), we have

$$\sum_{g \in \mathcal{G}_j} \left((n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{2,j,g}^*(z_{i,t}) \right)^2 = O_p(G_j T^{-1}),$$

which together with Assumption 9, Lemma S1(ii, v) and (S81) implies that

$$\sum_{g \in \mathcal{G}_j} \left(\frac{1}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{1}{\Psi_{j,\gamma\gamma,g}} \right) (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} C_{2,j,g}^*(z_{i,t}) = O_p(G_j T^{-1}). \quad (\text{S82})$$

Next, observe that the first term on the right-hand side of (S80) can be rewritten as

$$\begin{aligned}
& \sum_{g \in \mathcal{G}_j} \left(\frac{1}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{1}{\Psi_{j,\gamma\gamma,g}} \right) (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})] \\
& = \sum_{g \in \mathcal{G}_j} \frac{(\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})] \\
& - \sum_{g \in \mathcal{G}_j} \frac{(\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})}{\Psi_{j,\gamma\gamma,g}^2} (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]. \quad (\text{S83})
\end{aligned}$$

Under Assumption 9, the first term on the RHS of (S83) can be bounded as follows:

$$\left| \sum_{g \in \mathcal{G}_j} \frac{(\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2}{\Psi_{j,\gamma\gamma,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})] \right| \leq \frac{K \sum_{g \in \mathcal{G}_j} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2}{\min_{g \in \mathcal{G}_j} (|\Psi_{j,\gamma\gamma,g}| \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g}))}.$$

Thus, by Lemma S1(ii, v), it follows that

$$\sum_{g \in \mathcal{G}_j} \frac{(\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2}{\Psi_{j,\gamma\gamma,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})] = O_p(G_j T^{-1}). \quad (\text{S84})$$

Combining this result with (S80), (S82) and (S83), we obtain:

$$\begin{aligned}
& T^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \left(\frac{\psi_{j,\gamma}^2(z_{i,t}; \phi_{j,g}^*)}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{\psi_{j,\gamma}^2(z_{i,t}; \phi_{j,g}^*)}{n_{j,g} \Psi_{j,\gamma\gamma,g}} \right) \\
&= \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} (\Psi_{j,\gamma\gamma,g} - \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})) + O_p(G_j T^{-1}). \tag{S85}
\end{aligned}$$

We now take care of the first term on the RHS of (S85). Under Assumptions 1, 8 and 9, we have the following:

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} (\Psi_{j,\gamma\gamma,g} - \hat{\Psi}_{j,\gamma\gamma}(\phi_{j,g}^*)) \right)^2 \right] \\
&= \sum_{g \in \mathcal{G}_j} \left(\frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \right)^2 \mathbb{E} [(\Psi_{j,\gamma\gamma,g} - \hat{\Psi}_{j,\gamma\gamma}(\phi_{j,g}^*))^2] \\
&\leq K \sum_{g \in \mathcal{G}_j} \mathbb{E} [(\Psi_{j,\gamma\gamma,g} - \hat{\Psi}_{j,\gamma\gamma}(\phi_{j,g}^*))^2] \leq K G_j T^{-1}.
\end{aligned}$$

Thus, by Markov's inequality, it follows that:

$$\sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} (\Psi_{j,\gamma\gamma,g} - \hat{\Psi}_{j,\gamma\gamma,g}(\phi_{j,g}^*)) = O_p((nT^{-1})^{1/2}). \tag{S86}$$

Applying a Taylor expansion, we obtain:

$$\begin{aligned}
& \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma,g}(\hat{\phi}_{j,g}) - \hat{\Psi}_{j,\gamma\gamma,g}(\phi_{j,g}^*)) \\
&= \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \hat{\Psi}_{j,\gamma\gamma,g}(\tilde{\phi}_{j,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*), \tag{S87}
\end{aligned}$$

where $\tilde{\phi}_{j,g}$ lies between $\hat{\phi}_{j,g}$ and $\phi_{j,g}^*$, and

$$\hat{\Psi}_{j,\gamma\gamma\phi}(\phi_{j,g}) \equiv (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma\gamma\phi}(z_{i,t}; \phi_{j,g}) \quad \text{and} \quad \psi_{j,\gamma\gamma\phi}(z_{i,t}; \phi_{j,g}) \equiv \frac{\partial \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g})}{\partial \phi_{j,g}}.$$

Using Assumptions 8 and 9, along with the triangle and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}
& \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma\phi}(\tilde{\phi}_{j,g}) - \hat{\Psi}_{j,\gamma\gamma\phi}(\phi_{j,g}^*)) (\hat{\phi}_{j,g} - \phi_{j,g}^*) \right| \\
& \leq \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \left\| \hat{\Psi}_{j,\gamma\gamma\phi}(\tilde{\phi}_{j,g}) - \hat{\Psi}_{j,\gamma\gamma\phi}(\phi_{j,g}^*) \right\| \|\hat{\phi}_{j,g} - \phi_{j,g}^*\| \\
& \leq \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} M(z_{i,t})}{(n_{j,g}T)} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 \\
& \leq K \left(\sum_{g \in \mathcal{G}_j} (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} M(z_{i,t})^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4 \right)^{1/2} = O_p(G_j T^{-1}), \quad (\text{S88})
\end{aligned}$$

where the last equality uses (S13) and Markov's inequality.

Let $\Psi_{j,\gamma\gamma\phi,g} \equiv \mathbb{E}[\hat{\Psi}_{j,\gamma\gamma\phi}(\phi_{j,g}^*)]$. By Assumptions 8 and 9, along with the triangle inequality and the Cauchy-Schwarz inequality:

$$\begin{aligned}
& \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma\phi}(\phi_{j,g}^*) - \Psi_{j,\gamma\gamma\phi,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*) \right| \\
& \leq K \sum_{g \in \mathcal{G}_j} \left| (\hat{\Psi}_{j,\gamma\gamma\phi}(\phi_{j,g}^*) - \Psi_{j,\gamma\gamma\phi,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*) \right| \\
& \leq K \left(\sum_{g \in \mathcal{G}_j} \left\| \hat{\Psi}_{j,\gamma\gamma\phi}(\phi_{j,g}^*) - \Psi_{j,\gamma\gamma\phi,g} \right\|^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 \right)^{1/2}. \quad (\text{S89})
\end{aligned}$$

Under Assumptions 1 and 8, the covariance inequality for mixing processes implies:

$$\mathbb{E} \left[\sum_{g \in \mathcal{G}_j} \left\| \hat{\Psi}_{j,\gamma\gamma\phi}(\phi_{j,g}^*) - \Psi_{j,\gamma\gamma\phi,g} \right\|^2 \right] \leq K G_j T^{-1},$$

which, combined with Markov's inequality, (S20) and (S89), establishes

$$\sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma\phi}(\phi_{j,g}^*) - \Psi_{j,\gamma\gamma\phi,g}) (\hat{\phi}_{j,g} - \phi_{j,g}^*) = O_p(G_j T^{-1}). \quad (\text{S90})$$

Combining the results in (S87), (S88) and (S90), we have

$$\begin{aligned}
& \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \hat{\Psi}_{j,\gamma\gamma}(\phi_{j,g}^*)) \\
& = \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma\phi,g} (\hat{\phi}_{j,g} - \phi_{j,g}^*) + O_p(G_j T^{-1}). \quad (\text{S91})
\end{aligned}$$

Using the triangle inequality, we can write:

$$\begin{aligned}
& \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g}(\hat{\phi}_{j,g} - \phi_{j,g}^*) \right| \\
& \leq \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g}(\hat{\theta}_{j,g} - \theta_{j,g}^*) \right| \\
& + \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g}(\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}) \right| \\
& + \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g} \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g} \right|, \tag{S92}
\end{aligned}$$

where $\psi_{j,\gamma\gamma a}(z_{i,t}; \phi_{j,g}) \equiv \partial \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g}) / \partial a$ for $a \in \{\theta, \gamma\}$. By Assumptions 8 and 9, and Lemma S1(iii),

$$\begin{aligned}
& \left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g}(\hat{\theta}_{j,g} - \theta_{j,g}^*) \right| \\
& \leq \frac{K \sum_{g \in \mathcal{G}_j} (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{\min_{g \in \mathcal{G}_j} \Psi_{j,\gamma\gamma,g}^2} \|\hat{\theta}_{j,g} - \theta_{j,g}^*\| \\
& \leq Kn \|\hat{\theta}_{j,g} - \theta_{j,g}^*\| = O_p((nT^{-1})^{1/2}). \tag{S93}
\end{aligned}$$

By Assumptions 8 and 9, and Lemma S1(vi),

$$\left| \sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g}(\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}) \right| = O_p((nT^{-1})^{1/2}). \tag{S94}$$

Moreover, under Assumptions 1, 8 and 9, and Lemma S1(viii), it follows that

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g} \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g} \right)^2 \right] \\
& = \sum_{g \in \mathcal{G}_j} \left(\frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g} \Psi_{j,\gamma\gamma,g}^{-1} \right)^2 \mathbb{E}[\hat{\Psi}_{j,\gamma,g}^2] \leq K \sum_{g \in \mathcal{G}_j} \mathbb{E}[\hat{\Psi}_{j,\gamma,g}^2] \leq G_j T^{-1},
\end{aligned}$$

which together with Markov's inequality implies that

$$\sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g} \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g} = O_p((nT^{-1})^{1/2}). \tag{S95}$$

From (S92)-(S95), it follows that

$$\sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} \Psi_{j,\gamma\gamma,g}(\hat{\phi}_{j,g} - \phi_{j,g}^*) = O_p((nT^{-1})^{1/2}). \tag{S96}$$

Finally, combining (S91) and (S96), we obtain

$$\sum_{g \in \mathcal{G}_j} \frac{\sum_{i \in I_{j,g}} \sum_{t \leq T} \mathbb{E}[C_{2,j,g}(z_{i,t})]}{(n_{j,g}T) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \hat{\Psi}_{j,\gamma\gamma}(\phi_{j,g}^*)) = O_p((nT^{-1})^{1/2} + G_j T^{-1}), \quad (\text{S97})$$

which, along with (S85) and (S86), establishes the lemma's claim. $Q.E.D.$

Lemma S5. *Under Assumptions 1 and 2, we have*

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(\hat{\mathbb{E}}_T[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) = O_p((G_j T^{-1})^{1/2}). \quad (\text{S98})$$

PROOF OF LEMMA S5. The term on the LHS of (S98) can be written as

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) \\ &= \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) \\ &+ \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2 - (\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right). \end{aligned} \quad (\text{S99})$$

The first term on the RHS of (S99) can be expressed as

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) \\ &= \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) \\ &- \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2 (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2}. \end{aligned} \quad (\text{S100})$$

By Assumption 9, Lemma S1(ii, v), we have

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2 (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} \right| \\ & \leq \frac{K}{\min_{g \in \mathcal{G}_j} |\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2|} \sum_{g \in \mathcal{G}_j} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 = O_p(G_j T^{-1}). \end{aligned} \quad (\text{S101})$$

By the similar arguments for deriving (S97), we can also show that

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) = O_p((G_j T^{-1})^{1/2}),$$

which along with (S100) and (S101) implies that

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) = O_p((G_j T^{-1})^{1/2}). \quad (\text{S102})$$

The second term on the RHS of (S99) can be expressed as

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2 - (\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) \right| \\ & \leq \left| \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}))^2}{n_{j,g} |\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})|} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) \right| \\ & + 2 \left| \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{\mathbb{E}[\psi_{j,\gamma}(z_{i,t})] T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t})}{n_{j,g} |\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})| |\Psi_{j,\gamma\gamma,g}|^{1/2}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) \right| \end{aligned} \quad (\text{S103})$$

with probability approaching 1. By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}))^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) \right| \\ & \leq \frac{\left(\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} (T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}))^4 \right)^{1/2}}{\min_{g \in \mathcal{G}_j} |\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}|} \left(\sum_{g \in \mathcal{G}_j} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 \right)^{1/2}. \end{aligned} \quad (\text{S104})$$

Under Assumptions 1, 2(iii), 8 and 9, we can apply Rosenthal's inequality to obtain

$$\mathbb{E} \left[\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} (T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}))^4 \right] \leq K G_j T^{-1}, \quad (\text{S105})$$

which together with Markov's inequality, and Lemma S1(ii, v) shows that

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}))^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) = O_p(G_j T^{-1}). \quad (\text{S106})$$

Similarly, we can show that

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{\mathbb{E}[\psi_{j,\gamma}(z_{i,t})] T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t})}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) (-\Psi_{j,\gamma\gamma,g})^{1/2}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) \right| \\ & \leq \frac{\max_{g \in \mathcal{G}_j} |\mathbb{E}[\psi_{j,\gamma}(z_{i,t})]| \left(\sum_{g \in \mathcal{G}_j} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 \right)^{1/2}}{\min_{g \in \mathcal{G}_j} |\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) (-\Psi_{j,\gamma\gamma,g})^{1/2}|} \\ & \times \left(\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} (T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}))^2 \right)^{1/2} = O_p(G_j T^{-1}), \end{aligned}$$

which along with (S103) and (S106) implies that

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \left(\frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2 - (\mathbb{E}[\psi_{j,\gamma}(z_{i,t}))]^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \right) = O_p(G_j T^{-1}). \quad (\text{S107})$$

The claim of the lemma now follows from (S99), (S102) and (S107). $Q.E.D.$

Lemma S6. *Under Assumptions 1 and 2, we have*

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\hat{\mathbb{E}}_T[\psi_{j,\gamma}^2(z_{i,t})] - \mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})]}{n_{j,g} \Psi_{j,\gamma\gamma,g}} = O_p((G_j T^{-1})^{1/2}) \quad (\text{S108})$$

and

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(\hat{\mathbb{E}}_T[\psi_{j,\gamma}(z_{i,t})])^2 - (\mathbb{E}_T[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \Psi_{j,\gamma\gamma,g}} = O_p((G_j T^{-1})^{1/2}). \quad (\text{S109})$$

PROOF OF LEMMA S6. Recall that $\psi_{j,\gamma}^*(z_{i,t}) \equiv \psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]$. Using this expression, we have

$$\sum_{t \leq T} \psi_{j,\gamma}(z_{i,t})^2 = \sum_{t \leq T} \psi_{j,\gamma}^*(z_{i,t})^2 + 2 \sum_{t \leq T} \psi_{j,\gamma}^*(z_{i,t}) \mathbb{E}[\psi_{j,\gamma}(z_{i,t})] + \sum_{t \leq T} (\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2$$

and

$$\sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}(z_{i,t})^2] = \sum_{t \leq T} \mathbb{E}[\psi_{j,\gamma}^*(z_{i,t})^2] + \sum_{t \leq T} (\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2.$$

Therefore,

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t})^2 - \mathbb{E}[\psi_{j,\gamma}(z_{i,t})^2])}{n_{j,g} \Psi_{j,\gamma\gamma,g}} \\ &= \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}^*(z_{i,t})^2 - \mathbb{E}[\psi_{j,\gamma}^*(z_{i,t})^2])}{n_{j,g} \Psi_{j,\gamma\gamma,g}} + 2 \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{T^{-1} \sum_{t \leq T} \psi_{j,\gamma}^*(z_{i,t}) \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]}{n_{j,g} \Psi_{j,\gamma\gamma,g}}. \end{aligned} \quad (\text{S110})$$

By Assumptions 1, 2, 8(iii) and 9, it follows that

$$\begin{aligned} & \text{Var} \left(\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}^*(z_{i,t})^2 - \mathbb{E}[\psi_{j,\gamma}^*(z_{i,t})^2])}{n_{j,g} \Psi_{j,\gamma\gamma,g}} \right) \\ &= \sum_{g \in \mathcal{G}_j} \frac{\text{Var} \left(\sum_{i \in I_{j,g}} \sum_{t \leq T} (\psi_{j,\gamma}^*(z_{i,t})^2 - \mathbb{E}[\psi_{j,\gamma}^*(z_{i,t})^2]) \right)}{T^2 n_{j,g}^2 \Psi_{j,\gamma\gamma,g}^2} \\ &\leq K \sum_{g \in \mathcal{G}_j} \frac{\mathbb{E} \left[\left| (n_{j,g} T)^{-1/2} \sum_{i \in I_{j,g}} \sum_{t \leq T} (\psi_{j,\gamma}^*(z_{i,t})^2 - \mathbb{E}[\psi_{j,\gamma}^*(z_{i,t})^2]) \right|^2 \right]}{T n_{j,g}} \leq K G_j T^{-1}, \end{aligned} \quad (\text{S111})$$

where the second inequality follows by (S553) in Lemma S47. Similarly,

$$\begin{aligned}
& \text{Var} \left(\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{T^{-1} \sum_{t \leq T} \psi_{j,\gamma}^*(z_{i,t}) \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]}{n_{j,g} \Psi_{j,\gamma\gamma,g}} \right) \\
&= \sum_{g \in \mathcal{G}_j} \frac{\text{Var} \left(\sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma}^*(z_{i,t}) \mathbb{E}[\psi_{j,\gamma}(z_{i,t})] \right)}{T^2 n_{j,g}^2 \Psi_{j,\gamma\gamma,g}^2} \\
&\leq K \sum_{g \in \mathcal{G}_j} \frac{\mathbb{E} \left[\left| (n_{j,g} T)^{-1/2} \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma}^*(z_{i,t}) \mathbb{E}[\psi_{j,\gamma}(z_{i,t})] \right|^2 \right]}{T n_{j,g}} \leq K G_j T^{-1} \tag{S112}
\end{aligned}$$

which, combined with (S110), (S111) and Markov's inequality, shows the claim in (S108).

To show the claim in (S109), we first express the term on its LHS as

$$\begin{aligned}
& \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2 - T^{-1} \sum_{t \leq T} (\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{n_{j,g} \Psi_{j,\gamma\gamma,g}} \\
&= \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}))^2}{n_{j,g}} + 2 \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]}{n_{j,g} (-\Psi_{j,\gamma\gamma,g})^{1/2}}. \tag{S113}
\end{aligned}$$

Under Assumptions 1, 2, 8 and 9, we can apply Markov's inequality to obtain:

$$\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^2 = O_p(G_j T^{-1}), \tag{S114}$$

and

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{T^{-1} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]}{n_{j,g} (-\Psi_{j,\gamma\gamma,g})^{1/2}} = O_p((G_j T^{-1})^{1/2}). \tag{S115}$$

The claim in (S109) now follows from (S113), (S114) and (S115). Q.E.D.

Lemma S7. *Under Assumptions 1 and 2, we have*

$$\frac{(MQLR_{n,T})^2 - (\overline{QLR}_{n,T})^2}{(nT)^{1/2} \omega_{n,T}^2} = O_p(1).$$

PROOF OF LEMMA S7. First, observe that by the definitions of $MQLR_{n,T}$ and $\overline{QLR}_{n,T}$, and the triangle inequality,

$$\begin{aligned}
\frac{|MQLR_{n,T} - \overline{QLR}_{n,T}|}{\omega_{n,T}} &\leq \frac{|MQLR_{n,T} - MQLR_{n,T}^*|}{\omega_{n,T}} + \frac{|MQLR_{n,T}^* - \overline{QLR}_{n,T}|}{\omega_{n,T}} \\
&= \frac{|R_{j,n,T}(\hat{\phi}_j) - T^{1/2} \sum_{i \leq n} \mathbb{E}[\tilde{V}_{j,i}]|}{(nT)^{1/2} \omega_{n,T}} + O_p(1) = O_p(1), \tag{S116}
\end{aligned}$$

where the first equality follows from Theorem 1, and the second equality follows from Theorem 2. By the definition of $\omega_{n,T}$, Assumption 2(ii), the triangle inequality and Hölder's inequality, we have

$$\frac{|\overline{QLR}_{n,T}|}{(nT)^{1/2}\omega_{n,T}} \leq (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \frac{\mathbb{E}[|\Delta\psi(z_{i,t})|]}{\omega_{n,T}} \leq K. \quad (\text{S117})$$

Combining the results in (S116) and (S117), we conclude that

$$\begin{aligned} \left| \frac{(MQLR_{n,T})^2 - (\overline{QLR}_{n,T})^2}{(nT)^{1/2}\omega_{n,T}^2} \right| &\leq \frac{(MQLR_{n,T} - \overline{QLR}_{n,T})^2}{(nT)^{1/2}\omega_{n,T}^2} \\ &\quad + \frac{2\overline{QLR}_{n,T}|MQLR_{n,T} - \overline{QLR}_{n,T}|}{(nT)^{1/2}\omega_{n,T}^2} = O_p(1), \end{aligned} \quad (\text{S118})$$

which shows the claim of the lemma. Q.E.D.

Lemma S8. *Under Assumptions 1 and 2, we have*

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \psi_{j,\phi}(z_{i,t}) (\hat{\phi}_{j,i} - \phi_{j,i}^*) = O_p(\omega_{n,T}(nT)^{-1/2}),$$

where $\psi_{j,\phi}(z_{i,t}) \equiv \psi_{j,\phi}(z_{i,t}, \phi_{j,i}^*)$ and $\psi_{j,\phi}(z_{i,t}, \phi_{j,i}) \equiv \partial\psi_j(z_{i,t}, \phi_{j,i})/\partial\phi_{j,i}$.

PROOF OF LEMMA S8. By the definition of $\psi_{j,\phi}(z_{i,t})$, we have

$$\psi_{j,\phi}(z_{i,t}) (\hat{\phi}_{j,i} - \phi_{j,i}^*) = \psi_{j,\theta}(z_{i,t}) (\hat{\theta}_j - \theta_j^*) + \psi_{j,\gamma}(z_{i,t}) (\hat{\gamma}_{j,i} - \gamma_{j,i}^*), \quad (\text{S119})$$

which leads to the expression:

$$\begin{aligned} (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \psi_{j,\phi}(z_{i,t}) (\hat{\phi}_{j,i} - \phi_{j,i}^*) &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \psi_{j,\theta}(z_{i,t}) (\hat{\theta}_j - \theta_j^*) \\ &\quad + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t}) (\hat{\gamma}_{j,i} - \gamma_{j,i}^*), \end{aligned} \quad (\text{S120})$$

where $\psi_{j,a}(z_{i,t}) \equiv \psi_{j,a}(z_{i,t}, \phi_{j,i}^*)$ and $\psi_{j,a}(z_{i,t}, \phi_{j,i}) \equiv \partial\psi_j(z_{i,t}, \phi_{j,i})/\partial a$ for $a \in \{\theta, \gamma\}$.

By the triangle inequality and the Cauchy-Schwarz inequality, the first term after the equality in (S120) can be bounded as:

$$\left| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \psi_{j,\theta}(z_{i,t}) (\hat{\theta}_j - \theta_j^*) \right| \leq \frac{\|\hat{\theta}_j - \theta_j^*\|}{nT} \sum_{i \leq n} \sum_{t \leq T} |\Delta\psi(z_{i,t})| \|\psi_{j,\theta}(z_{i,t})\|. \quad (\text{S121})$$

Under Assumptions 2(ii) and 8,

$$\sum_{i \leq n} \sum_{t \leq T} \frac{\mathbb{E}[|\Delta\psi(z_{i,t})| \|\psi_{j,\theta}(z_{i,t})\|]}{nT} \leq \sum_{i \leq n} \sum_{t \leq T} \frac{\left(\mathbb{E}[\Delta\psi(z_{i,t})^2] \mathbb{E}[\|\psi_{j,\theta}(z_{i,t})\|^2] \right)^{1/2}}{nT} \leq K\omega_{n,T},$$

which, along with Markov's inequality, Lemma S1(iii) and (S121), establishes

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \psi_{j,\theta}(z_{i,t}) (\hat{\theta}_j - \theta_j^*) = O_p(\omega_{n,T}(nT)^{-1/2}). \quad (\text{S122})$$

Next, we rewrite the second term after the equality in (S120) as:

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t}) (\hat{\gamma}_{j,i} - \gamma_{j,i}^*) \\ &= (nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\mathbb{E}[\Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t})]}{-\Psi_{j,\gamma\gamma,g}} \hat{\Psi}_{j,\gamma,g} \\ &+ (nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t})]}{-\Psi_{j,\gamma\gamma,g}} \hat{\Psi}_{j,\gamma,g} \\ &+ (nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t}) (\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}). \end{aligned} \quad (\text{S123})$$

Using Assumptions 2(ii), 8 and 9, we bound the second moment of the first term:

$$\begin{aligned} & \mathbb{E} \left[\left((nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\mathbb{E}[\Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t})]}{-\Psi_{j,\gamma\gamma,g}} \hat{\Psi}_{j,\gamma,g} \right)^2 \right] \\ &= \omega_{n,T}^2 (nT)^{-2} \sum_{g \in \mathcal{G}_j} \mathbb{E}[\hat{\Psi}_{j,\gamma,g}^2] \left(\sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\mathbb{E}[\Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t})]}{-\Psi_{j,\gamma\gamma,g} \omega_{n,T}} \right)^2 \\ &\leq K \omega_{n,T}^2 (nT)^{-2} \sum_{g \in \mathcal{G}_j} (n_{j,g} T)^2 \mathbb{E}[\hat{\Psi}_{j,\gamma,g}^2] \leq K \omega_{n,T}^2 (nT)^{-1}, \end{aligned}$$

where the second inequality follows from Lemma S1(viii). Hence by Markov's inequality and Assumption 1(i), the first term on the right of (S123) satisfies:

$$(nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\mathbb{E}[\Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t})]}{-\Psi_{j,\gamma\gamma,g}} \hat{\Psi}_{j,\gamma,g} = O_p(\omega_{n,T}(nT)^{-1/2}). \quad (\text{S124})$$

For the second term in the RHS of (S123), we can apply the triangle inequality and Holder's inequality to obtain:

$$\begin{aligned} & \mathbb{E} \left[\left| (nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t})]}{-\Psi_{j,\gamma\gamma,g}} \hat{\Psi}_{j,\gamma,g} \right|^2 \right] \\ &\leq K (nT)^{-1} \sum_{g \in \mathcal{G}_j} \sqrt{\mathbb{E}[\hat{\Psi}_{j,\gamma,g}^2]} \sqrt{\mathbb{E} \left[\left| \sum_{i \in I_{j,g}} \sum_{t \leq T} \Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\Delta\psi(z_{i,t}) \psi_{j,\gamma}(z_{i,t})] \right|^2 \right]}. \end{aligned} \quad (\text{S125})$$

Under Assumptions 1 and 2(ii, iii), it follows that

$$\begin{aligned}
& \mathbb{E} \left[\left| \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t})]}{\omega_{n,T}} \right|^2 \right] \\
&= \sum_{i \in I_{j,g}} \mathbb{E} \left[\left| \sum_{t \leq T} \frac{\Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t})]}{\omega_{n,T}} \right|^2 \right] \\
&\leq KT \sum_{i \in I_{j,g}} \mathbb{E} \left[\left| \frac{\Delta\psi(z_{i,t})}{\omega_{n,T}} \psi_{j,\gamma}(z_{i,t}) \right|^{2+\delta/2} \right] \\
&\leq KT \sum_{i \in I_{j,g}} \left(\mathbb{E} \left[\left| \frac{\Delta\psi(z_{i,t})}{\omega_{n,T}} \right|^{4+\delta} \right] \mathbb{E} \left[|\psi_{j,\gamma}(z_{i,t})|^{4+\delta} \right] \right)^{1/2} \leq K(n_{j,g}T),
\end{aligned}$$

where $\delta > 0$ is defined in Assumption 2(ii). The first inequality follows from Assumption 1 together with a covariance inequality for strongly mixing processes, and the second inequality follows from Hölder's inequality. This together with Assumption 1(i), Lemma S1(viii) and (S125), and Markov's inequality implies that the first term on the right of (S123) satisfies

$$(nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \frac{\Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t})]}{-\Psi_{j,\gamma\gamma,g}} \hat{\Psi}_{j,\gamma,g} = O_p(\omega_{n,T}(nT)^{-1/2}). \quad (\text{S126})$$

By the Cauchy-Schwarz inequality, the third term after the equality in (S123) can be bounded as

$$\begin{aligned}
& \left| (nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t})(\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}) \right| \\
&\leq n^{-1} \left(\sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g})^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_j} T^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} (\Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t}))^2 \right)^{1/2}. \quad (\text{S127})
\end{aligned}$$

By Assumptions 2(iii) and 8, and Markov's inequality,

$$\sum_{g \in \mathcal{G}_j} T^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} (\Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t}))^2 = O_p(\omega_{n,T}^2 n). \quad (\text{S128})$$

Combining this with Assumption 1(i), (S127) and Lemma S1(vii), we establish a bound of the third term on the right of (S123):

$$(nT)^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \sum_{t \leq T} \Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t})(\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}) = O_p(\omega_{n,T}(nT)^{-1/2}). \quad (\text{S129})$$

Collecting results from (S123), (S124), (S126) and (S129), we obtain:

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t})\psi_{j,\gamma}(z_{i,t})(\hat{\gamma}_{j,i} - \gamma_{j,i}^*) = O_p(\omega_{n,T}(nT)^{-1/2}). \quad (\text{S130})$$

The lemma follows from (S120), (S122) and (S130).

Q.E.D.

Lemma S9. *Under Assumptions 1 and 2, we have*

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t})(\hat{\gamma}_{j,i} - \gamma_{j,i}^*))^2 \\ &= (nT)^{-1} \sum_{g \in \mathcal{G}_j} n_{j,g}^{-2} \left(\sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \right) \sum_{i \in I_{j,g}} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^2 \right] + O_p(G_j^{1/2}(nT)^{-1}). \end{aligned}$$

PROOF OF LEMMA S9. We start with the following characterization:

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t})(\hat{\gamma}_{j,i} - \gamma_{j,i}^*))^2 \\ &= (nT)^{-1} \sum_{i \leq n} (\hat{\gamma}_{j,i} - \gamma_{j,i}^*)^2 \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]) + n^{-1} \sum_{i \leq n} \mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})](\hat{\gamma}_{j,i} - \gamma_{j,i}^*)^2. \end{aligned} \quad (\text{S131})$$

By the Cauchy-Schwarz inequality, the first term after the equality in (S131) can be bounded as:

$$\begin{aligned} & \left| \sum_{i \leq n} (\hat{\gamma}_{j,i} - \gamma_{j,i}^*)^2 \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]) \right| \\ &= \left| \sum_{g \in \mathcal{G}_j} (\hat{\gamma}_{j,g} - \gamma_{j,g}^*)^2 \sum_{i \in I_{j,g}} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]) \right| \\ &\leq \left(\sum_{g \in \mathcal{G}_j} n_{j,g}^2 (\hat{\gamma}_{j,g} - \gamma_{j,g}^*)^4 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_j} \left(n_{j,g}^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]) \right)^2 \right)^{1/2}. \end{aligned} \quad (\text{S132})$$

Using Assumptions 1, 2(iii) and the covariance inequality for strongly mixing processes, we can derive:

$$\begin{aligned} & \mathbb{E} \left[\sum_{g \in \mathcal{G}_j} \left(n_{j,g}^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]) \right)^2 \right] \\ &= \sum_{g \in \mathcal{G}_j} n_{j,g}^{-2} \sum_{i \in I_{j,g}} \mathbb{E} \left[\left(\sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]) \right)^2 \right] \\ &\leq KT \sum_{g \in \mathcal{G}_j} n_{j,g}^{-2} \sum_{i \in I_{j,g}} \mathbb{E} [|\psi_{j,\gamma}(z_{i,t})|^{4+\delta}] \leq KG_j T. \end{aligned}$$

Applying Markov's inequality, we obtain:

$$\sum_{g \in \mathcal{G}_j} \left(n_{j,g}^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]) \right)^2 = O_p(G_j T).$$

Combining this result with (S132) and Lemma S1(iv), we establish the bound for the first term on the right of (S131)

$$(nT)^{-1} \sum_{i \leq n} (\hat{\gamma}_{j,i} - \gamma_{j,i}^*)^2 \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]) = O_p(G_j(nT^{3/2})^{-1}). \quad (\text{S133})$$

Collecting the results from (S131) and (S133), we obtain:

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t})(\hat{\gamma}_{j,i} - \gamma_{j,i}^*))^2 = n^{-1} \sum_{i \leq n} \mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})](\hat{\gamma}_{j,i} - \gamma_{j,i}^*)^2 + O_p(G_j(nT^{3/2})^{-1}),$$

which we rewrite as:

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t})(\hat{\gamma}_{j,i} - \gamma_{j,i}^*))^2 \\ &= n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})]}{\Psi_{j,\gamma\gamma,g}^2} \hat{\Psi}_{j,\gamma,g}^2 - 2n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})]}{\Psi_{j,\gamma\gamma,g}} \hat{\Psi}_{j,\gamma,g} (\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}) \\ &+ n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})](\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g})^2 + O_p(G_j(nT^{3/2})^{-1}). \end{aligned} \quad (\text{S134})$$

Recall that $s_{j,\gamma,i}^2 \equiv \mathbb{E}_T[\tilde{\psi}_{j,\gamma}^2(z_{i,t})]$ and $\tilde{\psi}_{j,\gamma}(z_{i,t}) \equiv \psi_{j,\gamma}(z_{i,t}) (-\Psi_{j,\gamma\gamma,g})^{-1/2}$. The first term after the second equality in (S134) can be written as

$$\begin{aligned} n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})] \frac{\hat{\Psi}_{j,\gamma,g}^2}{\Psi_{j,\gamma\gamma,g}^2} &= n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \mathbb{E} \left[\frac{\hat{\Psi}_{j,\gamma,g}^2}{-\Psi_{j,\gamma\gamma,g}} \right] \\ &+ n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \left(\frac{\hat{\Psi}_{j,\gamma,g}^2}{-\Psi_{j,\gamma\gamma,g}} - \mathbb{E} \left[\frac{\hat{\Psi}_{j,\gamma,g}^2}{-\Psi_{j,\gamma\gamma,g}} \right] \right). \end{aligned} \quad (\text{S135})$$

Under Assumption 1, we know that

$$\mathbb{E} \left[\frac{\hat{\Psi}_{j,\gamma,g}^2}{-\Psi_{j,\gamma\gamma,g}} \right] = \mathbb{E} \left[\left((n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^2 \right] = (n_{j,g}^2T)^{-1} \sum_{i \in I_{j,g}} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^2 \right].$$

Thus, we have:

$$n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \mathbb{E} \left[\frac{\hat{\Psi}_{j,\gamma,g}^2}{-\Psi_{j,\gamma\gamma,g}} \right] = (nT)^{-1} \sum_{g \in \mathcal{G}_j} n_{j,g}^{-2} \left(\sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \right) \sum_{i \in I_{j,g}} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^2 \right], \quad (\text{S136})$$

where

$$\mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^2 \right] \leq K \quad (\text{S137})$$

by arguments analogous to those used to establish (S1). Using Assumptions 1, 8 and 9, and Lemma S1(viii), we also derive:

$$\begin{aligned} & \mathbb{E} \left[\left| n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \left(\frac{\hat{\Psi}_{j,\gamma,g}^2}{-\Psi_{j,\gamma\gamma,g}} - \mathbb{E} \left[\frac{\hat{\Psi}_{j,\gamma,g}^2}{-\Psi_{j,\gamma\gamma,g}} \right] \right) \right|^2 \right] \\ & \leq n^{-2} \sum_{g \in \mathcal{G}_j} \left(\sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \right)^2 \mathbb{E} \left[\frac{\hat{\Psi}_{j,\gamma,g}^4}{\Psi_{j,\gamma\gamma,g}^2} \right] \leq K n^{-2} \sum_{g \in \mathcal{G}_j} n_{j,g}^2 \mathbb{E}[\hat{\Psi}_{j,\gamma,g}^4] \leq K G_j (nT)^{-2}, \end{aligned}$$

which along with Markov's inequality implies that

$$n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \left(\frac{\hat{\Psi}_{j,\gamma,g}^2}{-\Psi_{j,\gamma\gamma,g}} - \mathbb{E} \left[\frac{\hat{\Psi}_{j,\gamma,g}^2}{-\Psi_{j,\gamma\gamma,g}} \right] \right) = O_p(G_j^{1/2} (nT)^{-1}). \quad (\text{S138})$$

Combining the results from (S135), (S136) and (S138), we establish a bound for the first term on the right of (S134):

$$\begin{aligned} n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})] \frac{\hat{\Psi}_{j,\gamma,g}^2}{\Psi_{j,\gamma\gamma,g}^2} &= (nT)^{-1} \sum_{g \in \mathcal{G}_j} n_{j,g}^{-2} \left(\sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \right) \sum_{i \in I_{j,g}} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^2 \right] \\ &+ O_p(G_j^{1/2} (nT)^{-1}) = O_p(G_j (nT)^{-1}), \end{aligned} \quad (\text{S139})$$

where the second equality follows from Assumption 8 and (S137).

By Assumption 8 and Lemma S1(vii), we obtain a bound for the third term on the right of (S134):

$$\begin{aligned} & n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})] (\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g})^2 \\ & \leq n^{-1} \sum_{g \in \mathcal{G}_j} n_{j,g} (\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g})^2 = O_p((nT)^{-1}). \end{aligned} \quad (\text{S140})$$

From (S139) and (S140), and the Cauchy-Schwarz inequality, we establish a bound for the second term on the right of (S134):

$$n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\mathbb{E}_T[\psi_{j,\gamma}^2(z_{i,t})]}{\Psi_{j,\gamma\gamma,g}} \hat{\Psi}_{j,\gamma,g} (\hat{\gamma}_{j,g} - \gamma_{j,g}^* + \Psi_{j,\gamma\gamma,g}^{-1} \hat{\Psi}_{j,\gamma,g}) = O_p(G_j^{1/2} (nT)^{-1}),$$

which along with (S134), (S139) and (S140), completes the proof of the lemma.. Q.E.D.

Lemma S10. *Under Assumptions 1 and 2 we have*

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*) (\hat{\gamma}_{2,i} - \gamma_{2,i}^*) \sum_{t \leq T} \psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) \\ & = (nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{12,\gamma,i} \left(T^{-1} \sum_{t,t' \leq T} \mathbb{E}[\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})] \right) + O_p(G_2^{1/2} (nT)^{-1}). \end{aligned} \quad (\text{S141})$$

PROOF OF LEMMA S10. First, we observe that the term on the LHS of (S141) can be written as:

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*) (\hat{\gamma}_{2,i} - \gamma_{2,i}^*) \sum_{t \leq T} \psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) \\
&= n^{-1} \sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*) (\hat{\gamma}_{2,i} - \gamma_{2,i}^*) \mathbb{E}_T[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})] \\
&+ (nT)^{-1} \sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*) (\hat{\gamma}_{2,i} - \gamma_{2,i}^*) \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]). \quad (\text{S142})
\end{aligned}$$

By the Cauchy-Schwarz inequality, we can bound the second term after the equality in (S142) as:

$$\begin{aligned}
& \left| (nT)^{-1} \sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*) (\hat{\gamma}_{2,i} - \gamma_{2,i}^*) \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]) \right| \\
&\leq (nT)^{-1} \left(\sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*)^4 \right)^{1/4} \times \left(\sum_{i \leq n} (\hat{\gamma}_{2,i} - \gamma_{2,i}^*)^4 \right)^{1/4} \\
&\times \left(\sum_{i \leq n} \left(\sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]) \right)^2 \right)^{1/2}. \quad (\text{S143})
\end{aligned}$$

By Assumptions 1, 2(iii), the covariance inequality for strongly mixing processes and Hölder's inequality,

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]) \right)^2 \right] \\
&\leq KT \max_{t \leq T} \|\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})\|_{2+\delta/2}^2 \leq KT \max_{t \leq T} \|\psi_{1,\gamma}(z_{i,t})\|_{4+\delta}^2 \|\psi_{2,\gamma}(z_{i,t})\|_{4+\delta}^2 \leq KT.
\end{aligned}$$

Hence by Markov's inequality,

$$\sum_{i \leq n} \left(\sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]) \right)^2 = O_p(nT),$$

which together with (S143) and Lemma S1(iv) leads to:

$$(nT)^{-1} \sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*) (\hat{\gamma}_{2,i} - \gamma_{2,i}^*) \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]) = O_p(T^{-3/2}). \quad (\text{S144})$$

The first order condition for $\gamma_{1,i}^*$ implies that $\mathbb{E}_T[\psi_{1,\gamma}(z_{i,t})] = 0$ for any i . Consequently,

$$\mathbb{E}_T[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})] = \mathbb{E}_T[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})] - \mathbb{E}_T[\psi_{1,\gamma}(z_{i,t})] \mathbb{E}_T[\psi_{2,\gamma}(z_{i,t})] = \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})]$$

for any i . Using this expression, we can rewrite the first term after the equality in (S142) as:

$$\begin{aligned}
& n^{-1} \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^*) \sum_{i \in I_{2,g}} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*) \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \\
&= n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\hat{\Psi}_{2,\gamma\gamma,g}} \sum_{i \in I_{2,g}} \frac{\hat{\Psi}_{1,\gamma,i}}{\hat{\Psi}_{1,\gamma\gamma,i}} \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \\
&+ n^{-1} \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^* + \hat{\Psi}_{2,\gamma\gamma,g}^{-1} \hat{\Psi}_{2,\gamma,g}) \sum_{i \in I_{2,g}} (\hat{\gamma}_{1,i} - \gamma_{1,i}^* + \hat{\Psi}_{1,\gamma\gamma,i}^{-1} \hat{\Psi}_{1,\gamma,i}) \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \\
&- n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\hat{\Psi}_{2,\gamma\gamma,g}} \sum_{i \in I_{2,g}} (\hat{\gamma}_{1,i} - \gamma_{1,i}^* + \hat{\Psi}_{1,\gamma\gamma,i}^{-1} \hat{\Psi}_{1,\gamma,i}) \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \\
&- n^{-1} \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^* + \hat{\Psi}_{2,\gamma\gamma,g}^{-1} \hat{\Psi}_{2,\gamma,g}) \sum_{i \in I_{2,g}} \frac{\hat{\Psi}_{1,\gamma,i}}{\hat{\Psi}_{1,\gamma\gamma,i}} \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})], \tag{S145}
\end{aligned}$$

where

$$\hat{\Psi}_{j,\gamma\gamma,g} \equiv (n_{j,g}T)^{-1} \sum_{i \in I_{j,g}} \sum_{t \leq T} \psi_{j,\gamma\gamma}(z_{i,t}; \phi_{j,g}^*),$$

for $j = 1, 2$, and any $g \in \mathcal{G}_j$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \sum_{g \in \mathcal{G}_2} n_{2,g}^2 \left(\frac{\hat{\Psi}_{2,\gamma,g}}{\hat{\Psi}_{2,\gamma\gamma,g}} - \frac{\hat{\Psi}_{2,\gamma,g}}{\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}]} \right)^2 \\
&= \sum_{g \in \mathcal{G}_2} \frac{n_{2,g}^2 \hat{\Psi}_{2,\gamma,g}^2}{\hat{\Psi}_{2,\gamma\gamma,g}^2 (\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}])^2} (\hat{\Psi}_{2,\gamma\gamma,g} - \mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}])^2 \\
&\leq \frac{\left(\sum_{g \in \mathcal{G}_2} n_{2,g}^2 \hat{\Psi}_{2,\gamma,g}^4 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^2 (\hat{\Psi}_{2,\gamma\gamma,g} - \mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}])^4 \right)^{1/2}}{\min_{g \in \mathcal{G}_2} \hat{\Psi}_{2,\gamma\gamma,g}^2 (\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}])^2}, \tag{S146}
\end{aligned}$$

By Lemma S1(viii) and Markov's inequality, we have

$$\sum_{g \in \mathcal{G}_2} n_{2,g}^2 \hat{\Psi}_{2,\gamma,g}^4 = O_p(G_2 T^{-2}). \tag{S147}$$

Similarly, we can show that

$$\sum_{g \in \mathcal{G}_2} n_{2,g}^2 (\hat{\Psi}_{2,\gamma\gamma,g} - \mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}])^4 = O_p(G_2 T^{-2}). \tag{S148}$$

Combining the results from Assumption 9 and Lemma S1(ii), (S146), (S147) and (S148), we establish:

$$\sum_{g \in \mathcal{G}_2} n_{2,g}^2 \left(\frac{\hat{\Psi}_{2,\gamma,g}}{\hat{\Psi}_{2,\gamma\gamma,g}} - \frac{\hat{\Psi}_{2,\gamma,g}}{\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}]} \right)^2 = O_p(G_2 T^{-2}). \tag{S149}$$

Similarly, we can show that

$$\begin{aligned}
& \sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \left(\frac{\hat{\Psi}_{1,\gamma,i}}{\hat{\Psi}_{1,\gamma\gamma,i}} - \frac{\hat{\Psi}_{1,\gamma,i}}{\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}]} \right) \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \right)^2 \\
& \leq \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \left(\frac{\hat{\Psi}_{1,\gamma,i}}{\hat{\Psi}_{1,\gamma\gamma,i}} - \frac{\hat{\Psi}_{1,\gamma,i}}{\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}]} \right)^2 (\mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})])^2 \\
& \leq \frac{K \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \hat{\Psi}_{1,\gamma,i}^2 (\hat{\Psi}_{1,\gamma\gamma,i} - \mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}])^2}{\min_{i \leq n} \hat{\Psi}_{1,\gamma\gamma,i}^2 (\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}])^2} = O_p(G_2 T^{-2}). \tag{S150}
\end{aligned}$$

By Assumptions 1, 2(iii), 9, and Markov's inequality, we have

$$\sum_{g \in \mathcal{G}_2} n_{2,g}^2 \frac{\hat{\Psi}_{2,\gamma,g}^2}{(\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}])^2} = O_p(n T^{-1}) \tag{S151}$$

and

$$\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \left(\sum_{i \in I_{2,g}} \frac{\mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \hat{\Psi}_{1,\gamma,i}}{\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}]} \right)^2 = O_p(G_2 T^{-1}). \tag{S152}$$

Collecting the results in (S149), (S150), (S151) and (S152), and applying the Cauchy-Schwarz inequality, we obtain an approximation to the first term on the right of (S145):

$$\begin{aligned}
& n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\hat{\Psi}_{2,\gamma\gamma,g}} \sum_{i \in I_{2,g}} \frac{\hat{\Psi}_{1,\gamma,i}}{\hat{\Psi}_{1,\gamma\gamma,i}} \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \\
& = n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}]} \sum_{i \in I_{2,g}} \frac{\hat{\Psi}_{1,\gamma,i}}{\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}]} \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \\
& + n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}]} \sum_{i \in I_{2,g}} \left(\frac{\hat{\Psi}_{1,\gamma,i}}{\hat{\Psi}_{1,\gamma\gamma,i}} - \frac{\hat{\Psi}_{1,\gamma,i}}{\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}]} \right) \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \\
& + n^{-1} \sum_{g \in \mathcal{G}_2} \left(\frac{\hat{\Psi}_{2,\gamma,g}}{\hat{\Psi}_{2,\gamma\gamma,g}} - \frac{\hat{\Psi}_{2,\gamma,g}}{\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}]} \right) \sum_{i \in I_{2,g}} \frac{\mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \hat{\Psi}_{1,\gamma,i}}{\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}]} \\
& + n^{-1} \sum_{g \in \mathcal{G}_2} \left(\frac{\hat{\Psi}_{2,\gamma,g}}{\hat{\Psi}_{2,\gamma\gamma,g}} - \frac{\hat{\Psi}_{2,\gamma,g}}{\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}]} \right) \sum_{i \in I_{2,g}} \left(\frac{\hat{\Psi}_{1,\gamma,i}}{\hat{\Psi}_{1,\gamma\gamma,i}} - \frac{\hat{\Psi}_{1,\gamma,i}}{\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}]} \right) \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \\
& = n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}]} \sum_{i \in I_{2,g}} \frac{\mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \hat{\Psi}_{1,\gamma,i}}{\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}]} + O_p(G_2^{1/2} (nT)^{-1}). \tag{S153}
\end{aligned}$$

Since $\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}] = \Psi_{2,\gamma\gamma,g}$ and $\mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}] = \Psi_{2,\gamma\gamma,i}$, by the definition of $\sigma_{12,\gamma,i}$, we have

$$\sigma_{12,\gamma,i} = \frac{\mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})]}{(\mathbb{E}[\hat{\Psi}_{2,\gamma\gamma,g}] \mathbb{E}[\hat{\Psi}_{1,\gamma\gamma,i}])^{1/2}}.$$

Let $\tilde{\Psi}_{j,\gamma,i}^* \equiv T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t})$. Then the leading term on the far RHS of (S153) can be expressed as

$$\begin{aligned}
& n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\mathbb{E}[\hat{\Psi}_{2,\gamma,g}]} \sum_{i \in I_{2,g}} \frac{\mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \hat{\Psi}_{1,\gamma,i}}{\mathbb{E}[\hat{\Psi}_{1,\gamma,i}]} \\
&= n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g} - \mathbb{E}[\hat{\Psi}_{2,\gamma,g}]}{(\mathbb{E}[-\hat{\Psi}_{2,\gamma,g}])^{1/2}} \sum_{i \in I_{2,g}} \frac{\hat{\Psi}_{1,\gamma,i} - \mathbb{E}[\hat{\Psi}_{1,\gamma,i}]}{(\mathbb{E}[-\hat{\Psi}_{1,\gamma,i}])^{1/2}} \sigma_{12,\gamma,i} \\
&= (nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \tilde{\Psi}_{2,\gamma,i}^* \sum_{i' \in I_{2,g}} \tilde{\Psi}_{1,\gamma,i'}^* \sigma_{12,\gamma,i'} \\
&= (nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{\sigma_{12,\gamma,i} \tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i}^*}{n_{2,g}} + (nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \sum_{i' \in I_{2,g}, i' \neq i} \frac{\sigma_{12,\gamma,i} \tilde{\Psi}_{1,\gamma,i'}^* \tilde{\Psi}_{2,\gamma,i'}^*}{n_{2,g}}. \quad (\text{S154})
\end{aligned}$$

By Assumption 1, Hölder's inequality and Lemma S1(viii), it follows that

$$\begin{aligned}
& \mathbb{E} \left[\left((nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{\sigma_{12,\gamma,i}}{n_{2,g}} (\tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i}^* - \mathbb{E}[\tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i}^*]) \right)^2 \right] \\
& \leq (nT)^{-2} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{\sigma_{12,\gamma,i}^2}{n_{2,g}^2} \mathbb{E}[(\tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i}^*)^2] \leq KG_2(nT)^{-2},
\end{aligned}$$

where we write $\tilde{\Psi}_{2,\gamma,i}^* \equiv \tilde{\Psi}_{2,\gamma,g}^*$ for any $i \in I_{2,g}$ and any $g \in \mathcal{G}_2$. This along with Markov's inequality implies that

$$(nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{\sigma_{12,\gamma,i}}{n_{2,g}} (\tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i}^* - \mathbb{E}[\tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i}^*]) = O_p(G_2^{1/2}(nT)^{-1}). \quad (\text{S155})$$

Since $\mathbb{E}[\tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i}^*] = T^{-1} \sum_{t,t' \leq T} \mathbb{E}[\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})]$, from (S155) it follows that the first term on the far RHS of (S154) can be alternatively written as:

$$\begin{aligned}
& (nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{\sigma_{12,\gamma,i} \tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i}^*}{n_{2,g}} \\
&= (nT^2)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{12,\gamma,i} \left(\sum_{t,t' \leq T} \mathbb{E}[\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})] \right) + O_p(G_2^{1/2}(nT)^{-1}). \quad (\text{S156})
\end{aligned}$$

For the second term on the RHS of the second equality in (S154), we can write it as

$$\begin{aligned}
& (nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \sum_{i' \in I_{2,g}, i' \neq i} \frac{\sigma_{12,\gamma,i} \tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i'}^*}{n_{2,g}} \\
&= (nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \sum_{i' \in I_{2,g}, i' < i} \left(\frac{\sigma_{12,\gamma,i} \tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i'}^*}{n_{2,g}} + \frac{\sigma_{12,\gamma,i'} \tilde{\Psi}_{1,\gamma,i'}^* \tilde{\Psi}_{2,\gamma,i}^*}{n_{2,g}} \right). \quad (\text{S157})
\end{aligned}$$

By Assumption 1, Lemma S1(viii) and Hölder's inequality, we have

$$\begin{aligned} & \mathbb{E} \left[\left((nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \sum_{i' \in I_{2,g}, i' < i} \frac{\sigma_{12,\gamma,i} \tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i'}^*}{n_{2,g}} \right)^2 \right] \\ &= (nT)^{-2} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \sum_{i' \in I_{2,g}, i' < i} \frac{\sigma_{12,\gamma,i}^2 \mathbb{E}[(\tilde{\Psi}_{1,\gamma,i}^* \tilde{\Psi}_{2,\gamma,i'}^*)^2]}{n_{2,g}^2} \leq G_2 (nT)^{-2}, \end{aligned}$$

which together with Markov's inequality implies that

$$(nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \sum_{i' \in I_{2,g}, i' < i} \frac{\sigma_{12,\gamma,i} \tilde{\Psi}_{1,\gamma,i'}^* \tilde{\Psi}_{2,\gamma,i}^*}{n_{2,g}} = O_p(G_2^{1/2} (nT)^{-1}). \quad (\text{S158})$$

Similarly, we can show that

$$(nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \sum_{i' \in I_{2,g}, i' < i} \frac{\sigma_{12,\gamma,i'} \tilde{\Psi}_{1,\gamma,i'}^* \tilde{\Psi}_{2,\gamma,i}^*}{n_{2,g}} = O_p(G_2^{1/2} (nT)^{-1}). \quad (\text{S159})$$

Collecting the results from (S154), and (S156)-(S159) yields an approximation to the first term on the right of (S145):

$$\begin{aligned} & n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\hat{\Psi}_{2,\gamma\gamma,g}} \sum_{i \in I_{2,g}} \frac{\hat{\Psi}_{1,\gamma,i}}{\hat{\Psi}_{1,\gamma\gamma,i}} \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \\ &= (nT^2)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{12,\gamma,i} \left(\sum_{t,t' \leq T} \mathbb{E}[\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})] \right) + O_p(G_2^{1/2} (nT)^{-1}). \end{aligned} \quad (\text{S160})$$

We now consider the second term on the right of (S145). By the Cauchy-Schwarz inequality, Assumption 8 and Lemma S1(vii), we have

$$\begin{aligned} & \left| n^{-1} \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^* + \hat{\Psi}_{2,\gamma\gamma,g}^{-1} \hat{\Psi}_{2,\gamma,g}) \sum_{i \in I_{2,g}} (\hat{\gamma}_{1,i} - \gamma_{1,i}^* + \hat{\Psi}_{1,\gamma\gamma,i}^{-1} \hat{\Psi}_{1,\gamma,i}) \mathbb{E}_T[\psi_{1,\gamma}^*(z_{i,t}) \psi_{2,\gamma}^*(z_{i,t})] \right| \\ & \leq K n^{-1} \left(\sum_{g \in \mathcal{G}_2} n_{2,g} (\hat{\gamma}_{2,g} - \gamma_{2,g}^* + \hat{\Psi}_{2,\gamma\gamma,g}^{-1} \hat{\Psi}_{2,\gamma,g})^2 \right)^{1/2} \\ & \times \left(\sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^* + \hat{\Psi}_{1,\gamma\gamma,i}^{-1} \hat{\Psi}_{1,\gamma,i})^2 \right)^{1/2} = O_p((nT)^{-1}). \end{aligned} \quad (\text{S161})$$

As for the third term on the right of (S145), we have by Assumption 8, Lemma S1(ii, vi, viii),

and Markov's inequality:

$$\begin{aligned}
& \left| n^{-1} \sum_{g \in \mathcal{G}_2} \frac{\hat{\Psi}_{2,\gamma,g}}{\hat{\Psi}_{2,\gamma\gamma,g}} \sum_{i \in I_{2,g}} (\hat{\gamma}_{1,i} - \gamma_{1,i}^* + \hat{\Psi}_{1,\gamma\gamma,i}^{-1} \hat{\Psi}_{1,\gamma,i}) \sigma_{12,\gamma,i} \right| \\
& \leq n^{-1} \left(\sum_{g \in \mathcal{G}_2} \frac{n_{2,g} \hat{\Psi}_{2,\gamma,g}^2}{\hat{\Psi}_{2,\gamma\gamma,g}^2} \right)^{1/2} \left(\sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^* + \hat{\Psi}_{1,\gamma\gamma,i}^{-1} \hat{\Psi}_{1,\gamma,i})^2 \sigma_{12,\gamma,i}^2 \right)^{1/2} \\
& = O_p(G_2^{1/2} (nT)^{-1}).
\end{aligned} \tag{S162}$$

Finally, the last term on the right of (S145) can be bounded by

$$\begin{aligned}
& \left| n^{-1} \sum_{g \in \mathcal{G}_2} (\hat{\gamma}_{2,g} - \gamma_{2,g}^* + \hat{\Psi}_{2,\gamma\gamma,g}^{-1} \hat{\Psi}_{2,\gamma,g}) \sum_{i \in I_{2,g}} \frac{\hat{\Psi}_{1,\gamma,i}}{\hat{\Psi}_{1,\gamma\gamma,i}} \sigma_{12,\gamma,i} \right| \\
& \leq n^{-1} \left(\sum_{g \in \mathcal{G}_2} n_{2,g} (\hat{\gamma}_{2,g} - \gamma_{2,g}^* + \hat{\Psi}_{2,\gamma\gamma,g}^{-1} \hat{\Psi}_{2,\gamma,g})^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{\hat{\Psi}_{1,\gamma,i}^2}{\hat{\Psi}_{1,\gamma\gamma,i}^2} \sigma_{12,\gamma,i}^2 \right)^{1/2} \\
& = O_p(n^{-1} T^{-1/2}).
\end{aligned} \tag{S163}$$

The claim of the lemma now follows from the results in (S145), (S160), (S161), (S162) and (S163). *Q.E.D.*

Lemma S11. *Under Assumptions 1 and 2, we have*

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_{1,\phi}(z_{i,t})(\hat{\phi}_{1,i} - \phi_{1,i}^*) - \psi_{2,\phi}(z_{i,t})(\hat{\phi}_{2,i} - \phi_{2,i}^*))^2 \\
& = (nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(\sum_{i \in I_{2,g}} \sigma_{1,\gamma,i}^4 + n_{2,g}^{-2} \sum_{i \in I_{j,g}} s_{j,\gamma,i}^2 \sum_{i \in I_{j,g}} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^2 \right] \right. \\
& \quad \left. - 2n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{12,\gamma,i} \left(T^{-1} \sum_{t,t' \leq T} \mathbb{E}[\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})] \right) \right) + O_p(T^{-3/2}).
\end{aligned}$$

PROOF OF LEMMA S11. From the expression of $\psi_{j,\phi}(z_{i,t})(\hat{\phi}_{j,i} - \phi_{j,i}^*)$ in (S119), we have

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_{1,\phi}(z_{i,t})(\hat{\phi}_{1,i} - \phi_{1,i}^*) - \psi_{2,\phi}(z_{i,t})(\hat{\phi}_{2,i} - \phi_{2,i}^*))^2 \\
& = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\Delta\psi_\theta(z_{i,t})(\hat{\theta} - \theta^*))^2 + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\Delta\psi_\gamma(z_{i,t})(\hat{\gamma}_i - \gamma_i^*))^2 \\
& \quad + 2(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\theta} - \theta^*)^\top \Delta\psi_\theta(z_{i,t})^\top \Delta\psi_\gamma(z_{i,t})(\hat{\gamma}_i - \gamma_i^*),
\end{aligned} \tag{S164}$$

where

$$\begin{aligned}
\Delta\psi_\theta(z_{i,t})(\hat{\theta} - \theta^*) & \equiv \psi_{1,\theta}(z_{i,t})(\hat{\theta}_1 - \theta_1^*) - \psi_{2,\theta}(z_{i,t})(\hat{\theta}_2 - \theta_2^*), \\
\Delta\psi_\gamma(z_{i,t})(\hat{\gamma}_i - \gamma_i^*) & \equiv \psi_{1,\gamma}(z_{i,t})(\hat{\gamma}_{1,i} - \gamma_{1,i}^*) - \psi_{2,\gamma}(z_{i,t})(\hat{\gamma}_{2,i} - \gamma_{2,i}^*).
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_{j,\theta}(z_{i,t})(\hat{\theta}_j - \theta_j^*))^2 \leq \|\hat{\theta}_j - \theta_j^*\|^2 (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\psi_{j,\theta}(z_{i,t})\|^2. \quad (\text{S165})$$

Since $(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\psi_{j,\theta}(z_{i,t})\|^2 = O_p(1)$ by Assumption 8 and Markov's inequality, it follows from Lemma S1(iii) and (S165) that for both $j = 1, 2$:

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_{j,\theta}(z_{i,t})(\hat{\theta}_j - \theta_j^*))^2 = O_p((nT)^{-1}),$$

which establishes:

$$(nT)^{-1} \sum_{t \leq T} (\Delta \psi_\theta(z_{i,t})(\hat{\theta} - \theta^*))^2 = O_p((nT)^{-1}). \quad (\text{S166})$$

For $j, j' = 1, 2$, we can apply the triangle inequality and Cauchy-Schwarz inequality to get

$$\begin{aligned} & \left| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \psi_{j,\theta}(z_{i,t})(\hat{\theta}_j - \theta_j^*) \psi_{j',\gamma}(z_{i,t})(\hat{\gamma}_{j',i} - \gamma_{j',i}^*) \right| \\ & \leq \|\hat{\theta}_j - \theta_j^*\| (nT)^{-1} \sum_{i \leq n} |\hat{\gamma}_{j',i} - \gamma_{j',i}^*| \sum_{t \leq T} \|\psi_{j,\theta}(z_{i,t}) \psi_{j',\gamma}(z_{i,t})\| \\ & \leq \|\hat{\theta}_j - \theta_j^*\| \left(n^{-1} \sum_{i \leq n} |\hat{\gamma}_{j',i} - \gamma_{j',i}^*|^2 \right)^{1/2} \left((nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\psi_{j,\theta}(z_{i,t}) \psi_{j',\gamma}(z_{i,t})\|^2 \right)^{1/2}. \quad (\text{S167}) \end{aligned}$$

Since by Assumption 8 and Markov's inequality $(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\psi_{j,\theta}(z_{i,t}) \psi_{j',\gamma}(z_{i,t})\|^2 = O_p(1)$, it follows from (S167), Lemma S1(iii, iv) that

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \psi_{j,\theta}(z_{i,t})(\hat{\theta}_j - \theta_j^*) \psi_{j',\gamma}(z_{i,t})(\hat{\gamma}_{j',i} - \gamma_{j',i}^*) = O_p((nT^2)^{-1/2}). \quad (\text{S168})$$

Combining the results in (S164), (S166) and (S168), we obtain

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_{1,\phi}(z_{i,t})(\hat{\phi}_{1,i} - \phi_{1,i}^*) - \psi_{2,\phi}(z_{i,t})(\hat{\phi}_{2,i} - \phi_{2,i}^*))^2 \\ & = (nT)^{-1} \sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*)^2 \sum_{t \leq T} \psi_{1,\gamma}^2(z_{i,t}) + \sum_{i \leq n} (\hat{\gamma}_{2,i} - \gamma_{2,i}^*)^2 \sum_{t \leq T} \psi_{2,\gamma}^2(z_{i,t}) \\ & \quad - 2(nT)^{-1} \sum_{i \leq n} (\hat{\gamma}_{1,i} - \gamma_{1,i}^*)(\hat{\gamma}_{2,i} - \gamma_{2,i}^*) \sum_{t \leq T} \psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) + O_p((nT^2)^{-1/2}) \quad (\text{S169}) \end{aligned}$$

which together with Lemma S9 and Lemma S10, and Assumption 1(i) establishes the claim of the lemma. *Q.E.D.*

Lemma S12. *Under Assumptions 1 and 2, we have*

$$\begin{aligned}
(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}, \hat{\phi}_i)^2 &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\Delta\psi(z_{i,t})^2] \\
&\quad + 2(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \Delta\psi_\phi(z_{i,t}; \phi_i^*) (\hat{\phi}_i - \phi_i^*) \\
&\quad + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_i - \phi_i^*)^\top \Delta\psi_\phi(z_{i,t}; \phi_i^*)^\top \Delta\psi_\phi(z_{i,t}; \phi_i^*) (\hat{\phi}_i - \phi_i^*) \\
&\quad + O_p(T^{-3/2} + \omega_{n,T} T^{-1} + \omega_{n,T}^2 (nT)^{-1/2}) \\
&= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\Delta\psi(z_{i,t})^2] + O_p(T^{-1/2} + \omega_{n,T} T^{-1} + \omega_{n,T}^2 (nT)^{-1/2}).
\end{aligned} \tag{S170}$$

PROOF OF LEMMA S12. Recall that $\Delta\psi(z_{i,t}; \phi_i) \equiv \psi_1(z_{i,t}; \phi_{1,i}) - \psi_2(z_{i,t}; \phi_{2,i})$ where $\phi_i \equiv (\phi_{1,i}^\top, \phi_{2,i}^\top)^\top$. Define

$$\Delta\psi_\phi(z_{i,t}; \phi_i) \equiv \frac{\partial \Delta\psi_\phi(z_{i,t}; \phi_i)}{\partial \phi_i} \quad \text{and} \quad \Delta\psi_{\phi\phi}(z_{i,t}; \phi_i) \equiv \frac{\partial^2 \Delta\psi_\phi(z_{i,t}; \phi_i)}{\partial \phi_i \partial \phi_i^\top}.$$

Applying the Taylor expansion, we obtain

$$\begin{aligned}
&(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_1(z_{i,t}; \hat{\phi}_{1,i}) - \psi_2(z_{i,t}; \hat{\phi}_{2,i}))^2 \\
&= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t})^2 + 2(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \Delta\psi_\phi(z_{i,t}; \phi_i^*) (\hat{\phi}_i - \phi_i^*) \\
&\quad + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_i - \phi_i^*)^\top \Delta\psi_\phi(z_{i,t}; \tilde{\phi}_i)^\top \Delta\psi_\phi(z_{i,t}; \tilde{\phi}_i) (\hat{\phi}_i - \phi_i^*) \\
&\quad + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}, \tilde{\phi}_i) (\hat{\phi}_i - \phi_i^*)^\top \Delta\psi_{\phi\phi}(z_{i,t}; \tilde{\phi}_i) (\hat{\phi}_i - \phi_i^*),
\end{aligned} \tag{S171}$$

where $\tilde{\phi}_i$ is a mean value in between ϕ_i^* and $\hat{\phi}_i$. By Assumptions 1 and 2(ii), it follows that

$$\begin{aligned}
&\mathbb{E} \left[\left(\frac{(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\Delta\psi(z_{i,t})^2 - \mathbb{E}[\Delta\psi(z_{i,t})^2])}{\omega_{n,T}^2} \right)^2 \right] \\
&= (nT)^{-2} \sum_{i \leq n} \mathbb{E} \left[\left(\sum_{t \leq T} \frac{\Delta\psi(z_{i,t})^2 - \mathbb{E}[\Delta\psi(z_{i,t})^2]}{\omega_{n,T}^2} \right)^2 \right] \\
&\leq K(n^2 T)^{-1} \sum_{i \leq n} \max_{t \leq T} \|(\Delta\psi(z_{i,t})/\omega_{n,T})^2\|_{2+\delta/2} \leq K(nT)^{-1},
\end{aligned}$$

which together with Markov's inequality implies that

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t})^2 = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\Delta\psi(z_{i,t})^2] + O_p(\omega_{n,T}^2 (nT)^{-1/2}). \tag{S172}$$

In view of (S171) and (S172), the first equality in (S170) follows if

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_i - \phi_i^*)^\top \Delta \psi_\phi(z_{i,t}; \tilde{\phi}_i)^\top \Delta \psi_\phi(z_{i,t}; \tilde{\phi}_i) (\hat{\phi}_i - \phi_i^*) \\ &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_i - \phi_i^*)^\top \Delta \psi_\phi(z_{i,t}; \phi_i^*)^\top \Delta \psi_\phi(z_{i,t}; \phi_i^*) (\hat{\phi}_i - \phi_i^*) + O_p(T^{-3/2}), \end{aligned} \quad (\text{S173})$$

and

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta \psi(z_{i,t}; \tilde{\phi}_i) (\hat{\phi}_i - \phi_i^*)^\top \Delta \psi_\phi(z_{i,t}; \tilde{\phi}_i) (\hat{\phi}_i - \phi_i^*) = O_p(T^{-3/2} + \omega_{n,T} T^{-1}). \quad (\text{S174})$$

By the definition of $\Delta \psi_\phi(z_{i,t}, \phi_i)$, we can write:

$$\Delta \psi_\phi(z_{i,t}; \tilde{\phi}_i) (\hat{\phi}_i - \phi_i^*) = \psi_{1,\phi}(z_{i,t}; \tilde{\phi}_{1,i}) (\hat{\phi}_{1,i} - \phi_{1,i}^*) - \psi_{2,\phi}(z_{i,t}; \tilde{\phi}_{2,i}) (\hat{\phi}_{2,i} - \phi_{2,i}^*).$$

Therefore, (S173) follows if for $j, j' = 1, 2$ the following hold:

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_{j,i} - \phi_{j,i}^*)^\top \begin{pmatrix} \psi_{j,\phi}(z_{i,t}; \tilde{\phi}_{j,i})^\top \psi_{j',\phi}(z_{i,t}; \tilde{\phi}_{j',i}) \\ -\psi_{j,\phi}(z_{i,t}) \psi_{j',\phi}(z_{i,t}) \end{pmatrix} (\hat{\phi}_{j',i} - \phi_{j',i}^*) = O_p(T^{-3/2}). \quad (\text{S175})$$

We next verify (S175). By Lemma S1(iii, iv),

$$n^{-1} \sum_{i \leq n} \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^4 \leq K \left(\|\hat{\theta}_j - \theta_j^*\|^4 + n^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_g} \|\hat{\gamma}_{j,g} - \gamma_{j,g}^*\|^4 \right) = O_p(T^{-2}). \quad (\text{S176})$$

By the triangle inequality and Assumption 8(i), we have

$$\begin{aligned} & \left| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_{j,i} - \phi_{j,i}^*)^\top \begin{pmatrix} \psi_{j,\phi}(z_{i,t}; \tilde{\phi}_{j,i})^\top \psi_{j',\phi}(z_{i,t}; \tilde{\phi}_{j',i}) \\ -\psi_{j,\phi}(z_{i,t}) \psi_{j',\phi}(z_{i,t}) \end{pmatrix} (\hat{\phi}_{j',i} - \phi_{j',i}^*) \right| \\ & \leq (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_3(z_{i,t}) \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^2 \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \\ & + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_3(z_{i,t}) \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \|\hat{\phi}_{j,i} - \phi_{j,i}^*\| \\ & + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_3(z_{i,t}) \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^2 \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|, \end{aligned} \quad (\text{S177})$$

where $\tilde{M}_3(z_{i,t}) \equiv M(z_{i,t})(M(z_{i,t}) + \|\psi_{j,\phi}(z_{i,t})\| + \|\psi_{j',\phi}(z_{i,t})\|)$. By Assumption 2(iii) and Markov's inequality

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_3(z_{i,t})^4 = O_p(1). \quad (\text{S178})$$

From the Cauchy-Schwarz inequality, (S176) and (S178), we have

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_3(z_{i,t}) \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \|\hat{\phi}_{j,i} - \phi_{j,i}^*\| \\
& \leq \left((nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^4 \right)^{1/2} \times \left((nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_3(z_{i,t})^4 \right)^{1/4} \\
& \quad \times \left((nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^4 \right)^{1/4} = O_p(T^{-3/2}). \tag{S179}
\end{aligned}$$

Similarly,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_3(z_{i,t}) \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^2 \|\hat{\phi}_{j',i} - \phi_{j',i}^*\| = O_p(T^{-3/2}). \tag{S180}$$

Moreover, by Lemma S1(i, iii) and (S180),

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_3(z_{i,t}) \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^2 \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \\
& \leq \max_{i \leq n} \|\hat{\phi}_{j',i} - \phi_{j',i}^*\| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_3(z_{i,t}) \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \|\hat{\phi}_{j,i} - \phi_{j,i}^*\| = O_p(T^{-3/2}), \tag{S181}
\end{aligned}$$

which, together with (S177), (S179) and (S180) shows (S175).

We next turn to verify (S174). For $j, j' = 1, 2$, by the triangle inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_j(z_{i,t}; \tilde{\phi}_{j,i}) - \psi_j(z_{i,t}; \phi_{j,i}^*)) (\hat{\phi}_{j',i} - \phi_{j',i}^*)^\top \psi_{j',\phi\phi}(z_{i,t}; \tilde{\phi}_{j',i}) (\hat{\phi}_{j',i} - \phi_{j',i}^*) \right| \\
& \leq (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} M(z_{i,t}) \|\psi_{j',\phi\phi}(z_{i,t}; \tilde{\phi}_{j',i})\| \times \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \|\hat{\phi}_{j,i} - \phi_{j,i}^*\| \\
& \leq K(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} M^2(z_{i,t}) \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^3 \|\hat{\phi}_{j,i} - \phi_{j,i}^*\| \\
& + K(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} M(z_{i,t}) \|\psi_{j',\phi\phi}(z_{i,t}; \phi_{j',i}^*)\| \times \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \|\hat{\phi}_{j,i} - \phi_{j,i}^*\| \\
& \leq K \left(1 + \max_{i \leq n} \|\hat{\phi}_{j',i} - \phi_{j',i}^*\| \right) (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_4(z_{i,t}) \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|, \tag{S182}
\end{aligned}$$

where $\tilde{M}_4(z_{i,t}) \equiv M(z_{i,t})(M(z_{i,t}) + \|\psi_{j',\phi\phi}(z_{i,t}; \phi_{j',i}^*)\|)$. By the similar arguments for deriving (S179), we can show that

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{M}_4(z_{i,t}) \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \|\hat{\phi}_{j,i} - \phi_{j,i}^*\| = O_p(T^{-3/2})$$

which along with (S182) and Lemma S1(i, iii) establishes

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\psi_j(z_{i,t}; \tilde{\phi}_{j,i}) - \psi_j(z_{i,t}; \phi_{j,i}^*)) (\hat{\phi}_{j',i} - \phi_{j',i}^*)^\top \psi_{j',\phi\phi}(z_{i,t}; \tilde{\phi}_{j',i}) (\hat{\phi}_{j',i} - \phi_{j',i}^*) = O_p(T^{-3/2}). \quad (\text{S183})$$

By the triangle inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) (\hat{\phi}_{j,i} - \phi_{j,i}^*)^\top \psi_{j,\phi\phi}(z_{i,t}; \tilde{\phi}_{j,i}) (\hat{\phi}_{j,i} - \phi_{j,i}^*) \right| \\ & \leq K(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\Delta\psi(z_{i,t}) \psi_{j,\phi\phi}(z_{i,t}; \phi_{j,i}^*)\| \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^2 \\ & + K \max_{i \leq n} \|\hat{\phi}_{j,i} - \phi_{j,i}^*\| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} |\Delta\psi(z_{i,t}) M(z_{i,t})| \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^2. \end{aligned} \quad (\text{S184})$$

By Assumption 2(ii) and 8, and Markov's inequality,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\Delta\psi(z_{i,t}, \phi_i^*) \psi_{j,\phi\phi}(z_{i,t}; \phi_{j,i}^*)\|^2 = O_p(\omega_{n,T}^2) \quad (\text{S185})$$

and

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} |\Delta\psi(z_{i,t}; \phi_i^*) M(z_{i,t})|^2 = O_p(\omega_{n,T}^2), \quad (\text{S186})$$

which together with Lemma S1(i, iii), (S176) and (S184) shows that

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) (\hat{\phi}_{j,i} - \phi_{j,i}^*)^\top \psi_{j,\phi\phi}(z_{i,t}; \tilde{\phi}_{j,i}) (\hat{\phi}_{j,i} - \phi_{j,i}^*) = O_p(\omega_{n,T} T^{-1}). \quad (\text{S187})$$

The desired result in (S174) follows from (S183) and (S187).

The second equality in (S170) holds if

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \Delta\psi_\phi(z_{i,t}; \phi_i^*) (\hat{\phi}_i - \phi_i^*) = O_p(T^{-1/2}), \quad (\text{S188})$$

and

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_{j,i} - \phi_{j,i}^*)^\top \psi_{j,\phi}(z_{i,t}) \psi_{j',\phi}(z_{i,t}) (\hat{\phi}_{j',i} - \phi_{j',i}^*) = O_p(T^{-1}). \quad (\text{S189})$$

To show (S188), we first observe that by the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}) \Delta\psi_\phi(z_{i,t}; \phi_i^*) (\hat{\phi}_i - \phi_i^*) \right|^2 \\ & \leq K(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t})^2 \\ & \times \sum_{j=1,2} \left(n^{-1} \sum_{i \leq n} \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^4 \right)^{1/2} \left((nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\psi_{j,\phi}(z_{i,t})\|^4 \right)^{1/2}. \end{aligned} \quad (\text{S190})$$

By Assumptions 2(ii) and 8(iii), and Markov's inequality,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta \psi(z_{i,t})^2 = O_p(1) \quad \text{and} \quad (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\psi_{j,\phi}(z_{i,t})\|^4 = O_p(1), \quad (\text{S191})$$

which together with (S176) and (S190) establishes (S188).

Finally, we verify (S189). From Assumption 2(iii) and Markov's inequality, it follows that

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\psi_{j,\phi}(z_{i,t}) \psi_{j',\phi}(z_{i,t})\|^2 = O_p(1). \quad (\text{S192})$$

Therefore, by the triangle inequality and Cauchy-Schwarz inequality,

$$\begin{aligned} & \left| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\phi}_{j,i} - \phi_{j,i}^*)^\top \psi_{j,\phi}(z_{i,t}) \psi_{j',\phi}(z_{i,t}) (\hat{\phi}_{j',i} - \phi_{j',i}^*) \right| \\ & \leq (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\hat{\phi}_{j,i} - \phi_{j,i}^*\| \times \|\hat{\phi}_{j',i} - \phi_{j',i}^*\| \times \|\psi_{j,\phi}(z_{i,t}) \psi_{j',\phi}(z_{i,t})\| \\ & \leq \left(n^{-1} \sum_{i \leq n} \|\hat{\phi}_{j,i} - \phi_{j,i}^*\|^2 \|\hat{\phi}_{j',i} - \phi_{j',i}^*\|^2 \right)^{1/2} \\ & \quad \times \left((nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\psi_{j,\phi}(z_{i,t}) \psi_{j',\phi}(z_{i,t})\|^2 \right)^{1/2} = O_p(T^{-1}), \end{aligned} \quad (\text{S193})$$

which establishes (S189).

Q.E.D.

Lemma S13. *Under Assumptions 1 and 2, we have $\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} (\hat{s}_{j,\gamma,i}^2 - s_{j,\gamma,i}^2)^2 = O_p(1)$.*

PROOF OF LEMMA S13. By the definitions of $\hat{s}_{j,\gamma,i}^2$ and $s_{j,\gamma,i}^2$, and Lemma S1(ii), we obtain the following expression:

$$\begin{aligned} \hat{s}_{j,\gamma,i}^2 - s_{j,\gamma,i}^2 &= \frac{T^{-1} \sum_{t \leq T} \psi_{j,\gamma}^2(z_{i,t}; \hat{\phi}_{j,g})}{-\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{\mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]}{-\Psi_{j,\gamma\gamma,g}} \\ &= \frac{T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}; \hat{\phi}_{j,g}) - \psi_{j,\gamma}^2(z_{i,t}; \phi_{j,g}^*))}{-\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} \\ &\quad + \frac{T^{-1} \sum_{t \leq T} \psi_{j,\gamma}^2(z_{i,t}; \phi_{j,g}^*)}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \\ &\quad + \frac{T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}; \phi_{j,g}^*) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})])}{-\Psi_{j,\gamma\gamma,g}}, \end{aligned}$$

wpa1. From the expression above, it is evident that the claim of the lemma follows if the following results hold:

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}; \hat{\phi}_{j,g}) - \psi_{j,\gamma}^2(z_{i,t})))^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} = O_p(1), \quad (\text{S194})$$

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}^2(z_{i,t}))^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 = O_p(1), \quad (\text{S195})$$

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t}; \phi_{j,g}^*)]))^2}{n_{j,g} \Psi_{j,\gamma\gamma,g}^2} = O_p(1). \quad (\text{S196})$$

We now proceed to verify (S194), (S195) and (S196).

By the Cauchy-Schwarz inequality, the term before the equality in (S194) can be bounded as follows:

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}; \hat{\phi}_{j,g}) - \psi_{j,\gamma}^2(z_{i,t})))^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \\ & \leq KT^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\left(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}; \hat{\phi}_{j,g}) - \psi_{j,\gamma}(z_{i,t}))^2 \right)^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \\ & + KT^{-1} \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\left(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}; \hat{\phi}_{j,g}) - \psi_{j,\gamma}(z_{i,t})) \psi_{j,\gamma}(z_{i,t}) \right)^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})}. \end{aligned} \quad (\text{S197})$$

By Assumption 8 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\left(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}; \hat{\phi}_{j,g}) - \psi_{j,\gamma}(z_{i,t}))^2 \right)^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \\ & \leq \frac{\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} M(z_{i,t})^2 \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2 \right)^2}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \\ & \leq K \frac{\max_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} (M(z_{i,t})^2 - \mathbb{E}[M(z_{i,t})^2]) \right)^2 \\ & + K \frac{\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4 n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} (\mathbb{E}[M(z_{i,t})^2]) \right)^2}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})}. \end{aligned} \quad (\text{S198})$$

Applying Assumptions 1, 8, and the covariance inequality for strong mixing processes, we obtain:

$$\mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} (M(z_{i,t})^2 - \mathbb{E}[M(z_{i,t})^2]) \right)^2 \right] \leq K. \quad (\text{S199})$$

Combining this with Lemma S1(i, ii, iii) and Markov's inequality, we establish

$$\frac{\max_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} (M(z_{i,t})^2 - \mathbb{E}[M(z_{i,t})^2]) \right)^2 = o_p(G_j T^{-1}). \quad (\text{S200})$$

Since $T^{-1} \sum_{t \leq T} (\mathbb{E}[M(z_{i,t})^2]) \leq K$, from Lemma S1(ii) and (S13) it follows that

$$\frac{\sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4 n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} (\mathbb{E}[M(z_{i,t})^2]) \right)^2}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} = O_p(G_j T^{-2}), \quad (\text{S201})$$

which along with (S198) and (S200) verifies (S194).

To verify (S195), we first observe that by Lemma S1(ii), the term on its LHS satisfies:

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}^2(z_{i,t}))^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 \\ & \leq \frac{\sum_{g \in \mathcal{G}_j} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} \max_{i \leq n} \left(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}^2(z_{i,t}) \right)^2. \end{aligned} \quad (\text{S202})$$

By the similar arguments for showing (S61), we can establish

$$\max_{i \leq n} T^{-1} \sum_{t \leq T} \psi_{j,\gamma}^2(z_{i,t}) = O_p(1).$$

Combining this with (S202), Assumption 9 and Lemma S1(ii, v), we obtain:

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}^2(z_{i,t}))^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 = O_p(G_j T^{-1}). \quad (\text{S203})$$

To verify (S196), we observe that by Assumptions 1, 8 and 9, we have

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\mathbb{E} \left[(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}^2(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}^2(z_{i,t})]))^2 \right]}{n_{j,g} \Psi_{j,\gamma\gamma,g}^2} \\ & \leq K \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{\max_{t \leq T} \left\| \psi_{j,\gamma}^2(z_{i,t}) \right\|_{2+\delta/2}^2}{n_{j,g} T \Psi_{j,\gamma\gamma,g}^2} \leq K G_j T^{-1}. \end{aligned} \quad (\text{S204})$$

The condition in (S196) now follows from Assumptions 1(i), (S204) and Markov's inequality. *Q.E.D.*

Lemma S14. *Under Assumptions 1 and 2, we have*

$$\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(\frac{(\hat{\mathbb{E}}_T[\hat{\psi}_{j,\gamma}(z_{i,t})])^2}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{(\mathbb{E}_T[\psi_{j,\gamma}(z_{i,t})])^2}{\Psi_{j,\gamma\gamma,g}} \right)^2 = O_p(1). \quad (\text{S205})$$

PROOF OF LEMMA S14. Since we can write

$$\begin{aligned}
& \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}; \hat{\phi}_{j,g}))^2}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{(\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{\Psi_{j,\gamma\gamma,g}} \\
&= \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}; \hat{\phi}_{j,g}))^2 - (T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} \\
&\quad - \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g}) \\
&\quad + \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2 - (\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2}{\Psi_{j,\gamma\gamma,g}},
\end{aligned}$$

the claim in (S205) follows if the following conditions hold:

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{((T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}; \hat{\phi}_{j,g}))^2 - (T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2)^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} = O_p(1), \quad (\text{S206})$$

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^4}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 = O_p(1), \quad (\text{S207})$$

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{((T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2 - (\mathbb{E}[\psi_{j,\gamma}(z_{i,t}; \phi_{j,g}^*)])^2)^2}{n_{j,g} \Psi_{j,\gamma\gamma,g}^2} = O_p(1). \quad (\text{S208})$$

We now proceed to verify (S206), (S207) and (S208).

By the Cauchy-Schwarz inequality, the term before the equality in (S206) can be bounded as follows:

$$\begin{aligned}
& \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{((T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}; \hat{\phi}_{j,g}))^2 - (T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2)^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \\
&\leq K \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}; \hat{\phi}_{j,g}) - \psi_{j,\gamma}(z_{i,t})))^4}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \\
&\quad + K \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}; \hat{\phi}_{j,g}) - \psi_{j,\gamma}(z_{i,t})))^2 (T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \\
&\leq \frac{K \max_{i \leq n} (T^{-1} \sum_{t \leq T} M(z_{i,t}))^4 \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \\
&\quad + \frac{K \max_{i \leq n} (T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2 (T^{-1} \sum_{t \leq T} M(z_{i,t}))^2 \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})}. \quad (\text{S209})
\end{aligned}$$

By the similar arguments for showing (S61), we can show that

$$\max_{i \leq n} T^{-1} \sum_{t \leq T} M(z_{i,t}) = O_p(1), \quad (\text{S210})$$

which together with Markov's inequality, Assumption 8 and Lemma S1(i, ii, iii) implies that

$$\frac{\max_{i \leq n} (T^{-1} \sum_{t \leq T} M(z_{i,t}))^4 \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^4}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} = O_p(G_j T^{-2}). \quad (\text{S211})$$

From Assumption 8, Lemma S1(ii), (S61) and (S210), we also have

$$\frac{\max_{i \leq n} (T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2 (T^{-1} \sum_{t \leq T} M(z_{i,t}))^2 \sum_{g \in \mathcal{G}_j} \|\hat{\phi}_{j,g} - \phi_{j,g}^*\|^2}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} = O_p(G_j T^{-1}),$$

which together with Assumption 1(i), (S209) and (S211) verifies (S206).

To verify (S207), we first observe that the term on its LHS satisfies:

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^4}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 \\ & \leq \frac{\max_{i \leq n} (T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^4}{\min_{g \in \mathcal{G}_j} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} \sum_{g \in \mathcal{G}_j} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2. \end{aligned}$$

This combined with Assumption 9, Lemma S1(ii, v) and (S61) implies that

$$\sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{(T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^4}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g}) \Psi_{j,\gamma\gamma,g}^2} (\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g}) - \Psi_{j,\gamma\gamma,g})^2 = O_p(G_j T^{-1}). \quad (\text{S212})$$

The claim in (S207) follows from Assumption 1(i) and (S212).

To verify (S208), we first observe that by Assumption 9, the term on its LHS satisfies:

$$\begin{aligned} & \sum_{g \in \mathcal{G}_j} \sum_{i \in I_{j,g}} \frac{((T^{-1} \sum_{t \leq T} \psi_{j,\gamma}(z_{i,t}))^2 - (\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2)^2}{n_{j,g} \hat{\Psi}_{j,\gamma\gamma}^2(\hat{\phi}_{j,g})} \\ & \leq K \sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]) \right)^4 \\ & \quad + K \sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]) \right)^2 (\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2. \end{aligned} \quad (\text{S213})$$

Applying Rosenthal's inequality and Assumption 8, we get

$$\mathbb{E} \left[\left(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]) \right)^4 \right] \leq K T^{-2},$$

which together with Markov's inequality implies that

$$\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]) \right)^4 = O_p(T^{-1}). \quad (\text{S214})$$

Similarly, we can show that

$$\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(T^{-1} \sum_{t \leq T} (\psi_{j,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{j,\gamma}(z_{i,t})]) \right)^2 (\mathbb{E}[\psi_{j,\gamma}(z_{i,t})])^2 = O_p(1),$$

which together with (S213) and (S214) verifies (S208).

Q.E.D.

Lemma S15. *Under Assumptions 1 and 2, we have*

$$\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{12,\gamma,i} - \sigma_{12,\gamma,i})^2 = O_p(1). \quad (\text{S215})$$

PROOF OF LEMMA S15. By the definitions of $\hat{\sigma}_{12,\gamma,i}^2$ and $\sigma_{12,\gamma,i}^2$, we can write

$$\begin{aligned} \hat{\sigma}_{12,\gamma,i} - \sigma_{12,\gamma,i} &= \frac{\hat{\mathbb{E}}_T[\psi_{1,\gamma}(z_{i,t}; \hat{\phi}_{1,i}) \psi_{2,\gamma}(z_{i,t}; \hat{\phi}_{2,i})]}{|\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^{1/2}} - \frac{\mathbb{E}_T[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]}{(\Psi_{1,\gamma\gamma,i} \Psi_{2,\gamma\gamma,i})^{1/2}} \\ &= \frac{T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}; \hat{\phi}_{1,i}) \psi_{2,\gamma}(z_{i,t}; \hat{\phi}_{2,i}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})])}{|\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^{1/2}} \\ &\quad + \frac{\mathbb{E}_T[\tilde{\psi}_{1,\gamma}(z_{i,t}) \tilde{\psi}_{2,\gamma}(z_{i,t})] \left(|\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^{1/2} - (\Psi_{1,\gamma\gamma,i} \Psi_{2,\gamma\gamma,i})^{1/2} \right)}{|\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^{1/2}}. \end{aligned} \quad (\text{S216})$$

From the expression above, the claim of the lemma follows if

$$\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}; \hat{\phi}_{1,i}) \psi_{2,\gamma}(z_{i,t}; \hat{\phi}_{2,i}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]))^2}{n_{2,g} \hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})} = O_p(1) \quad (\text{S217})$$

and

$$\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(\mathbb{E}[\tilde{\psi}_{1,\gamma}(z_{i,t}) \tilde{\psi}_{2,\gamma}(z_{i,t})] (|\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^{1/2} - (\Psi_{1,\gamma\gamma,i} \Psi_{2,\gamma\gamma,i})^{1/2}))^2}{n_{2,g} \hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})} = O_p(1). \quad (\text{S218})$$

To verify (S217), we first observe that by Assumption 8 and the triangle inequality:

$$\begin{aligned}
& \left| T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}; \hat{\phi}_{1,i}) \psi_{2,\gamma}(z_{i,t}; \hat{\phi}_{2,i}) - \psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})) \right| \\
& \leq \left| T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}; \hat{\phi}_{1,i}) - \psi_{1,\gamma}(z_{i,t})) (\psi_{2,\gamma}(z_{i,t}; \hat{\phi}_{2,i}) - \psi_{2,\gamma}(z_{i,t})) \right| \\
& + \left| T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}; \hat{\phi}_{1,i}) - \psi_{1,\gamma}(z_{i,t})) \psi_{2,\gamma}(z_{i,t}) \right| \\
& + \left| T^{-1} \sum_{t \leq T} (\psi_{2,\gamma}(z_{i,t}; \hat{\phi}_{2,i}) - \psi_{2,\gamma}(z_{i,t})) \psi_{1,\gamma}(z_{i,t}) \right| \\
& \leq K(1 + \max_{i \leq n} \|\hat{\phi}_{1,i} - \phi_{1,i}^*\|)(\|\hat{\phi}_{1,i} - \phi_{1,i}^*\| + \|\hat{\phi}_{2,i} - \phi_{2,i}^*\|) T^{-1} \sum_{t \leq T} M_5(z_{i,t})^2, \tag{S219}
\end{aligned}$$

where $M_5(z_{i,t}) \equiv M(z_{i,t}) + \psi_{1,\gamma}(z_{i,t}) + \psi_{2,\gamma}(z_{i,t})$. Thus, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left| \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}; \hat{\phi}_{1,i}) \psi_{2,\gamma}(z_{i,t}; \hat{\phi}_{2,i}) - \psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})))^2}{n_{2,g} \hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})} \right| \\
& \leq \frac{K(1 + \max_{i \leq n} \|\hat{\phi}_{1,i} - \phi_{1,i}^*\|^2)}{\min_{i \leq n} |\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|} \\
& \times \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\|\hat{\phi}_{1,i} - \phi_{1,i}^*\| + \|\hat{\phi}_{2,i} - \phi_{2,i}^*\|)^2 \left(T^{-1} \sum_{t \leq T} M_1(z_{i,t})^2 \right)^2 \\
& \leq \frac{K(1 + \max_{i \leq n} \|\hat{\phi}_{1,i} - \phi_{1,i}^*\|) \max_{i \leq n} \left(T^{-1} \sum_{t \leq T} M_1(z_{i,t})^2 \right)^2}{\min_{i \leq n} |\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|} \\
& \times \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\|\hat{\phi}_{1,i} - \phi_{1,i}^*\|^2 + \|\hat{\phi}_{2,i} - \phi_{2,i}^*\|^2) \right). \tag{S220}
\end{aligned}$$

By the similar arguments for showing (S61), we can show that

$$\max_{i \leq n} T^{-1} \sum_{t \leq T} M_1(z_{i,t})^2 = O_p(1),$$

which, along with Assumptions 1(i) and 8, Lemma S1(ii, iii, iv) and (S220) implies that

$$\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}; \hat{\phi}_{1,i}) \psi_{2,\gamma}(z_{i,t}; \hat{\phi}_{2,i}) - \psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})))^2}{n_{2,g} \hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})} = O_p(1). \tag{S221}$$

Next, by Assumptions 1 and 8,

$$\begin{aligned}
& \mathbb{E} \left[\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \left(T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]) \right)^2 \right] \\
&= \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \mathbb{E} \left[\left(T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]) \right)^2 \right] \\
&\leq KT^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \max_{t \leq T} \|\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})\|_{2+\delta/2}^2 \leq K,
\end{aligned}$$

which along with Markov's inequality, Assumptions 1(i) and Lemma S1(ii) shows that

$$\sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(T^{-1} \sum_{t \leq T} (\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t}) - \mathbb{E}[\psi_{1,\gamma}(z_{i,t}) \psi_{2,\gamma}(z_{i,t})]))^2}{n_{2,g} \hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})} = O_p(1). \quad (\text{S222})$$

The desired result in (S217) follows from (S221) and (S222).

To verify (S218), we first notice that the term on its LHS satisfies:

$$\begin{aligned}
& \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \frac{(\mathbb{E}[\tilde{\psi}_{1,\gamma}(z_{i,t}) \tilde{\psi}_{2,\gamma}(z_{i,t})] (|\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^{1/2} - (\Psi_{1,\gamma\gamma,i} \Psi_{2,\gamma\gamma,i})^{1/2}))^2}{n_{2,g} |\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|} \\
&\leq \frac{K \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (|\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^{1/2} - (\Psi_{1,\gamma\gamma,i} \Psi_{2,\gamma\gamma,i})^{1/2})^2}{\min_{i \leq n} |\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|} \\
&\leq \frac{K \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i}) - \Psi_{1,\gamma\gamma,i} \Psi_{2,\gamma\gamma,i})^2}{\min_{i \leq n} |\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^2} \\
&\leq K \max_{i \leq n} (\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{2,i}) - \Psi_{1,\gamma\gamma,i})^2 \frac{\sum_{g \in \mathcal{G}_2} (\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i}) - \Psi_{2,\gamma\gamma,i})^2}{\min_{i \leq n} |\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^2} \\
&+ \frac{K \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) - \Psi_{1,\gamma\gamma,i})^2}{\min_{i \leq n} |\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^2} + \frac{K \sum_{g \in \mathcal{G}_2} (\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{1,i}) - \Psi_{2,\gamma\gamma,i})^2}{\min_{i \leq n} |\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,i}) \hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,i})|^2},
\end{aligned}$$

which together with Lemma S1(ii, v) shows (S218).

Q.E.D.

Lemma S16. *Under Assumptions 1, 2 and 3, we have:*

$$\omega_{n,T}^2 = \sigma_{n,T}^2 + (2nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \left(\sigma_{1,\gamma,i}^4 + n_{2,g}^{-2} \sigma_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}} \sigma_{2,\gamma,i'}^2 - 2n_{2,g}^{-1} \sigma_{12,\gamma,i}^2 \right) + O(\omega_{n,T} T^{-1}).$$

PROOF OF LEMMA S16. To prove the claim in the lemma, we first observe that by Assumption 1(iii),

$$\text{Cov}(\tilde{\Psi}_i, \tilde{U}_i) = 0 \quad \text{and} \quad \text{Cov}(\tilde{V}_i, \tilde{U}_i) = 0.$$

Therefore,

$$\text{Var}(\tilde{\Psi}_i + \tilde{V}_i + \tilde{U}_i) = \text{Var}(\tilde{\Psi}_i + \tilde{V}_i) + \text{Var}(\tilde{U}_i).$$

Together with the definition of $\omega_{n,T}^2$, this yields

$$\begin{aligned} \omega_{n,T}^2 &= n^{-1} \sum_{i \leq n} (\text{Var}(\tilde{\Psi}_i + \tilde{V}_i) + \text{Var}(\tilde{U}_i)) \\ &= n^{-1} \sum_{i \leq n} (\text{Var}(\tilde{\Psi}_i) + \text{Var}(\tilde{V}_i) + 2\text{Cov}(\tilde{\Psi}_i, \tilde{V}_i) + \text{Var}(\tilde{U}_i)) \\ &= \sigma_{n,T}^2 + n^{-1} \sum_{i \leq n} (\text{Var}(\tilde{V}_i) + \text{Var}(\tilde{U}_i)) + 2n^{-1} \sum_{i \leq n} \text{Cov}(\tilde{\Psi}_i, \tilde{V}_i). \end{aligned} \quad (\text{S223})$$

We next study the second and the third terms in the far RHS of (S223).

For model 1, $\tilde{U}_{1,i} = 0$ and

$$\begin{aligned} \tilde{V}_{1,i} &= (2T^{1/2})^{-1} \tilde{\Psi}_{1,\gamma,i}^2 = (2T^{1/2})^{-1} \left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{1,\gamma}^*(z_{i,t}) \right)^2 \\ &= (2T^{3/2})^{-1} \sum_{t \leq T} \tilde{\psi}_{1,\gamma}^*(z_{i,t})^2 + T^{-3/2} \sum_{t=2}^T \sum_{t'=1}^{t-1} \tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{1,\gamma}^*(z_{i,t'}), \end{aligned} \quad (\text{S224})$$

while for model 2, we have

$$\begin{aligned} \tilde{V}_{2,i} &= (2n_{2,g}T^{1/2})^{-1} \tilde{\Psi}_{2,\gamma,i}^2 = (2n_{2,g}T^{1/2})^{-1} \left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \right)^2 \\ &= (2T^{3/2})^{-1} \sum_{t \leq T} n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t})^2 + T^{-3/2} \sum_{t=2}^T \sum_{t'=1}^{t-1} n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'}), \end{aligned} \quad (\text{S225})$$

and

$$\tilde{U}_{2,i} = (n_{2,g}T^{3/2})^{-1} \sum_{\{i' \in I_{2,g}; i' < i\}} \left(\sum_{t \leq T} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \right) \left(\sum_{t' \leq T} \tilde{\psi}_{2,\gamma}^*(z_{i',t'}) \right). \quad (\text{S226})$$

Subtracting (S225) from (S224) yields

$$\begin{aligned} \tilde{V}_{1,i} - \tilde{V}_{2,i} &= (2T^{3/2})^{-1} \sum_{t \leq T} (\tilde{\psi}_{1,\gamma}^*(z_{i,t})^2 - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t})^2) \\ &\quad + T^{-3/2} \sum_{t=2}^T \sum_{t'=1}^{t-1} (\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{1,\gamma}^*(z_{i,t'}) - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})). \end{aligned} \quad (\text{S227})$$

For any $h \geq 1$ with $t - h \geq 1$ and $j, j' \in \{1, 2\}$,

$$\text{Cov} \left(\tilde{\psi}_{j,\gamma}^*(z_{i,t-h})^2, \tilde{\psi}_{j',\gamma}^*(z_{i,t}) \sum_{t'=1}^{t-1} \tilde{\psi}_{j',\gamma}^*(z_{i,t'}) \right) = \sum_{t'=1}^{t-1} \mathbb{E} \left[\tilde{\psi}_{j,\gamma}^*(z_{i,t-h})^2 \tilde{\psi}_{j',\gamma}^*(z_{i,t'}) \tilde{\psi}_{j',\gamma}^*(z_{i,t}) \right] = 0, \quad (\text{S228})$$

where the second equality is by Assumption 3. For any $h \geq 0$ with $t + h \leq T$ and $j, j' \in \{1, 2\}$, we have

$$\begin{aligned}
& \left| \text{Cov} \left(\tilde{\psi}_{j,\gamma}^* (z_{i,t+h})^2, \tilde{\psi}_{j',\gamma}^* (z_{i,t}) \sum_{t'=1}^{t-1} \tilde{\psi}_{j',\gamma}^* (z_{i,t'}) \right) \right| \\
& \leq K \alpha_i^{1/4} \left\| \tilde{\psi}_{j,\gamma}^* (z_{i,t+h})^2 \right\|_4 \left\| \tilde{\psi}_{j',\gamma}^* (z_{i,t}) \sum_{t'=1}^{t-1} \tilde{\psi}_{j',\gamma}^* (z_{i,t'}) \right\|_2 \\
& \leq K \alpha_i^{1/4}(h) \left\| \tilde{\psi}_{j,\gamma}^* (z_{i,t+h})^2 \right\|_4 \left\| \tilde{\psi}_{j',\gamma}^* (z_{i,t}) \right\|_4 \left\| \sum_{t'=1}^{t-1} \tilde{\psi}_{j',\gamma}^* (z_{i,t'}) \right\|_4 \leq K \alpha_i^{1/4}(h) t^{1/2}, \quad (\text{S229})
\end{aligned}$$

where we use the covariance inequality for strong mixing processes, Hölder's inequality, Assumption 2(iii), and a Rosenthal-type inequality $\left\| \sum_{t'=1}^{t-1} \tilde{\psi}_{j',\gamma}^* (z_{i,t'}) \right\|_4 \leq K t^{1/2}$ (cf. (S556) in the proof of Lemma S47). Combining (S228) and (S229) yields

$$\begin{aligned}
& \left| \text{Cov} \left(\sum_{t \leq T} (\tilde{\psi}_{1,\gamma}^* (z_{i,t})^2 - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^* (z_{i,t})^2), \sum_{t=2}^T \sum_{t'=1}^{t-1} \tilde{\psi}_{1,\gamma}^* (z_{i,t}) \tilde{\psi}_{1,\gamma}^* (z_{i,t'}) - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^* (z_{i,t}) \tilde{\psi}_{2,\gamma}^* (z_{i,t'}) \right) \right| \\
& \leq K \sum_{j \leq 2} \sum_{j' \leq 2} \sum_{0 \leq h \leq T-1} \sum_{t \leq T-h} \text{Cov} \left(\tilde{\psi}_{j,\gamma}^* (z_{i,t+h})^2, \sum_{t'=1}^{t-1} \tilde{\psi}_{j',\gamma}^* (z_{i,t}) \tilde{\psi}_{j',\gamma}^* (z_{i,t'}) \right) \\
& \leq K \sum_{j \leq 2} \sum_{j' \leq 2} \sum_{0 \leq h \leq T-1} \sum_{t \leq T-h} \alpha_i^{1/4}(h) t^{1/2} \leq K \sum_{j \leq 2} \sum_{j' \leq 2} \sum_{0 \leq h \leq T} \alpha_i^{1/4}(h) (T-h)^{3/2} \leq K T^{3/2}. \quad (\text{S230})
\end{aligned}$$

The last inequality follows since

$$\sum_{0 \leq h \leq T} \alpha_i^{1/4}(h) (T-h)^{3/2} = T^{3/2} \sum_{0 \leq h \leq T} \alpha_i^{1/4}(h) (1-h/T)^{3/2} \leq T^{3/2} \sum_{0 \leq h \leq T} \alpha_i^{1/4}(h) \leq K T^{3/2}, \quad (\text{S231})$$

where the final inequality is implied by Assumption 1(iv). Using (S227) and (S230),

$$\begin{aligned}
n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_i) &= n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_{1,i} - \tilde{V}_{2,i}) \\
&= n^{-1} \sum_{i \leq n} \text{Var} \left(T^{-3/2} \sum_{t=2}^T \sum_{t'=1}^{t-1} (\tilde{\psi}_{1,\gamma}^* (z_{i,t}) \tilde{\psi}_{1,\gamma}^* (z_{i,t'}) - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^* (z_{i,t}) \tilde{\psi}_{2,\gamma}^* (z_{i,t'})) \right) \\
&\quad + n^{-1} \sum_{i \leq n} \text{Var} \left((2T^{3/2})^{-1} \sum_{t \leq T} (\tilde{\psi}_{1,\gamma}^* (z_{i,t})^2 - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^* (z_{i,t})^2) \right) + O_p(T^{-3/2}). \quad (\text{S232})
\end{aligned}$$

From Assumptions 1 and 2(iii) and the covariance inequality of strong mixing process, it follows

that

$$\begin{aligned}
& n^{-1} \sum_{i \leq n} \text{Var} \left((2T^{3/2})^{-1} \sum_{t \leq T} (\tilde{\psi}_{1,\gamma}^*(z_{i,t})^2 - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t})^2) \right) \\
& \leq K(nT^3)^{-1} \sum_{j \leq 2} \sum_{i \leq n} \text{Var} \left(\sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t})^2 \right) \leq KT^{-2}.
\end{aligned} \tag{S233}$$

Let $\sigma_{j,\gamma,i,t} \equiv (\mathbb{E}[\tilde{\psi}_{j,\gamma}^*(z_{i,t})^2])^{1/2}$ and $\sigma_{12,\gamma,i,t} \equiv \mathbb{E}[\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t})]$. A direct calculation using Assumption 3 yields

$$\begin{aligned}
& \text{Var} \left(T^{-3/2} \sum_{t=2}^T \sum_{t'=1}^{t-1} (\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{1,\gamma}^*(z_{i,t'}) - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})) \right) \\
& = T^{-3} \sum_{t=2}^T \sum_{t'=1}^{t-1} \text{Var}(\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{1,\gamma}^*(z_{i,t'}) - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})) \\
& = T^{-3} \sum_{t=2}^T \sum_{t'=1}^{t-1} (\text{Var}(\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{1,\gamma}^*(z_{i,t'})) + n_{2,g}^{-2} \text{Var}(\tilde{\psi}_{2,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'}))) \\
& \quad - 2T^{-3} \sum_{t=2}^T \sum_{t'=1}^{t-1} \text{Cov}(\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{1,\gamma}^*(z_{i,t'}), n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})) \\
& = T^{-3} \sum_{t=2}^T \sum_{t'=1}^{t-1} (\sigma_{1,\gamma,i,t}^2 \sigma_{1,\gamma,i,t'}^2 + n_{2,g}^{-2} \sigma_{2,\gamma,i,t}^2 \sigma_{2,\gamma,i,t'}^2 - 2n_{2,g}^{-1} \sigma_{12,\gamma,i,t} \sigma_{12,\gamma,i,t'}) \\
& = (2T^3)^{-1} \left[\left(\sum_{t \leq T} \sigma_{1,\gamma,i,t}^2 \right)^2 + n_{2,g}^{-2} \left(\sum_{t \leq T} \sigma_{2,\gamma,i,t}^2 \right)^2 - 2n_{2,g}^{-1} \left(\sum_{t \leq T} \sigma_{12,\gamma,i,t} \right)^2 \right] \\
& \quad - T^{-3} \left(\sum_{t \leq T} \sigma_{1,\gamma,i,t}^4 + n_{2,g}^{-2} \sum_{t \leq T} \sigma_{2,\gamma,i,t}^4 - 2n_{2,g}^{-1} \sum_{t \leq T} \sigma_{12,\gamma,i,t}^2 \right)
\end{aligned} \tag{S234}$$

Under Assumption 2(iii),

$$T^{-3} \left| \sum_{t \leq T} \sigma_{1,\gamma,i,t}^4 + n_{2,g}^{-2} \sum_{t \leq T} \sigma_{2,\gamma,i,t}^4 - 2n_{2,g}^{-1} \sum_{t \leq T} \sigma_{12,\gamma,i,t}^2 \right| \leq KT^{-2},$$

which along with (S234) and the definitions of $\sigma_{j,\gamma,i}^2$ and $\sigma_{12,\gamma,i}$ implies that

$$\begin{aligned}
& n^{-1} \sum_{i \leq n} \text{Var} \left(T^{-3/2} \sum_{t=2}^T \sum_{t'=1}^{t-1} (\tilde{\psi}_{1,\gamma}^*(z_{i,t}) \tilde{\psi}_{1,\gamma}^*(z_{i,t'}) - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \tilde{\psi}_{2,\gamma}^*(z_{i,t'})) \right) \\
& = (2nT)^{-1} \sum_{i \leq n} (\sigma_{1,\gamma,i}^4 + n_{2,g}^{-2} \sigma_{2,\gamma,i}^4 - 2n_{2,g}^{-1} \sigma_{12,\gamma,i}^2) + O(T^{-2}).
\end{aligned} \tag{S235}$$

Using (S232), (S233) and (S235), we obtain

$$n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_i) = (2nT)^{-1} \sum_{i \leq n} (\sigma_{1,\gamma,i}^4 + n_{2,g}^{-2} \sigma_{2,\gamma,i}^4 - 2n_{2,g}^{-1} \sigma_{12,\gamma,i}^2) + O(T^{-3/2}). \quad (\text{S236})$$

Similarly, by Assumption 3 we have

$$\begin{aligned} n^{-1} \sum_{i \leq n} \text{Var}(\tilde{U}_i) &= n^{-1} \sum_{i \leq n} \text{Var}(\tilde{U}_{2,i}) = (nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-2} \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^2 \sum_{\{i' \in I_{2,g}: i' < i\}} \sigma_{2,\gamma,i'}^2 \\ &= (2nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-2} \left[\left(\sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^2 \right)^2 - \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^4 \right], \end{aligned} \quad (\text{S237})$$

which, along with (S236) shows that

$$\begin{aligned} n^{-1} \sum_{i \leq n} (\text{Var}(\tilde{V}_i) + \text{Var}(\tilde{U}_i)) \\ = (2nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \left(\sigma_{1,\gamma,i}^4 + n_{2,g}^{-2} \sigma_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}} \sigma_{2,\gamma,i'}^2 - 2n_{2,g}^{-1} \sigma_{12,\gamma,i}^2 \right) + O(T^{-2}). \end{aligned} \quad (\text{S238})$$

To bound the third term in the far RHS of (S223), we observe that by Assumptions 1, 2(ii, iii) and the covariance inequality of strong mixing process,

$$\begin{aligned} & \left| \text{Cov} \left(\sum_{t \leq T} \Delta \psi(z_{i,t}), \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t})^2 \right) \right| \\ & \leq \sum_{h \leq T} \sum_{t \leq T-h} \left(\left| \text{Cov} \left(\Delta \psi(z_{i,t+h}), \tilde{\psi}_{j,\gamma}^*(z_{i,t})^2 \right) \right| + \left| \text{Cov} \left(\Delta \psi(z_{i,t}), \tilde{\psi}_{j,\gamma}^*(z_{i,t+h})^2 \right) \right| \right) \\ & \leq \sum_{h \leq T} \sum_{t \leq T-h} \alpha_i^{\delta/(2+\delta)}(h) \left(\left\| \Delta \psi(z_{i,t+h}) \right\|_{2+\delta} \left\| \tilde{\psi}_{j,\gamma}^*(z_{i,t})^2 \right\|_{2+\delta} + \left\| \Delta \psi(z_{i,t}) \right\|_{2+\delta} \left\| \tilde{\psi}_{j,\gamma}^*(z_{i,t+h})^2 \right\|_{2+\delta} \right) \\ & \leq K\omega_{n,T} \sum_{h \leq T} (T-h) \alpha_i^{\delta/(2+\delta)}(h) \leq K\omega_{n,T} T, \end{aligned}$$

where the final inequality follows by arguments analogous to those used to establish (S231). This implies

$$\begin{aligned} \left| n^{-1} \sum_{i \leq n} \text{Cov}(\tilde{\Psi}_i, \tilde{V}_i) \right| &\leq n^{-1} \sum_{i \leq n} \left| \text{Cov} \left(\tilde{\Psi}_i, (2T^{3/2})^{-1} \sum_{t \leq T} (\tilde{\psi}_{1,\gamma}^*(z_{i,t})^2 - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t})^2) \right) \right| \\ &= (2nT^2)^{-1} \sum_{i \leq n} \left| \text{Cov} \left(\sum_{t \leq T} \Delta \psi(z_{i,t}), \sum_{t \leq T} (\tilde{\psi}_{1,\gamma}^*(z_{i,t})^2 - n_{2,g}^{-1} \tilde{\psi}_{2,\gamma}^*(z_{i,t})^2) \right) \right| \\ &\leq K\omega_{n,T} T^{-1}. \end{aligned} \quad (\text{S239})$$

The claim of the lemma follows from Assumption 2(v), (S223), (S238) and (S239).

Q.E.D.

Lemma S17. *Under Assumptions 1 and 2, we have $\omega_{n,T}^2 \leq K$.*

PROOF OF LEMMA S17. From the definition of $\omega_{n,T}^2$, we have

$$\omega_{n,T}^2 = n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_i + \tilde{V}_i + \tilde{U}_i) \leq K n^{-1} \sum_{i \leq n} (\text{Var}(\tilde{\Psi}_i) + \text{Var}(\tilde{V}_i) + \text{Var}(\tilde{U}_i)). \quad (\text{S240})$$

Under Assumptions 1 and 8(iii), we can use covariance inequality for strong mixing processes to obtain

$$n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_i) = n^{-1} \sum_{i \leq n} \text{Var} \left(T^{-1/2} \sum_{t \leq T} \Delta \psi(z_{i,t}) \right) \leq K n^{-1} \sum_{i \leq n} \max_{t \leq T} \|\Delta \psi(z_{i,t})\|_{2+\delta}^2 \leq K, \quad (\text{S241})$$

where the first inequality follows from arguments analogous to those used in establishing (S1).

From the definition of $\tilde{V}_i$, we have

$$\begin{aligned} n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_i) &\leq n^{-1} \sum_{i \leq n} \text{Var} \left((2T^{1/2})^{-1} \tilde{\Psi}_{1,\gamma,i}^2 - (2n_{2,g}T^{1/2})^{-1} \tilde{\Psi}_{2,\gamma,i}^2 \right) \\ &\leq (2T)^{-1} \max_{j=1,2} n^{-1} \sum_{i \leq n} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{j,\gamma}^*(z_{i,t}) \right)^4 \right] \leq KT^{-1}, \end{aligned} \quad (\text{S242})$$

where the last inequality is by Assumptions 1 and 8(iii), and Rosenthal's inequality. Similarly,

$$\begin{aligned} \text{Var}(\tilde{U}_i) &= n_{2,g}^{-2} T^{-1} \sum_{\{i' \in I_{2,g}: i' < i\}} \text{Var} \left(\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \right) \left(T^{-1/2} \sum_{t' \leq T} \tilde{\psi}_{2,\gamma}^*(z_{i',t'}) \right) \right) \\ &\leq n_{2,g}^{-2} T^{-1} \sum_{\{i' \in I_{2,g}: i' < i\}} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \tilde{\psi}_{2,\gamma}^*(z_{i,t}) \right)^2 \right] \mathbb{E} \left[\left(T^{-1/2} \sum_{t' \leq T} \tilde{\psi}_{2,\gamma}^*(z_{i',t'}) \right)^2 \right] \leq K n_{2,g}^{-1} T^{-1} \end{aligned}$$

which implies that

$$n^{-1} \sum_{i \leq n} \text{Var}(\tilde{U}_i) \leq n^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_g} K n_{2,g}^{-1} T^{-1} \leq KT^{-1}. \quad (\text{S243})$$

The claim of the lemma follows from (S240)-(S243).

Q.E.D.

Lemma S18. *Under Assumptions 1 and 2, we have:*

$$\frac{(2nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(\left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \hat{\sigma}_{2,\gamma,i}^2 \right)^2 - \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^2 \right)^2 \right)}{\omega_{n,T}^2} = O_p(n^{-1/2}).$$

PROOF OF LEMMA S18. Next note that

$$\begin{aligned}
& \left| \frac{(2nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(\left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \hat{\sigma}_{2,\gamma,i}^2 \right)^2 - \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^2 \right)^2 \right)}{\omega_{n,T}^2} \right| \\
&= \left| \frac{(2nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2) \right) \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 + \sigma_{2,\gamma,i}^2) \right)}{\omega_{n,T}^2} \right| \\
&\leq \frac{(2nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2) \right)^2}{\omega_{n,T}^2} \\
&+ \left| \frac{(nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2) \right) \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^2 \right)}{\omega_{n,T}^2} \right| \tag{S244}
\end{aligned}$$

From Lemma S13 and (S205) in Lemma S14, we have

$$\begin{aligned}
\sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} (\hat{\sigma}_{j,\gamma,i}^2 - \sigma_{j,\gamma,i}^2)^2 &\leq 2 \sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} \left(\frac{(\hat{\mathbb{E}}_T[\hat{\psi}_{j,\gamma}(z_{i,t})])^2}{\hat{\Psi}_{j,\gamma\gamma}(\hat{\phi}_{j,g})} - \frac{(\mathbb{E}_T[\psi_{j,\gamma}(z_{i,t})])^2}{\Psi_{j,\gamma\gamma,g}} \right)^2 \\
&+ 2 \sum_{g \in \mathcal{G}_j} n_{j,g}^{-1} \sum_{i \in I_{j,g}} (\hat{s}_{j,\gamma,i}^2 - s_{j,\gamma,i}^2)^2 = O_p(1). \tag{S245}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{(2nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2) \right)^2}{\omega_{n,T}^2} \\
&\leq \frac{(2nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2)^2}{\omega_{n,T}^2} = O_p(n^{-1}). \tag{S246}
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left| \frac{(nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2) \right) \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^2 \right)}{\omega_{n,T}^2} \right| \\
&\leq \frac{(nT)^{-1} \left(\sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2) \right)^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^2 \right)^2 \right)^{1/2}}{\omega_{n,T}^2} \\
&\leq \frac{(nT)^{-1} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2)^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^4 \right)^{1/2}}{\omega_{n,T}^2} \\
&\leq \frac{KG_2^{1/2} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2)^2 \right)^{1/2}}{(nT)\omega_{n,T}^2} \\
&= O_p(n^{-1/2}(G_2n)^{-1/2}(T\omega_{n,T}^2)^{-1}) = O_p(n^{-1/2}), \tag{S247}
\end{aligned}$$

where the third inequality follows by Assumption 8(iii), and the equality follows from (S245) and Assumptions 1(i) and 2(v). Combining the results in (S244), (S246) and (S247), we get the desired conclusion. Q.E.D.

Lemma S19. *Under Assumptions 1 and 2, we have:*

$$\frac{(nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-2} \sum_{i \in I_{2,g}} \left(\hat{\sigma}_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}} \hat{s}_{2,\gamma,i'}^2 - \sigma_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}} s_{2,\gamma,i'}^2 \right)}{\omega_{n,T}^2} = O_p(n^{-1/2}).$$

PROOF OF LEMMA S19. We apply the triangle inequality along with the Cauchy-Schwarz inequality, and obtain

$$\begin{aligned} & \left| (nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-2} \left(\sum_{i \in I_{2,g}} \hat{\sigma}_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}} \hat{s}_{2,\gamma,i'}^2 - \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}} s_{2,\gamma,i'}^2 \right) \right| \\ & \leq \left| (nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-2} \sum_{i, i' \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2)(\hat{s}_{2,\gamma,i'}^2 - s_{2,\gamma,i'}^2) \right| \\ & + \left| (nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-2} \sum_{i, i' \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2) s_{2,\gamma,i'}^2 \right| + \left| (nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-2} \sum_{i, i' \in I_{2,g}} \sigma_{2,\gamma,i}^2 (\hat{s}_{2,\gamma,i'}^2 - s_{2,\gamma,i'}^2) \right| \\ & \leq (nT)^{-1} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2)^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i' \in I_{2,g}} (\hat{s}_{2,\gamma,i'}^2 - s_{2,\gamma,i'}^2)^2 \right)^{1/2} \\ & + (nT)^{-1} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{\sigma}_{2,\gamma,i}^2 - \sigma_{2,\gamma,i}^2)^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i' \in I_{2,g}} s_{2,\gamma,i'}^4 \right)^{1/2} \\ & + (nT)^{-1} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^4 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i' \in I_{2,g}} (\hat{s}_{2,\gamma,i'}^2 - s_{2,\gamma,i'}^2)^2 \right)^{1/2}, \end{aligned}$$

which along with Assumptions 8 and 9, (S245) and Lemma S13 implies that

$$(nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-2} \left(\sum_{i \in I_{2,g}} \hat{\sigma}_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}} \hat{s}_{2,\gamma,i'}^2 - \sum_{i \in I_{2,g}} \sigma_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}} s_{2,\gamma,i'}^2 \right) = O_p((n^{1/2}T)^{-1}). \quad (\text{S248})$$

The claim of the lemma follows from (S248) and Assumption 2(v). Q.E.D.

Lemma S20. *Under Assumptions 1 and 2, we have: (i) $\hat{\sigma}_{U,n,T}^2 \leq K$ and $\hat{\sigma}_{n,T}^2 \leq K$ wpa1; and (ii) $\hat{\sigma}_{S,n,T}^2 \geq 0$.*

PROOF OF LEMMA S20. (i) Since $|\hat{\sigma}_{12,\gamma,i}| \leq \hat{\sigma}_{1,\gamma,i}^2 \hat{\sigma}_{2,\gamma,i}^2$ for all i , from (31), it follows that

$$\begin{aligned}
\hat{\sigma}_{U,n,T}^2 &\leq (2nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} (\hat{\sigma}_{1,\gamma,i}^4 + 2n_{2,g}^{-1} \hat{\sigma}_{1,\gamma,i}^2 \hat{\sigma}_{2,\gamma,i}^2 + n_{2,g}^{-2} \hat{\sigma}_{2,\gamma,i}^4) \\
&\quad + (2nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-2} \sum_{i \in I_{2,g}} \sum_{i' \in I_{2,g}, i' \neq i} \hat{\sigma}_{2,\gamma,i}^2 \hat{\sigma}_{2,\gamma,i'}^2 \\
&\leq (nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \left(\hat{\sigma}_{1,\gamma,i}^4 + n_{2,g}^{-2} \hat{\sigma}_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}} \hat{\sigma}_{2,\gamma,i'}^2 \right) \\
&\leq (nT)^{-1} \sum_{i \leq n} \hat{s}_{1,\gamma,i}^4 + (nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \hat{s}_{2,\gamma,i}^2 \right)^2. \tag{S249}
\end{aligned}$$

By Assumptions 8 and 9, $s_{j,\gamma,i}^2 \leq K$. Together with Lemma S13, this implies

$$\begin{aligned}
(nT)^{-1} \sum_{i \leq n} \hat{s}_{1,\gamma,i}^4 &\leq 2 \left((nT)^{-1} \sum_{i \leq n} (\hat{s}_{1,\gamma,i}^2 - s_{1,\gamma,i}^2)^2 + (nT)^{-1} \sum_{i \leq n} s_{1,\gamma,i}^4 \right) \\
&= 2(nT)^{-1} \sum_{i \leq n} s_{1,\gamma,i}^4 + O_p((nT)^{-1}) \leq K, \tag{S250}
\end{aligned}$$

wpa1. Similarly,

$$\begin{aligned}
(nT)^{-1} \sum_{g \in \mathcal{G}_2} \left(n_{2,g}^{-1} \sum_{i \in I_{2,g}} \hat{s}_{2,\gamma,i}^2 \right)^2 &\leq 2(nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} s_{2,\gamma,i}^4 \\
&\quad + 2(nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} (\hat{s}_{2,\gamma,i}^2 - s_{2,\gamma,i}^2)^2 \\
&\leq 2(nT)^{-1} \sum_{g \in \mathcal{G}_2} n_{2,g}^{-1} \sum_{i \in I_{2,g}} s_{2,\gamma,i}^4 + O_p((nT)^{-1}) \leq K, \tag{S251}
\end{aligned}$$

with probability approaching 1. Combining (S249), (S250) and (S251) yields $\hat{\sigma}_{U,n,T}^2 \leq K$ with probability approaching 1.

Next, by Lemma S12 and Lemma S17,

$$\hat{\sigma}_{n,T}^2 \leq (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \Delta\psi(z_{i,t}, \hat{\phi}_i)^2 = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\Delta\psi(z_{i,t})^2] + O_p(T^{-1/2}). \tag{S252}$$

Since $\mathbb{E}[\Delta\psi(z_{i,t})^2] \leq K$ by Assumption 8(iii), from (S252), we obtain $\hat{\sigma}_{n,T}^2 \leq K$ wpa1.

Finally, since $\hat{s}_{2,\gamma,i}^2 \geq \hat{\sigma}_{2,\gamma,i}^2 \geq 0$ for all $i \leq n$, the definition of $\hat{\sigma}_{S,n,T}^2$ implies

$$\begin{aligned}
\hat{\sigma}_{S,n,T}^2 &= (2nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} (\hat{\sigma}_{1,\gamma,i}^4 + n_{2,g}^{-2} \hat{\sigma}_{2,\gamma,i}^2 \hat{s}_{2,\gamma,i}^2 - 2n_{2,g}^{-1} \hat{\sigma}_{12,\gamma,i}^2) \\
&\quad + (2nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} \left(n_{2,g}^{-2} \hat{\sigma}_{2,\gamma,i}^2 \sum_{i' \in I_{2,g}, i' \neq i} \hat{s}_{2,\gamma,i'}^2 \right) \\
&\geq (2nT)^{-1} \sum_{g \in \mathcal{G}_2} \sum_{i \in I_{2,g}} (\hat{\sigma}_{1,\gamma,i}^4 + n_{2,g}^{-2} \hat{\sigma}_{2,\gamma,i}^4 - 2n_{2,g}^{-1} \hat{\sigma}_{12,\gamma,i}^2). \tag{S253}
\end{aligned}$$

Next, note that

$$\hat{\sigma}_{1,\gamma,i}^4 + n_{2,g}^{-2} \hat{\sigma}_{2,\gamma,i}^4 - 2n_{2,g}^{-1} \hat{\sigma}_{12,\gamma,i}^2 = (\hat{\sigma}_{1,\gamma,i}^2 - n_{2,g}^{-1} \hat{\sigma}_{2,\gamma,i}^2)^2 + 2n_{2,g}^{-1} (\hat{\sigma}_{1,\gamma,i}^2 \hat{\sigma}_{2,\gamma,i}^2 - \hat{\sigma}_{12,\gamma,i}^2). \quad (\text{S254})$$

Moreover, since $\widehat{\mathbb{E}}_T[\hat{\psi}_{1,\gamma}(z_{i,t})] = 0$ by the first-order condition of $(\hat{\gamma}_{j,i})_{i \leq n}$, we may write

$$\hat{\sigma}_{12,\gamma,i}^2 = \frac{(\widehat{\mathbb{E}}_T[(\hat{\psi}_{1,\gamma}(z_{i,t}) - \widehat{\mathbb{E}}_T[\hat{\psi}_{1,\gamma}(z_{i,t})])(\hat{\psi}_{2,\gamma}(z_{i,t}) - \widehat{\mathbb{E}}_T[\hat{\psi}_{2,\gamma}(z_{i,t})]))^2}{|\hat{\Psi}_{1,\gamma\gamma}(\hat{\phi}_{1,g})\hat{\Psi}_{2,\gamma\gamma}(\hat{\phi}_{2,g})|}.$$

Together with the definitions of $\hat{\sigma}_{j,\gamma,i}^2$ and the Cauchy-Schwarz inequality, this yields

$$\hat{\sigma}_{12,\gamma,i}^2 \leq \hat{\sigma}_{1,\gamma,i}^2 \hat{\sigma}_{2,\gamma,i}^2. \quad (\text{S255})$$

Combining (S253), (S254) and (S255) therefore establishes that $\hat{\sigma}_{S,n,T}^2 \geq 0$. *Q.E.D.*

2 Proofs for Lemmas in Appendix B of Hahn, Liao, Menzel, and Vuong (2026)

Throughout the proof, we let $x_{j,i,t}$, $\bar{x}_{j,i}$, $\bar{x}_{j,t}$ and $\bar{x}_j$ denote the j th entries of $x_{i,t}$, $\bar{x}_i$, $\bar{x}_t$ and $\bar{x}$ for $j = 1, \dots, d_x$ respectively.

Lemma S21. *Under the conditions of Theorem 6, we have $\sum_{i \leq n} \xi_{nT,i} \rightarrow_d N(0, 1)$, where $\xi_{nT,i}$ is defined in (85) in the proof of Theorem 6.*

PROOF OF LEMMA S21. The claim of the lemma follows by arguments analogous to those used in the proof of Theorem A4, provided that Lemma S69, Lemma S70 and Lemma S71 still hold. We next sketch the verification of these lemmas.

First, we observe that $\psi_j(z_{i,t}; \phi_{j,i,t}^*) = \varepsilon_{j,i,t}^2$ and $\psi_{j\gamma}(z_{i,t}; \phi_{j,i,t}^*) = \varepsilon_{j,i,t}$ for $j = 1, 2$. This shows that Assumptions 11(i) and 12(i) are maintained in Assumptions 5(ii, iii), respectively. Moreover, Assumption 12(ii) holds due to Lemma S34(ii) and Lemma S40(iii). Additionally, Assumptions 10(ii) and 12(iii) hold for model 1. The only hindering factors for the direct application of Theorem A4 are Assumptions 10(ii) and 12(iii) for model 2. This is because the two-way fixed effect model involves n and T groups across i and T separately.

However as there is only one group across i in the definition of $\tilde{U}_{2,i}$, Assumptions 10(ii) and 12(iii) hold for model 2 with $\mathcal{G}_2 = \{1\}$ and $\mathcal{M}_2 = \{1 \dots, T\}$. Consequently, we can apply Lemma S70 and Lemma S71 to obtain

$$(n\omega_{n,T}^2)^{-1} \sum_{i \leq n} \left(\mathbb{E}[\tilde{U}_i^2 | \mathcal{F}_{nT,i-1}] - \mathbb{E}[\tilde{U}_i^2] \right) = O_p \left(T^{-1/2} + \max_{g \in \mathcal{G}_1} n_g^{-1/2} \right), \quad (\text{S256})$$

and

$$\max_{j=1,2} \max_{g \in \mathcal{G}_j} \max_{i \in I_g} \mathbb{E} \left[\left| n_g^{1/2} \tilde{U}_{j,i} \right|^4 \right] \leq K, \quad (\text{S257})$$

respectively. Applying similar reasoning as presented in the proof of Lemma S69,

$$(n\omega_{n,T}^2)^{-1} \sum_{i \leq n} \mathbb{E} \left[(\tilde{\Psi}_i + \tilde{V}_i^*) \tilde{U}_{j,i} | \mathcal{F}_{nT,i-1} \right] = O_p \left(T^{-1/2} + \max_{g \in \mathcal{G}_1} n_g^{-1} \right). \quad (\text{S258})$$

This expression serves as the counterpart of (S711). The detailed proof for (S258) is provided at the end of this proof. In view of (S256), (S257) and (S258), we can verify (105) and (106) using the same arguments in the proof of Theorem A4.

We proceed to verify (S258). Applying Lemma S69 with $\mathcal{G}_1 = \{1, \dots, G_1\}$, $\mathcal{M}_1 = \{1, \dots, T\}$, $\mathcal{G}_2 = \{1, \dots, n\}$ and $\mathcal{M}_2 = \{1\}$ obtains

$$(n\omega_{n,T}^2)^{-1} \sum_{i \leq n} \mathbb{E} \left[(\tilde{\Psi}_i + \tilde{V}_i^*) \tilde{U}_{1,i} | \mathcal{F}_{nT,i-1} \right] = O_p \left(T^{-1/2} + \max_{g \in \mathcal{G}_1} n_g^{-1} \right). \quad (\text{S259})$$

By the independence of $\{z_{i,t}\}_{t \leq T}$ across i we have

$$\frac{\sum_{i \leq n} \mathbb{E} \left[(\tilde{\Psi}_i + \tilde{V}_i^*) \tilde{U}_{2,i} | \mathcal{F}_{nT,i-1} \right]}{n\omega_{n,T}^2} = \sum_{i=1}^{n-1} \frac{C_{2,i}^\top e_{2,i}}{n^2 T^{1/2} \omega_{n,T}} \quad (\text{S260})$$

where $C_{2,i} \equiv \sum_{i'=i+1}^n c_{j,i'}$, $c_{j,i'} \equiv \mathbb{E}[(\tilde{\Psi}_{i'} + \tilde{V}_{i'}^*) e_{2,i'} / \omega_{n,T}]$ and $e_{2,i} = (\varepsilon_{2,i,t})_{t \leq T}$. Moreover,

$$\mathbb{E} \left[\left| \sum_{i \leq n} \frac{C_{2,i}^\top e_{2,i}}{n^2 T^{1/2} \omega_{n,T}} \right|^2 \right] \leq \frac{K \max_{i \leq n} \lambda_{\max}(\text{Cov}(e_{2,i})) \sum_{i \leq n} C_{2,i}^\top C_{2,i}}{n^4 T \omega_{n,T}^2} \leq \frac{K \sum_{i \leq n} C_{2,i}^\top C_{2,i}}{n^3 T}, \quad (\text{S261})$$

where the second inequality is by Assumption 5(ii) and (S441). By the definition of $\tilde{V}_i^*$,

$$\mathbb{E}[|\tilde{V}_i^*|^2] = \frac{\mathbb{E} \left[\left((\sum_{t \leq T} \varepsilon_{2,i,t})^2 - \mathbb{E}[(\sum_{t \leq T} \varepsilon_{2,i,t})^2] \right)^2 \right]}{(2T^{3/2})^2} \leq \frac{\mathbb{E} \left[(T^{-1/2} \sum_{t \leq T} \varepsilon_{2,i,t})^4 \right]}{4T} \leq KT^{-1}, \quad (\text{S262})$$

where the second inequality is by Assumption 1(ii, iv) and Rosenthal's inequality of strong mixing processes. In view of Assumptions 5(ii, iii) and (S262), we can use the same arguments for proving (S720) to show that

$$\max_{i \leq n} |c_{j,i}|^2 \leq K \left(1 + M_2 \max_{g \in \mathcal{G}_2} n_g^{-2} \right) \leq K. \quad (\text{S263})$$

This, along with the definition of $C_{2,i}$, implies that

$$\max_{i \leq n} \frac{C_{2,i}^\top C_{2,i}}{n^2} \leq \max_{i \leq n} \left(\frac{\sum_{i'=i+1}^n |c_{j,i'}|}{n} \right)^2 \leq K. \quad (\text{S264})$$

Therefore by Assumption 5(ii), (S261) and (S264),

$$\mathbb{E} \left[\left| \sum_{i \leq n} \frac{C_{2,i}^\top e_{2,i}}{T^{1/2} n^2 \omega_{n,T}} \right|^2 \right] \leq KT^{-1}$$

which, together with the Markov inequality and (S260), implies that

$$\frac{\sum_{i \leq n} \mathbb{E} \left[(\tilde{\Psi}_i + \tilde{V}_i^*) \tilde{U}_{2,i} | \mathcal{F}_{nT,i-1} \right]}{n\omega_{n,T}^2} = O_p(T^{-1/2}) \quad (\text{S265})$$

The claim in (S258) now follows from (S259) and the above result. $Q.E.D.$

PROOF OF LEMMA 6. (a) We first observe that

$$\begin{aligned} \mathbb{E}[\varphi_{n,T}^{2\text{-side}}(p)] &= \mathbb{P} \left(\frac{|MQLR_{n,T}|}{\hat{\omega}_{n,T}} > z_{1-p/2} \right) \\ &\geq \mathbb{P} \left(|\overline{QLR}_{n,T}| > \hat{\omega}_{n,T} z_{1-p/2} + \left| QLR_{n,T} - |\widehat{\mathbb{E}[S_{n,T}]} - QLR_{n,T}^* \right| \right). \end{aligned} \quad (\text{S266})$$

From Lemma 10, it follows that

$$\hat{\omega}_{n,T} z_{1-p/2} + |\widehat{\mathbb{E}[S_{n,T}]} - \mathbb{E}[S_{n,T}]| = O_p(G_1). \quad (\text{S267})$$

By Theorem 6 and Lemma 10,

$$QLR_{n,T} - \mathbb{E}[S_{n,T}] - \overline{QLR}_{n,T} = O_p(\omega_{n,T}) = O_p(1). \quad (\text{S268})$$

Therefore, by the triangle inequality

$$\hat{\omega}_{n,T} z_{1-p/2} + \left| QLR_{n,T} - |\widehat{\mathbb{E}[S_{n,T}]} - \overline{QLR}_{n,T} \right| = O_p(1).$$

This along with $|\overline{QLR}_{n,T}| \succ G_1$ and (S266) implies that

$$\mathbb{E}[\varphi_{n,T}^{2\text{-side}}(p)] \geq \mathbb{P}(|QLR_{n,T}^*| > O_p(G_1)) \rightarrow 1,$$

as $n, T \rightarrow \infty$. This establishes the claim in this part of the lemma.

(b) The proof for the one-sided test follows by analogous arguments and is therefore omitted.

$Q.E.D.$

PROOF OF LEMMA 7. From the definitions of $\tilde{V}_{j,i}$ and $\tilde{U}_{j,i}$, we have

$$\tilde{V}_i = \tilde{V}_{1,i} - \tilde{V}_{2,i} = (2T^{1/2})^{-1} \sum_{t \leq T} (n_g^{-1} \varepsilon_{1,i,t}^{*2} - (n^{-1} + T^{-1}) \varepsilon_{2,i,t}^{*2}) - T^{-3/2} \sum_{t=2}^T \sum_{t'=1}^{t-1} \varepsilon_{2,i,t}^* \varepsilon_{2,i,t'}^*, \quad (\text{S269})$$

and

$$\tilde{U}_i = \tilde{U}_{1,i} - \tilde{U}_{2,i} = \sum_{i'=N_g+1}^{i-1} T^{-1/2} \sum_{t \leq T} (n_g^{-1} \varepsilon_{1,i,t}^* \varepsilon_{1,i',t}^* - n^{-1} \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^*) - \sum_{i'=1}^{N_g} T^{-1/2} \sum_{t \leq T} n^{-1} \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^*. \quad (\text{S270})$$

Recall that $\tilde{\Psi}_i = (2T^{1/2})^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)$. From Assumption 6(i), it follows that

$$\begin{aligned}
\text{Cov}(\tilde{\Psi}_i, \tilde{V}_i) &= (4T)^{-1} \text{Cov} \left(\sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2), \sum_{t \leq T} (n_g^{-1} \varepsilon_{1,i,t}^{*2} - (n^{-1} + T^{-1}) \varepsilon_{2,i,t}^{*2}) \right) \\
&\quad - (2T^2)^{-1} \text{Cov} \left(\sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2), \sum_{t=2}^T \sum_{t'=1}^{t-1} \varepsilon_{2,i,t}^* \varepsilon_{2,i,t'}^* \right) \\
&= (4n_g T)^{-1} \text{Cov} \left(\sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2), \sum_{t \leq T} \varepsilon_{1,i,t}^{*2} \right) \\
&\quad - (4T)^{-1} (n^{-1} + T^{-1}) \text{Cov} \left(\sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2), \sum_{t \leq T} \varepsilon_{2,i,t}^{*2} \right). \tag{S271}
\end{aligned}$$

Using Assumptions 1 and 5(ii), Lemma S34(ii), Lemma S40(iii), and the covariance inequality of strong mixing process,

$$\left| \text{Cov} \left(\sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2), \sum_{t \leq T} \varepsilon_{1,i,t}^{*2} \right) \right| + \left| \text{Cov} \left(\sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2), \sum_{t \leq T} \varepsilon_{2,i,t}^{*2} \right) \right| < K \omega_{n,T} T,$$

which together with Assumption 1(i) and (S271) implies that

$$\begin{aligned}
\left| n^{-1} \sum_{i \leq n} \text{Cov}(\tilde{\Psi}_i, \tilde{V}_i) \right| &\leq K \omega_{n,T} n^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} (n_g^{-1} + n^{-1} + T^{-1}) \\
&= K \omega_{n,T} n^{-1} \sum_{g \in \mathcal{G}_1} (1 + n_g(n^{-1} + T^{-1})) \leq K G_1 \omega_{n,T} n^{-1}. \tag{S272}
\end{aligned}$$

Here the last inequality holds by $n_g(n^{-1} + T^{-1}) \leq K$. Under Assumption 6(i) and some direct calculation, we can show that

$$\text{Cov}(\tilde{\Psi}_i, \tilde{U}_i) = 0 \quad \text{and} \quad \text{Cov}(\tilde{V}_i, \tilde{U}_i) = 0. \tag{S273}$$

Since $\sigma_{n,T}^2 = n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_i)$, combining (S272) and (S273) yields

$$\begin{aligned}
\omega_{n,T}^2 &= n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_i + \tilde{V}_i + \tilde{U}_i) \\
&= n^{-1} \sum_{i \leq n} (\text{Var}(\tilde{\Psi}_i) + \text{Var}(\tilde{V}_i) + \text{Var}(\tilde{U}_i) + 2\text{Cov}(\tilde{\Psi}_i, \tilde{V}_i)) \\
&= \sigma_{n,T}^2 + n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_i) + n^{-1} \sum_{i \leq n} \text{Var}(\tilde{U}_i) + O(G_1 \omega_{n,T} n^{-1}). \tag{S274}
\end{aligned}$$

To finish the proof, it remains to derive the approximations for $n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_i)$ and $n^{-1} \sum_{i \leq n} \text{Var}(\tilde{U}_i)$.

Using Assumption 6(i), we have

$$\begin{aligned}
\text{Var}(\tilde{U}_i) &= \sum_{i'=N_g+1}^{i-1} T^{-1} \text{Var} \left(\sum_{t \leq T} (n_g^{-1} \varepsilon_{1,i,t}^* \varepsilon_{1,i',t}^* - n^{-1} \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^*) \right) + \sum_{i'=1}^{N_g} T^{-1} \text{Var} \left(\sum_{t \leq T} n^{-1} \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^* \right) \\
&= \sum_{i'=N_g+1}^{i-1} T^{-1} \sum_{t \leq T} \text{Var} (n_g^{-1} \varepsilon_{1,i,t}^* \varepsilon_{1,i',t}^* - n^{-1} \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^*) + \sum_{i'=1}^{N_g} T^{-1} \sum_{t \leq T} \text{Var} (n^{-1} \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^*) \\
&= \sum_{i'=N_g+1}^{i-1} T^{-1} \sum_{t \leq T} (n_g^{-2} \sigma_{1,i}^2 \sigma_{1,i'}^2 + n^{-2} \sigma_{2,i}^2 \sigma_{2,i'}^2 - 2n_g^{-1} n^{-1} \sigma_{1,2,i} \sigma_{1,2,i'}) + \sum_{i'=1}^{N_g} T^{-1} \sum_{t \leq T} n^{-2} \sigma_{2,i}^2 \sigma_{2,i'}^2 \\
&= n_g^{-2} \sum_{i'=N_g+1}^{i-1} \sigma_{1,i}^2 \sigma_{1,i'}^2 + n^{-2} \sum_{i'=1}^{i-1} \sigma_{2,i}^2 \sigma_{2,i'}^2 - 2(n n_g)^{-1} \sum_{i'=N_g+1}^{i-1} \sigma_{1,2,i} \sigma_{1,2,i'}.
\end{aligned}$$

This implies that

$$\begin{aligned}
n^{-1} \sum_{i \leq n} \text{Var}(\tilde{U}_i) &= n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-2} \sum_{i \in I_g} \sum_{i'=N_g+1}^{i-1} \sigma_{1,i}^2 \sigma_{1,i'}^2 + n^{-3} \sum_{i \leq n} \sum_{i'=1}^{i-1} \sigma_{2,i}^2 \sigma_{2,i'}^2 \\
&\quad - 2n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{i'=N_g+1}^{i-1} \sigma_{1,2,i} \sigma_{1,2,i'}. \\
&= (2n)^{-1} \sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} \sigma_{1,i}^2 \right)^2 + (2n)^{-1} \left(n^{-1} \sum_{i \leq n} \sigma_{2,i}^2 \right)^2 - n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sigma_{1,2,i} \right)^2 \\
&\quad - (2n)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-2} \sum_{i \in I_g} \sigma_{1,i}^4 - (2n^3)^{-1} \sum_{i \leq n} \sigma_{2,i}^4 + n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sigma_{1,2,i}^2. \tag{S275}
\end{aligned}$$

Since $\sigma_{1,i}^4 + \sigma_{2,i}^4 + \sigma_{1,2,i}^2 < K$ by Lemma S34(ii), Lemma S40(iii), it follows that

$$(2n)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-2} \sum_{i \in I_g} \sigma_{1,i}^4 + (2n^3)^{-1} \sum_{i \leq n} \sigma_{2,i}^4 + n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sigma_{1,2,i}^2 = O \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \right). \tag{S276}$$

Similarly,

$$\begin{aligned}
\text{Var}(\tilde{V}_i) &= (2T^{1/2})^{-2} \sum_{t \leq T} \text{Var}(n_g^{-1} \varepsilon_{1,i,t}^{*2} - (n^{-1} + T^{-1}) \varepsilon_{2,i,t}^{*2}) + T^{-3} \sum_{t=2}^T \sum_{t'=1}^{t-1} \text{Var}(\varepsilon_{2,i,t}^* \varepsilon_{2,i,t'}^*) \\
&= T^{-3} \sum_{t=2}^T \sum_{t'=1}^{t-1} \sigma_{2,i}^4 + (2T^{1/2})^{-2} \sum_{t \leq T} \text{Var}(n_g^{-1} \varepsilon_{1,i,t}^{*2} - (n^{-1} + T^{-1}) \varepsilon_{2,i,t}^{*2}), \\
&= (2T)^{-1} \sigma_{2,i}^4 + (2T^{1/2})^{-2} \sum_{t \leq T} \text{Var}(n_g^{-1} \varepsilon_{1,i,t}^{*2} - (n^{-1} + T^{-1}) \varepsilon_{2,i,t}^{*2}) - (2T^2)^{-1} \sigma_{2,i}^4. \tag{S277}
\end{aligned}$$

Since

$$\text{Var}(n_g^{-1} \varepsilon_{1,i,t}^{*2} - (n^{-1} + T^{-1}) \varepsilon_{2,i,t}^{*2}) \leq K n_g^{-2} \quad \text{and} \quad \sigma_{2,i}^4 \leq K,$$

from (S277) and Assumption 1(i), it follows that

$$n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_i) = (2nT)^{-1} \sum_{i \leq n} \sigma_{2,i}^4 + O \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \right). \quad (\text{S278})$$

Collecting the results in (S274), (S275), (S276) and (S278), we have

$$\begin{aligned} \omega_{n,T}^2 &= \sigma_{n,T}^2 + (2nT)^{-1} \sum_{i \leq n} \sigma_{2,i}^4 + (2n)^{-1} \left(n^{-1} \sum_{i \leq n} \sigma_{2,i}^2 \right)^2 + (2n)^{-1} \sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} \sigma_{1,i}^2 \right)^2 \\ &\quad - n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sigma_{1,2,i} \right)^2 + O \left(G_1 \omega_{n,T} n^{-1} + n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \right). \end{aligned} \quad (\text{S279})$$

The claim of the lemma follows from (S279), the definition of $\sigma_{U,n,T}^2$ and Assumption 6(ii). *Q.E.D.*

PROOF OF THEOREM 8. From the definition of $\hat{\sigma}_{U,n,T}^2$, we can write

$$\begin{aligned} \hat{\sigma}_{U,n,T}^2 &= (2nT)^{-1} \sum_{i \leq n} \hat{\sigma}_{2,i}^4 + (2n)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-2} \left(\sum_{i \in I_g} \hat{\sigma}_{1,i}^2 \right)^2 \\ &\quad + (2n^3)^{-1} \left(\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right)^2 - n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \hat{\sigma}_{1,2,i} \right)^2 \\ &= (2nT)^{-1} \sum_{i \leq n} \hat{\sigma}_{2,i}^4 + n^{-3} \sum_{g=2}^{G_1} \sum_{g'=1}^{g-1} \left(\sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right) \left(\sum_{i \in I_{g'}} \hat{\sigma}_{2,i}^2 \right) \\ &\quad + n^{-1} \sum_{g \in \mathcal{G}_1} \left(n_g^{-2} \left(\sum_{i \in I_g} \hat{\sigma}_{1,i}^2 \right)^2 + n^{-2} \left(\sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right)^2 - 2(n_g n)^{-1} \left(\sum_{i \in I_g} \hat{\sigma}_{1,2,i} \right)^2 \right). \end{aligned} \quad (\text{S280})$$

Thus it suffices to show that, for each $g \in \mathcal{G}_1$,

$$n_g^{-2} \left(\sum_{i \in I_g} \hat{\sigma}_{1,i}^2 \right)^2 + n^{-2} \left(\sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right)^2 - 2n^{-1} n_g^{-1} \left(\sum_{i \in I_g} \hat{\sigma}_{1,2,i} \right)^2 \geq 0. \quad (\text{S281})$$

By the triangle inequality and the Cauchy-Schwarz inequality,

$$\left(\sum_{i \in I_g} \hat{\sigma}_{1,2,i} \right)^2 \leq \left(\sum_{i \in I_g} \hat{\sigma}_{1,i} \hat{\sigma}_{2,i} \right)^2 \leq \left(\sum_{i \in I_g} \hat{\sigma}_{1,i}^2 \right) \left(\sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right).$$

Therefore for any $g \in \mathcal{G}_1$

$$\begin{aligned}
& n_g^{-2} \left(\sum_{i \in I_g} \hat{\sigma}_{1,i}^2 \right)^2 + n^{-2} \left(\sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right)^2 - 2n^{-1} n_g^{-1} \left(\sum_{i \in I_g} \hat{\sigma}_{1,2,i}^2 \right)^2 \\
& \geq n_g^{-2} \left(\sum_{i \in I_g} \hat{\sigma}_{1,i}^2 \right)^2 + n^{-2} \left(\sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right)^2 - 2n^{-1} n_g^{-1} \left(\sum_{i \in I_g} \hat{\sigma}_{1,i}^2 \right) \left(\sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right) \\
& = \left(n_g^{-1} \sum_{i \in I_g} \hat{\sigma}_{1,i}^2 - n^{-1} \sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right)^2 \geq 0,
\end{aligned}$$

which establishes (S281) and hence the lemma. Q.E.D.

PROOF OF LEMMA 9. The first claim of the lemma follows directly from Lemma S28. We now establish the second claim. By the definition of $\hat{\omega}_{n,T}^2$, we have the deterministic bounds

$$\hat{\sigma}_{n,T}^2 - \hat{\sigma}_{U,n,T}^2 \leq \hat{\omega}_{n,T}^2 \leq \max \{ \hat{\sigma}_{n,T}^2 - \hat{\sigma}_{U,n,T}^2, \hat{\sigma}_{U,n,T}^2 + \sigma_{n,T}^2 \}.$$

Therefore,

$$\frac{|\hat{\omega}_{n,T}^2 - \omega_{n,T}^2|}{\omega_{n,T}^2} \leq \frac{|\hat{\sigma}_{n,T}^2 - \hat{\sigma}_{U,n,T}^2 - \omega_{n,T}^2|}{\omega_{n,T}^2} + \frac{|\hat{\sigma}_{U,n,T}^2 + \sigma_{n,T}^2 - \omega_{n,T}^2|}{\omega_{n,T}^2}. \quad (\text{S282})$$

For the first term, by the triangle inequality,

$$\begin{aligned}
\frac{|\hat{\sigma}_{n,T}^2 - \hat{\sigma}_{U,n,T}^2 - \omega_{n,T}^2|}{\omega_{n,T}^2} & \leq \frac{|\hat{\sigma}_{n,T}^2 - \hat{\sigma}_{U,n,T}^2 - (\sigma_{n,T}^2 + \sigma_{U,n,T}^2)|}{\omega_{n,T}^2} + \frac{|\omega_{n,T}^2 - (\sigma_{n,T}^2 + \sigma_{U,n,T}^2)|}{\omega_{n,T}^2} \\
& = \frac{|\hat{\sigma}_{n,T}^2 - \sigma_{n,T}^2 - 2\sigma_{U,n,T}^2 - (\hat{\sigma}_{U,n,T}^2 - \sigma_{U,n,T}^2)|}{\omega_{n,T}^2} + o_p(1) \\
& \leq \frac{|\hat{\sigma}_{n,T}^2 - \sigma_{n,T}^2 - 2\sigma_{U,n,T}^2|}{\omega_{n,T}^2} + \frac{|\hat{\sigma}_{U,n,T}^2 - \sigma_{U,n,T}^2|}{\omega_{n,T}^2} + o_p(1), \quad (\text{S283})
\end{aligned}$$

where the equality is by Lemma 7. Under Assumption 6, we have $\sigma_{U,n,T}^2 = s_{U,n,T}^2$, where $s_{U,n,T}^2$ is defined in Lemma S32. It therefore follows from Lemma S32 that

$$\hat{\sigma}_{U,n,T}^2 - \sigma_{U,n,T}^2 = O_p \left(n^{-1} \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} \right)^{1/2} \right) = O_p \left(\omega_{n,T}^2 \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} \right)^{1/2} \right) = o_p(\omega_{n,T}^2). \quad (\text{S284})$$

Here, the second equality follows from Assumption 5(i), while the third equality is implied by Assumption 6(ii). Combining (S283), (S284) and Lemma S30 yields:

$$|\hat{\sigma}_{n,T}^2 - \hat{\sigma}_{U,n,T}^2 - \omega_{n,T}^2| = o_p(\omega_{n,T}^2). \quad (\text{S285})$$

For the second term in (S282), Lemma 7 and (S284) imply

$$\frac{|\hat{\sigma}_{U,n,T}^2 + \sigma_{n,T}^2 - \omega_{n,T}^2|}{\omega_{n,T}^2} \leq \frac{|\hat{\sigma}_{U,n,T}^2 - \sigma_{U,n,T}^2|}{\omega_{n,T}^2} + \frac{|\omega_{n,T}^2 - \sigma_{n,T}^2 - \sigma_{U,n,T}^2|}{\omega_{n,T}^2} = o_p(1).$$

Substituting these bounds into (S282) gives

$$|\hat{\omega}_{n,T}^2 - \omega_{n,T}^2| = o_p(\omega_{n,T}^2),$$

which is equivalent to the second claim in (87). Q.E.D.

PROOF OF LEMMA 10. By Lemma S22,

$$\begin{aligned} \hat{\sigma}_{n,T}^2 &\leq (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2)^2 \\ &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^2 - \dot{\varepsilon}_{1,i,t}^2)^2 + O_p(\omega_{n,T} n^{-1} + G_1^{1/2} n^{-3/2}) \\ &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^2 - \dot{\varepsilon}_{1,i,t}^2)^2 + O_p(n^{-1}) \\ &\leq 2(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^4 + \dot{\varepsilon}_{1,i,t}^4) + o_p(1), \end{aligned}$$

where the third line uses the facts that $\omega_{n,T} \leq K$ (established below), and $G_1^{1/2} n^{-3/2} = (G_1 n^{-1})^{1/2} n^{-1} \leq K n^{-1}$. By Lemma S34(ii) and Lemma S40(iii), (S310), and Markov's inequality,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^4 + \dot{\varepsilon}_{1,i,t}^4) = O_p(1)$$

and thus,

$$\hat{\sigma}_{n,T}^2 = O_p(1). \tag{S286}$$

Next by Assumptions 1(i) and 5(iii), and Lemma S32,

$$\hat{\sigma}_{U,n,T}^2 - s_{U,n,T}^2 = o_p(1). \tag{S287}$$

Moreover, by Assumption 1(i), Lemma S34(ii) and Lemma S40(iii),

$$\begin{aligned} s_{U,n,T}^2 &\leq (2nT)^{-1} \sum_{i \leq n} \sigma_{2,i}^2 s_{2,i}^2 + (2n)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-2} \left(\sum_{i \in I_g} s_{1,i}^2 \right)^2 \\ &\quad + (2n^3)^{-1} \left(\sum_{i \leq n} s_{2,i}^2 \right)^2 + n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} s_{1,2,i} \right)^2 \leq K, \end{aligned}$$

and hence, by (S287),

$$\hat{\sigma}_{U,n,T}^2 = O_p(1). \tag{S288}$$

Combining (S286) and (S288) gives

$$\hat{\omega}_{n,T}^2 \leq \hat{\sigma}_{n,T}^2 + \hat{\sigma}_{U,n,T}^2 = O_p(1)$$

which proves the first claim of the lemma.

Next note that

$$\begin{aligned} \widehat{\mathbb{E}[S_{n,T}]} &= (2nT)^{-1/2} \left(\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} T n_g^{-1} \hat{\sigma}_{1,i}^2 - (1 + T n^{-1}) \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \hat{\sigma}_{2,i}^2 \right) \\ &= (2nT)^{-1/2} \left(\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} T n_g^{-1} s_{1,i}^2 - (1 + T n^{-1}) \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} s_{2,i}^2 \right) \\ &\quad + (2nT)^{-1/2} \left(\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} T n_g^{-1} (\hat{\sigma}_{1,i}^2 - s_{1,i}^2) - (1 + T n^{-1}) \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2) \right) \\ &= (2nT)^{-1/2} \left(\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} T n_g^{-1} s_{1,i}^2 - (1 + T n^{-1}) \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} s_{2,i}^2 \right) + O_p \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} \right), \end{aligned} \quad (\text{S289})$$

where the last step follows from Lemma S38 and Lemma S44. By Assumption 1(i) and Lemma S34(ii),

$$(2nT)^{-1/2} (1 + T n^{-1}) \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} s_{2,i}^2 \leq K, \quad (\text{S290})$$

and by Assumption 1(i) and Lemma S40(iii),

$$(2nT)^{-1/2} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} T n_g^{-1} s_{1,i}^2 \leq K G_1. \quad (\text{S291})$$

Combining (S289), (S290) and (S291) yields

$$\left| \widehat{\mathbb{E}[S_{n,T}]} \right| \leq K G_1 + O_p \left(n^{-1} + \sum_{g \in \mathcal{G}_1} n_g^{-1} \right) = O_p(G_1), \quad (\text{S292})$$

where the final equality uses Assumptions 1(i) and 5(iii). This proves the second claim of the lemma. In addition, since $\sigma_{2,i}^2 \leq s_{2,i}^2$ and $\sigma_{1,i}^2 \leq s_{1,i}^2$ for any i , the bound $|\mathbb{E}[S_{n,T}]| \leq K G_1$ follows from the definition of $\mathbb{E}[S_{n,T}]$, (S290), (S291) and the triangle inequality.

Finally, to show $\omega_{n,T}^2 \leq K$, note that

$$\omega_{n,T}^2 = n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_i + \tilde{V}_i + \tilde{U}_i) \leq K n^{-1} \sum_{i \leq n} (\text{Var}(\tilde{\Psi}_i) + \text{Var}(\tilde{V}_i) + \text{Var}(\tilde{U}_i)). \quad (\text{S293})$$

By Assumption 1, Lemma S34(ii) and Lemma S40(iii),

$$\text{Var}(\tilde{\Psi}_i) = \text{Var} \left((2T^{1/2})^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \right) \leq K \max_{t \leq T} \|\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2\|_{2+\delta}^2 \leq K. \quad (\text{S294})$$

Similarly,

$$\begin{aligned}
\text{Var}(\tilde{V}_i) &\leq 2(\text{Var}(\tilde{V}_{1,i}) + \text{Var}(\tilde{V}_{2,i})) \\
&\leq K \left(n_g^{-2} \text{Var} \left(T^{-1/2} \sum_{t \leq T} \varepsilon_{1,i,t}^{*2} \right) + \text{Var} \left(T^{-3/2} \left(\sum_{t \leq T} \varepsilon_{2,i,t}^* \right)^2 \right) + n^{-2} \text{Var} \left(T^{-1/2} \sum_{t \leq T} \varepsilon_{2,i,t}^{*2} \right) \right) \\
&\leq K \left(n_g^{-2} + \mathbb{E} \left[T^{-3} \left(\sum_{t \leq T} \varepsilon_{2,i,t}^* \right)^4 \right] \right) \leq K(n_g^{-2} + T^{-1}), \tag{S295}
\end{aligned}$$

and

$$\begin{aligned}
\text{Var}(\tilde{U}_i) &\leq 2(\text{Var}(\tilde{U}_{1,i}) + \text{Var}(\tilde{U}_{2,i})) \\
&= 2 \left(n_g^{-2} \sum_{i'=N_g+1}^{i-1} \text{Var} \left(T^{-1/2} \sum_{t \leq T} \varepsilon_{1,i,t}^* \varepsilon_{1,i',t}^* \right) + n^{-2} \sum_{i'=1}^{i-1} \text{Var} \left(T^{-1/2} \sum_{t \leq T} \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^* \right) \right) \\
&\leq K(n_g^{-1} + n^{-1}). \tag{S296}
\end{aligned}$$

Substituting (S294), (S295) and (S296) into (S293) yields $\omega_{n,T}^2 \leq K(1 + n_g^{-1} + n^{-1} + T^{-1}) \leq K$, which proves the final claim. *Q.E.D.*

2.1 Auxiliary Lemmas

Lemma S22. *Suppose Assumptions 1, 4 and 5 hold. Then*

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2)^2 = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^2 - \dot{\varepsilon}_{1,i,t}^2)^2 + O_p(\omega_{n,T} n^{-1} + G_1^{1/2} n^{-3/2}).$$

PROOF OF LEMMA S22. By definition $\hat{\varepsilon}_{2,i,t} = \ddot{\varepsilon}_{2,i,t} - \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*)$ where $\ddot{\varepsilon}_{2,i,t} \equiv \varepsilon_{2,i,t} - \bar{\varepsilon}_{2,i} - \bar{\varepsilon}_{2,t} + \bar{\varepsilon}_2$, which implies

$$\hat{\varepsilon}_{2,i,t}^2 = \ddot{\varepsilon}_{2,i,t}^2 - 2\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) + (\hat{\theta}_2 - \theta_2^*)^\top \ddot{x}_{i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*). \tag{S297}$$

Similarly, from the definition of $\hat{\varepsilon}_{1,i,t}$,

$$\hat{\varepsilon}_{1,i,t}^2 = \dot{\varepsilon}_{1,i,t}^2 - 2\dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) + (\hat{\theta}_1 - \theta_1^*)^\top \dot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*). \tag{S298}$$

Subtracting the population quantity and using (S297) and (S298), we obtain:

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2)^2 - (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{2,i,t}^2 - \tilde{\varepsilon}_{1,i,t}^2)^2 \\
= & 4(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \tilde{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*))^2 \\
& + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} ((\hat{\theta}_2 - \theta_2^*)^\top \ddot{x}_{i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - (\hat{\theta}_1 - \theta_1^*)^\top \dot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*))^2 \\
& - 4(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{2,i,t}^2 - \tilde{\varepsilon}_{1,i,t}^2) (\tilde{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \tilde{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) \\
& + 2(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{2,i,t}^2 - \tilde{\varepsilon}_{1,i,t}^2) ((\hat{\theta}_2 - \theta_2^*)^\top \ddot{x}_{i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - (\hat{\theta}_1 - \theta_1^*)^\top \dot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) \\
& - 4(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \begin{pmatrix} \tilde{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) \\ -\tilde{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \end{pmatrix} \begin{pmatrix} (\hat{\theta}_2 - \theta_2^*)^\top \ddot{x}_{i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) \\ -(\hat{\theta}_1 - \theta_1^*)^\top \dot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \end{pmatrix}. \quad (\text{S299})
\end{aligned}$$

Under Assumptions 1 and 4(ii), Lemma S34(ii) and Lemma S40(iii), and Markov's inequality,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\varepsilon}_{1,i,t}^4 + \tilde{\varepsilon}_{2,i,t}^4 + \|\dot{x}_{i,t}\|^4 + \|\ddot{x}_{i,t}\|^4) = O_p(1). \quad (\text{S300})$$

Combining this with Lemmas S36 and S42, and applying the Cauchy-Schwarz inequality yields

$$\sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \tilde{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*))^2 = O_p(1), \quad (\text{S301})$$

$$\sum_{i \leq n} \sum_{t \leq T} ((\hat{\theta}_2 - \theta_2^*)^\top \ddot{x}_{i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - (\hat{\theta}_1 - \theta_1^*)^\top \dot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*))^2 = O_p((nT)^{-1}), \quad (\text{S302})$$

$$\sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{2,i,t}^2 - \tilde{\varepsilon}_{1,i,t}^2) \begin{pmatrix} (\hat{\theta}_2 - \theta_2^*)^\top \ddot{x}_{i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) \\ -(\hat{\theta}_1 - \theta_1^*)^\top \dot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \end{pmatrix} = O_p(1), \quad (\text{S303})$$

$$\sum_{i \leq n} \sum_{t \leq T} \begin{pmatrix} \tilde{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) \\ -\tilde{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \end{pmatrix} \begin{pmatrix} (\hat{\theta}_2 - \theta_2^*)^\top \ddot{x}_{i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) \\ -(\hat{\theta}_1 - \theta_1^*)^\top \dot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \end{pmatrix} = O_p((nT)^{-1/2}). \quad (\text{S304})$$

Therefore, using Assumption 1(i) and (S299), (S301)-(S304), we obtain

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2)^2 - (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{2,i,t}^2 - \tilde{\varepsilon}_{1,i,t}^2)^2 \\
= & -4(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{2,i,t}^2 - \tilde{\varepsilon}_{1,i,t}^2) (\tilde{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \tilde{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) + O_p(n^{-2}). \quad (\text{S305})
\end{aligned}$$

It therefore suffices to show that

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{2,i,t}^2 - \tilde{\varepsilon}_{1,i,t}^2) (\tilde{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \tilde{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) = O_p(\omega_{n,T} n^{-1} + G_1^{1/2} n^{-3/2}), \quad (\text{S306})$$

which together with (S305) completes the proof.

From the definitions of $\ddot{\varepsilon}_{2,i,t}$ and $\dot{\varepsilon}_{1,i,t}$, we have

$$\ddot{\varepsilon}_{2,i,t}^2 = (\varepsilon_{2,i,t} - \bar{\varepsilon}_{2,i} - \bar{\varepsilon}_{2,t} + \bar{\varepsilon}_2)^2 = \varepsilon_{2,i,t}^2 - 2\varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) + (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 \quad (\text{S307})$$

and

$$\dot{\varepsilon}_{1,i,t}^2 = (\varepsilon_{1,i,t} - \bar{\varepsilon}_{1,g,t})^2 = (\varepsilon_{1,i,t} - \bar{\varepsilon}_{1,g,t}^*)^2 = \varepsilon_{1,i,t}^2 - 2\varepsilon_{1,i,t}\bar{\varepsilon}_{1,g,t}^* + \bar{\varepsilon}_{1,g,t}^{*2}. \quad (\text{S308})$$

Substituting these expansions yields

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^2 - \dot{\varepsilon}_{1,i,t}^2) (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) \\ = & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) \\ & - 2(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t}\bar{\varepsilon}_{1,g,t}^*) (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) \\ & + (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} ((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2}) (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)). \quad (\text{S309}) \end{aligned}$$

By Assumption 1, Lemma S34(ii) and Lemma S40(iii),

$$\mathbb{E}[\bar{\varepsilon}_{2,i}^{*4} + \bar{\varepsilon}_{2,t}^{*4}] \leq Kn^{-2}, \quad \mathbb{E}[\bar{\varepsilon}_2^{*4}] \leq Kn^{-4} \quad \text{and} \quad \mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*4}] \leq Kn_g^{-2}. \quad (\text{S310})$$

Combining this with Assumption 1(i), (S301), the Cauchy–Schwarz inequality, and Markov’s inequality leads to

$$\begin{aligned} & \left| (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} ((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2}) (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) \right|^2 \\ \leq & (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} ((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2})^2 \\ & \times (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*))^2 = O_p \left(n^{-3} \sum_{g \in \mathcal{G}_1} n_g^{-1} \right) \quad (\text{S311}) \end{aligned}$$

and

$$\begin{aligned} & \left| (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t}\bar{\varepsilon}_{1,g,t}^*) (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) \right|^2 \\ \leq & K(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 + \varepsilon_{1,i,t}^2 \bar{\varepsilon}_{1,g,t}^{*2}) \\ & \times (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*))^2 = O_p(G_1 n^{-3}). \quad (\text{S312}) \end{aligned}$$

Moreover, by Assumption 5(ii), (S301), and Markov's inequality,

$$\begin{aligned}
& \left| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) \right|^2 \\
& \leq \omega_{n,T}^2 \times (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \frac{(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2}{\omega_{n,T}^2} \\
& \quad \times (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*))^2 = O_p(\omega_{n,T}^2 n^{-2}). \quad (\text{S313})
\end{aligned}$$

Combining (S309), (S311), (S312) and (S313), we conclude that

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^2 - \dot{\varepsilon}_{1,i,t}^2) (\ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) - \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)) \\
& = O_p \left(\omega_{n,T} n^{-1} + n^{-3/2} \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} \right)^{1/2} + G_1^{1/2} n^{-3/2} \right),
\end{aligned}$$

which, together with $\sum_{g \in \mathcal{G}_1} n_g^{-1} \leq G_1 \max_{g \in \mathcal{G}_1} n_g^{-1}$, establishes the claim in (S306). *Q.E.D.*

Lemma S23. *Suppose Assumptions 1, 4 and 5 hold. Then*

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^2 - \dot{\varepsilon}_{1,i,t}^2)^2 - (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2 \\
& = 4(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*)^2 + O_p \left(\omega_{n,T} G_1 n^{-1} + n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1/2} \right).
\end{aligned}$$

PROOF OF LEMMA S23. Using the decompositions in (S307) and (S308), we obtain

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^2 - \dot{\varepsilon}_{1,i,t}^2)^2 - (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2 \\
& = 4(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*)^2 \\
& \quad + (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} ((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2})^2 \\
& \quad - 4(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*) ((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2}) \\
& \quad - 4(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) (\varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*) \\
& \quad + 2(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) ((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2}). \quad (\text{S314})
\end{aligned}$$

By (S310), together with the Cauchy–Schwarz and Markov’s inequalities,

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} ((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2})^2 = O_p \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \right). \quad (\text{S315})$$

Next, note that

$$\begin{aligned} & (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t}\bar{\varepsilon}_{1,g,t}^*)((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2}) \\ = & (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^3 - (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)\bar{\varepsilon}_{1,g,t}^{*2} \\ & - (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t}\bar{\varepsilon}_{1,g,t}^*(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 + (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t}\bar{\varepsilon}_{1,g,t}^{*3}. \end{aligned} \quad (\text{S316})$$

From Assumption 1(i), Lemma S34(ii), Lemma S40(iii), and (S310),

$$\mathbb{E}[|\varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^3|] \leq \|\varepsilon_{2,i,t}\|_4 \|(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^3\|_{4/3} \leq K n^{-3/2}.$$

Hence, by Markov’s inequality,

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^3 = O_p(n^{-3/2}). \quad (\text{S317})$$

Similarly, the remaining three terms in (S316) satisfy:

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)\bar{\varepsilon}_{1,g,t}^{*2} = O_p \left(G_1 n^{-3/2} \right), \quad (\text{S318})$$

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t}\bar{\varepsilon}_{1,g,t}^*(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 = O_p \left(n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{1/2} \right), \quad (\text{S319})$$

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t}\bar{\varepsilon}_{1,g,t}^{*3} = O_p \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1/2} \right). \quad (\text{S320})$$

Combining (S316)-(S320) yields

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t}\bar{\varepsilon}_{1,g,t}^*)((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2}) = O_p \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1/2} \right). \quad (\text{S321})$$

Let $\bar{u}_{1,i} \equiv T^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t}$ and $\bar{u}_{1,i}^* \equiv \bar{u}_{1,i} - \mathbb{E}[\bar{u}_{1,i}]$. By Assumption 5(ii), Lemma S34(ii) and Hölder’s inequality

$$\begin{aligned} |\mathbb{E}[\bar{u}_{1,i}]| & \leq T^{-1} \sum_{t \leq T} \mathbb{E}[|(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t}|] \\ & \leq \omega_{n,T} T^{-1} \sum_{t \leq T} \|(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) / \omega_{n,T}\|_2 \|\varepsilon_{2,i,t}\|_2 \leq K \omega_{n,T}. \end{aligned} \quad (\text{S322})$$

Together with Assumption 1 and (S310), this implies

$$\mathbb{E} \left[\left| n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{u}_{1,i}] \bar{\varepsilon}_{2,i}^* \right|^2 \right] = n^{-2} \sum_{i \leq n} (\mathbb{E}[\bar{u}_{1,i}])^2 \mathbb{E}[\bar{\varepsilon}_{2,i}^{*2}] \leq K \omega_{n,T}^2 n^{-2}.$$

Therefore, by Markov's inequality,

$$n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{u}_{1,i}] \bar{\varepsilon}_{2,i}^* = O_p(\omega_{n,T} n^{-1}). \quad (\text{S323})$$

Next, by Assumptions 1 and 5(ii), together with the covariance inequality for strong mixing processes,

$$\begin{aligned} \mathbb{E}[\bar{u}_{1,i}^{*2}] &= \text{Var} \left(T^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t} \right) \\ &\leq 2T^{-2} \sum_{0 \leq h \leq T} \sum_{t \leq T-h} |\text{Cov}((\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t}, (\varepsilon_{2,i,t+h}^2 - \varepsilon_{1,i,t+h}^2) \varepsilon_{2,i,t+h})| \\ &\leq 2T^{-2} \max_{t \leq T} \|(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t}\|_{2+\delta/3}^2 \sum_{0 \leq h \leq T} \sum_{t \leq T-h} \alpha_i^{\delta/(6+\delta)}(h) \\ &\leq KT^{-1} \max_{t \leq T} \|(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t}\|_{2+\delta/3}^2. \end{aligned} \quad (\text{S324})$$

Here the last inequality follows since, by Assumption 1(iv),

$$T^{-1} \sum_{0 \leq h \leq T} \sum_{t \leq T-h} \alpha_i^{\delta/(6+\delta)}(h) \leq K \sum_{0 \leq h \leq T} (1 - h/T) a^{\delta h/(6+\delta)} \leq K.$$

By Hölder's inequality

$$\begin{aligned} \|(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t}\|_{2+\delta/3}^{2+\delta/3} &= \mathbb{E}[|(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t}|^{2+\delta/3}] \\ &\leq (\mathbb{E}[|\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2|^{3+\delta/2}])^{2/3} (E[|\varepsilon_{2,i,t}|^{6+\delta}])^{1/3} \\ &\leq K \omega_{n,T}^{2+\delta/3} (\mathbb{E}[|(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)/\omega_{n,T}|^{3+\delta/2}])^{2/3} \leq K \omega_{n,T}^{2+\delta/3}, \end{aligned} \quad (\text{S325})$$

where the second inequality follows from Lemma S34(ii) and the last from Assumption 5(ii). Combining (S323) and (S324) yields

$$\mathbb{E}[\bar{u}_{1,i}^{*2}] \leq K \omega_{n,T}^2 T^{-1}. \quad (\text{S326})$$

Finally, together with (S310), the triangle inequality, and Hölder's inequality

$$\mathbb{E} \left[\left| n^{-1} \sum_{i \leq n} \bar{u}_{1,i}^* \bar{\varepsilon}_{2,i}^* \right|^2 \right] \leq n^{-1} \sum_{i \leq n} \mathbb{E}[|\bar{u}_{1,i}^* \bar{\varepsilon}_{2,i}^*|] \leq n^{-1} \sum_{i \leq n} \|\bar{u}_{1,i}^*\|_2 \|\bar{\varepsilon}_{2,i}^*\|_2 \leq K \omega_{n,T} (nT)^{-1/2}.$$

Therefore, by Assumption 1(i) and Markov's inequality,

$$n^{-1} \sum_{i \leq n} \bar{u}_{1,i}^* \bar{\varepsilon}_{2,i}^* = O_p(\omega_{n,T} n^{-1}). \quad (\text{S327})$$

Combining (S323) and (S327), we obtain

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t} \bar{\varepsilon}_{2,i}^* = n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{u}_{1,i}] \bar{\varepsilon}_{2,i}^* + n^{-1} \sum_{i \leq n} \bar{u}_{1,i}^* \bar{\varepsilon}_{2,i}^* = O_p(\omega_{n,T} n^{-1}). \quad (\text{S328})$$

Similarly,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t} (\bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) = O_p(\omega_{n,T} n^{-1})$$

which, along with (S328), yields:

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) = O_p(\omega_{n,T} n^{-1}). \quad (\text{S329})$$

Next, define $\bar{u}_{2,g,t} \equiv n_g^{-1} \sum_{i \in I_g} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{1,i,t}$ and $\bar{u}_{2,g,t}^* \equiv \bar{u}_{2,g,t} - \mathbb{E}[\bar{u}_{2,g,t}]$. By arguments analogous to those used to establish (S322),

$$|\mathbb{E}[\bar{u}_{2,g,t}]| \leq K \omega_{n,T} \quad \text{and} \quad \mathbb{E}[\bar{u}_{2,g,t}^{*2}] \leq K \omega_{n,T}^2 n_g^{-1}.$$

Hence by (S310), Holder's inequality and Markov's inequality,

$$\begin{aligned} & (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^* \\ &= (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \mathbb{E}[\bar{u}_{2,g,t}] \bar{\varepsilon}_{1,g,t}^* + (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \bar{\varepsilon}_{1,g,t}^* \bar{u}_{2,g,t}^* = O_p(\omega_{n,T} G_1 n^{-1}) \end{aligned} \quad (\text{S330})$$

By (S329) and (S330), the fourth term in (S314) satisfies

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) (\varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*) = O_p(\omega_{n,T} G_1 n^{-1}). \quad (\text{S331})$$

Moreover, by Assumption 5(i), (S310), and Holder's inequality

$$\begin{aligned} & (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \mathbb{E}[|(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2|] \\ & \leq \omega_{n,T} (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \|(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) / \omega_{n,T}\|_2 \|(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2\|_2 \leq K \omega_{n,T} n^{-1}. \end{aligned}$$

Together with Markov's inequality, this implies

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 = O_p(\omega_{n,T} n^{-1}). \quad (\text{S332})$$

Similarly,

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) \bar{\varepsilon}_{1,g,t}^{*2} = O_p(\omega_{n,T} G_1 n^{-1}). \quad (\text{S333})$$

Combining (S332) and (S333) yields

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2) ((\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 - \bar{\varepsilon}_{1,g,t}^{*2}) = O_p(\omega_{n,T} G_1 n^{-1}). \quad (\text{S334})$$

The claim of the lemma follows immediately from (S314), (S315), (S321), (S331) and (S334). *Q.E.D.*

Lemma S24. *Suppose Assumptions 1, 4 and 5 hold. Then*

$$\begin{aligned} & (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*)^2 \\ &= n^{-1} \sum_{i \leq n} s_{2,i}^2 \mathbb{E}[\bar{\varepsilon}_{2,i}^{*2}] + n^{-3} \sum_{i \leq n} \sum_{i' \leq n} \left(T^{-1} \sum_{t \leq T} s_{2,i,t}^2 \sigma_{2,i',t}^2 \right) \\ & \quad + (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} s_{1,i,t}^2 \right) \left(n_g^{-1} \sum_{i \in I_g} \sigma_{1,i,t}^2 \right) \\ & \quad - 2(n^2 T)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g s_{1,2,g,t} \sigma_{1,2,g,t} + O_p \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1/2} \right), \end{aligned}$$

where

$$s_{j,i,t}^2 \equiv \mathbb{E}[\varepsilon_{j,i,t}^2], \quad s_{j,i}^2 = T^{-1} \sum_{t \leq T} s_{j,i,t}^2, \quad s_{1,2,g,t} \equiv n_g^{-1} \sum_{i \in I_g} \mathbb{E}[\varepsilon_{1,i,t} \varepsilon_{2,i,t}] \text{ and } \sigma_{1,2,g,t} \equiv n_g^{-1} \sum_{i \in I_g} \mathbb{E}[\varepsilon_{1,i,t}^* \varepsilon_{2,i,t}^*].$$

PROOF OF LEMMA S24. By Lemma S34(ii) and Lemma S40(iii), (S310) and Hölder's inequality,

$$\mathbb{E}[|(\varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*) \varepsilon_{2,i,t}|] \leq \|\varepsilon_{2,i,t}^2\|_2 \|\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*\|_2 + \|\varepsilon_{1,i,t} \varepsilon_{2,i,t}\|_2 \|\bar{\varepsilon}_{1,g,t}^*\|_2 \leq K n_g^{-1/2}.$$

Therefore, by Markov's inequality,

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*) \varepsilon_{2,i,t} = O_p \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{1/2} \right). \quad (\text{S335})$$

By Assumption 1(i), (S310) and Markov's inequality,

$$\bar{\varepsilon}_2^* = O_p(n^{-1}), \quad (\text{S336})$$

which together with Assumptions 5(i) and (S335) implies that

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*) \varepsilon_{2,i,t} \bar{\varepsilon}_2^* = O_p \left(n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{1/2} \right). \quad (\text{S337})$$

Therefore,

$$\begin{aligned}
& (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*)^2 \\
&= (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{2,i,t}(\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*) - \varepsilon_{1,i,t} \bar{\varepsilon}_{1,g,t}^*)^2 + O_p \left(n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{1/2} \right) \\
&= (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{2,i,t}^2 (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*)^2 + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{1,i,t}^2 \bar{\varepsilon}_{1,g,t}^{*2} \\
&\quad - 2(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t} \varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*) \bar{\varepsilon}_{1,g,t}^* + O_p \left(n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{1/2} \right). \quad (\text{S338})
\end{aligned}$$

The claim of the lemma follows from (S338), Lemmas S25-S27 below.

Q.E.D.

Lemma S25. *Suppose Assumptions 1, 4 and 5 hold. Then*

$$\begin{aligned}
(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*)^2 &= n^{-1} \sum_{i \leq n} s_{2,i}^2 \mathbb{E}[\bar{\varepsilon}_{2,i}^{*2}] \\
&\quad + n^{-3} \sum_{i \leq n} \sum_{i' \leq n} \left(T^{-1} \sum_{t \leq T} s_{2,i,t}^2 \sigma_{2,i',t}^2 \right) + O_p(n^{-3/2}).
\end{aligned}$$

PROOF OF LEMMA S25. We begin by decomposing

$$\begin{aligned}
(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*)^2 &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^{*2} + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,t}^{*2} \\
&\quad + 2(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^* \bar{\varepsilon}_{2,t}^*. \quad (\text{S339})
\end{aligned}$$

We analyze the three terms on the right-hand side in turn.

Using Assumption 1, Lemma S34(ii), (S310), the Cauchy-Schwarz inequality and Markov's inequality, we obtain

$$\begin{aligned}
& n^{-1} \sum_{i \leq n} \mathbb{E} \left[\left| T^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) \right| \bar{\varepsilon}_{2,i}^{*2} \right] \\
&\leq n^{-1} \sum_{i \leq n} \left(\mathbb{E} \left[\left(T^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) \right)^2 \right] \mathbb{E}[\bar{\varepsilon}_{2,i}^{*4}] \right)^{1/2} \leq Kn^{-3/2}. \quad (\text{S340})
\end{aligned}$$

This, together with Markov's inequality, implies that

$$\begin{aligned}
(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^{*2} &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\varepsilon_{2,i,t}^2] \bar{\varepsilon}_{2,i}^{*2} + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) \bar{\varepsilon}_{2,i}^{*2} \\
&= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} s_{2,i,t}^2 \bar{\varepsilon}_{2,i}^{*2} + O_p(n^{-3/2}) \\
&= n^{-1} \sum_{i \leq n} s_{2,i}^2 \bar{\varepsilon}_{2,i}^{*2} + O_p(n^{-3/2}),
\end{aligned} \tag{S341}$$

where $s_{2,i}^2 = T^{-1} \sum_{t \leq T} s_{2,i,t}^2$. Using Assumption 1, Lemma S34(ii), and (S310), we further have

$$\mathbb{E} \left[\left(n^{-1} \sum_{i \leq n} s_{2,i}^2 (\bar{\varepsilon}_{2,i}^{*2} - \mathbb{E}[\bar{\varepsilon}_{2,i}^{*2}]) \right)^2 \right] = n^{-2} \sum_{i \leq n} \mathbb{E}[s_{2,i}^4 (\bar{\varepsilon}_{2,i}^{*2} - \mathbb{E}[\bar{\varepsilon}_{2,i}^{*2}])^2] \leq Kn^{-2} \sum_{i \leq n} \mathbb{E}[\bar{\varepsilon}_{2,i}^{*4}] \leq Kn^{-3},$$

which implies

$$n^{-1} \sum_{i \leq n} s_{2,i}^2 \bar{\varepsilon}_{2,i}^{*2} = n^{-1} \sum_{i \leq n} s_{2,i}^2 \mathbb{E}[\bar{\varepsilon}_{2,i}^{*2}] + O_p(n^{-3/2}). \tag{S342}$$

Combining (S341) and (S342) yields

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^{*2} = n^{-1} \sum_{i \leq n} s_{2,i}^2 \mathbb{E}[\bar{\varepsilon}_{2,i}^{*2}] + O_p(n^{-3/2}). \tag{S343}$$

Using Assumption 1, (S310) and Lemma S34(ii), we have

$$\begin{aligned}
&T^{-1} \sum_{t \leq T} \mathbb{E} \left[\left| n^{-1} \sum_{i \leq n} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) \right| \bar{\varepsilon}_{2,t}^{*2} \right] \\
&\leq T^{-1} \sum_{t \leq T} \left(\mathbb{E} \left[\left(n^{-1} \sum_{i \leq n} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) \right)^2 \right] \mathbb{E}[\bar{\varepsilon}_{2,t}^{*4}] \right)^{1/2} \leq Kn^{-3/2}.
\end{aligned}$$

Together with Markov's inequality, this implies that

$$\begin{aligned}
(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,t}^{*2} &= T^{-1} \sum_{t \leq T} \left(n^{-1} \sum_{i \leq n} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) \right) \bar{\varepsilon}_{2,t}^{*2} \\
&\quad + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\varepsilon_{2,i,t}^2] \bar{\varepsilon}_{2,t}^{*2} \\
&= T^{-1} \sum_{t \leq T} s_{2,t}^2 \bar{\varepsilon}_{2,t}^{*2} + O_p(n^{-3/2}),
\end{aligned} \tag{S344}$$

where $s_{2,t}^2 \equiv n^{-1} \sum_{i \leq n} s_{2,i,t}^2$. By the definition of $\bar{\varepsilon}_{2,t}^{*2}$, we can write

$$\bar{\varepsilon}_{2,t}^{*2} = n^{-2} \sum_{i \leq n} \varepsilon_{2,i,t}^2 + 2n^{-2} \sum_{i=2}^n \sum_{i'=1}^{i-1} \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^*. \tag{S345}$$

Using this expression, we may write

$$T^{-1} \sum_{t \leq T} s_{2,t}^2 \bar{\varepsilon}_{2,t}^{*2} = (n^2 T)^{-1} \sum_{t \leq T} \sum_{i \leq n} s_{2,t}^2 \varepsilon_{2,i,t}^{*2} + 2(n^2 T)^{-1} \sum_{t \leq T} \sum_{i=2}^n \sum_{i'=1}^{i-1} s_{2,t}^2 \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^*. \quad (\text{S346})$$

Under Assumption 1 and Lemma S34(ii), we have

$$\begin{aligned} & \mathbb{E} \left[\left((n^2 T)^{-1} \sum_{i \leq n} \sum_{t \leq T} s_{2,t}^2 (\varepsilon_{2,i,t}^{*2} - \sigma_{2,i,t}^2) \right)^2 \right] \\ &= (n^2 T)^{-2} \sum_{i \leq n} \mathbb{E} \left[\left(\sum_{t \leq T} s_{2,t}^2 (\varepsilon_{2,i,t}^{*2} - \sigma_{2,i,t}^2) \right)^2 \right] \\ &\leq K(n^4 T)^{-1} \sum_{i \leq n} \max_{t \leq T} \|s_{2,t}^2 (\varepsilon_{2,i,t}^{*2} - \sigma_{2,i,t}^2)\|_{2+\delta}^2 \leq K n^{-4} \end{aligned} \quad (\text{S347})$$

and

$$\begin{aligned} & \mathbb{E} \left[\left((n^2 T)^{-1} \sum_{i=2}^n \sum_{i'=1}^{i-1} \sum_{t \leq T} s_{2,t}^2 \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^* \right)^2 \right] \\ &= (n^2 T)^{-2} \sum_{i=2}^n \sum_{i'=1}^{i-1} \mathbb{E} \left[\left(\sum_{t \leq T} s_{2,t}^2 \varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^* \right)^2 \right] \\ &\leq K(n^4 T)^{-1} \sum_{i=2}^n \sum_{i'=1}^{i-1} \max_{t \leq T} \|\varepsilon_{2,i,t}^* \varepsilon_{2,i',t}^*\|_{2+\delta}^2 \\ &\leq K(n^4 T)^{-1} \sum_{i=2}^n \sum_{i'=1}^{i-1} \max_{t \leq T} \|\varepsilon_{2,i,t}^*\|_{4+2\delta}^2 \|\varepsilon_{2,i',t}^*\|_{4+2\delta}^2 \leq K n^{-3}. \end{aligned} \quad (\text{S348})$$

Therefore, by (S346)-(S348), and Markov's inequality,

$$T^{-1} \sum_{t \leq T} s_{2,t}^2 \bar{\varepsilon}_{2,t}^{*2} = (n^2 T)^{-1} \sum_{t \leq T} \sum_{i \leq n} s_{2,t}^2 \sigma_{2,i,t}^2 + O_p(n^{-3/2}). \quad (\text{S349})$$

Combining (S344), (S345) and (S349) yields

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,t}^{*2} = n^{-3} \sum_{i \leq n} \sum_{i' \leq n} \left(T^{-1} \sum_{t \leq T} s_{2,i,t}^2 \sigma_{2,i',t}^2 \right) + O_p(n^{-3/2}). \quad (\text{S350})$$

The third term in (S339) can be written as

$$\begin{aligned} (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^* \bar{\varepsilon}_{2,t}^* &= (nT)^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{i \leq n} \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*] \\ &\quad + (nT)^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{i \leq n} (\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^* - \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*]). \end{aligned} \quad (\text{S351})$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left((nT)^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{i \leq n} (\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^* - \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*]) \right)^2 \\ & \leq T^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^{*2} \times T^{-1} \sum_{t \leq T} \left(n^{-1} \sum_{i \leq n} (\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^* - \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*]) \right)^2. \end{aligned} \quad (\text{S352})$$

From Assumption 1, Lemma S34(ii) and (S310), it follows that

$$\begin{aligned} T^{-1} \sum_{t \leq T} \mathbb{E} \left[\left(n^{-1} \sum_{i \leq n} (\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^* - \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*]) \right)^2 \right] & \leq (n^2 T)^{-1} \sum_{t \leq T} \sum_{i \leq n} \mathbb{E}[\varepsilon_{2,i,t}^4 \bar{\varepsilon}_{2,i}^{*2}] \\ & \leq (n^2 T)^{-1} \sum_{t \leq T} \sum_{i \leq n} (\mathbb{E}[\varepsilon_{2,i,t}^8] \mathbb{E}[\bar{\varepsilon}_{2,i}^{*4}])^{1/2} \leq K n^{-2}. \end{aligned}$$

Hence by Markov's inequality,

$$T^{-1} \sum_{t \leq T} \left(n^{-1} \sum_{i \leq n} (\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^* - \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*]) \right)^2 = O_p(n^{-2}). \quad (\text{S353})$$

Since $T^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^{*2} = O_p(n^{-1})$ by (S310), from (S352) and (S353) it follows that

$$(nT)^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{i \leq n} (\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^* - \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*]) = O_p(n^{-3/2}). \quad (\text{S354})$$

Let $v_{2,t} \equiv \sum_{i \leq n} \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*]$. Then we may write:

$$(nT)^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{i \leq n} \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*] = (nT)^{-1} \sum_{t \leq T} v_{2,t} \bar{\varepsilon}_{2,t}^* = (n^2 T)^{-1} \sum_{i \leq n} \sum_{t \leq T} v_{2,t} \varepsilon_{2,i,t}^*.$$

Note that $\left| \sum_{i \leq n} \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*] \right| \leq K$ by Assumption 1(i), (S310) and Lemma S34(ii). Therefore,

$$\begin{aligned} \mathbb{E} \left[\left((nT)^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{i \leq n} \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*] \right)^2 \right] & = (n^2 T)^{-2} \mathbb{E} \left[\left(\sum_{i \leq n} \sum_{t \leq T} v_{2,t} \varepsilon_{2,i,t}^* \right)^2 \right] \\ & = (n^2 T)^{-2} \sum_{i \leq n} \mathbb{E} \left[\left(\sum_{t \leq T} v_{2,t} \varepsilon_{2,i,t}^* \right)^2 \right] \\ & \leq K (n^4 T)^{-1} \sum_{i \leq n} \max_{t \leq T} \|v_{2,t} \varepsilon_{2,i,t}^*\|_{2+\delta}^2 \leq K n^{-4}, \end{aligned}$$

which together with Markov's inequality implies that

$$(nT)^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{i \leq n} \mathbb{E}[\varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^*] = O_p(n^{-2}). \quad (\text{S355})$$

From (S351), (S354) and (S355), we obtain

$$(nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^* \bar{\varepsilon}_{2,t}^* = O_p(n^{-3/2}). \quad (\text{S356})$$

The claim of the lemma now follows from (S339), (S343), (S350) and (S355). *Q.E.D.*

Lemma S26. *Suppose Assumptions 1, 4 and 5. Then*

$$\begin{aligned} (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t}^2 \bar{\varepsilon}_{1,g,t}^{*2} &= (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} s_{1,i,t}^2 \right) \left(n_g^{-1} \sum_{i \in I_g} \sigma_{1,i,t}^2 \right) \\ &\quad + O_p \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1/2} \right). \end{aligned}$$

PROOF OF LEMMA S26. We begin the proof by writing

$$\begin{aligned} (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t}^2 \bar{\varepsilon}_{1,g,t}^{*2} &= (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \mathbb{E}[\varepsilon_{1,i,t}^2] \bar{\varepsilon}_{1,g,t}^{*2} \\ &\quad + (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{1,i,t}^2 - \mathbb{E}[\varepsilon_{1,i,t}^2]) \bar{\varepsilon}_{1,g,t}^{*2}. \end{aligned} \quad (\text{S357})$$

By Hölder's inequality, the second term in (S357) satisfies

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{1,i,t}^2 - \mathbb{E}[\varepsilon_{1,i,t}^2]) \bar{\varepsilon}_{1,g,t}^{*2} \right| \right] &= \mathbb{E} \left[\left| \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \bar{\varepsilon}_{1,g,t}^{*2} \sum_{i \in I_g} (\varepsilon_{1,i,t}^2 - \mathbb{E}[\varepsilon_{1,i,t}^2]) \right| \right] \\ &\leq \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \mathbb{E} \left[\left| \bar{\varepsilon}_{1,g,t}^{*2} \sum_{i \in I_g} (\varepsilon_{1,i,t}^2 - \mathbb{E}[\varepsilon_{1,i,t}^2]) \right| \right] \\ &\leq \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \|\bar{\varepsilon}_{1,g,t}^{*2}\|_2 \left\| \sum_{i \in I_g} (\varepsilon_{1,i,t}^2 - \mathbb{E}[\varepsilon_{1,i,t}^2]) \right\|_2 \\ &\leq KT \sum_{g \in \mathcal{G}_1} n_g^{-1/2}, \end{aligned} \quad (\text{S358})$$

where the last inequality holds by Assumption 1, Lemma S34(ii) and (S310). Therefore,

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{1,i,t}^2 - \mathbb{E}[\varepsilon_{1,i,t}^2]) \bar{\varepsilon}_{1,g,t}^{*2} = O_p \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1/2} \right). \quad (\text{S359})$$

The first term in (S357) satisfies

$$\begin{aligned}
\mathbb{E} \left[\left(\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \mathbb{E}[\varepsilon_{1,i,t}^2] (\bar{\varepsilon}_{1,g,t}^{*2} - \mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*2}]) \right)^2 \right] &= \mathbb{E} \left[\left(\sum_{g \in \mathcal{G}_1} \sum_{t \leq T} (\bar{\varepsilon}_{1,g,t}^{*2} - \mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*2}]) \sum_{i \in I_g} \mathbb{E}[\varepsilon_{1,i,t}^2] \right)^2 \right] \\
&= \sum_{g \in \mathcal{G}_1} \mathbb{E} \left[\left(\sum_{t \leq T} (\bar{\varepsilon}_{1,g,t}^{*2} - \mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*2}]) \sum_{i \in I_g} \mathbb{E}[\varepsilon_{1,i,t}^2] \right)^2 \right] \\
&\leq TK \sum_{g \in \mathcal{G}_1} \max_{t \leq T} \left\| (\bar{\varepsilon}_{1,g,t}^{*2} - \mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*2}]) \sum_{i \in I_g} \mathbb{E}[\varepsilon_{1,i,t}^2] \right\|_{2+\delta}^2 \\
&\leq KT \sum_{g \in \mathcal{G}_1} n_g^2 \|\bar{\varepsilon}_{1,g,t}^{*2}\|_{2+\delta}^2 \leq KG_1 T,
\end{aligned}$$

which, together with Assumption 1(i) and Markov's inequality, implies

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \mathbb{E}[\varepsilon_{1,i,t}^2] (\bar{\varepsilon}_{1,g,t}^{*2} - \mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*2}]) = O_p(G_1^{1/2} n^{-3/2}). \quad (\text{S360})$$

It is clear that $\mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*2}] = n_g^{-2} \sum_{i \in I_g} \sigma_{1,i,t}^2$. Hence

$$\begin{aligned}
(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\varepsilon_{1,i,t}^2] \mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*2}] &= (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*2}] \sum_{i \in I_g} \mathbb{E}[\varepsilon_{1,i,t}^2] \\
&= (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} s_{1,i,t}^2 \right) \left(n_g^{-1} \sum_{i \in I_g} \sigma_{1,i,t}^2 \right). \quad (\text{S361})
\end{aligned}$$

The claim of the lemma follows from (S357), (S359), (S360) and (S361). $Q.E.D.$

Lemma S27. *Suppose Assumptions 1, 4 and 5 hold. Then*

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t} \varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^*) \bar{\varepsilon}_{1,g,t}^* = (n^2 T)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g s_{1,2,g,t} \sigma_{1,2,g,t} + O_p(G_1 n^{-3/2}).$$

PROOF OF LEMMA S27. Let $\bar{v}_{1,g,t} \equiv n_g^{-1} \sum_{i \in I_g} \varepsilon_{1,i,t} \varepsilon_{2,i,t} \bar{\varepsilon}_{2,i}^*$ and $\bar{v}_{1,g,t}^* \equiv \bar{v}_{1,g,t} - \mathbb{E}[\bar{v}_{1,g,t}]$. Then

$$\begin{aligned}
\sum_{g \in \mathcal{G}_1} \sum_{t \leq T} \sum_{i \in I_g} \varepsilon_{1,i,t} \varepsilon_{2,i,t} \bar{\varepsilon}_{2,i}^* \bar{\varepsilon}_{1,g,t}^* &= \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \bar{v}_{1,g,t} \bar{\varepsilon}_{1,g,t}^* \\
&= \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \mathbb{E}[\bar{v}_{1,g,t}] \bar{\varepsilon}_{1,g,t}^* + \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \bar{v}_{1,g,t}^* \bar{\varepsilon}_{1,g,t}^*. \quad (\text{S362})
\end{aligned}$$

From Assumption 1, Lemma S34(ii) and Lemma S40(iii),

$$\mathbb{E}[\bar{v}_{1,g,t}^{*2}] \leq n_g^{-2} \sum_{i \in I_g} \mathbb{E}[\varepsilon_{1,i,t}^2 \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{2,i}^{*2}] \leq K n_g^{-2} \sum_{i \in I_g} \|\bar{\varepsilon}_{2,i}^{*2}\|_2 \leq K n_g^{-1} T^{-1}. \quad (\text{S363})$$

This, together with (S310) and Hölder's inequality, implies that

$$\sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \mathbb{E}[|\bar{v}_{1,g,t}^* \bar{\varepsilon}_{1,g,t}^*|] \leq \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \|\bar{v}_{1,g,t}^*\|_2 \|\bar{\varepsilon}_{1,g,t}^*\|_2 \leq K G_1 T^{1/2}.$$

Therefore, by Assumption 1(i) and Markov's inequality,

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \bar{v}_{1,g,t}^* \bar{\varepsilon}_{1,g,t}^* = O_p(G_1 n^{-3/2}). \quad (\text{S364})$$

From Assumption 1, Lemma S34(ii) and Lemma S40(iii),

$$|\mathbb{E}[\bar{v}_{1,g,t}]| \leq n_g^{-1} \sum_{i \in I_g} \mathbb{E}[|\varepsilon_{1,i,t} \varepsilon_{2,i,t} \bar{\varepsilon}_{2,i}^*|] \leq n_g^{-1} \sum_{i \in I_g} \|\varepsilon_{1,i,t} \varepsilon_{2,i,t}\|_2 \|\bar{\varepsilon}_{2,i}^*\|_2 \leq K T^{-1/2},$$

which, together with Assumption 1 and (S310), implies that

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \mathbb{E}[\bar{v}_{1,g,t}] \bar{\varepsilon}_{1,g,t}^* \right)^2 \right] &= \sum_{g \in \mathcal{G}_1} n_g^2 \mathbb{E} \left[\left(\sum_{t \leq T} \mathbb{E}[\bar{v}_{1,g,t}] \bar{\varepsilon}_{1,g,t}^* \right)^2 \right] \\ &= \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \mathbb{E} \left[\left(\sum_{t \leq T} \mathbb{E}[\bar{v}_{1,g,t}] \varepsilon_{1,i,t}^* \right)^2 \right] \\ &\leq K T \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \max_{t \leq T} \|\mathbb{E}[\bar{v}_{1,g,t}] \varepsilon_{1,i,t}^*\|_{2+\delta}^2 \leq K n. \end{aligned}$$

Therefore, by Assumption 1(i) and Markov's inequality,

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} n_g \mathbb{E}[\bar{v}_{1,g,t}] \bar{\varepsilon}_{1,g,t}^* = O_p(n^{-3/2}), \quad (\text{S365})$$

From (S362), (S364) and (S365), it follows that

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} \sum_{i \in I_g} \varepsilon_{1,i,t} \varepsilon_{2,i,t} \bar{\varepsilon}_{2,i}^* \bar{\varepsilon}_{1,g,t}^* = O_p(G_1 n^{-3/2}). \quad (\text{S366})$$

Let $\bar{v}_{2,g,t} \equiv n_g^{-1} \sum_{i \in I_g} \varepsilon_{1,i,t} \varepsilon_{2,i,t}$ and $\bar{v}_{2,g,t}^* \equiv \bar{v}_{2,g,t} - \mathbb{E}[\bar{v}_{2,g,t}]$. Then

$$\begin{aligned} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \varepsilon_{1,i,t} \varepsilon_{2,i,t} \bar{\varepsilon}_{2,t}^* \bar{\varepsilon}_{1,g,t}^* &= \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{g \in \mathcal{G}_1} n_g \bar{\varepsilon}_{1,g,t}^* \bar{v}_{2,g,t} \\ &= \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{g \in \mathcal{G}_1} n_g \bar{\varepsilon}_{1,g,t}^* \mathbb{E}[\bar{v}_{2,g,t}] + \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{g \in \mathcal{G}_1} n_g \bar{\varepsilon}_{1,g,t}^* \bar{v}_{2,g,t}^* \end{aligned} \quad (\text{S367})$$

Using Assumption 1, Lemma S34(ii) and Lemma S40(iii),

$$\mathbb{E}[\bar{v}_{2,g,t}^{*2}] \leq n_g^{-2} \sum_{i \in I_g} \mathbb{E}[|\varepsilon_{1,i,t} \varepsilon_{2,i,t}|^2] \leq K n_g^{-1}. \quad (\text{S368})$$

This, together with (S310) and Hölder's inequality, implies that

$$\sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g \mathbb{E}[|\bar{\varepsilon}_{2,t}^* \bar{\varepsilon}_{1,g,t}^* \bar{v}_{2,g,t}^*|] \leq \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g \|\bar{v}_{2,g,t}^*\|_2 \|\bar{\varepsilon}_{2,t}^*\|_4 \|\bar{\varepsilon}_{1,g,t}^*\|_4 \leq K G_1 T n^{-1/2}.$$

Therefore, by Assumption 1(i) and Markov's inequality, the second term in (S367) satisfies

$$(nT)^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{g \in \mathcal{G}_1} n_g \bar{\varepsilon}_{1,g,t}^* \bar{v}_{2,g,t}^* = O_p(G_1 n^{-3/2}). \quad (\text{S369})$$

Note that $\mathbb{E}[\bar{v}_{2,g,t}] = s_{1,2,g,t}$. Let $\tilde{\varepsilon}_{1,i,t}^* \equiv s_{1,2,g,t} \varepsilon_{1,i,t}^*$. Then the first term in (S367) can be written as

$$\begin{aligned} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^* \sum_{g \in \mathcal{G}_1} n_g \bar{\varepsilon}_{1,g,t}^* \mathbb{E}[\bar{v}_{2,g,t}] &= n^{-1} \sum_{t \leq T} \left(\sum_{i \leq n} \tilde{\varepsilon}_{1,i,t}^* \right) \left(\sum_{i \leq n} \varepsilon_{2,i,t}^* \right) \\ &= n^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i,t}^* \\ &\quad + n^{-1} \sum_{i'=2}^n \sum_{i=1}^{i-1} \sum_{t \leq T} (\tilde{\varepsilon}_{1,i',t}^* \varepsilon_{2,i,t}^* + \tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i',t}^*). \end{aligned} \quad (\text{S370})$$

Under Assumption 1, Lemma S34(ii) and Lemma S40(iii), we have

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{i \leq n} \sum_{t \leq T} (\tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i,t}^* - \mathbb{E}[\tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i,t}^*]) \right)^2 \right] &= \sum_{i \leq n} \mathbb{E} \left[\left(\sum_{t \leq T} (\tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i,t}^* - \mathbb{E}[\tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i,t}^*]) \right)^2 \right] \\ &\leq K T \sum_{i \leq n} \max_{t \leq T} \|\tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i,t}^*\|_{2+\delta}^2 \leq K n T, \end{aligned}$$

which together with Assumption 1(i) and Markov's inequality implies that

$$\begin{aligned} (n^2 T)^{-1} \sum_{i \leq n} \sum_{t \leq T} \tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i,t}^* &= (n^2 T)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i,t}^*] + O_p(n^{-2}) \\ &= (n^2 T)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g s_{1,2,g,t} \sigma_{1,2,g,t} + O_p(n^{-2}). \end{aligned} \quad (\text{S371})$$

Under Assumption 1, Lemma S34(ii) and Lemma S40(iii), we have

$$\mathbb{E} \left[\left(\sum_{i'=2}^n \sum_{i=1}^{i-1} \sum_{t \leq T} \tilde{\varepsilon}_{1,i',t}^* \varepsilon_{2,i,t}^* \right)^2 \right] = \sum_{i'=2}^n \sum_{i=1}^{i-1} \mathbb{E} \left[\left(\sum_{t \leq T} \tilde{\varepsilon}_{1,i',t}^* \varepsilon_{2,i,t}^* \right)^2 \right] \leq K n^2 T. \quad (\text{S372})$$

Similarly,

$$\mathbb{E} \left[\left(\sum_{i'=2}^n \sum_{i=1}^{i-1} \sum_{t \leq T} \tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i',t}^* \right)^2 \right] \leq K n^2 T$$

which together with Assumption 1(i), (S372) and Markov's inequality implies that

$$(n^2T)^{-1} \sum_{i'=2}^n \sum_{i=1}^{i-1} \sum_{t \leq T} (\tilde{\varepsilon}_{1,i',t}^* \varepsilon_{2,i,t}^* + \tilde{\varepsilon}_{1,i,t}^* \varepsilon_{2,i',t}^*) = O_p(n^{-3/2}). \quad (\text{S373})$$

From (S370), (S371) and (S373), it follows that

$$(nT)^{-1} \sum_{t \leq T} \tilde{\varepsilon}_{2,t}^* \sum_{g \in \mathcal{G}_1} n_g \tilde{\varepsilon}_{1,g,t}^* \mathbb{E}[\bar{v}_{2,g,t}] = (n^2T)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g s_{1,2,g,t} \sigma_{1,2,g,t} + O_p(n^{-3/2}).$$

This, along with (S369), establishes the claim of the lemma. Q.E.D.

Lemma S28. *Suppose Assumptions 1, 4, 5 and 6 hold. Then*

$$\frac{\widehat{\mathbb{E}[S_{n,T}]} - \mathbb{E}[S_{n,T}]}{\omega_{n,T}} = o_p(1). \quad (\text{S374})$$

PROOF OF LEMMA 9. Under Assumption 6(i),

$$\begin{aligned} \mathbb{E}[S_{n,T}] &= (2nT)^{-1/2} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \mathbb{E} \left[\sum_{t \leq T} (n_g^{-1} \varepsilon_{1,i,t}^{*2} - n^{-1} \varepsilon_{2,i,t}^{*2}) - T^{-1} \left(\sum_{t \leq T} \varepsilon_{2,i,t}^* \right)^2 \right] \\ &= (2nT)^{-1/2} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \left[\sum_{t \leq T} (n_g^{-1} \sigma_{1,i}^2 - n^{-1} \sigma_{2,i}^2) - \sigma_{2,i}^2 \right] \\ &= (2nT)^{-1/2} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} (Tn_g^{-1} \sigma_{1,i}^2 - (1 + Tn^{-1}) \sigma_{2,i}^2). \end{aligned}$$

Therefore,

$$\widehat{\mathbb{E}[S_{n,T}]} - \mathbb{E}[S_{n,T}] = (2nT)^{-1/2} \left(\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} Tn_g^{-1} (\hat{\sigma}_{1,i}^2 - \sigma_{1,i}^2) - (1 + Tn^{-1}) \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} (\hat{\sigma}_{2,i}^2 - \sigma_{2,i}^2) \right). \quad (\text{S375})$$

Under Assumption 6(i) and (S425), we have $\mathbb{E}[\varepsilon_{2,i,t}] = 0$. This implies that $\sigma_{2,i}^2 = s_{2,i}^2$. Similarly, under Assumption 6(i) and (S492), $\sigma_{1,i}^2 = s_{1,i}^2$. Therefore, by Assumption 1(i), Lemmas S38 and S44,

$$\widehat{\mathbb{E}[S_{n,T}]} - \mathbb{E}[S_{n,T}] = O_p \left(n^{-1} + \sum_{g \in \mathcal{G}_1} n_g^{-1} \right). \quad (\text{S376})$$

The claim of the lemma follows from Assumptions 5(i) and 6(ii), and (S376). Q.E.D.

Lemma S29. *Suppose Assumptions 1, 4, 5 and 6 hold. Then under the null hypothesis,*

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \frac{\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2}{2} - (nT)^{-1/2} \widehat{\mathbb{E}[S_{n,T}]} = O_p(n^{-1}).$$

PROOF OF LEMMA S29. From Theorem 6 and Lemma S28, we have

$$\begin{aligned}
& (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \frac{\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2}{2} - (nT)^{-1/2} \widehat{\mathbb{E}[S_{n,T}]} \\
&= \frac{\mathbb{E}[S_{n,T}] - \widehat{\mathbb{E}[S_{n,T}]} + \overline{QLR}_{n,T}}{(nT)^{1/2}} + \frac{QLR_{n,T} - \mathbb{E}[S_{n,T}] - \overline{QLR}_{n,T}}{(nT)^{1/2}} \\
&= \frac{\mathbb{E}[S_{n,T}] - \widehat{\mathbb{E}[S_{n,T}]} + \overline{QLR}_{n,T}}{(nT)^{1/2}} + O_p(\omega_{n,T}(nT)^{-1/2}) \\
&= (nT)^{-1/2} \overline{QLR}_{n,T} + O_p(\omega_{n,T}(nT)^{-1/2})
\end{aligned} \tag{S377}$$

where

$$\overline{QLR}_{n,T} = (nT)^{-1/2} \sum_{i \leq n} \sum_{t \leq T} \frac{s_{2,i,t}^2 - s_{1,i,t}^2}{2}.$$

Since $\overline{QLR}_{n,T} = 0$ under the null hypothesis, from (S377) it follows that

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \frac{\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2}{2} - (nT)^{-1/2} \widehat{\mathbb{E}[S_{n,T}]} = O_p(\omega_{n,T}(nT)^{-1/2}). \tag{S378}$$

Moreover, from Lemma 10(iii), we have $\omega_{n,T} = O(1)$. This along with Assumption 1(i) and (S378) establishes the claim of the lemma. Q.E.D.

Lemma S30. *Suppose Assumptions 1, 4, 5 and 6 hold. Then*

$$\frac{\hat{\sigma}_{n,T}^2 - (\sigma_{n,T}^2 + 2\sigma_{U,n,T}^2)}{\omega_{n,T}^2} = o_p(1),$$

under the null hypothesis.

PROOF OF LEMMA S30. Applying Assumptions 5(i) and 6(ii), Lemmas S22-S24, we obtain:

$$\begin{aligned}
(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2)^2 &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2 \\
&+ 4n^{-1} \sum_{i \leq n} s_{2,i}^2 \mathbb{E}[\varepsilon_{2,i}^{*2}] + 4n^{-3} \sum_{i \leq n} \sum_{i' \leq n} \left(T^{-1} \sum_{t \leq T} s_{2,i,t}^2 \sigma_{2,i',t}^2 \right) \\
&+ 4(nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} s_{1,i,t}^2 \right) \left(n_g^{-1} \sum_{i \in I_g} \sigma_{1,i,t}^2 \right) \\
&- 8(n^2 T)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g s_{1,2,g,t} \sigma_{1,2,g,t} + o_p(\omega_{n,T}^2).
\end{aligned} \tag{S379}$$

Under Assumption 6(i) and the null hypothesis, we have $s_{j,i,t}^2 = \sigma_{j,i,t}^2$ for $j = 1, 2$, $s_{1,2,g,t} = \sigma_{1,2,g,t}$ and $\mathbb{E}[\bar{\varepsilon}_{2,i}^{*2}] = T^{-1}\sigma_{2,i}^2$. Hence,

$$\begin{aligned}
(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2)^2 &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2 \\
&\quad + 4(nT)^{-1} \sum_{i \leq n} \sigma_{2,i}^4 + 4n^{-3} \left(\sum_{i \leq n} \sigma_{2,i}^2 \right)^2 + 4n^{-1} \sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} \sigma_{1,i}^2 \right)^2 \\
&\quad - 8n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sigma_{1,2,i} \right)^2 + o_p(\omega_{n,T}^2) \\
&= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2 + 8\sigma_{U,n,T}^2 + o_p(\omega_{n,T}^2). \tag{S380}
\end{aligned}$$

By Assumptions 1, 5(ii) and 6(i),

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2 = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2] + O_p(\omega_{n,T}^2 n^{-1}),$$

which, together with (S380), yields

$$(4nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2)^2 = (4n)^{-1} \sum_{i \leq n} \mathbb{E}[(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2] + 2\sigma_{U,n,T}^2 + o_p(\omega_{n,T}^2). \tag{S381}$$

Under the null hypothesis $\sigma_{n,T}^2 = (4n)^{-1} \sum_{i \leq n} \mathbb{E}[(\varepsilon_{2,i,t}^2 - \varepsilon_{1,i,t}^2)^2]$, so (S381) implies

$$(4nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2)^2 = \sigma_{n,T}^2 + 2\sigma_{U,n,T}^2 + o_p(\omega_{n,T}^2). \tag{S382}$$

Moreover, Lemma S29 implies

$$(nT)^{-1/2} MQLR_{n,T} = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \frac{\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2}{2} - (nT)^{-1/2} \widehat{\mathbb{E}[S_{n,T}]} = O_p(n^{-1}),$$

which, together with (S382), yields:

$$\begin{aligned}
\hat{\sigma}_{n,T}^2 &\equiv (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \frac{(\hat{\varepsilon}_{2,i,t}^2 - \hat{\varepsilon}_{1,i,t}^2)^2}{4} - (nT)^{-1} (MQLR_{n,T})^2 \\
&= \sigma_{n,T}^2 + 2\sigma_{U,n,T}^2 + O_p(n^{-2}) + o_p(\omega_{n,T}^2) = \sigma_{n,T}^2 + 2\sigma_{U,n,T}^2 + o_p(\omega_{n,T}^2) \tag{S383}
\end{aligned}$$

where the second equality follows from Assumption 5(i). The claim of the lemma follows immediately from (S383). Q.E.D.

Lemma S31. *Suppose Assumptions 1, 4 and 5 hold. Then*

$$n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} (\hat{\sigma}_{1,2,i} - s_{1,2,i}) \right)^2 = O_p(G_1 n^{-1}).$$

PROOF OF LEMMA S31. By the definition of $\hat{\sigma}_{1,2,i}$,

$$\begin{aligned}
\hat{\sigma}_{1,2,i} - s_{1,2,i} &= T^{-1} \sum_{t \leq T} \hat{\varepsilon}_{1,i,t} \hat{\varepsilon}_{2,i,t} - T^{-1} \sum_{t \leq T} \mathbb{E}[\varepsilon_{1,i,t} \varepsilon_{2,i,t}] \\
&= T^{-1} \sum_{t \leq T} (\dot{\varepsilon}_{1,i,t} \ddot{\varepsilon}_{2,i,t} - \mathbb{E}[\varepsilon_{1,i,t} \varepsilon_{2,i,t}]) \\
&\quad - T^{-1} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) - T^{-1} \sum_{t \leq T} \dot{\varepsilon}_{1,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) \\
&\quad + (\hat{\theta}_2 - \theta_2^*)^\top \left(T^{-1} \sum_{t \leq T} \ddot{x}_{i,t} \dot{x}_{i,t}^\top \right) (\hat{\theta}_1 - \theta_1^*)
\end{aligned} \tag{S384}$$

It follows that

$$\begin{aligned}
n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} (\hat{\sigma}_{1,2,i} - s_{1,2,i}) \right)^2 &\leq K(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} (\dot{\varepsilon}_{1,i,t} \ddot{\varepsilon}_{2,i,t} - \mathbb{E}[\varepsilon_{1,i,t} \varepsilon_{2,i,t}]) \right)^2 \\
&\quad + K(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \right)^2 \\
&\quad + K(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \dot{\varepsilon}_{1,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) \right)^2 \\
&\quad + K(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left((\hat{\theta}_2 - \theta_2^*)^\top \sum_{i \in I_g} \sum_{t \leq T} \ddot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \right)^2.
\end{aligned} \tag{S385}$$

By Assumptions 1(i) and 4(ii), Lemma S36 and Lemma S42, the Cauchy-Schwarz inequality and Markov's inequality,

$$\begin{aligned}
&(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left((\hat{\theta}_2 - \theta_2^*)^\top \sum_{i \in I_g} \sum_{t \leq T} \ddot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \right)^2 \\
&\leq K \|\hat{\theta}_1 - \theta_1^*\|^2 \|\hat{\theta}_2 - \theta_2^*\|^2 (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \|\ddot{x}_{i,t} \dot{x}_{i,t}^\top\|^2 = O_p(n^{-4}).
\end{aligned} \tag{S386}$$

Similarly,

$$(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \right)^2 \leq \frac{\|\hat{\theta}_1 - \theta_1^*\|^2}{nT} \sum_{i \leq n} \sum_{t \leq T} \|\ddot{\varepsilon}_{2,i,t} \dot{x}_{i,t}^\top\|^2 = O_p(n^{-2}) \tag{S387}$$

and

$$(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \dot{\varepsilon}_{1,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) \right)^2 \leq \frac{\|\hat{\theta}_1 - \theta_1^*\|^2}{nT} \sum_{i \leq n} \sum_{t \leq T} \|\dot{\varepsilon}_{1,i,t} \ddot{x}_{i,t}^\top\|^2 = O_p(n^{-2}). \quad (\text{S388})$$

Next, using the definitions of $\dot{\varepsilon}_{1,i,t}$ and $\ddot{\varepsilon}_{2,i,t}$,

$$\begin{aligned} \sum_{t \leq T} \dot{\varepsilon}_{1,i,t} \ddot{\varepsilon}_{2,i,t} &= \sum_{t \leq T} (\varepsilon_{2,i,t} - \bar{\varepsilon}_{2,i} - \bar{\varepsilon}_{2,t} + \bar{\varepsilon}_2) (\varepsilon_{1,i,t} - \bar{\varepsilon}_{1,g,t}) \\ &= \sum_{t \leq T} \varepsilon_{1,i,t} \varepsilon_{2,i,t} - T^{-1} \sum_{t \leq T} \varepsilon_{1,i,t} (\bar{\varepsilon}_{2,i} + \bar{\varepsilon}_{2,t} - \bar{\varepsilon}_2) \\ &\quad - \sum_{t \leq T} \varepsilon_{2,i,t} \bar{\varepsilon}_{1,g,t} + T^{-1} \sum_{t \leq T} \bar{\varepsilon}_{1,g,t} (\bar{\varepsilon}_{2,i} + \bar{\varepsilon}_{2,t} - \bar{\varepsilon}_2), \end{aligned} \quad (\text{S389})$$

and hence

$$\begin{aligned} &(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} (\dot{\varepsilon}_{1,i,t} \ddot{\varepsilon}_{2,i,t} - \mathbb{E}[\varepsilon_{1,i,t} \varepsilon_{2,i,t}]) \right)^2 \\ &\leq K(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{1,i,t} \varepsilon_{2,i,t} - \mathbb{E}[\varepsilon_{1,i,t} \varepsilon_{2,i,t}]) \right)^2 \\ &\quad + K(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t} (\bar{\varepsilon}_{2,i} + \bar{\varepsilon}_{2,t} - \bar{\varepsilon}_2) \right)^2 \\ &\quad + K(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{2,i,t} \bar{\varepsilon}_{1,g,t} \right)^2 \\ &\quad + K(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \bar{\varepsilon}_{1,g,t} (\bar{\varepsilon}_{2,i} + \bar{\varepsilon}_{2,t} - \bar{\varepsilon}_2) \right)^2. \end{aligned} \quad (\text{S390})$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} &(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t} (\bar{\varepsilon}_{2,i} + \bar{\varepsilon}_{2,t} - \bar{\varepsilon}_2) \right)^2 \\ &= (nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) \right)^2 \\ &\leq (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t}^2 (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 = O_p(n^{-1}), \end{aligned} \quad (\text{S391})$$

where the first equality follows from (S425), and the last from Lemma S40(iii), (S310), Hölder's

inequality and Markov's inequality. Similarly,

$$\begin{aligned}
& (nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{2,i,t} \bar{\varepsilon}_{1,g,t} \right)^2 \\
&= (nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{2,i,t} \bar{\varepsilon}_{1,g,t}^* \right)^2 \\
&\leq (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{2,i,t}^2 \bar{\varepsilon}_{1,g,t}^{*2} = O_p(G_1 n^{-1})
\end{aligned} \tag{S392}$$

and

$$\begin{aligned}
& (nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \bar{\varepsilon}_{1,g,t} (\bar{\varepsilon}_{2,i} + \bar{\varepsilon}_{2,t} - \bar{\varepsilon}_2) \right)^2 \\
&= (nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} \bar{\varepsilon}_{1,g,t}^* (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) \right)^2 \\
&\leq (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \bar{\varepsilon}_{1,g,t}^{*2} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 \leq O_p(G_1 n^{-2}).
\end{aligned} \tag{S393}$$

Finally, by Assumption 1, Lemma S34(ii) and Lemma S40(iii),

$$(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{1,i,t} \varepsilon_{2,i,t} - \mathbb{E}[\varepsilon_{1,i,t} \varepsilon_{2,i,t}]) \right)^2 = O_p(G_1 n^{-2}). \tag{S394}$$

Combining (S390)-(S394) yields

$$(nT^2)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} \sum_{t \leq T} (\hat{\varepsilon}_{1,i,t} \hat{\varepsilon}_{2,i,t} - \mathbb{E}[\varepsilon_{1,i,t} \varepsilon_{2,i,t}]) \right)^2 = O_p(G_1 n^{-1}). \tag{S395}$$

The claim of the lemma follows from (S385)-(S388), and (S395).

Q.E.D.

Lemma S32. *Suppose Assumptions 1, 4 and 5 hold. Then $\hat{\sigma}_{U,n,T}^2 - s_{U,n,T}^2 = O_p(n^{-1}(\sum_{g \in \mathcal{G}_1} n_g^{-1})^{1/2})$, where*

$$\begin{aligned}
s_{U,n,T}^2 &\equiv (2nT)^{-1} \sum_{i \leq n} s_{2,i}^4 + (2n)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-2} \left(\sum_{i \in I_g} s_{1,i}^2 \right)^2 \\
&\quad + (2n^3)^{-1} \left(\sum_{i \leq n} s_{2,i}^2 \right)^2 - n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} s_{1,2,i} \right)^2.
\end{aligned}$$

PROOF OF LEMMA S32. From the definitions of $\hat{\sigma}_{U,n,T}^2$ and $\sigma_{U,n,T}^2$, we can write

$$\begin{aligned}
\hat{\sigma}_{U,n,T}^2 - s_{U,n,T}^2 &= (2nT)^{-1} \sum_{i \leq n} (\hat{\sigma}_{2,i}^4 - s_{2,i}^4) + (2n)^{-1} \left(\left(n^{-1} \sum_{i \leq n} \hat{\sigma}_{2,i}^2 \right)^2 - \left(n^{-1} \sum_{i \leq n} s_{2,i}^2 \right)^2 \right) \\
&\quad + (2n)^{-1} \sum_{g \in \mathcal{G}_1} \left(\left(n_g^{-1} \sum_{i \in I_g} \hat{\sigma}_{1,i}^2 \right)^2 - \left(n_g^{-1} \sum_{i \in I_g} s_{1,i}^2 \right)^2 \right) \\
&\quad + n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\left(\sum_{i \in I_g} \hat{\sigma}_{1,2,i} \right)^2 - \left(\sum_{i \in I_g} s_{1,2,i} \right)^2 \right). \tag{S396}
\end{aligned}$$

From Lemma S34(ii), $s_{2,i}^4 \leq K$. Hence, by Lemma S39 and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left| (nT)^{-1} \sum_{i \leq n} (\hat{\sigma}_{2,i}^4 - s_{2,i}^4) \right| &\leq 2 \left((nT)^{-1} \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2)^2 \right)^{1/2} \left((nT)^{-1} \sum_{i \leq n} s_{2,i}^4 \right)^{1/2} \\
&\quad + (nT)^{-1} \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2)^2 = O_p((nT^2)^{-1/2}). \tag{S397}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left| n^{-1} \left(\left(n^{-1} \sum_{i \leq n} \hat{\sigma}_{2,i}^2 \right)^2 - \left(n^{-1} \sum_{i \leq n} s_{2,i}^2 \right)^2 \right) \right| \\
&= n^{-3} \left| \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2) \times \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 + s_{2,i}^2) \right| \\
&\leq n^{-1} \left(n^{-1} \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2) \right)^2 \\
&\quad + 2n^{-3} \left| \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2) \times \sum_{i \leq n} s_{2,i}^2 \right| = O_p(n^{-2}). \tag{S398}
\end{aligned}$$

Next, by the triangle inequality, the Cauchy-Schwarz inequality, and Lemma S39,

$$\begin{aligned}
&\left| n^{-1} \sum_{g \in \mathcal{G}_1} \left(\left(n_g^{-1} \sum_{i \in I_g} \hat{\sigma}_{1,i}^2 \right)^2 - \left(n_g^{-1} \sum_{i \in I_g} s_{1,i}^2 \right)^2 \right) \right| \\
&\leq 2n^{-1} \left(\sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} (\hat{\sigma}_{1,i}^2 - s_{1,i}^2) \right)^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} s_{1,i}^2 \right)^2 \right)^{1/2} \\
&\quad + n^{-1} \sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} (\hat{\sigma}_{1,i}^2 - s_{1,i}^2) \right)^2 = O_p \left(n^{-1} \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} \right)^{1/2} \right) \tag{S399}
\end{aligned}$$

Finally, by Lemma S31,

$$\begin{aligned}
& \left| n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\left(\sum_{i \in I_g} \hat{\sigma}_{1,2,i} \right)^2 - \left(\sum_{i \in I_g} s_{1,2,i} \right)^2 \right) \right| \\
& \leq 2n^{-1} \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} (\hat{\sigma}_{1,2,i} - s_{1,2,i}) \right)^2 \right)^{1/2} \left(n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} s_{1,2,i} \right)^2 \right)^{1/2} \\
& \quad + n^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \left(\sum_{i \in I_g} (\hat{\sigma}_{1,2,i} - s_{1,2,i}) \right)^2 = O_p(G_1^{1/2} n^{-3/2}). \tag{S400}
\end{aligned}$$

Since $\sum_{g \in \mathcal{G}_1} n_g^{-1} \geq G_1 n^{-1}$, the claim of the lemma follows from (S396)-(S400). Q.E.D.

2.2 Properties of Estimators in the Two Way Linear Model

Throughout this section, we let $x_{j,i,t}$, $\bar{x}_{j,i}$, $\bar{x}_{j,t}$ and $\bar{x}_j$ denote the j th entries of $x_{i,t}$, $\bar{x}_i$, $\bar{x}_t$ and $\bar{x}$ for $j = 1, \dots, d_x$ respectively.

Lemma S33. *Under Assumptions 1 and 4(ii), we have:*

- (i) $(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (x_{i,t} x_{i,t}^\top - \mathbb{E}[x_{i,t} x_{i,t}^\top]) = O_p((nT)^{-1/2})$;
- (ii) $n^{-1} \sum_{i \leq n} (\bar{x}_i \bar{x}_i^\top - \mathbb{E}[\bar{x}_i] \mathbb{E}[\bar{x}_i^\top]) = O_p((nT)^{-1/2})$;
- (iii) $T^{-1} \sum_{t \leq T} (\bar{x}_t \bar{x}_t^\top - \mathbb{E}[\bar{x}_t] \mathbb{E}[\bar{x}_t^\top]) = O_p((nT)^{-1/2})$;
- (iv) $\bar{x} \bar{x}^\top = \mathbb{E}[\bar{x}] \mathbb{E}[\bar{x}^\top] + O_p((nT)^{-1/2})$.

PROOF OF LEMMA S33. (i) Given Assumptions 1(ii, iii, iv), 4(ii) and Corollary A.2. in Hall and Heyde (1980), we obtain

$$\begin{aligned}
& \mathbb{E} \left[\left| (nT)^{-1/2} \sum_{i \leq n} \sum_{t \leq T} (x_{j,i,t} x_{j',i,t} - \mathbb{E}[x_{j,i,t} x_{j',i,t}]) \right|^2 \right] \\
& = n^{-1} \sum_{i \leq n} \mathbb{E} \left[\left| T^{-1/2} \sum_{t \leq T} (x_{j,i,t} x_{j',i,t} - \mathbb{E}[x_{j,i,t} x_{j',i,t}]) \right|^2 \right] \\
& \leq \max_{i \leq n} T^{-1} \sum_{t_1, t_2 \leq T} \mathbb{E} [(x_{j,i,t_1} x_{j',i,t_1} - \mathbb{E}[x_{j,i,t_1} x_{j',i,t_1}]) (x_{j,i,t_2} x_{j',i,t_2} - \mathbb{E}[x_{j,i,t_2} x_{j',i,t_2}])] \\
& \leq K \max_{i \leq n, t \leq T} \|x_{j,i,t} x_{j',i,t}\|_{2+\delta/2}^2 \sum_{h=0}^{\infty} |\alpha_i(h)|^{\delta/(4+\delta)} \leq K \max_{i \leq n, t \leq T} \|x_{j,i,t}\|_{4+\delta}^2 \|x_{j',i,t}\|_{4+\delta}^2 \leq K, \tag{S401}
\end{aligned}$$

for any $j, j' = 1, \dots, d_x$, which along with Markov's inequality, establishes the given claim.

(ii) We begin by observing that

$$\begin{aligned} n^{-1} \sum_{i \leq n} (\bar{x}_i \bar{x}_i^\top - \mathbb{E}[\bar{x}_i] \mathbb{E}[\bar{x}_i^\top]) &= n^{-1} \sum_{i \leq n} (\bar{x}_i - \mathbb{E}[\bar{x}_i]) (\bar{x}_i^\top - \mathbb{E}[\bar{x}_i^\top]) \\ &\quad + n^{-1} \sum_{i \leq n} \left(\mathbb{E}[\bar{x}_i] (\bar{x}_i^\top - \mathbb{E}[\bar{x}_i^\top]) + (\bar{x}_i - \mathbb{E}[\bar{x}_i]) \mathbb{E}[\bar{x}_i^\top] \right). \end{aligned} \quad (\text{S402})$$

Relying on Assumptions 1(ii, iv), 4(ii) and Rosenthal's inequality of strong mixing processes (e.g., (S556) in the proof of Lemma S47), we can proceed:

$$\begin{aligned} &\max_{i \leq n} \mathbb{E}[(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}])^4] \\ &\leq K \max_{i \leq n} \max \left\{ \left(T^{-2} \sum_{t \leq T} \|x_{j,i,t} - \mathbb{E}[x_{j,i,t}]\|_{2+\delta}^2 \right)^2, T^{-4} \sum_{t \leq T} \|x_{j,i,t} - \mathbb{E}[x_{j,i,t}]\|_{4+\delta}^4 \right\} \\ &\leq K \left(T^{-2} \max_{i \leq n, t \leq T} \|x_{j,i,t}\|_{2+\delta}^4 + T^{-3} \max_{i \leq n, t \leq T} \|x_{j,i,t}\|_{4+\delta}^4 \right) \leq KT^{-2}. \end{aligned} \quad (\text{S403})$$

Hence by the triangle inequality and Hölder's inequality,

$$\begin{aligned} &\mathbb{E} \left[\left| n^{-1} \sum_{i \leq n} (\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}]) (\bar{x}_{j',i} - \mathbb{E}[\bar{x}_{j',i}]) \right| \right] \\ &\leq n^{-1} \sum_{i \leq n} \mathbb{E} [|(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}]) (\bar{x}_{j',i} - \mathbb{E}[\bar{x}_{j',i}])|] \\ &\leq \max_{i \leq n} (\mathbb{E}[(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}])^2] \mathbb{E}[(\bar{x}_{j',i} - \mathbb{E}[\bar{x}_{j',i}])^2])^{1/2} \leq KT^{-1}, \end{aligned}$$

which together with Markov's inequality implies that the first term in the RHS of (S402) satisfies

$$n^{-1} \sum_{i \leq n} (\bar{x}_i - \mathbb{E}[\bar{x}_i]) (\bar{x}_i^\top - \mathbb{E}[\bar{x}_i^\top]) = O_p(T^{-1}). \quad (\text{S404})$$

We next analyze the term in the second line of (S402). Since by Assumption 4(ii) and the Cauchy-Schwarz inequality,

$$\max_{i \leq n, t \leq T} (|\mathbb{E}[\bar{x}_i]|^2 + |\mathbb{E}[\bar{x}_t]|^2 + |\mathbb{E}[\bar{x}]|^2) \leq 3 \max_{i \leq n, t \leq T} \mathbb{E}[|x_{i,t}|^2] \leq K, \quad (\text{S405})$$

we can employ Assumption 1(iii) and (S403) to show that

$$\begin{aligned} \mathbb{E} \left[\left| n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{x}_{j,i}] (\bar{x}_{j',i} - \mathbb{E}[\bar{x}_{j',i}]) \right|^2 \right] &= n^{-2} \sum_{i \leq n} (\mathbb{E}[\bar{x}_{j,i}]^2 \mathbb{E}[(\bar{x}_{j',i} - \mathbb{E}[\bar{x}_{j',i}])^2]) \\ &\leq n^{-1} \max_{i \leq n} ((\mathbb{E}[\bar{x}_{j,i}])^2 \mathbb{E}[(\bar{x}_{j',i} - \mathbb{E}[\bar{x}_{j',i}])^2]) \\ &\leq Kn^{-1} \max_{i \leq n} \mathbb{E}[(\bar{x}_{j',i} - \mathbb{E}[\bar{x}_{j',i}])^2] \leq K(nT)^{-1}. \end{aligned}$$

Hence by Markov's inequality,

$$n^{-1} \sum_{i \leq n} \left(\mathbb{E} [\bar{x}_i] (\bar{x}_i^\top - \mathbb{E} [\bar{x}_i^\top]) + (\bar{x}_i - \mathbb{E} [\bar{x}_i]) \mathbb{E} [\bar{x}_i^\top] \right) = O_p((nT)^{-1/2}). \quad (\text{S406})$$

Combining the results in (S402), (S404) and (S406), and applying Assumption 1(i) establishes the lemma claim in this part.

(iii) Let $x_{i,t}^* \equiv x_{i,t} - \mathbb{E} [x_{i,t}]$ and $x_{j,i,t}^*$ denote the j th entry of $x_{i,t}^*$. By the definition of $\bar{x}_t$, we can express

$$(\bar{x}_{j,t} - \mathbb{E} [\bar{x}_{j,t}]) (\bar{x}_{j',t} - \mathbb{E} [\bar{x}_{j',t}]) = n^{-2} \sum_{i \leq n} x_{j,i,t}^* x_{j',i,t}^* + n^{-2} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} (x_{j,i_1 t}^* x_{j',i_2 t}^* + x_{j',i_1 t}^* x_{j,i_2 t}^*). \quad (\text{S407})$$

Therefore

$$\begin{aligned} & T^{-1} \sum_{t \leq T} (\bar{x}_{j,t} - \mathbb{E} [\bar{x}_{j,t}]) (\bar{x}_{j',t} - \mathbb{E} [\bar{x}_{j',t}]) \\ &= (n^2 T)^{-1} \sum_{i \leq n} X_{j,i}^{*\top} X_{j',i}^* + (n^2 T)^{-1} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} (X_{j,i_1}^{*\top} X_{j',i_2}^* + X_{j',i_1}^{*\top} X_{j,i_2}^*), \end{aligned} \quad (\text{S408})$$

where $X_{j,i}^* \equiv (x_{j,i,t}^*)_{t \leq T}$ for $j = 1, \dots, d_x$. By Assumption 4(ii), the triangle inequality and Hölder's inequality,

$$\begin{aligned} \mathbb{E} \left[\left| (n^2 T)^{-1} \sum_{i \leq n} X_{j,i}^{*\top} X_{j',i}^* \right| \right] &\leq (n^2 T)^{-1} \sum_{i \leq n} \mathbb{E} [|X_{j,i}^{*\top} X_{j',i}^*|] \\ &\leq n^{-1} \max_{i \leq n, t \leq T} (\mathbb{E} [x_{j,i,t}^2] \mathbb{E} [x_{j',i,t}^2])^{1/2} \leq K n^{-1}. \end{aligned}$$

Hence by Markov's inequality,

$$(n^2 T)^{-1} \sum_{i \leq n} X_{j,i}^{*\top} X_{j',i}^* = O_p(n^{-1}). \quad (\text{S409})$$

We next turn to the second term in the RHS of (S408). By Assumption 1(iii),

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} X_{j,i_1}^{*\top} X_{j',i_2}^* \right|^2 \right] &= \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \mathbb{E} [X_{j,i_1}^{*\top} \text{Cov}(X_{j',i_2}^*) X_{j,i_1}^*] \\ &\leq K n^2 \left(\max_{i \leq n} \lambda_{\max}(\text{Cov}(X_{j',i}^*)) \right) \left(\max_{i \leq n} \mathbb{E} [\|X_{j,i}^*\|^2] \right), \end{aligned} \quad (\text{S410})$$

where

$$\max_{j \leq d_x} \max_{i \leq n} \mathbb{E} [\|X_{j,i}^*\|^2] \leq \max_{j \leq d_x} \max_{i \leq n} \sum_{t \leq T} \mathbb{E} [x_{j,i,t}^2] \leq K T \quad (\text{S411})$$

by Assumption 4(ii). By Assumptions 1(ii), (iv), and 4(ii), and Corollary A.2. in Hall and Heyde (1980),

$$\max_{i \leq n} \max_{t \leq T} \sum_{t'=1}^T |\mathbb{E}[x_{j,i,t}(x_{j,i,t'} - \mathbb{E}[x_{j,i,t'}])]| \leq \max_{i \leq n} \max_{t \leq T} \|x_{j,i,t}\|_{2+\delta/2}^2 \sum_{h=0}^{\infty} |\alpha_i(h)|^{\delta/(4+\delta)} \leq K.$$

Hence by Geršgorin's disc theorem

$$\max_{j \leq d_x} \max_{i \leq n} \lambda_{\max}(\text{Cov}(X_{j,i}^*)) \leq K, \quad (\text{S412})$$

which along with (S410) and (S411), and Markov's inequality shows that for any $j, j' = 1, \dots, d_x$

$$(n^2 T)^{-1} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} X_{j,i_1}^{*\top} X_{j',i_2}^* = O_p((n^2 T)^{-1/2}). \quad (\text{S413})$$

Combining the results in (S408), (S409) and (S413) yields

$$T^{-1} \sum_{t \leq T} (\bar{x}_{j,t} - \mathbb{E}[\bar{x}_{j,t}])(\bar{x}_{j',t} - \mathbb{E}[\bar{x}_{j',t}]) = O_p(n^{-1}). \quad (\text{S414})$$

Next note that

$$T^{-1} \sum_{t \leq T} \mathbb{E}[\bar{x}_{j,t}] (\bar{x}_{j',t} - \mathbb{E}[\bar{x}_{j',t}]) = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\bar{x}_{j,t}] x_{j',i,t}^*. \quad (\text{S415})$$

By Assumptions 1(ii, iii, iv) and 4(ii), (S405) and Corollary A.2. in Hall and Heyde (1980),

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\bar{x}_{j,t}] x_{j',i,t}^* \right|^2 \right] &= \sum_{i \leq n} \mathbb{E} \left[\left| \sum_{t \leq T} \mathbb{E}[\bar{x}_{j,t}] x_{j',i,t}^* \right|^2 \right] \\ &\leq n \max_{i \leq n} \sum_{t_1, t_2=1}^T \mathbb{E}[\bar{x}_{j,t_1}] \mathbb{E}[\bar{x}_{j,t_2}] \mathbb{E}[x_{j',i,t_1}^* x_{j',i,t_2}^*] \\ &\leq n \max_{j \leq d_x} \max_{t \leq T} (\mathbb{E}[\bar{x}_{j,t}])^2 \sum_{t_1, t_2=1}^T |\mathbb{E}[x_{j',i,t_1}^* x_{j',i,t_2}^*]| \\ &\leq K n T \max_{j \leq d_x} \max_{i \leq n, t \leq T} \|x_{j,i,t}\|_{2+\delta/2}^2 \sum_{h=0}^{\infty} |\alpha_i(h)|^{\delta/(4+\delta)} \leq K n T, \end{aligned} \quad (\text{S416})$$

which together with (S415) and Markov's inequality implies that

$$T^{-1} \sum_{t \leq T} \mathbb{E}[\bar{x}_{j,t}] (\bar{x}_{j',t} - \mathbb{E}[\bar{x}_{j',t}]) = O_p((nT)^{-1/2}). \quad (\text{S417})$$

Since

$$\begin{aligned} T^{-1} \sum_{t \leq T} (\bar{x}_t \bar{x}_t^\top - \mathbb{E}[\bar{x}_t] \mathbb{E}[\bar{x}_t^\top]) &= T^{-1} \sum_{t \leq T} \mathbb{E}[\bar{x}_t] (\bar{x}_t^\top - \mathbb{E}[\bar{x}_t^\top]) + T^{-1} \sum_{t \leq T} (\bar{x}_t - \mathbb{E}[\bar{x}_t]) \mathbb{E}[\bar{x}_t^\top] \\ &\quad + T^{-1} \sum_{t \leq T} (\bar{x}_t - \mathbb{E}[\bar{x}_t]) (\bar{x}_t^\top - \mathbb{E}[\bar{x}_t^\top]), \end{aligned}$$

the claim of the lemma in this part follows from Assumption 1(i), (S414) and (S417).

(iv) By the similar arguments as in the proofs of (S401) and (S406), we obtain

$$\text{Var}(\bar{x}_j) = O((nT)^{-1/2}) \quad \text{and} \quad \mathbb{E}[\bar{x}_j] (\bar{x}_j - \mathbb{E}[\bar{x}_j]) = O_p((nT)^{-1/2}) \quad (\text{S418})$$

for any $j = 1, \dots, d_x$. Therefore, using (S418), Hölder's inequality and Markov's inequality,

$$\begin{aligned} \bar{x}\bar{x}^\top - \mathbb{E}[\bar{x}]\mathbb{E}[\bar{x}^\top] &= (\bar{x} - \mathbb{E}[\bar{x}])(\bar{x} - \mathbb{E}[\bar{x}])^\top + \mathbb{E}[\bar{x}](\bar{x} - \mathbb{E}[\bar{x}])^\top \\ &\quad + (\bar{x} - \mathbb{E}[\bar{x}])\mathbb{E}[\bar{x}^\top] = O_p((nT)^{-1/2}), \end{aligned}$$

which establishes the claim of the lemma in this part. Q.E.D.

Lemma S34. *Under Assumptions 1 and 4(ii), we have:*

- (i) $\|\theta_2^*\| + \max_{i \leq n} |\gamma_{2,i}^*| + \max_{t \leq T} |\gamma_{2,t}^*| \leq K$;
- (ii) $\max_{i \leq n, t \leq T} \mathbb{E}[\varepsilon_{2,i,t}^8] \leq K$.

PROOF OF LEMMA S34. (i) By the definition of θ^* and Assumption 4,

$$\begin{aligned} \|\theta_2^*\|^2 &= \left(\sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[y_{i,t} \ddot{x}_{i,t}^{*\top}] \right) \left(\sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\ddot{x}_{i,t}^* \ddot{x}_{i,t}^{*\top}] \right)^{-2} \left(\sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\ddot{x}_{i,t}^* y_{i,t}] \right) \\ &\leq \lambda_{\max} \left(\left((nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\ddot{x}_{i,t}^* \ddot{x}_{i,t}^{*\top}] \right)^{-1} \right) (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[y_{i,t}^2] \\ &\leq K \max_{i \leq n, t \leq T} \mathbb{E}[y_{i,t}^2] \leq K. \end{aligned} \quad (\text{S419})$$

By Assumption 4(ii) and the triangle inequality, we have

$$\max_{i \leq n, t \leq T} (|\mathbb{E}[\bar{y}_i]|^2 + |\mathbb{E}[\bar{y}_t]|^2 + |\mathbb{E}[\bar{y}]|^2) \leq 3 \max_{i \leq n, t \leq T} \mathbb{E}[y_{i,t}^2] \leq K. \quad (\text{S420})$$

Combining the results in (S405), (S419), (S420) yields

$$\max_{i \leq n} |\gamma_{2,i}^*| \leq \max_{i \leq n} |\mathbb{E}[\bar{y}_i]| + \|\theta_2^*\| \max_{i \leq n} \|\mathbb{E}[\bar{x}_i]\| \leq K \quad (\text{S421})$$

and

$$\max_{t \leq T} |\gamma_{2,t}^*| \leq \max_{t \leq T} |\mathbb{E}[\bar{y}_t - \bar{y}]| + \|\theta_2^*\| \max_{i \leq n} \|\mathbb{E}[\bar{x}_t - \bar{x}]\| \leq K. \quad (\text{S422})$$

This completes the proof of part (i) of the lemma.

(ii) By Assumption 4(ii), (S419), (S421), (S422), the triangle inequality, the c_r -inequality and the Cauchy-Schwarz inequality

$$\max_{i \leq n, t \leq T} \mathbb{E}[\varepsilon_{2,i,t}^8] \leq K \max_{i \leq n, t \leq T} \mathbb{E} \left[|y_{i,t}|^8 + |x_{i,t}|^8 \|\theta^*\|^8 + |\gamma_{1,i}^*|^8 + |\gamma_{1,t}^*|^8 \right] \leq K. \quad (\text{S423})$$

This completes the proof of part (ii) of the lemma. Q.E.D.

Lemma S35. *Under Assumptions 1 and 4, we have*

$$(nT)^{-1/2} \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \varepsilon_{2,i,t} = (nT)^{-1/2} \sum_{i \leq n} \sum_{t \leq T} (\ddot{x}_{i,t}^* \varepsilon_{2,i,t} - \mathbb{E}[(\bar{x}_i + \bar{x}_t) \varepsilon_{2,i,t}]) + o_p(1) = O_p(1), \quad (\text{S424})$$

where $\varepsilon_{2,i,t} \equiv y_{i,t} - x_{i,t}^\top \theta_2^* - \gamma_{2,i}^* - \gamma_{2,t}^*$.

PROOF OF LEMMA S35. We begin by providing some axillary results which are useful for the proof. First by the first-order conditions of θ^* , $\gamma_{1,i}^*$ and $\gamma_{2,t}^*$, we obtain

$$\sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\ddot{x}_{i,t}^* \varepsilon_{2,i,t}] = 0 \quad \text{and} \quad \mathbb{E}[\bar{\varepsilon}_{2,i}] = \mathbb{E}[\bar{\varepsilon}_{2,t}] = 0. \quad (\text{S425})$$

Moreover in view of Lemma S34(i), Assumptions 1(ii), (iii), (iv) apply to $(x_{i,t}^\top, \varepsilon_{2,i,t})$.

We next prove (S424). By the definition of $\ddot{x}_{i,t}$ and (S425), we can express:

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \varepsilon_{2,i,t} = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} x_{i,t} \varepsilon_{2,i,t} - n^{-1} \sum_{i \leq n} \bar{x}_i \bar{\varepsilon}_{2,i} - T^{-1} \sum_{t \leq T} \bar{x}_t \bar{\varepsilon}_{2,t} + \bar{x} \bar{\varepsilon}_2. \quad (\text{S426})$$

Starting with the first term in the RHS of the equality in (S426), we can use similar arguments as in (S401) to show that

$$\mathbb{E} \left[\left| (nT)^{-1/2} \sum_{i \leq n} \sum_{t \leq T} (x_{j,i,t} \varepsilon_{2,i,t} - \mathbb{E}[x_{j,i,t} \varepsilon_{2,i,t}]) \right|^2 \right] \leq K \max_{i \leq n, t \leq T} \|x_{j,i,t}\|_{4+\delta}^2 \|\varepsilon_{2,i,t}\|_{4+\delta}^2 \leq K, \quad (\text{S427})$$

where the last inequality follows from Assumption 4(ii) and Lemma S34(ii). Therefore, by (S425) and Markov's inequality,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} x_{i,t} \varepsilon_{2,i,t} = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (x_{i,t} \varepsilon_{2,i,t} - \mathbb{E}[x_{i,t} \varepsilon_{2,i,t}]) = O_p((nT)^{-1/2}). \quad (\text{S428})$$

To analyze the second term in the RHS of the equality in (S426), we first express:

$$n^{-1} \sum_{i \leq n} \bar{x}_i \bar{\varepsilon}_{2,i} = n^{-1} \sum_{i \leq n} ((\bar{x}_i - \mathbb{E}[\bar{x}_i]) \bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{x}_i \bar{\varepsilon}_{2,i}]) + n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{x}_i] \bar{\varepsilon}_{2,i} + n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{x}_i \bar{\varepsilon}_{2,i}]. \quad (\text{S429})$$

We next study the three terms in the RHS of the equation in (S429). Since $\mathbb{E}[(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}]) \bar{\varepsilon}_{2,i}] = \mathbb{E}[\bar{x}_{j,i} \bar{\varepsilon}_{2,i}]$ in view of (S425), by Assumption 1(i) we obtain

$$\begin{aligned} \mathbb{E} \left[\left| n^{-1} \sum_{i \leq n} ((\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}]) \bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{x}_{j,i} \bar{\varepsilon}_{2,i}]) \right|^2 \right] &\leq n^{-1} \max_{i \leq n} \mathbb{E}[(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}]) \bar{\varepsilon}_{2,i}]^2 \\ &\leq n^{-1} \max_{i \leq n} (\mathbb{E}[(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}])^4] \mathbb{E}[\bar{\varepsilon}_{2,i}^4])^{1/2}. \end{aligned} \quad (\text{S430})$$

Using similar arguments as in deriving (S403), we can show that

$$\max_{i \leq n} \mathbb{E}[\bar{\varepsilon}_{2,i}^4] = \max_{i \leq n} \mathbb{E}[(\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{\varepsilon}_{2,i}])^4] \leq KT^{-2}, \quad (\text{S431})$$

which together with (S403), (S430) and Markov's inequality implies that

$$n^{-1} \sum_{i \leq n} ((\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}])\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{x}_{j,i}\bar{\varepsilon}_{2,i}]) = O_p((nT^2)^{-1/2}). \quad (\text{S432})$$

For the second and the third terms in the RHS of equation in (S429), we can use Assumption 1(iii), (S405), (S425) and (S431) to show that

$$\mathbb{E} \left[\left| n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{x}_{j,i}] \bar{\varepsilon}_{2,i} \right|^2 \right] = n^{-2} \sum_{i \leq n} (\mathbb{E}[\bar{x}_{j,i}]^2 \mathbb{E}[\bar{\varepsilon}_{2,i}^2]) \leq K(nT)^{-1}, \quad (\text{S433})$$

and use Hölder's inequality, (S403), (S425) and (S431) to obtain

$$\max_{i \leq n} |\mathbb{E}[\bar{x}_{j,i} \bar{\varepsilon}_{2,i}]| = \max_{i \leq n} |\mathbb{E}[(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}])\bar{\varepsilon}_{2,i}]| \leq \max_{i \leq n} (\mathbb{E}[(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}])^2] \mathbb{E}[\bar{\varepsilon}_{2,i}^2])^{1/2} \leq K(nT)^{-1}. \quad (\text{S434})$$

Combining the results in (S429), (S432)-(S434), and applying Markov's inequality yields

$$n^{-1} \sum_{i \leq n} \bar{x}_i \bar{\varepsilon}_{2,i} = n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{x}_i] \bar{\varepsilon}_{2,i} + n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{x}_i \bar{\varepsilon}_{2,i}] + O_p((nT^2)^{-1/2}) = O_p((nT)^{-1/2}). \quad (\text{S435})$$

We next turn to the third term in the RHS of the equality in (S426). By (S425), and the definitions of $\bar{x}_t$ and $\bar{\varepsilon}_{2,t}$, we can express:

$$\bar{x}_t \bar{\varepsilon}_{2,t} = \bar{x}_t (\bar{\varepsilon}_{2,t} - \mathbb{E}[\bar{\varepsilon}_{2,t}]) = n^{-2} \sum_{i \leq n} x_{i,t} \varepsilon_{2,i,t}^* + n^{-2} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} (x_{i_1 t} \varepsilon_{2,i_2,t}^* + \varepsilon_{2,i_1 t}^* x_{i_2,t}),$$

where $\varepsilon_{i,t}^* \equiv \varepsilon_{i,t} - \mathbb{E}[\varepsilon_{i,t}]$. Therefore,

$$n^2 \sum_{t \leq T} (\bar{x}_t \bar{\varepsilon}_{2,t} - \mathbb{E}[\bar{x}_t \bar{\varepsilon}_{2,t}]) = \sum_{i \leq n} \sum_{t \leq T} (x_{i,t} \varepsilon_{2,i,t}^* - \mathbb{E}[x_{i,t} \varepsilon_{2,i,t}^*]) + \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \sum_{t \leq T} (x_{i_1 t} \varepsilon_{2,i_2,t}^* + \varepsilon_{2,i_1 t}^* x_{i_2,t}). \quad (\text{S436})$$

By employing analogous arguments as demonstrated in the proof of (S401), we can establish that

$$(n^2 T)^{-1} \sum_{i \leq n} \sum_{t \leq T} (x_{j,i,t} \varepsilon_{2,i,t}^* - \mathbb{E}[x_{j,i,t} \varepsilon_{2,i,t}^*]) = O_p((n^3 T)^{-1/2}). \quad (\text{S437})$$

Let $e_i \equiv (\varepsilon_{2,i,t}^*)_{t \leq T}$ and $X_{j,i} \equiv (x_{j,i,t})_{t \leq T}$. Then

$$\sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \sum_{t \leq T} x_{j,i_1 t} \varepsilon_{2,i_2,t}^* = \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} (X_{j,i_1}^\top - \mathbb{E}[X_{j,i_1}^\top]) e_{i_2} + \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \mathbb{E}[X_{j,i_1}^\top] e_{i_2}. \quad (\text{S438})$$

By Assumption 1(iii), (S412) and Lemma S34(ii),

$$\begin{aligned}
& \mathbb{E} \left[\left| (n^2 T)^{-1} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} (X_{j,i_1}^\top - \mathbb{E}[X_{j,i_1}^\top]) e_{i_2} \right|^2 \right] \\
&= (n^2 T)^{-2} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \mathbb{E}[e_{i_2}^\top \text{Cov}(X_{j,i_1}) e_{i_2}] \\
&\leq (nT)^{-2} \max_{j \leq d_x} \max_{i \leq n} \lambda(\text{Cov}(X_{j,i})) \max_{i \leq n} \mathbb{E}[\|e_i\|^2] \leq K(n^2 T)^{-1}.
\end{aligned} \tag{S439}$$

Consider the expression for the second term on the RHS of the equation in (S438):

$$\sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \mathbb{E}[X_{j,i_1}^\top] e_{i_2} = \sum_{i=1}^{n-1} \left(\sum_{i'=i+1}^n \mathbb{E}[X_{j,i'}^\top] \right) e_i. \tag{S440}$$

Using arguments similar to those employed in deriving (S412) we deduce

$$\max_{i \leq n} \lambda(\text{Cov}(e_i)) \leq K. \tag{S441}$$

Therefore by Assumptions 1(iii) and 4(ii), (S440) and (S441),

$$\begin{aligned}
\mathbb{E} \left[\left| \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \mathbb{E}[X_{j,i_1}^\top] e_{i_2} \right|^2 \right] &= \sum_{i=1}^{n-1} \left(\sum_{i'=i+1}^n \mathbb{E}[X_{j,i'}^\top] \right) \text{Cov}(e_i) \left(\sum_{i'=i+1}^n \mathbb{E}[X_{j,i'}] \right) \\
&\leq K \sum_{i=1}^{n-1} \sum_{i'_1, i'_2=i+1}^n \mathbb{E}[X_{j,i'_1}^\top] \mathbb{E}[X_{j,i'_2}] \\
&\leq K \sum_{i=1}^{n-1} \left(\sum_{i'=i+1}^n \|\mathbb{E}[X_{j,i'}]\| \right)^2 \leq K n^3 T,
\end{aligned} \tag{S442}$$

where the last inequality is by $\|\mathbb{E}[X_{j,i'}]\|^2 \leq \sum_{t \leq T} \mathbb{E}[x_{j,i',t}^2] \leq KT$. Combining the results from (S438), (S439) and (S442), and applying Markov's inequality yields:

$$\begin{aligned}
(n^2 T)^{-1} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \sum_{t \leq T} x_{j,i_1,t} \varepsilon_{2,i_2,t}^* &= (n^2 T)^{-1} \sum_{i=1}^{n-1} \left(\sum_{i'=i+1}^n \mathbb{E}[X_{j,i'}^\top] \right) e_i \\
&\quad + O_p((n^2 T)^{-1/2}) = O_p((nT)^{-1/2}).
\end{aligned} \tag{S443}$$

Next consider the following expression:

$$\sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \sum_{t \leq T} \varepsilon_{2,i_1,t}^* x_{i_2,t} = \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} e_{i_1}^\top (X_{j,i_2} - \mathbb{E}[X_{j,i_2}]) + \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} e_{i_1}^\top \mathbb{E}[X_{j,i_2}]. \tag{S444}$$

Using the same arguments as in the proof of (S439), we can establish:

$$\mathbb{E} \left[\left| (n^2 T)^{-1} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} e_{i_1}^\top (X_{j,i_2} - \mathbb{E}[X_{j,i_2}]) \right|^2 \right] \leq K(n^2 T)^{-1}. \tag{S445}$$

By employing arguments similar to the proof of (S442),

$$\begin{aligned}
\mathbb{E} \left[\left| (n^2 T)^{-1} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \mathbb{E}[X_{j,i_2}^\top] e_{i_1} \right|^2 \right] &= (n^2 T)^{-2} \sum_{i=2}^n \left(\sum_{i'=1}^{i-1} \mathbb{E}[X_{j,i'}^\top] \right) \text{Cov}(e_i) \left(\sum_{i'=1}^{i-1} \mathbb{E}[X_{j,i'}] \right) \\
&\leq K (n^2 T)^{-2} \sum_{i=2}^n \sum_{i'_1, i'_2=1}^{i-1} \mathbb{E}[X_{j,i'_1}^\top] \mathbb{E}[X_{j,i'_2}] \\
&\leq K \sum_{i=2}^n \left(\sum_{i'=1}^{i-1} \|\mathbb{E}[X_{j,i'}]\| \right)^2 \leq K (nT)^{-1}, \tag{S446}
\end{aligned}$$

which along with (S444), (S445) and (S446), and Markov's inequality implies that

$$(n^2 T)^{-1} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \sum_{t \leq T} \varepsilon_{2,i_1,t}^* x_{i_2,t} = (n^2 T)^{-1} \sum_{i=2}^n \sum_{i'=1}^{i-1} \mathbb{E}[X_{j,i'}^\top] e_i + O_p((n^2 T)^{-1/2}) = O_p((nT)^{-1/2}). \tag{S447}$$

By Assumptions 1(iii) and 4(ii), and (S441),

$$\begin{aligned}
\mathbb{E} \left[\left| (n^2 T)^{-1} \sum_{i \leq n} \mathbb{E}[X_{j,i}^\top] e_i \right|^2 \right] &= (n^2 T)^{-2} \sum_{i \leq n} \mathbb{E}[X_{j,i}^\top] \text{Cov}(e_i) \mathbb{E}[X_{j,i}] \\
&\leq (n^2 T)^{-2} \max_{i \leq n} \lambda(\text{Cov}(e_i)) \sum_{i \leq n} \|\mathbb{E}[X_{j,i}]\|^2 \leq K (n^3 T)^{-1},
\end{aligned}$$

which together with Markov's inequality implies that

$$(n^2 T)^{-1} \sum_{i \leq n} \mathbb{E}[X_{j,i}^\top] e_i = O_p((n^3 T)^{-1/2}). \tag{S448}$$

Combining (S436), (S437), (S443), (S447) and (S448) yields

$$\begin{aligned}
T^{-1} \sum_{t \leq T} (\bar{x}_{j,t} \bar{\varepsilon}_{2,t} - \mathbb{E}[\bar{x}_{j,t} \bar{\varepsilon}_{2,t}]) &= (n^2 T)^{-1} \sum_{i=1}^{n-1} \left(\sum_{i'=i+1}^n \mathbb{E}[X_{j,i'}^\top] \right) e_i \\
&\quad + (n^2 T)^{-1} \sum_{i=2}^n \sum_{i'=1}^{i-1} \mathbb{E}[X_{j,i'}^\top] e_i + O_p((n^2 T)^{-1/2}) \\
&= (n^2 T)^{-1} \sum_{i \leq n} \sum_{i'=1}^n \mathbb{E}[X_{j,i'}^\top] e_i + O_p((n^2 T)^{-1/2}) = O_p((nT)^{-1/2}). \tag{S449}
\end{aligned}$$

By Assumptions 1(i, iii) and 4(ii), (S425) and Hölder's inequality,

$$\begin{aligned}
\max_{t \leq T} |\mathbb{E}[\bar{x}_{j,t} \bar{\varepsilon}_{2,t}]| &= \max_{t \leq T} |\mathbb{E}[(\bar{x}_{j,t} - \mathbb{E}[\bar{x}_{j,t}]) \bar{\varepsilon}_{2,t}]| \leq \max_{i \leq n} (\mathbb{E}[(\bar{x}_{j,t} - \mathbb{E}[\bar{x}_{j,t}])^2] \mathbb{E}[\bar{\varepsilon}_{2,t}^2])^{1/2} \\
&\leq n^{-1} \max_{i \leq n, t \leq T} (\mathbb{E}[x_{j,i,t}^2])^{1/2} \max_{i \leq n, t \leq T} (\mathbb{E}[\varepsilon_{2,i,t}^2])^{1/2} \leq K (nT)^{-1/2}. \tag{S450}
\end{aligned}$$

Collecting the results in (S448), (S449) and (S450) yields

$$\begin{aligned}
T^{-1} \sum_{t \leq T} \bar{x}_{j,t} \bar{\varepsilon}_{2,t} &= T^{-1} \sum_{t \leq T} \mathbb{E}[\bar{x}_{j,t} \bar{\varepsilon}_{2,t}] + (n^2 T)^{-1} \sum_{i \leq n} \sum_{i'=1}^n \mathbb{E}[X_{j,i'}^\top] e_i + O_p((n^2 T)^{-1/2}) \\
&= T^{-1} \sum_{t \leq T} \mathbb{E}[\bar{x}_{j,t} \bar{\varepsilon}_{2,t}] + T^{-1} \sum_{t \leq T} \mathbb{E}[\bar{x}_{j,t}] \bar{\varepsilon}_{2,t} + O_p((n^2 T)^{-1/2}) = O_p((nT)^{-1/2}).
\end{aligned} \tag{S451}$$

We next turn to the last term in the RHS of the equality in (S426). Since $\mathbb{E}[\bar{\varepsilon}_2] = 0$ by (S425), we can express

$$\begin{aligned}
(\bar{x} - \mathbb{E}[\bar{x}]) \bar{\varepsilon}_2 &= (\bar{x} - \mathbb{E}[\bar{x}]) (\bar{\varepsilon}_2 - \mathbb{E}[\bar{\varepsilon}_2]) = n^{-2} \sum_{i_1, i_2=1}^n (\bar{x}_{i_1} - \mathbb{E}[\bar{x}_{i_1}]) (\bar{\varepsilon}_{2,i_2} - \mathbb{E}[\bar{\varepsilon}_{2,i_2}]) \\
&= n^{-2} \sum_{i \leq n} (\bar{x}_i - \mathbb{E}[\bar{x}_i]) (\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{\varepsilon}_{2,i}]) + n^{-2} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} (\bar{x}_{i_1} - \mathbb{E}[\bar{x}_{i_1}]) (\bar{\varepsilon}_{2,i_2} - \mathbb{E}[\bar{\varepsilon}_{2,i_2}]) \\
&\quad + n^{-2} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} (\bar{\varepsilon}_{2,i_1} - \mathbb{E}[\bar{\varepsilon}_{2,i_1}]) (\bar{x}_{i_2} - \mathbb{E}[\bar{x}_{i_2}]).
\end{aligned} \tag{S452}$$

By the triangle inequality and Hölder's inequality, we can show that

$$\begin{aligned}
\mathbb{E} \left[\left| n^{-2} \sum_{i \leq n} (\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}]) (\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{\varepsilon}_{2,i}]) \right| \right] &\leq n^{-2} \sum_{i \leq n} \mathbb{E} [|(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}]) (\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{\varepsilon}_{2,i}])|] \\
&\leq n^{-1} \max_{i \leq n} (\mathbb{E} [(\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}])^2] \mathbb{E} [(\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{\varepsilon}_{2,i}])^2])^{1/2} \\
&\leq K(nT)^{-1},
\end{aligned} \tag{S453}$$

where the last inequality is by (S403) and (S431). Similarly, we can show that

$$\begin{aligned}
&\mathbb{E} \left[\left| n^{-2} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} (\bar{x}_{j,i_1} - \mathbb{E}[\bar{x}_{j,i_1}]) (\bar{\varepsilon}_{2,i_2} - \mathbb{E}[\bar{\varepsilon}_{2,i_2}]) \right|^2 \right] \\
&= n^{-4} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} \mathbb{E} [|(\bar{x}_{j,i_1} - \mathbb{E}[\bar{x}_{j,i_1}]) (\bar{\varepsilon}_{2,i_2} - \mathbb{E}[\bar{\varepsilon}_{2,i_2}])|^2] \\
&\leq n^{-2} \left(\max_{i \leq n} (\mathbb{E} [|\bar{x}_{j,i} - \mathbb{E}[\bar{x}_{j,i}] |^4])^{1/2} \max_{i \leq n} (\mathbb{E} [|\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{\varepsilon}_{2,i}] |^4])^{1/2} \right) \leq K(nT)^{-2}
\end{aligned} \tag{S454}$$

and

$$\mathbb{E} \left[\left| n^{-2} \sum_{i_1=2}^n \sum_{i_2=1}^{i_1-1} (\bar{\varepsilon}_{2,i_1} - \mathbb{E}[\bar{\varepsilon}_{2,i_1}]) (\bar{x}_{j,i_2} - \mathbb{E}[\bar{x}_{j,i_2}]) \right|^2 \right] \leq K(nT)^{-2}. \tag{S455}$$

From (S452)-(S455) and Markov's inequality, we deduce that

$$(\bar{x} - \mathbb{E}[\bar{x}]) \bar{\varepsilon}_2 = O_p((nT)^{-1}). \tag{S456}$$

Since $\bar{\varepsilon}_2 = n^{-1} \sum_{i \leq n} (\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{\varepsilon}_{2,i}])$ by (S425), using Assumption 1(iii) we get

$$\mathbb{E}[|\bar{\varepsilon}_2|^2] = n^{-2} \sum_{i \leq n} \mathbb{E} \left[|\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{\varepsilon}_{2,i}]|^2 \right] \leq n^{-1} \max_{i \leq n} \mathbb{E} \left[|\bar{\varepsilon}_{2,i} - \mathbb{E}[\bar{\varepsilon}_{2,i}]|^2 \right] \leq K(nT)^{-1}, \quad (\text{S457})$$

where the second inequality is by (S431) and Hölder's inequality. Therefore, by (S405), (S457) and Markov's inequality, we get

$$\mathbb{E}[\bar{x}] \bar{\varepsilon}_2 = O_p((nT)^{-1/2}).$$

This, combined with (S456) shows that

$$\bar{x} \bar{\varepsilon}_2 = n^{-1} \sum_{i \leq n} \mathbb{E}[\bar{x}] \bar{\varepsilon}_{2,i} + O_p((nT)^{-1}) = O_p((nT)^{-1/2}). \quad (\text{S458})$$

The claim of the lemma now follows from Assumption 1(i), (S426), (S428), (S435), (S451) and (S458). Q.E.D.

Lemma S36. *Under Assumptions 1 and 4, we have:*

$$\hat{\theta}_2 - \theta_2^* = \frac{\Sigma_{\ddot{x}}^{-1}}{nT} \sum_{i \leq n} \sum_{t \leq T} (\ddot{x}_{i,t}^* \varepsilon_{2,i,t} - \mathbb{E}[(\bar{x}_i + \bar{x}_t) \varepsilon_{2,i,t}]) + o_p((nT)^{-1/2}) = O_p((nT)^{-1/2}), \quad (\text{S459})$$

where $\varepsilon_{2,i,t} \equiv y_{i,t} - x_{i,t}^\top \theta_2^* - \gamma_{2,i}^* - \gamma_{2,t}^*$.

PROOF OF LEMMA S36. By the definitions of $\hat{\theta}_2$, θ_2^* and $\varepsilon_{2,i,t}$, we can express:

$$\begin{aligned} \hat{\theta}_2 - \theta_2^* &= \hat{\Sigma}_{\ddot{x}}^{-1} (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} y_{i,t} - \theta^* = \hat{\Sigma}_{\ddot{x}}^{-1} (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \varepsilon_{2,i,t} \\ &= \Sigma_{\ddot{x}}^{-1} (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \varepsilon_{2,i,t} + (\hat{\Sigma}_{\ddot{x}}^{-1} - \Sigma_{\ddot{x}}^{-1}) (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \varepsilon_{2,i,t}. \end{aligned} \quad (\text{S460})$$

By the definition of $\hat{\Sigma}_{\ddot{x}}$ and Lemma S33,

$$\begin{aligned} \hat{\Sigma}_{\ddot{x}} - \Sigma_{\ddot{x}} &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (x_{i,t} x_{i,t}^\top - \mathbb{E}[x_{i,t} x_{i,t}^\top]) - n^{-1} \sum_{i \leq n} (\bar{x}_i \bar{x}_i^\top - \mathbb{E}[\bar{x}_i] \mathbb{E}[\bar{x}_i^\top]) \\ &\quad - T^{-1} \sum_{t \leq T} (\bar{x}_t \bar{x}_t^\top - \mathbb{E}[\bar{x}_t] \mathbb{E}[\bar{x}_t^\top]) + (\bar{x} \bar{x}^\top - \mathbb{E}[\bar{x}] \mathbb{E}[\bar{x}^\top]) = O_p((nT)^{-1/2}). \end{aligned} \quad (\text{S461})$$

This, combined with Assumption 4(i), further implies that

$$\lambda_{\min}(\hat{\Sigma}_{\ddot{x}}) \geq K^{-1}, \quad \text{wpa1}. \quad (\text{S462})$$

Summing up the results in Lemma S35, (S461) and (S462) yields

$$\left\| \frac{\hat{\Sigma}_{\ddot{x}}^{-1} - \Sigma_{\ddot{x}}^{-1}}{nT} \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \varepsilon_{2,i,t} \right\|^2 \leq \frac{\|\hat{\Sigma}_{\ddot{x}} - \Sigma_{\ddot{x}}\|^2}{\lambda_{\min}(\hat{\Sigma}_{\ddot{x}}) \lambda_{\min}(\Sigma_{\ddot{x}})} \left\| (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \varepsilon_{2,i,t} \right\|^2 = O_p((nT)^{-2}).$$

This together with (S460) shows that

$$\hat{\theta} - \theta^* = \Sigma_{\ddot{x}}^{-1} (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \varepsilon_{2,i,t} + O_p((nT)^{-1}). \quad (\text{S463})$$

The claim of the lemma now follows from Assumption 4(i), Lemma S35 and (S463). *Q.E.D.*

Lemma S37. *Under Assumptions 1 and 4, we have:*

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \frac{\varepsilon_{2,i,t}^2 - (y_{i,t} - x_{i,t}^\top \hat{\theta}_2 - \hat{\gamma}_{2,i,t})^2}{2} = \frac{n^{-1} \sum_{i \leq n} (\bar{\varepsilon}_{2,i}^*)^2 + T^{-1} \sum_{t \leq T} (\bar{\varepsilon}_{2,t}^*)^2}{2} + O_p((nT)^{-1})$$

where $\varepsilon_{2,i,t} \equiv y_{i,t} - x_{i,t}^\top \theta_2^* - \gamma_{2,i}^* - \gamma_{2,t}^*$, $\bar{\varepsilon}_{2,i}^* \equiv T^{-1} \sum_{t \leq T} \varepsilon_{2,i,t}^*$, $\bar{\varepsilon}_{2,t}^* \equiv n^{-1} \sum_{i \leq n} \varepsilon_{2,i,t}^*$ and $\varepsilon_{2,i,t}^* \equiv \varepsilon_{2,i,t} - \mathbb{E}[\varepsilon_{2,i,t}]$.

PROOF OF LEMMA S37. First note that $y_{i,t} - x_{i,t}^\top \hat{\theta}_2 - \hat{\gamma}_{2,i,t} = \ddot{y}_{i,t} - \ddot{x}_{i,t}^\top \hat{\theta}_2 = \ddot{\varepsilon}_{2,i,t} - \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*)$, where $\ddot{\varepsilon}_{2,i,t} \equiv \varepsilon_{2,i,t} - \bar{\varepsilon}_{2,i} - \bar{\varepsilon}_{2,t} + \bar{\varepsilon}_2$. Therefore

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (-2)^{-1} \left[(y_{i,t} - x_{i,t}^\top \hat{\theta}_2 - \hat{\gamma}_{2,i,t})^2 - \varepsilon_{2,i,t}^2 \right] \\ &= -(2nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t}^2 - \varepsilon_{2,i,t}^2) + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) \\ & \quad - (2nT)^{-1} (\hat{\theta}_2 - \theta_2^*)^\top \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*). \end{aligned} \quad (\text{S464})$$

By Lemma S35 and Lemma S36,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) = O_p((nT)^{-1}). \quad (\text{S465})$$

By Assumption 4(ii) and (S405), $\|\Sigma_{\ddot{x}}\| \leq K$ which together with (S461) implies that

$$\hat{\Sigma}_{\ddot{x}} = O_p(1). \quad (\text{S466})$$

This along with Lemma S36 implies that

$$(2nT)^{-1} (\hat{\theta}_2 - \theta_2^*)^\top \sum_{i \leq n} \sum_{t \leq T} \ddot{x}_{i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) = O_p((nT)^{-1}). \quad (\text{S467})$$

By the definition of $\ddot{\varepsilon}_{i,t}$, we can express

$$\begin{aligned} (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t}^2 &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t} \varepsilon_{2,i,t} \\ &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 - n^{-1} \sum_{i \leq n} \bar{\varepsilon}_{2,i}^2 - T^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^2 + \bar{\varepsilon}_2^2. \end{aligned} \quad (\text{S468})$$

Therefore by (S457) and Markov's inequality

$$-(2nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \varepsilon_{2,i,t}) = (2n)^{-1} \sum_{i \leq n} \bar{\varepsilon}_{2,i}^2 + (2T)^{-1} \sum_{t \leq T} \bar{\varepsilon}_{2,t}^2 + O_p(nT)^{-1}. \quad (\text{S469})$$

The claim and the lemma now follows from (S425), (S464), (S465), (S467) and (S469). *Q.E.D.*

Lemma S38. *Under Assumptions 1 and 4, $n^{-1} \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2) = O_p(n^{-1})$, where $s_{2,i}^2 \equiv T^{-1} \sum_{t \leq T} \mathbb{E}[\varepsilon_{2,i,t}^2]$.*

PROOF OF LEMMA S38. Using Lemma S37 and the definition of $\hat{\sigma}_{\varepsilon_{2,i}}^2$, we have

$$\begin{aligned} (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 - n^{-1} \sum_{i \leq n} \hat{\sigma}_{2,i}^2 &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \hat{\varepsilon}_{2,i,t}^2) \\ &= n^{-1} \sum_{i \leq n} (\bar{\varepsilon}_{2,i}^*)^2 + T^{-1} \sum_{t \leq T} (\bar{\varepsilon}_{2,t}^*)^2 + O_p((nT)^{-1}). \end{aligned} \quad (\text{S470})$$

By Assumption 1 and Lemma S34(ii)

$$\mathbb{E}[\bar{\varepsilon}_{2,i}^{*4} + \bar{\varepsilon}_{2,t}^{*4}] \leq Kn^{-2} \quad \text{and} \quad \mathbb{E}[\bar{\varepsilon}_2^{*4}] \leq Kn^{-4}. \quad (\text{S471})$$

Hence by Assumption 1(i) and Markov's inequality,

$$n^{-1} \sum_{i \leq n} (\bar{\varepsilon}_{2,i}^*)^2 + T^{-1} \sum_{t \leq T} (\bar{\varepsilon}_{2,t}^*)^2 = O_p(n^{-1}). \quad (\text{S472})$$

From (S470) and (S472), it follows that

$$n^{-1} \sum_{i \leq n} \hat{\sigma}_{2,i}^2 = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 + O_p(n^{-1}). \quad (\text{S473})$$

Similarly, Assumption 1, Lemma S34(ii), and Markov's inequality imply

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) = O_p(n^{-1}). \quad (\text{S474})$$

Combining (S473), and (S471) with the definition of $s_{\varepsilon_{2,i}}^2$ establishes the claim of the lemma. *Q.E.D.*

Lemma S39. *Under Assumptions 1 and 4, $n^{-1} \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2)^2 = O_p(n^{-1})$.*

PROOF OF LEMMA S39. By the definition of $\hat{\sigma}_{2,i}^2$,

$$\begin{aligned} \hat{\sigma}_{2,i}^2 &= T^{-1} \sum_{t \leq T} \hat{\varepsilon}_{2,i,t}^2 = T^{-1} \sum_{t \leq T} (\ddot{\varepsilon}_{2,i,t} - \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*))^2 \\ &= T^{-1} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t}^2 - 2T^{-1} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top (\hat{\theta}_2 - \theta_2^*) + (\hat{\theta}_2 - \theta_2^*)^\top \left(T^{-1} \sum_{t \leq T} \ddot{x}_{i,t} \ddot{x}_{i,t}^\top \right) (\hat{\theta}_2 - \theta_2^*). \end{aligned}$$

Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
n^{-1} \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2)^2 &= n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} \hat{\varepsilon}_{2,i,t}^2 - s_{\varepsilon_2,i}^2 \right)^2 \\
&\leq K n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t}^2 - s_{\varepsilon_2,i}^2 \right)^2 + K(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \left\| \ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top \right\|^2 \|\hat{\theta}_2 - \theta_2^*\|^2 \\
&\quad + K(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \left\| \ddot{x}_{i,t} \ddot{x}_{i,t}^\top \right\|^2 \|\hat{\theta}_2 - \theta_2^*\|^4. \tag{S475}
\end{aligned}$$

From Assumptions 1 and 4(ii), and Lemma S34(ii), it follows that

$$\mathbb{E}[\|\ddot{x}_{i,t}\|^4] + \mathbb{E}[\|\ddot{\varepsilon}_{2,i,t}\|^4] \leq K. \tag{S476}$$

This together with Markov's inequality implies

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \left\| \ddot{x}_{i,t} \ddot{x}_{i,t}^\top \right\|^2 + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \left\| \ddot{\varepsilon}_{2,i,t} \ddot{x}_{i,t}^\top \right\|^2 = O_p(1). \tag{S477}$$

Combining this with Assumption 1(i), Lemma S36, and (S475) yields

$$n^{-1} \sum_{i \leq n} (\hat{\sigma}_{2,i}^2 - s_{2,i}^2)^2 \leq K n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t}^2 - s_{\varepsilon_2,i}^2 \right)^2 + O_p(n^{-2}). \tag{S478}$$

Using (S425) and the definition of $\ddot{\varepsilon}_{2,i,t}$, we may write

$$\ddot{\varepsilon}_{2,i,t} = \varepsilon_{2,i,t} - \bar{\varepsilon}_{2,i}^* - \bar{\varepsilon}_{2,t}^* + \bar{\varepsilon}_2^*.$$

Hence

$$\begin{aligned}
T^{-1} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t}^2 - s_{\varepsilon_2,i}^2 &= T^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) + T^{-1} \sum_{t \leq T} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 \\
&\quad - 2T^{-1} \sum_{t \leq T} \varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*).
\end{aligned}$$

Therefore,

$$\begin{aligned}
n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} \ddot{\varepsilon}_{2,i,t}^2 - s_{\varepsilon_2,i}^2 \right)^2 &\leq K n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) \right)^2 \\
&\quad + K n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 \right)^2 \\
&\quad + K n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} \varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) \right)^2. \tag{S479}
\end{aligned}$$

From (S471) and Markov's inequality,

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\bar{\varepsilon}_{2,i}^{*4} + \bar{\varepsilon}_{2,t}^{*4}) = O_p(n^{-2}) \quad \text{and} \quad \bar{\varepsilon}_2^{*4} = O_p(n^{-4}). \quad (\text{S480})$$

Together with Assumption 1(i) and the Cauchy–Schwarz inequality, this implies

$$n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 \right)^2 \leq K(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\bar{\varepsilon}_{2,i}^{*4} + \bar{\varepsilon}_{2,t}^{*4} + \bar{\varepsilon}_2^{*4}) = O_p(n^{-2}). \quad (\text{S481})$$

Similarly,

$$\begin{aligned} & n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} \varepsilon_{2,i,t} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*) \right)^2 \\ & \leq (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^2 (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^2 \\ & \leq \sqrt{(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{2,i,t}^4} \sqrt{(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\bar{\varepsilon}_{2,i}^* + \bar{\varepsilon}_{2,t}^* - \bar{\varepsilon}_2^*)^4} = O_p(n^{-1}). \end{aligned} \quad (\text{S482})$$

Finally, by Assumption 1 and Lemma S34(ii),

$$\mathbb{E} \left[n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) \right)^2 \right] \leq K(nT)^{-1} \sum_{i \leq n} \max_{t \leq T} \|\varepsilon_{2,i,t}^2\|_{2+\delta}^2 \leq KT^{-1}.$$

Thus, by Assumption 1(i) and Markov's inequality

$$n^{-1} \sum_{i \leq n} \left(T^{-1} \sum_{t \leq T} (\varepsilon_{2,i,t}^2 - \mathbb{E}[\varepsilon_{2,i,t}^2]) \right)^2 = O_p(n^{-1}). \quad (\text{S483})$$

The claim of the lemma follows from Assumption 1(i), (S478), (S479), (S481)–(S483). *Q.E.D.*

2.3 Properties of Estimators from the Heterogeneous Time Effects Model

Lemma S40. *Suppose that Assumptions 1, 4(ii) and 5(i) hold. Then we have*

- (i) $(nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g (\bar{x}_{g,t} x_{g,t}^\top - \mathbb{E}[\bar{x}_{g,t}] \mathbb{E}[\bar{x}_{g,t}^\top]) = O_p((nT)^{-1/2})$;
- (ii) $\|\theta_1^*\| + \max_{g \in \mathcal{G}_1, t \leq T} |\gamma_{1,g,t}^*| \leq K$;
- (iii) $\max_{i \leq n, t \leq T} \mathbb{E}[\varepsilon_{1,i,t}^8] \leq K$.

PROOF OF LEMMA S40. (i) By Assumptions 1(iii) and 4(ii), we can establish

$$\max_{g \in \mathcal{G}_1, t \leq T} (|\mathbb{E}[\bar{x}_{g,t}]| + |\mathbb{E}[\bar{y}_{g,t}]|) \leq K, \quad (\text{S484})$$

and

$$\max_{g \in \mathcal{G}_1, t \leq T} \mathbb{E}[n_g(\bar{x}_{j,g,t} - \mathbb{E}[\bar{x}_{j,g,t}])^2] = \max_{g \in \mathcal{G}_1, t \leq T} n_g^{-1} \sum_{i \in I_g} \mathbb{E}[x_{j,i,t}^2] \leq K. \quad (\text{S485})$$

Consider the expression:

$$\begin{aligned} \bar{x}_{g,t} x_{g,t}^\top - \mathbb{E}[\bar{x}_{g,t}] \mathbb{E}[\bar{x}_{g,t}^\top] &= (\bar{x}_{g,t} - \mathbb{E}[\bar{x}_{g,t}]) (\bar{x}_{g,t}^\top - \mathbb{E}[\bar{x}_{g,t}^\top]) \\ &\quad + \mathbb{E}[\bar{x}_{g,t}] (\bar{x}_{g,t}^\top - \mathbb{E}[\bar{x}_{g,t}^\top]) + (\bar{x}_{g,t} - \mathbb{E}[\bar{x}_{g,t}]) \mathbb{E}[\bar{x}_{g,t}^\top]. \end{aligned} \quad (\text{S486})$$

By the triangle inequality, Holder's inequality and (S485),

$$\begin{aligned} &\mathbb{E} \left[\left| (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g (\bar{x}_{j,g,t} - \mathbb{E}[\bar{x}_{j,g,t}]) (\bar{x}_{j',g,t} - \mathbb{E}[\bar{x}_{j',g,t}]) \right| \right] \\ &\leq (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g \mathbb{E} [|(\bar{x}_{j,g,t} - \mathbb{E}[\bar{x}_{j,g,t}]) (\bar{x}_{j',g,t} - \mathbb{E}[\bar{x}_{j',g,t}])|] \\ &\leq (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \max_{j \leq d_x} \mathbb{E}[n_g(\bar{x}_{j,g,t} - \mathbb{E}[\bar{x}_{j,g,t}])^2] \\ &\leq G_1 n^{-1} \max_{j \leq d_x} \max_{g \in \mathcal{G}_1, t \leq T} \mathbb{E}[n_g(\bar{x}_{j,g,t} - \mathbb{E}[\bar{x}_{j,g,t}])^2] \leq K n^{-1}. \end{aligned} \quad (\text{S487})$$

By the definition of $\bar{x}_{g,t}$, we can write

$$(nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g \mathbb{E}[\bar{x}_{g,t}] (\bar{x}_{g,t}^\top - \mathbb{E}[\bar{x}_{g,t}^\top]) = (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{t \leq T} \mathbb{E}[\bar{x}_{g,t}] (x_{i,t}^\top - \mathbb{E}[x_{i,t}^\top]).$$

Hence by Assumption 1(iii) and the Cauchy-Schwarz inequality,

$$\begin{aligned} &\mathbb{E} \left[\left| (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g \mathbb{E}[\bar{x}_{j,g,t}] (\bar{x}_{j',g,t} - \mathbb{E}[\bar{x}_{j',g,t}]) \right|^2 \right] \\ &= (nT)^{-2} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \mathbb{E} \left[\left| \sum_{t \leq T} \mathbb{E}[\bar{x}_{j,g,t}] (x_{j',i,t} - \mathbb{E}[x_{j',i,t}]) \right|^2 \right] \\ &\leq (n^2 T)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} n_g^{-1} \sum_{i' \in I_g} \mathbb{E} \left[\left| T^{-1/2} \sum_{t \leq T} \mathbb{E}[x_{j,i't}] (x_{j',i,t} - \mathbb{E}[x_{j',i,t}]) \right|^2 \right] \\ &\leq (nT)^{-1} \max_{i, i' \leq n} \mathbb{E} \left[\left| T^{-1/2} \sum_{t \leq T} \mathbb{E}[x_{j,i't}] (x_{j',i,t} - \mathbb{E}[x_{j',i,t}]) \right|^2 \right]. \end{aligned} \quad (\text{S488})$$

By Assumptions 1(ii, iv) and 4(ii), and Corollary A.2. in Hall and Heyde (1980), we obtain

$$\max_{i, i' \leq n} \mathbb{E} \left[\left| T^{-1/2} \sum_{t \leq T} \mathbb{E}[x_{j,i't}] (x_{j',i,t} - \mathbb{E}[x_{j',i,t}]) \right|^2 \right] \leq K$$

which along with (S488) shows that

$$\mathbb{E} \left[\left| (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g \mathbb{E}[\bar{x}_{j,g,t}] (\bar{x}_{j',g,t} - \mathbb{E}[\bar{x}_{j',g,t}]) \right|^2 \right] \leq K(nT)^{-1} \quad (\text{S489})$$

for any $j, j' = 1, \dots, d_x$. The claim of the lemma in this part now follows from (S486), (S487), (S489), Assumption 1(i) and Markov's inequality.

(ii) By Assumptions 4(ii) and 5(i), we can use similar arguments in (S419) to show that $\|\theta_1^*\| \leq K$. This together with (S484) further implies that $\max_{g \in \mathcal{G}_1, t \leq T} |\gamma_{1,g,t}^*| \leq K$.

(iii) This part can be proved using similar arguments in the proof of Lemma S34, and its proof is hence omitted. Q.E.D.

Lemma S41. *Suppose that Assumptions 1, 4(ii) and 5(i) hold. Then we have*

$$(nT)^{-1/2} \sum_{i \leq n} \sum_{t \leq T} \dot{x}_{i,t} \varepsilon_{1,i,t} = (nT)^{-1/2} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} (\dot{x}_{i,t}^* \varepsilon_{1,i,t} - \mathbb{E}[\bar{x}_{g,t} \varepsilon_{1,i,t}]) + O_p(n^{-1/2}), \quad (\text{S490})$$

where

$$(nT)^{-1/2} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} (\dot{x}_{i,t}^* \varepsilon_{1,i,t} - \mathbb{E}[\bar{x}_{g,t} \varepsilon_{1,i,t}]) = O_p(1). \quad (\text{S491})$$

PROOF OF LEMMA S41. By the first-order conditions of θ_1^* and $\gamma_{1,g,t}^*$, we have

$$\sum_{i \leq n} \sum_{t \leq T} \mathbb{E}[\dot{x}_{i,t}^* \varepsilon_{1,i,t}] = 0 \quad \text{and} \quad \mathbb{E}[\bar{\varepsilon}_{1,g,t}] = 0 \quad (\text{S492})$$

where $\bar{\varepsilon}_{1,g,t} \equiv n_g^{-1} \sum_{i \in I_g} \varepsilon_{1,i,t}$. Moreover in view of Lemma S40(ii), Assumptions 1(ii, iii, iv) apply to $(x_{i,t}^\top, \varepsilon_{1,i,t})$.

Let $x_{i,t}^* \equiv x_{i,t} - \mathbb{E}[x_{i,t}]$, $X_i^* \equiv (x_{i,t}^*)_{t \leq T}$ and $e_{1,i} \equiv (\varepsilon_{1,i,t})_{t \leq T}$. Then

$$(\bar{x}_{g,t} - \mathbb{E}[\bar{x}_{g,t}]) \bar{\varepsilon}_{1,g,t} = n_g^{-2} \sum_{i \in I_g} x_{i,t}^* \varepsilon_{1,i,t} + n_g^{-2} \sum_{i=N_g+2}^{N_g+n_g} \sum_{i'=N_g+1}^i (x_{i,t}^* \varepsilon_{1,i',t} + x_{i',t}^* \varepsilon_{1,i,t}).$$

Since $\mathbb{E}[(\bar{x}_{g,t} - \mathbb{E}[\bar{x}_{g,t}]) \bar{\varepsilon}_{1,g,t}] = \mathbb{E}[\bar{x}_{g,t} \bar{\varepsilon}_{1,g,t}]$ by (S492), and $\mathbb{E}[x_{i,t}^* \varepsilon_{1,i',t}] = 0$ and $\mathbb{E}[x_{i',t}^* \varepsilon_{1,i,t}] = 0$ for any $i \neq i'$, we can use the above expression to obtain

$$\begin{aligned} & \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g [(\bar{x}_{j,g,t} - \mathbb{E}[\bar{x}_{j,g,t}]) \bar{\varepsilon}_{1,g,t} - \mathbb{E}[\bar{x}_{g,t} \bar{\varepsilon}_{1,g,t}]] \\ &= \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \frac{X_{j,i}^{*\top} e_{1,i} - \mathbb{E}[X_{j,i}^{*\top} e_{1,i}]}{n_g} + \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g} \sum_{i'=N_g+1}^i \frac{X_{j,i}^{*\top} e_{1,i'} + X_{j,i'}^{*\top} e_{1,i}}{n_g}. \end{aligned} \quad (\text{S493})$$

By Assumptions 1(ii, iii, iv) and 4(ii), and Lemma S40(iii), we can establish

$$\begin{aligned}
& \mathbb{E} \left[\left| (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \frac{X_{j,i}^{*\top} e_{1,i} - \mathbb{E}[X_{j,i}^{*\top} e_{1,i}]}{n_g} \right|^2 \right] \\
&= (nT)^{-2} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \frac{\mathbb{E} \left[\left| X_{j,i}^{*\top} e_{1,i} - \mathbb{E}[X_{j,i}^{*\top} e_{1,i}] \right|^2 \right]}{n_g^2} \\
&\leq G_1 (nT)^{-2} \max_{g \in \mathcal{G}_1} n_g^{-1} \max_{i \leq n} \mathbb{E} \left[\left| X_{j,i}^{*\top} e_{1,i} - \mathbb{E}[X_{j,i}^{*\top} e_{1,i}] \right|^2 \right] \leq K (n^2 T)^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1} \quad (\text{S494})
\end{aligned}$$

where the last inequality is by

$$\max_{i \leq n} \mathbb{E} \left[T^{-1} \left| X_{j,i}^{*\top} e_{1,i} - \mathbb{E}[X_{j,i}^{*\top} e_{1,i}] \right|^2 \right] \leq K \quad (\text{S495})$$

which can be verified using Assumptions 1(ii, iv) and 4(ii), and Lemma S40(iii) and similar arguments in (S401). Therefore by (S494) and Markov's inequality, the first term in the second line of (S493) satisfies

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \frac{X_{j,i}^{*\top} e_{1,i} - \mathbb{E}[X_{j,i}^{*\top} e_{1,i}]}{n_g} = O_p((n^2 T)^{-1/2} \max_{g \in \mathcal{G}_1} n_g^{-1/2}). \quad (\text{S496})$$

We next study the second term in the second line of (S493). By Assumption 1(iii), Lemma S40(iii) and (S412),

$$\begin{aligned}
& \mathbb{E} \left[\left| (nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g} \sum_{i'=N_g+1}^i \frac{X_{j,i}^{*\top} e_{1,i'}}{n_g} \right|^2 \right] \\
&= (nT)^{-2} \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g} \sum_{i'=N_g+1}^i \frac{\mathbb{E} \left[e_{1,i'}^\top X_{j,i}^* X_{j,i}^{*\top} e_{1,i'} \right]}{n_g^2} \\
&\leq \max_{j \leq d_x} \max_{i \leq n} \lambda_{\max}(\text{Cov}(X_{j,i}^*)) (nT)^{-2} \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g} \sum_{i'=N_g+1}^i \frac{\mathbb{E} [\|e_{1,i'}\|^2]}{n_g^2} \\
&\leq K G_1 (nT)^{-2} \max_{i \leq n} \mathbb{E} [\|e_{1,i}\|^2] \leq K (n^2 T)^{-1}. \quad (\text{S497})
\end{aligned}$$

Therefore by Markov inequality, we have

$$(nT)^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g} \sum_{i'=N_g+1}^i \frac{X_{j,i}^{*\top} e_{1,i'}}{n_g} = O_p((n^2 T)^{-1/2}). \quad (\text{S498})$$

Combining (S493), (S496) and (S498) yields

$$(nT)^{-1/2} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g [(\bar{x}_{j,g,t} - \mathbb{E}[\bar{x}_{j,g,t}]) \bar{\varepsilon}_{1,g,t} - \mathbb{E}[\bar{x}_{j,g,t} \bar{\varepsilon}_{1,g,t}]] = O_p(n^{-1/2}). \quad (\text{S499})$$

Since

$$\sum_{i \leq n} \sum_{t \leq T} \dot{x}_{i,t} \varepsilon_{1,i,t} = \sum_{i \leq n} \sum_{t \leq T} \dot{x}_{i,t}^* \varepsilon_{1,i,t} - \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g (\bar{x}_{g,t} - \mathbb{E}[\bar{x}_{g,t}]) \bar{\varepsilon}_{1,g,t},$$

the claim in (S490) follows from (S499).

To show (S491), we note that the combination of (S495) and Markov's inequality leads to

$$(nT)^{-1/2} \sum_{i \leq n} \sum_{t \leq T} \dot{x}_{i,t}^* \varepsilon_{1,i,t} = O_p(1). \quad (\text{S500})$$

Next, by Assumptions 1(iii), 4(ii) and Lemma S40(iii),

$$\begin{aligned} \max_{t \leq T} \max_{g \in \mathcal{G}_1} |n_g \mathbb{E}[\bar{x}_{j,g,t} \bar{\varepsilon}_{1,g,t}]| &= \max_{t \leq T} \max_{g \in \mathcal{G}_1} |n_g \mathbb{E}[(\bar{x}_{j,g,t} - \mathbb{E}[\bar{x}_{j,g,t}]) \bar{\varepsilon}_{1,g,t}]| \\ &\leq \max_{t \leq T} \max_{g \in \mathcal{G}_1} (n_g^2 \mathbb{E}[(\bar{x}_{j,g,t} - \mathbb{E}[\bar{x}_{j,g,t}])^2] \mathbb{E}[\bar{\varepsilon}_{1,g,t}^2])^{1/2} \\ &\leq \max_{t \leq T} \max_{g \in \mathcal{G}_1} (\mathbb{E}[x_{j,i,t}^2] \mathbb{E}[\varepsilon_{1,i,t}^2])^{1/2} \leq K, \end{aligned}$$

which together with Assumption 1(i) implies that

$$\left| (nT)^{-1/2} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g \mathbb{E}[\bar{x}_{g,t} \bar{\varepsilon}_{1,g,t}] \right| \leq K G_1. \quad (\text{S501})$$

Since G_1 is fixed, the claim in (S491) follows from (S500) and (S501). *Q.E.D.*

Lemma S42. *Suppose that Assumptions 1, 4(ii) and 5(i) hold. Then we have*

$$(\hat{\theta}_1 - \theta_1^*) = \frac{\Sigma_{\dot{x}}^{-1}}{nT} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} (\dot{x}_{i,t}^* \varepsilon_{1,i,t} - \mathbb{E}[\bar{x}_{g,t} \varepsilon_{1,i,t}]) + o_p((nT)^{-1/2}) = O_p((nT)^{-1/2}).$$

PROOF OF LEMMA S42. By the definition of $\dot{x}_{i,t}$, we obtain

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \dot{x}_{i,t} \dot{x}_{i,t}^\top = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} x_{i,t} x_{i,t}^\top - (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} n_g \bar{x}_{g,t} \bar{x}_{i,t}^\top.$$

Hence by Assumption 5(i), Lemma S33(i) and Lemma S40(i),

$$\hat{\Sigma}_{\dot{x}} - \Sigma_{\dot{x}} = O_p((nT)^{-1/2}) = o_p(1) \quad (\text{S502})$$

which together with Assumption 5(i) implies that

$$\lambda_{\min}(\hat{\Sigma}_{\dot{x}}) \geq K^{-1}, \quad \text{wpa1}. \quad (\text{S503})$$

The rest of the proof follows similar arguments in the proof of Lemma S36 and hence is omitted.

Q.E.D.

Lemma S43. Suppose that Assumptions 1, 4(ii) and 5(i) hold. Then we have

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \frac{\varepsilon_{1,i,t}^2 - (y_{i,t} - x_{i,t}^\top \hat{\theta}_1 - \hat{\gamma}_{1,i,t})^2}{2} = (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \frac{n_g \bar{\varepsilon}_{1,g,t}^{*2}}{2} + O_p((nT)^{-1})$$

where $\bar{\varepsilon}_{1,g,t}^* \equiv n_g^{-1} \sum_{i \in I_g} \varepsilon_{1,i,t}^*$.

PROOF OF LEMMA S43. First, observe that $y_{i,t} - x_{i,t}^\top \hat{\theta}_1 - \hat{\gamma}_{1,i,t} = \dot{\varepsilon}_{1,i,t} - \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*)$, where $\dot{\varepsilon}_{1,i,t} \equiv \varepsilon_{1,i,t} - \bar{\varepsilon}_{1,g,t}$. Therefore,

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (-2)^{-1} \left[(y_{i,t} - x_{i,t}^\top \hat{\theta}_1 - \hat{\gamma}_{1,i,t})^2 - \varepsilon_{1,i,t}^2 \right] \\ &= -(2nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (\dot{\varepsilon}_{1,i,t}^2 - \varepsilon_{1,i,t}^2) + (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) \\ & \quad - (2nT)^{-1} (\hat{\theta}_1 - \theta_1^*)^\top \sum_{i \leq n} \sum_{t \leq T} \dot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*). \end{aligned} \quad (\text{S504})$$

Using Lemma S41 and Lemma S42, and similar arguments as those used to derive (S465) and (S467), we can show that

$$(nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) = (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} \varepsilon_{1,i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) = O_p((nT)^{-1}) \quad (\text{S505})$$

and

$$(2nT)^{-1} (\hat{\theta}_1 - \theta_1^*)^\top \sum_{i \leq n} \sum_{t \leq T} \dot{x}_{i,t} \dot{x}_{i,t}^\top (\hat{\theta}_1 - \theta_1^*) = O_p((nT)^{-1}). \quad (\text{S506})$$

Combining (S504)-(S506) yields

$$\begin{aligned} & (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (-2)^{-1} \left[(y_{i,t} - x_{i,t}^\top \hat{\theta}_1 - \hat{\gamma}_{1,i,t})^2 - \varepsilon_{1,i,t}^2 \right] \\ &= (nT)^{-1} \sum_{i \leq n} \sum_{t \leq T} (-2)^{-1} (\dot{\varepsilon}_{1,i,t} \varepsilon_{1,i,t} - \varepsilon_{1,i,t}^2) + O_p((nT)^{-1}) \\ &= (nT)^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}_1} \frac{n_g \bar{\varepsilon}_{1,g,t}^2}{2} + O_p((nT)^{-1}), \end{aligned}$$

which together with (S492) shows the claim of the lemma. Q.E.D.

Lemma S44. Under Assumptions 1, 4(ii) and 5(i), $\sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} (\hat{\sigma}_{1,i}^2 - s_{1,i}^2) = O_p(\sum_{g \in \mathcal{G}_1} n_g^{-1})$, where $s_{1,i}^2 \equiv T^{-1} \sum_{t \leq T} \mathbb{E}[\varepsilon_{1,i,t}^2]$.

PROOF OF LEMMA S44. Using the definition of $\hat{\sigma}_{\varepsilon_1, i}^2$, we have

$$\begin{aligned}
\hat{\sigma}_{1, i}^2 - T^{-1} \sum_{t \leq T} \varepsilon_{1, i, t}^2 &= T^{-1} \sum_{t \leq T} (\hat{\varepsilon}_{1, i, t}^2 - \varepsilon_{1, i, t}^2) \\
&= T^{-1} \sum_{t \leq T} (\hat{\varepsilon}_{1, i, t}^2 - \varepsilon_{1, i, t}^2) - 2T^{-1} \sum_{t \leq T} \dot{\varepsilon}_{1, i, t} \dot{x}_{i, t}^\top (\hat{\theta}_1 - \theta_1^*) \\
&\quad + (\hat{\theta}_1 - \theta_1^*)^\top \left(T^{-1} \sum_{t \leq T} \dot{x}_{i, t} \dot{x}_{i, t}^\top \right) (\hat{\theta}_1 - \theta_1^*). \tag{S507}
\end{aligned}$$

From this expression, we may write

$$\begin{aligned}
\sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \left(\hat{\sigma}_{1, i}^2 - T^{-1} \sum_{t \leq T} \varepsilon_{1, i, t}^2 \right) &= T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} (\hat{\varepsilon}_{1, i, t}^2 - \varepsilon_{1, i, t}^2) \\
&= T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} (\hat{\varepsilon}_{1, i, t}^2 - \varepsilon_{1, i, t}^2) \\
&\quad - 2T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} \dot{\varepsilon}_{1, i, t} \dot{x}_{i, t}^\top (\hat{\theta}_1 - \theta_1^*) \\
&\quad + T^{-1} (\hat{\theta}_1 - \theta_1^*)^\top \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} \dot{x}_{i, t} \dot{x}_{i, t}^\top (\hat{\theta}_1 - \theta_1^*). \tag{S508}
\end{aligned}$$

By Assumptions 1 and 4(ii) and Lemma S40(iii)

$$\mathbb{E}[|\dot{x}_{i, t}|^4] \leq K \quad \text{and} \quad \mathbb{E}[|\dot{\varepsilon}_{1, i, t}|^4] \leq K. \tag{S509}$$

Together with Lemma S42, the triangle inequality, the Cauchy-Schwarz inequality and Markov's inequality, this implies that the quadratic term in (S508) satisfies

$$\begin{aligned}
&\left| (\hat{\theta}_1 - \theta_1^*)^\top \left(T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} \dot{x}_{i, t} \dot{x}_{i, t}^\top \right) (\hat{\theta}_1 - \theta_1^*) \right| \\
&\leq \|\hat{\theta}_1 - \theta_1^*\|^2 \times T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} \|\dot{x}_{i, t}\|^2 = O_p(G_1(nT)^{-1}). \tag{S510}
\end{aligned}$$

Similarly,

$$\left| T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} \dot{\varepsilon}_{1, i, t} \dot{x}_{i, t}^\top (\hat{\theta}_1 - \theta_1^*) \right| \leq \|\hat{\theta}_1 - \theta_1^*\| \times T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} \|\dot{\varepsilon}_{1, i, t} \dot{x}_{i, t}^\top\| = O_p(G_1 n^{-1}). \tag{S511}$$

Next, since $\dot{\varepsilon}_{1, i, t} = \varepsilon_{1, i, t} - \bar{\varepsilon}_{1, g, t}$, we can decompose

$$T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} \dot{\varepsilon}_{1, i, t}^2 = T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1, i, t}^2 - T^{-1} \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} \bar{\varepsilon}_{1, g, t}^2. \tag{S512}$$

By Assumptions 1, Lemma S40(iii), and (S492),

$$\mathbb{E}[\bar{\varepsilon}_{1,g,t}^4] = \mathbb{E}[\bar{\varepsilon}_{1,g,t}^{*4}] \leq K n_g^{-2}. \quad (\text{S513})$$

Hence by (S512) and Markov's inequality,

$$T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} (\hat{\varepsilon}_{1,i,t}^2 - \varepsilon_{1,i,t}^2) = T^{-1} \sum_{g \in \mathcal{G}_1} \sum_{t \leq T} \bar{\varepsilon}_{1,g,t}^2 = O_p \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} \right). \quad (\text{S514})$$

Combining (S508), (S510), (S511) and (S514), we obtain

$$\sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \left(\hat{\sigma}_{1,i}^2 - T^{-1} \sum_{t \leq T} \varepsilon_{1,i,t}^2 \right) = O_p \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} + G_1 n^{-1} \right). \quad (\text{S515})$$

Finally, by Assumption 1, Lemma S40(iii), and Markov's inequality,

$$T^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \sum_{t \leq T} (\varepsilon_{1,i,t}^2 - \mathbb{E}[\varepsilon_{1,i,t}^2]) = O_p \left(n^{-1/2} \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} \right)^{1/2} \right).$$

Together with (S515), this implies

$$\sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \left(\hat{\sigma}_{1,i}^2 - T^{-1} \sum_{t \leq T} \mathbb{E}[\varepsilon_{1,i,t}^2] \right) = O_p \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} + G_1 n^{-1} + n^{-1/2} \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} \right)^{1/2} \right). \quad (\text{S516})$$

Since $\sum_{g \in \mathcal{G}_1} n_g^{-1} \geq G_1 n^{-1} \geq n^{-1}$, the claim of the lemma follows from (S516) and the definition of $s_{\varepsilon_{1,i}}^2$. Q.E.D.

Lemma S45. Under Assumptions 1, 4(ii) and 5(i), $\sum_{g \in \mathcal{G}_1} (n_g^{-1} \sum_{i \in I_g} (\hat{\sigma}_{1,i}^2 - s_{1,i}^2))^2 = O_p(\sum_{g \in \mathcal{G}_1} n_g^{-1})$.

PROOF OF LEMMA S45. Using 1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{g \in \mathcal{G}_1} \left(n_g^{-1} \sum_{i \in I_g} (\hat{\sigma}_{1,i}^2 - s_{1,i}^2) \right)^2 &\leq K \sum_{g \in \mathcal{G}_1} (n_g T)^{-1} \sum_{i \in I_g} \sum_{t \leq T} (\hat{\varepsilon}_{1,i,t}^2 - \varepsilon_{1,i,t}^2)^2 \\ &\quad + K \|\hat{\theta}_1 - \theta_1^*\|^2 \sum_{g \in \mathcal{G}_1} (n_g T)^{-1} \sum_{i \in I_g} \sum_{t \leq T} \left\| \hat{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top \right\|^2 \\ &\quad + K \|\hat{\theta}_1 - \theta_1^*\|^4 \sum_{g \in \mathcal{G}_1} (n_g T)^{-1} \sum_{i \in I_g} \sum_{t \leq T} \|\dot{x}_{i,t} \dot{x}_{i,t}^\top\|^2. \end{aligned} \quad (\text{S517})$$

By the similar arguments for showing (S510) and (S511), we have

$$\|\hat{\theta}_1 - \theta_1^*\|^4 \sum_{g \in \mathcal{G}_1} (n_g T)^{-1} \sum_{i \in I_g} \sum_{t \leq T} \|\dot{x}_{i,t} \dot{x}_{i,t}^\top\|^2 = O_p(G_1 n^{-4}), \quad (\text{S518})$$

and

$$\|\hat{\theta}_1 - \theta_1^*\|^2 \sum_{g \in \mathcal{G}_1} (n_g T)^{-1} \sum_{i \in I_g} \sum_{t \leq T} \left\| \dot{\varepsilon}_{1,i,t} \dot{x}_{i,t}^\top \right\|^2 = O_p(G_1 n^{-2}). \quad (\text{S519})$$

Next, since $\dot{\varepsilon}_{1,i,t} = \varepsilon_{1,i,t} - \bar{\varepsilon}_{1,g,t}$, we obtain

$$\begin{aligned} \sum_{g \in \mathcal{G}_1} (n_g T)^{-1} \sum_{i \in I_g} \sum_{t \leq T} (\dot{\varepsilon}_{1,i,t}^2 - \varepsilon_{1,i,t}^2)^2 &\leq 2 \sum_{g \in \mathcal{G}_1} (n_g T)^{-1} \sum_{i \in I_g} \sum_{t \leq T} \varepsilon_{1,i,t}^2 \bar{\varepsilon}_{1,g,t}^2 \\ &\quad + 2 \sum_{g \in \mathcal{G}_1} (n_g T)^{-1} \sum_{i \in I_g} \sum_{t \leq T} \bar{\varepsilon}_{1,g,t}^4 = O_p \left(\sum_{g \in \mathcal{G}_1} n_g^{-1} \right) \end{aligned} \quad (\text{S520})$$

where the last equality holds by Lemma S40(iii), (S513) and Markov's inequality. The claim of the lemma now follows from (S517)-(S520). *Q.E.D.*

3 Proofs for Appendix C of Hahn, Liao, Menzel, and Vuong (2026)

PROOF OF THEOREM A1. By (S612) in Lemma S57, Assumptions 8 and 10(i), and Lemma S58,

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left| (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \partial^{|\nu|} \psi(z_{i,t}, \tilde{\phi}_{g,m}) / (\partial \phi_1^{\nu_1} \cdots \partial \phi_k^{\nu_k}) \right| = O_p(1) \quad (\text{S521})$$

for $|\nu| \leq 3$. Applying the Taylor expansion to the second equation (S622) obtains

$$\begin{aligned} 0_{d_\theta \times 1} &= \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left[\hat{\Psi}_{\theta,g,m} + \hat{\Psi}_{\theta\theta,g,m}(\hat{\theta} - \theta^*) + \hat{\Psi}_{\theta\gamma,g,m}(\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \right] \\ &\quad + 2^{-1} \left(\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \hat{\Psi}_{\theta_j \phi \phi, g, m} (\hat{\phi}_{g,m} - \phi_{g,m}^*) \right)_{j=1, \dots, d_\theta} \\ &\quad + O_p \left(\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^3 \right), \end{aligned} \quad (\text{S522})$$

where the last term is by (S521) and the Cauchy-Schwarz inequality. By Lemma S62 and (S537) in Lemma S1

$$\begin{aligned} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^3 &\leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 \\ &= O_p \left((GM)^{1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right) O_p \left(GM + (nT) \|\hat{\theta} - \theta^*\|^2 \right) \\ &= O_p \left((GM)^{1+1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right), \end{aligned} \quad (\text{S523})$$

which together with Lemma S62 implies that

$$\begin{aligned} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^3 &\leq K \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^3 + K(nT) \|\hat{\theta} - \theta^*\|^3 \\ &= O_p \left((GM)^{1+1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right). \end{aligned} \quad (\text{S524})$$

By (S621) in Lemma S60 and Lemma S62,

$$(nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \hat{\Psi}_{\theta\theta,g,m}(\hat{\theta} - \theta^*) = (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \Psi_{\theta\theta,g,m}(\hat{\theta} - \theta^*) + o_p \left((nT)^{-1/2} \right). \quad (\text{S525})$$

Combining the results in Lemma S63, Lemma S64, (S522), (S524) and (S525), and applying Assumption 10(ii) we get

$$\begin{aligned} 0_{d_\theta \times 1} &= (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta,g,m} - \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m}) \\ &\quad + (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (\Psi_{\theta\theta,g,m} - \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \Psi_{\gamma\theta,g,m}) (\hat{\theta} - \theta^*) \\ &\quad - (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{i \in I_g} (n_g T_m) \mathbb{E} \left[\hat{U}_{\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] \\ &\quad - \frac{(nT)^{-1}}{2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \left(\mathbb{E} \left[\hat{\Psi}_{\gamma,g,m}^\top \Psi_{\gamma\gamma,g,m}^{-1} \Psi_{\gamma_j\gamma\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] \right)_{j \leq d_\gamma} \\ &\quad + \frac{(nT)^{-1}}{2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left(\mathbb{E} \left[\hat{\Psi}_{\gamma,g,m}^\top \Psi_{\gamma\gamma,g,m}^{-1} \Psi_{\theta_j\gamma\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] \right)_{j \leq d_\theta} + o_p \left((nT)^{-1/2} \right). \end{aligned}$$

The assertion of the lemma follows from Assumption 9 and the above equation.

Q.E.D.

PROOF OF THEOREM A2. First note that by Lemma S67 and Lemma S68

$$\begin{aligned} &(nT)^{-1/2} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \hat{\Psi}_{\phi\phi,g,m} (\hat{\phi}_{g,m} - \phi_{g,m}^*) \\ &= (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma,g,m}^\top \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} + o_p((GM)^{1/2} (nT)^{-1/2}). \end{aligned}$$

which together with Lemma S66 implies that

$$\begin{aligned} &(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left[\hat{\Psi}_{g,m}(\hat{\phi}_{g,m}) - \hat{\Psi}_{g,m}(\phi_{g,m}^*) \right] \\ &= (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\phi,g,m}(\hat{\phi}_{g,m} - \phi_{g,m}^*) \\ &\quad + \frac{(nT)^{-1/2}}{2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma,g,m}^\top \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} + o_p((GM)^{1/2} (nT)^{-1/2}). \end{aligned} \quad (\text{S526})$$

By Lemma S62 and (S553) in Lemma S47,

$$\left| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta, g, m} (\hat{\theta} - \theta^*) \right| \leq \|\hat{\theta} - \theta^*\| \left\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta, g, m} \right\| = O_p((nT)^{-1/2})$$

which together with (S526) implies that

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left[\hat{\Psi}_{g, m}(\hat{\phi}_{g, m}) - \hat{\Psi}_{g, m}(\phi_{g, m}^*) \right] \\ &= (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}(\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \\ &+ \frac{(nT)^{-1/2}}{2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} + o_p((GM)^{1/2}(nT)^{-1/2}). \end{aligned} \quad (\text{S527})$$

The assertion of the lemma follows from (S706) in the proof of Lemma S67 and (S527). *Q.E.D.*

PROOF OF THEOREM A4. By the martingale CLT (e.g., Corollary 3.1 of Hall and Heyde (1980)), to verify Assumption 11(ii) it is sufficient to show that (105) and (106) hold.

To show (105), we first notice that

$$\mathbb{E}[\xi_{nT, i}^2 | \mathcal{F}_{nT, i-1}] = \frac{\mathbb{E}[(\tilde{\Psi}_i^* + \tilde{V}_i^*)^2]}{n\omega_{n, T}^2} + \frac{\mathbb{E}[\tilde{U}_i^2 | \mathcal{F}_{nT, i-1}]}{n\omega_{n, T}^2} + \frac{2\mathbb{E}[(\tilde{\Psi}_i^* + \tilde{V}_i^*)\tilde{U}_i | \mathcal{F}_{nT, i-1}]}{n\omega_{n, T}^2}. \quad (\text{S528})$$

By Assumptions 11(i) and 12(iii), and Lemma S70,

$$\frac{\sum_{i \leq n} \mathbb{E}[\tilde{U}_i^2 | \mathcal{F}_{nT, i-1}]}{n\omega_{n, T}^2} = \frac{\sum_{i \leq n} \mathbb{E}[\tilde{U}_i^2]}{n\omega_{n, T}^2} + o_p(1). \quad (\text{S529})$$

By the independence of $\{z_{i, t}\}_{t \leq T}$ across i , $\tilde{\Psi}_i^* + \tilde{V}_i^*$ and $\tilde{U}_i$ are uncorrelated which implies that

$$n\omega_{n, T}^2 = \sum_{i \leq n} \text{Var}(\tilde{\Psi}_i + \tilde{V}_i + \tilde{U}_i) = \sum_{i \leq n} (\text{Var}(\tilde{\Psi}_i + \tilde{V}_i) + \text{Var}(\tilde{U}_i)) = \sum_{i \leq n} \mathbb{E}[(\tilde{\Psi}_i^* + \tilde{V}_i^*)^2 + \tilde{U}_i^2] \quad (\text{S530})$$

where the third equality is by $\mathbb{E}[\tilde{U}_i] = 0$ for all i . By (S528), (S529) and (S530) we deduce that

$$\begin{aligned} \sum_{i \leq n} \mathbb{E}[\xi_{nT, i}^2 | \mathcal{F}_{nT, i-1}] &= \frac{\sum_{i \leq n} \mathbb{E}[(\tilde{\Psi}_i^* + \tilde{V}_i^*)^2 + \tilde{U}_i^2]}{n\omega_{n, T}^2} \\ &+ \frac{2 \sum_{i \leq n} \mathbb{E}[(\tilde{\Psi}_i^* + \tilde{V}_i^*)\tilde{U}_i | \mathcal{F}_{nT, i-1}]}{n\omega_{n, T}^2} + o_p(1) \\ &= \frac{\sum_{i \leq n} \mathbb{E}[(\tilde{\Psi}_i^* + \tilde{V}_i^*)^2 + \tilde{U}_i^2]}{n\omega_{n, T}^2} + o_p(1) = 1 + o_p(1), \end{aligned} \quad (\text{S531})$$

where the second equality follows by Lemma S69 and Assumption 12(iii). The first sufficient condition of the martingale CLT, i.e., (105) now follows from (S531).

To verify (106), we notice that for any $\varepsilon > 0$,

$$\begin{aligned} \sum_{i \leq n} \mathbb{E}[\xi_{nT,i}^2 I\{|\xi_{nT,i}| > \varepsilon\}] &\leq \varepsilon^{-2} \sum_{i \leq n} \mathbb{E}[|\xi_{nT,i}|^4] \\ &\leq K\varepsilon^{-2} \frac{\sum_{i \leq n} \mathbb{E}[(\tilde{\Psi}_i^*)^4] + \sum_{i \leq n} \mathbb{E}[(\tilde{V}_i^*)^4] + \sum_{i \leq n} \mathbb{E}[\tilde{U}_i^4]}{(n\omega_{n,T}^2)^2}. \end{aligned} \quad (\text{S532})$$

By (S718) in the proof of Lemma S69,

$$\frac{\sum_{i \leq n} \mathbb{E}[(\tilde{\Psi}_i^*)^4]}{(n\omega_{n,T}^2)^2} = \frac{\sum_{i \leq n} \mathbb{E}[|\tilde{\Psi}_{1,i}^* - \tilde{\Psi}_{2,i}^*|^4]}{(n\omega_{n,T}^2)^2} \leq n^{-1} \max_{i \leq n} \mathbb{E} \left[\left(\frac{\tilde{\Psi}_{1,i}^* - \tilde{\Psi}_{2,i}^*}{\omega_{n,T}} \right)^4 \right] \leq Kn^{-1}. \quad (\text{S533})$$

By Assumption 11(i) and (S719) in the proof of Lemma S69

$$\begin{aligned} \frac{\sum_{i \leq n} \mathbb{E}[(\tilde{V}_{j,i}^*)^4]}{(n\omega_{n,T}^2)^2} &= \frac{\sum_{g \in \mathcal{G}_j} \sum_{i \in I_g} \mathbb{E}[(\tilde{V}_{j,i}^*)^4]}{(n\omega_{n,T}^2)^2} \leq K \sum_{g \in \mathcal{G}_j} \sum_{i \in I_g} (M_j^2 n_g^{-4} T^{-2}) \\ &\leq KM_j^2 T^{-2} \sum_{g \in \mathcal{G}_j} n_g^{-3} \leq KM_j^2 G_j T^{-2} \max_{g \in \mathcal{G}_j} n_g^{-3} \leq KG_j^{-1} \max_{g \in \mathcal{G}_j} n_g^{-3} \end{aligned} \quad (\text{S534})$$

where the last inequality is by

$$M_j^2 G_j T^{-2} = (M_j G_j (nT)^{-1/2})^2 (nT^{-1}) G_j^{-1} \leq KG_j^{-1}, \quad (\text{S535})$$

where the inequality follows since $nT^{-1} \leq K$ by Assumption 1(i) and $M_j G_j (nT)^{-1/2} \leq K$ by Assumption 10(ii). By Assumption 11(i) and Lemma S71,

$$\begin{aligned} \frac{\sum_{i \leq n} \mathbb{E}[|\tilde{U}_{j,i}|^4]}{(n\omega_{n,T}^2)^2} &= \frac{\sum_{g \in \mathcal{G}_j} \sum_{i \in I_g} \mathbb{E}[|\tilde{U}_{j,i}|^4]}{(n\omega_{n,T}^2)^2} \leq K \sum_{g \in \mathcal{G}_j} \sum_{i \in I_g} M_j^2 T^{-2} n_g^{-2} \\ &\leq KM_j^2 G_j T^{-2} \max_{g \in \mathcal{G}_j} n_g^{-1} \leq KG_j^{-1} \max_{g \in \mathcal{G}_j} n_g^{-1}, \end{aligned} \quad (\text{S536})$$

where the last equality is by (S535). Collecting the results in (S532)-(S536), we deduce that

$$\sum_{i \leq n} \mathbb{E}[\xi_{nT,i}^2 I\{|\xi_{nT,i}| > \varepsilon\}] \leq K\varepsilon^{-2} \left(n^{-1} + G_j^{-1} \max_{g \in \mathcal{G}_j} n_g^{-1} \right) \rightarrow 0, \text{ as } n, T \rightarrow \infty$$

which shows (106).

Q.E.D.

Theorem S1. *Under Assumptions 1, 7, 8, 9 and 10, we have*

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} = O_p(1), \quad (\text{S537})$$

and

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g,m} - \gamma_{g,m}^* + \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m}\|}{(GM)^{2/p} (n_g T_m)^{-1}} = O_p(1),$$

where $\hat{\Psi}_{\gamma,g,m} \equiv (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \psi_\gamma(z_{i,t})$.

PROOF OF THEOREM S1. By the triangle inequality,

$$\begin{aligned}
& \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \hat{\gamma}_{g,m} - \gamma_{g,m}^* + \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1} + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2} \\
& \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} \frac{\left\| \hat{\Psi}_{\gamma\gamma,g,m}(\hat{\gamma}_{g,m} - \gamma_{g,m}^*) + \hat{\Psi}_{\gamma\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1} + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2} \\
& \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} \frac{\left\| \hat{\Psi}_{\gamma\gamma,g,m}(\hat{\gamma}_{g,m} - \gamma_{g,m}^*) + \hat{\Psi}_{\gamma\theta,g,m}(\hat{\theta} - \theta^*) + \hat{\Psi}_{\gamma,g,m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1} + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2} \\
& \quad + \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} \frac{\left\| (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1}} \\
& \quad + \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} (nT)^{1/2} \left\| \hat{\Psi}_{\gamma\theta,g,m}(\hat{\theta} - \theta^*) \right\|. \tag{S538}
\end{aligned}$$

The first term on the far RHS of (S538) can be analyzed using (S626) of Lemma S62 and Lemma S60(ii) to yield

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{(\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} \left\| \hat{\Psi}_{\gamma\gamma,g,m}(\hat{\gamma}_{g,m} - \gamma_{g,m}^*) + \hat{\Psi}_{\gamma\theta,g,m}(\hat{\theta} - \theta^*) + \hat{\Psi}_{\gamma,g,m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1} + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2} = O_p(1). \tag{S539}$$

In order to analyze the second term on the far RHS of (S538), we note

$$\begin{aligned}
& \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} \frac{\left\| (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1}} \\
& \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2) \lambda_{\max}((\Psi_{\gamma\gamma,g,m}^{-1})^2))^{1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right\| \left\| \hat{\Psi}_{\gamma,g,m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1}}, \tag{S540}
\end{aligned}$$

where $\max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2) \lambda_{\max}((\Psi_{\gamma\gamma,g,m}^{-1})^2))^{1/2} = O_p(1)$ by Lemma S60(ii) and Assumption 9;

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} = O_p(1) \tag{S541}$$

by Lemma S59; and

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \hat{\Psi}_{\gamma,g,m} \right\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} = O_p(1) \tag{S542}$$

by (S563) in Lemma S49. By (S540), (S541) and (S542), we have

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} \frac{\left\| (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1}} = O_p(1). \tag{S543}$$

In order to analyze the third term on the far RHS of (S538), we use the Cauchy-Schwarz inequality, Lemma S62, Lemma S60(ii), and (S620) of Lemma S60, and conclude

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma, g, m}^{-1})^2))^{1/2} (nT)^{1/2} \left\| \Psi_{\gamma, g, m}(\hat{\theta} - \theta^*) \right\| = O_p(1). \quad (\text{S544})$$

From (S538), (S539), (S543) and (S544), we have

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1} + \|\hat{\phi}_{g, m} - \phi_{g, m}^*\|^2} = O_p(1). \quad (\text{S545})$$

By the triangle inequality,

$$\begin{aligned} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} &\leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} \\ &\quad + \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \Psi_{\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|}{(GM)^{1/p} (n_g T_m)^{-1/2}}, \end{aligned} \quad (\text{S546})$$

where

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \Psi_{\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} = O_p(1) \quad (\text{S547})$$

by Assumption 9 and (S542). Since $\|\hat{\phi}_{g, m} - \phi_{g, m}^*\|^2 = \|\hat{\theta} - \theta^*\|^2 + \|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|^2$ and $n_g T_m \leq nT$,

$$\begin{aligned} &\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{(GM)^{2/p} (n_g T_m)^{-1} + \|\hat{\phi}_{g, m} - \phi_{g, m}^*\|^2}{(GM)^{1/p} (n_g T_m)^{-1/2}} \\ &\leq (GM)^{1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} + \frac{(nT)^{1/2} \|\hat{\theta} - \theta^*\|^2}{(GM)^{1/p}} + \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|^2}{(GM)^{1/p} (n_g T_m)^{-1/2}} \\ &= \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\| \times \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} + o_p(1) \\ &= o_p(1) \times \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} + o_p(1) \end{aligned} \quad (\text{S548})$$

where the first equality is by Assumption 10(i) and Lemma S62, the second equality is by Lemma S58. By (S545) and (S548)

$$\begin{aligned} &\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} \\ &= \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|}{(GM)^{2/p} (n_g T_m)^{-1} + \|\hat{\phi}_{g, m} - \phi_{g, m}^*\|^2} \frac{(GM)^{2/p} (n_g T_m)^{-1} + \|\hat{\phi}_{g, m} - \phi_{g, m}^*\|^2}{(GM)^{1/p} (n_g T_m)^{-1/2}} \\ &= o_p(1) \times \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} + o_p(1). \end{aligned} \quad (\text{S549})$$

Collecting the results in (S546), (S547) and (S549), we get

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|}{(GM)^{1/p}(n_g T_m)^{-1/2}} \leq o_p(1) \times \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|}{(GM)^{1/p}(n_g T_m)^{-1/2}} + O_p(1),$$

which implies that (S537) holds. Therefore, by Lemma S62 and (S537),

$$\begin{aligned} & \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{(GM)^{2/p}(n_g T_m)^{-1} + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2}{(GM)^{2/p}(n_g T_m)^{-1}} \\ &= \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left(1 + \frac{(nT)^{-1}}{(GM)^{2/p}(n_g T_m)^{-1}} (nT) \|\hat{\theta} - \theta^*\|^2 + \frac{\|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2}{(GM)^{2/p}(n_g T_m)^{-1}} \right) = O_p(1). \end{aligned} \quad (\text{S550})$$

The assertion of the lemma now follows from (S545) and (S550). Q.E.D.

3.1 Auxiliary Lemmas

Lemma S46. Suppose X_1 and X_2 are zero mean random vectors with $\max_{j=1,2} \lambda_{\max}(\mathbb{E}[X_j X_j^\top]) < \infty$. Then

$$\lambda_{\max}(\mathbb{E}[X_1 X_2^\top] \mathbb{E}[X_2 X_1^\top]) \leq \lambda_{\max}(\mathbb{E}[X_1 X_1^\top]) \lambda_{\max}(\mathbb{E}[X_2 X_2^\top]). \quad (\text{S551})$$

PROOF OF LEMMA S46. Let $\Sigma_{X_j} \equiv \mathbb{E}[X_j X_j^\top]$ and k_j denote the dimension of X_j ($j = 1, 2$). For any $k_1 \times 1$ real vectors a_1 , we shall show

$$a_1^\top (\mathbb{E}[X_1 X_2^\top] \mathbb{E}[X_2 X_1^\top]) a_1 \leq \lambda_{\max}(\Sigma_{X_1}) \lambda_{\max}(\Sigma_{X_2}) a_1^\top a_1, \quad (\text{S552})$$

which directly shows (S551). Since (S552) holds trivially if $a_1^\top \Sigma_{X_1} a_1 = 0$, we only need to consider the case that $a_1^\top \Sigma_{X_1} a_1 > 0$. Let $(\Sigma_{X_2}^{1/2})^+$ denote Moore-Penrose inverse of $\Sigma_{X_2}^{1/2}$ and $\Sigma_{X_2}^{1/2}$ is the unique symmetric matrix square root of Σ_{X_2} . Then by Lemma E.1.(b, c) in Liao and Shi (2020),

$$\begin{aligned} a_1^\top (\mathbb{E}[X_1 X_2^\top] \mathbb{E}[X_2 X_1^\top]) a_1 &= (\text{Cov}(a_1^\top X_1, X_2))^\top (\text{Cov}(a_1^\top X_1, X_2)) \\ &= \left((\Sigma_{X_2}^{1/2})^+ \text{Cov}(a_1^\top X_1, X_2) \right)^\top \Sigma_{X_2} \left((\Sigma_{X_2}^{1/2})^+ \text{Cov}(a_1^\top X_1, X_2) \right) \\ &\leq \lambda_{\max}(\Sigma_{X_2}) \text{Var}(a_1^\top X_1) \leq \lambda_{\max}(\Sigma_{X_1}) \lambda_{\max}(\Sigma_{X_2}) a_1^\top a_1, \end{aligned}$$

which proves (S552). Q.E.D.

Lemma S47. Suppose that: (i) $\varphi(z_{i,t}; \phi)$ is a function indexed by the parameter $\phi \in \Phi$ where Φ is a compact, convex subset of $\mathbb{R}^k$ and $k \equiv \dim(\phi)$; (ii) there exists a function $D(z_{i,t})$ such that $|\varphi(z_{i,t}, \phi_1) - \varphi(z_{i,t}, \phi_2)| \leq D(z_{i,t}) \|\phi_1 - \phi_2\|$ for all $\phi_1, \phi_2 \in \Phi$; (iii) the function $D(z_{i,t})$ satisfies $\sup_{i,t} \mathbb{E} \left[|D(z_{i,t})|^{p+\delta} \right] < K$ for some integer $p \geq \max\{2, k + \delta\}$ and for some $\delta > 0$; (iv) $\sup_{i,t} \sup_{\phi \in \Phi} \mathbb{E} \left[|\varphi(z_{i,t}, \phi)|^{p+\delta} \right] < K$. Then under Assumption 1, we have

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \mathbb{E} \left[\left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right|^p \right] \leq K \quad (\text{S553})$$

for any $\phi \in \Phi$, where $\varphi^*(z_{i,t}; \phi) \equiv \varphi(z_{i,t}; \phi) - \mathbb{E}[\varphi(z_{i,t}; \phi)]$. Moreover for any $\phi_1, \phi_2 \in \Phi$

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \mathbb{E} \left[\left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} (\varphi^*(z_{i,t}; \phi_1) - \varphi^*(z_{i,t}; \phi_2)) \right|^p \right] \leq K \|\phi_1 - \phi_2\|^p. \quad (\text{S554})$$

PROOF OF LEMMA S47. Since $\{z_{i,t}, t = 1, 2, \dots\}$ are independent across i , we can use Rosenthal's inequality of independent random variables (see, e.g., Theorem 2.12 in Hall and Heyde (1980)) to obtain

$$\mathbb{E} \left[\left| \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right|^p \right] \leq K \left(\left(\sum_{i \in I_g} \left\| \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right\|_2^2 \right)^{p/2} + \sum_{i \in I_g} \left\| \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right\|_p^p \right). \quad (\text{S555})$$

For any $\tilde{p} \geq 2$, applying Rosenthal's inequality of strong mixing processes (see, e.g., Theorem 2 in Section 1.4 of Doukhan (1994)), we get a moment bound on $\sum_{t \in I_m} \varphi^*(z_{i,t}; \phi)$:

$$\mathbb{E} \left[\left| \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right|^{\tilde{p}} \right] \leq K \max \left\{ \left(\sum_{t \in I_m} \|\varphi^*(z_{i,t}; \phi)\|_{2+\delta}^2 \right)^{\tilde{p}/2}, \sum_{t \in I_m} \|\varphi^*(z_{i,t}; \phi)\|_{\tilde{p}+\delta}^{\tilde{p}} \right\} \quad (\text{S556})$$

as long as $\mathbb{E}[|\varphi(z_{i,t}; \phi)|^{\tilde{p}+\delta}] < \infty$. In the rest of the proof, we shall consider $\tilde{p} = 2$ or p . Since $\|\varphi^*(z_{i,t}; \phi)\|_{\tilde{p}+\delta} \leq K$ by condition (iv) of the lemma, (S556) further implies that

$$\sum_{i \in I_g} \mathbb{E} \left[\left| \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right|^{\tilde{p}} \right] \leq K(n_g T_m^{\tilde{p}/2}). \quad (\text{S557})$$

Combining the results in (S555) and (S557), we obtain

$$\mathbb{E} \left[\left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right|^p \right] \leq K \left(1 + n_g^{1-p/2} \right) \leq K \quad (\text{S558})$$

where the second inequality is due to $p \geq 2$. The assertion in (S553) now follows from (S558).

For ease of notation, let $u(z_{i,t}; \phi_1, \phi_2) \equiv \varphi^*(z_{i,t}; \phi_1) - \varphi^*(z_{i,t}; \phi_2)$. By the triangle inequality and condition (ii) of the lemma,

$$|u(z_{i,t}; \phi_1, \phi_2)| \leq (D(z_{i,t}) + \mathbb{E}[D(z_{i,t})]) \|\phi_1 - \phi_2\|,$$

which together with condition (iii) of the lemma implies that for $\tilde{p} = 2$ or p ,

$$\sup_{i,t} \|u(z_{i,t}; \phi_1, \phi_2)\|_{\tilde{p}+\delta} \leq \|\phi_1 - \phi_2\|. \quad (\text{S559})$$

By the same arguments for deriving (S555) and (S556), we can show that

$$\mathbb{E} \left[\left| \sum_{i \in I_g} \sum_{t \in I_m} u(z_{i,t}; \phi_1, \phi_2) \right|^p \right] \leq K \left(\left(\sum_{i \in I_g} \left\| \sum_{t \in I_m} u(z_{i,t}; \phi_1, \phi_2) \right\|_2^2 \right)^{p/2} + \sum_{i \in I_g} \left\| \sum_{t \in I_m} u(z_{i,t}; \phi_1, \phi_2) \right\|_p^p \right)$$

and

$$\mathbb{E} \left[\left| \sum_{t \in I_m} u(z_{i,t}; \phi_1, \phi_2) \right|^{\tilde{p}} \right] \leq K \max \left\{ \left(\sum_{t \in I_m} \|u(z_{i,t}; \phi_1, \phi_2)\|_{2+\delta}^2 \right)^{\tilde{p}/2}, \sum_{t \in I_m} \|u(z_{i,t}; \phi_1, \phi_2)\|_{\tilde{p}+\delta}^{\tilde{p}} \right\},$$

respectively, which combined with (S559) yields

$$\mathbb{E} \left[\left| \sum_{i \in I_g} \sum_{t \in I_m} u(z_{i,t}; \phi_1, \phi_2) \right|^p \right] \leq K \left(1 + n_g^{1-p/2} \right) (n_g T_m)^{p/2} \|\phi_1 - \phi_2\|^p. \quad (\text{S560})$$

The claim in (S554) follows directly from (S560).

Q.E.D.

Lemma S48. *Under Assumptions 1, 8, 9 and 10, we have*

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \mathbb{E} \left[\left\| (n_g T_m)^{1/2} (\hat{\Psi}_{\gamma, g, m} - \Psi_{\gamma, g, m}) \right\|^4 \right] \leq K \quad (\text{S561})$$

and

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \mathbb{E} \left[\left\| (n_g T_m)^{1/2} \hat{\Psi}_{\gamma, g, m} \right\|^2 \right] \leq K, \quad (\text{S562})$$

PROOF OF LEMMA S48 . Follows from Assumption 8(ii) and (S553) in Lemma S47.

Q.E.D.

Lemma S49. *Under the conditions of Lemma S47, we have*

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right| = O_p((GM)^{1/p}), \quad (\text{S563})$$

and for any $\phi' \in \text{int } \Phi$

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \frac{\left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} (\varphi(z_{i,t}; \phi) - \mathbb{E}[\varphi(z_{i,t}; \phi^*)]) \right|}{(GM)^{1/p} + \|\phi - \phi'\|} = O_p(1). \quad (\text{S564})$$

PROOF OF LEMMA S49. To prove the claim in (S563), we first show that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \sup_{\phi \in \Phi} (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right\|_p \leq K. \quad (\text{S565})$$

Note that by a maximal inequality under the L_p -norm (see, e.g., Lemma 2.2.2 of van der Vaart and Wellner (1996)) and (S565),

$$\begin{aligned} & \left\| \max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right\|_p \\ & \leq K(GM)^{1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \sup_{\phi \in \Phi} (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right\|_p \leq K(GM)^{1/p} \end{aligned} \quad (\text{S566})$$

which together with Markov's inequality implies that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) = O_p((GM)^{1/p}). \quad (\text{S567})$$

The result in (S567) is then used to show (S563).

For ease of notation, we set for each $g \in \mathcal{G}$ and $m \in \mathcal{M}$,

$$\pi_{g,m}(\phi) \equiv (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi), \quad \phi \in \Phi.$$

Construct nested sets $\Phi_0 \subset \Phi_1 \cdots \subset \Phi$ such that $\Phi_0 = \{\phi_0\}$ for any given $\phi_0 \in \Phi$, and for each $j \geq 1$, Φ_j is a maximal set of points such that each pair of distinct elements in Φ has distance greater than 2^{-j} . Since Φ is compact, the number of points in Φ_j is less than $K(2^j)^k$. Link every point $\phi_{j+1} \in \Phi_{j+1}$ to a unique $\phi_j \in \Phi_j$ such that $\|\phi_{j+1} - \phi_j\| \leq 2^{-j}$. Then for any $J \geq 0$ and $\phi_{J+1} \in \Phi_{J+1}$, we can construct a chain $\phi_{J+1}, \dots, \phi_0$, and hence, by the triangle inequality

$$|\pi_{g,m}(\phi_{J+1}) - \pi_{g,m}(\phi_0)| = \left| \sum_{j=0}^J [\pi_{g,m}(\phi_{j+1}) - \pi_{g,m}(\phi_j)] \right| \leq \sum_{j=0}^J \max |\pi_{g,m}(\phi_{j+1}) - \pi_{g,m}(\phi_j)| \quad (\text{S568})$$

where, for each j , the maximum is taken over all links (ϕ_{j+1}, ϕ_j) from Φ_{j+1} to Φ_j (with the total number less than $K(2^{j+1})^k$). We then observe

$$\begin{aligned} \left\| \max_{\phi \in \Phi_{J+1}} |\pi_{g,m}(\phi)| \right\|_p &\leq \sum_{j=0}^J \left\| \max |\pi_{g,m}(\phi_{j+1}) - \pi_{g,m}(\phi_j)| \right\|_p + \|\pi_{g,m}(\phi_0)\|_p \\ &\leq K \sum_{j=0}^J (2^{j+1})^{k/p} \max \|\pi_{g,m}(\phi_{j+1}) - \pi_{g,m}(\phi_j)\|_p + K \\ &\leq K \sum_{j=0}^J (2^j)^{k/p} 2^{-j} + \|\pi_{g,m}(\phi_0)\|_p \leq K, \end{aligned}$$

where the first inequality is by (S568); the second inequality is by the maximal inequality under the L_p -norm and (S553); the third inequality follows from (S554) in Lemma S47; and the last inequality holds because $\sum_j (2^j)^{k/p} 2^{-j} < \infty$ as implied by $p > k$. Since the stochastic process $\pi_{g,m}(\phi)$ indexed by ϕ is separable, by letting $J \rightarrow \infty$, we further have

$$\left\| \sup_{\phi \in \Phi} |\pi_{g,m}(\phi)| \right\|_p \leq K$$

for any $g \in \mathcal{G}$ and $m \in \mathcal{M}$, which shows (S565). Using the same arguments, we can show that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} -\varphi^*(z_{i,t}; \phi) = O_p((GM)^{1/p}). \quad (\text{S569})$$

Since

$$\left| \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right| \leq \max \left\{ \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi), \sum_{i \in I_g} \sum_{t \in I_m} -\varphi^*(z_{i,t}; \phi) \right\}, \quad (\text{S570})$$

the claim in (S563) follows from (S567), (S569) and (S570).

To show (S564), we note that

$$\begin{aligned} & \max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \frac{\left| (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} (\varphi(z_{i,t}; \phi) - \mathbb{E}[\varphi(z_{i,t}; \phi')]) \right|}{(n_g T_m)^{-1/2} (GM)^{1/p} + \|\phi - \phi'\|} \\ & \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \frac{\left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \varphi^*(z_{i,t}; \phi) \right|}{(GM)^{1/p}} \\ & \quad + \max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \frac{\left| (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \mathbb{E}[\varphi(z_{i,t}; \phi) - \varphi(z_{i,t}; \phi')] \right|}{(n_g T_m)^{-1/2} (GM)^{1/p} + \|\phi - \phi'\|} \\ & \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \frac{\sup_{i,t} |\mathbb{E}[D(z_{i,t})]| \|\phi - \phi'\|}{(n_g T_m)^{-1/2} (GM)^{1/p} + \|\phi - \phi'\|} + O_p(1) = O_p(1), \end{aligned}$$

where the first inequality is by the triangle inequality, the second inequality is by condition (ii) of Lemma S47 and (S563), the equality is by condition (iii) of Lemma S47. Q.E.D.

For the sake of simplicity in notation, we assume that the observations across i are arranged in a manner where the group assignment function $g(\cdot)$ is non-decreasing. This arrangement is inconsequential because of the independence assumption stated in Assumption 1.(iii).

Lemma S50. *Let $\varphi_j(\cdot): \mathcal{Z} \rightarrow \mathbb{R}$ be functions on the support $\mathcal{Z}$ of $z_{i,t}$ for $j = 1, 2$. Suppose that for $j = 1, 2$: (i) $\mathbb{E}[\varphi_j(z_{i,t})] = 0$; (ii) $\sup_{i,t} \mathbb{E}[|\varphi_j(z_{i,t})|^{4+\delta}] < K$ for some $\delta > 0$. Then under Assumption 1, we have*

$$\text{Var} \left(M^{-1/2} \sum_{m \in \mathcal{M}} T_m^{-1} \sum_{t_1 \in I_m} \sum_{t_2 \in I_m} \varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}) \right) \leq K, \quad (\text{S571})$$

and

$$\mathbb{E} \left[\left\| \sum_{g \in \mathcal{G}} n_g^{-1} \sum_{i_1 = i_g^{\min} + 1}^{i_g^{\max}} \sum_{i_2 = i_g^{\min}}^{i_1 - 1} \left(\sum_{m \in \mathcal{M}} T_m^{-1} \left(\sum_{t \in I_m} \varphi_1(z_{i_1,t}) \right) \left(\sum_{t \in I_m} \varphi_2(z_{i_2,t}) \right) \right) \right\|^2 \right] \leq K(GM), \quad (\text{S572})$$

where $i_g^{\max} \equiv \max(I_g)$ and $i_g^{\min} \equiv \min(I_g)$ for any $g \in \mathcal{G}$.

PROOF OF LEMMA S50. For ease of notation, let $Y_m \equiv T_m^{-1} \sum_{t_1 \in I_m} \sum_{t_2 \in I_m} \varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2})$ for any $m \in \mathcal{M}$ and $\delta' \equiv \delta/3$. For any i , let $\mathcal{F}_{-\infty}^{i,t}$ denote the sigma field generated by $\{z_{is}\}_{s=-\infty}^t$ and

$\mathcal{F}_t^{i,+\infty}$ denote the sigma field generated by $\{z_{is}\}_{s=t}^{+\infty}$. Consider any $m_1, m_2 \in \mathcal{M}$ with $m_2 > m_1$. It is clear that Y_{m_1} is $\mathcal{F}_{-\infty}^{i, t_{m_1}^{\max}}$ measurable and Y_{m_2} is $\mathcal{F}_{t_{m_2}^{\min}}^{i, +\infty}$ measurable where $t_m^{\max} \equiv \max(I_m)$ and $t_m^{\min} \equiv \min(I_m)$ for any $m \in \mathcal{M}$. By Corollary A.2. in Hall and Heyde (1980),

$$\begin{aligned} |\text{Cov}(Y_{m_1}, Y_{m_2})| &\leq K (\alpha_i(t_{m_2}^{\min} - t_{m_1}^{\max}))^{\delta'/(2+\delta')} \|Y_{m_1}\|_{2+\delta'} \|Y_{m_2}\|_{2+\delta'} \\ &\leq K a^{(t_{m_2}^{\min} - t_{m_1}^{\max})\delta'/(2+\delta')} \max_{m \in \mathcal{M}} \|Y_m\|_{2+\delta'}^2 \\ &\leq K a^{(m_2 - m_1)\delta'/(2+\delta')} \max_{m \in \mathcal{M}} \|Y_m\|_{2+\delta'}^2 \end{aligned}$$

where the second inequality is by Assumption 1(iv), and the last inequality is by $t_{m_2}^{\min} - t_{m_1}^{\max} \geq m_2 - m_1$ and $a \in (0, 1)$. Therefore

$$\begin{aligned} \text{Var}\left(M^{-1/2} \sum_{m \in \mathcal{M}} Y_m\right) &= M^{-1} \sum_{m \in \mathcal{M}} \text{Var}(Y_m) + 2M^{-1} \sum_{l=1}^{M-1} \sum_{j=1}^{M-l} \text{Cov}(Y_j, Y_{j+l}) \\ &\leq K \left(1 + 2 \sum_{l=1}^{M-1} (1 - lM^{-1}) a^{l\delta'/(2+\delta')}\right) \max_{m \in \mathcal{M}} \|Y_m\|_{2+\delta'}^2 \\ &\leq K \left(1 + 2 \sum_{l=1}^{M-1} a^{l\delta'/(2+\delta')}\right) \max_{m \in \mathcal{M}} \|Y_m\|_{2+\delta'}^2 \leq K \max_{m \in \mathcal{M}} \|Y_m\|_{2+\delta'}^2. \quad (\text{S573}) \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} \|Y_m\|_{2+\delta'}^2 &= \left(\mathbb{E} \left[\left| \left(T_m^{-1/2} \sum_{t \in I_m} \varphi_1(z_{i,t}) \right) \left(T_m^{-1/2} \sum_{t \in I_m} \varphi_2(z_{i,t}) \right) \right|^{2+\delta'} \right] \right)^{\frac{2}{2+\delta'}} \\ &\leq \left(\mathbb{E} \left[\left| T_m^{-1/2} \sum_{t \in I_m} \varphi_1(z_{i,t}) \right|^{4+2\delta'} \right] \mathbb{E} \left[\left| T_m^{-1/2} \sum_{t \in I_m} \varphi_2(z_{i,t}) \right|^{4+2\delta'} \right] \right)^{\frac{1}{2+\delta'}}. \quad (\text{S574}) \end{aligned}$$

By the same arguments for deriving (S556), we can show that

$$\mathbb{E} \left[\left| \sum_{t \in I_m} \varphi_j(z_{i,t}) \right|^{4+2\delta'} \right] \leq \max \left\{ \sum_{t \in I_m} \|\varphi_j(z_{i,t})\|_{4+3\delta'}^{4+2\delta'}, \left(\sum_{t \in I_m} \|\varphi_j(z_{i,t})\|_{2+\delta'}^2 \right)^{2+\delta'} \right\} \leq K T_m^{2+\delta'} \quad (\text{S575})$$

where the second inequality is by condition (ii) of the lemma. Combining the results in (S574) and (S575), we have $\max_{m \in \mathcal{M}} \|Y_m\|_{2+\delta'}^2 \leq K$, which together with (S573) shows (S571).

To show (S572), we first use Assumption 1(iii) and conditions (i, ii) of the lemma to observe

that

$$\begin{aligned}
& \mathbb{E} \left[\left| \sum_{g \in \mathcal{G}} n_g^{-1} \sum_{i_1=i_g^{\min}+1}^{i_g^{\max}} \sum_{i_2=i_g^{\min}}^{i_1-1} \left(\sum_{m \in \mathcal{M}} T_m^{-1} \left(\sum_{t \in I_m} \varphi_1(z_{i_1,t}) \right) \left(\sum_{t \in I_m} \varphi_2(z_{i_2,t}) \right) \right) \right|^2 \right] \\
&= \sum_{g \in \mathcal{G}} n_g^{-2} \mathbb{E} \left[\left| \sum_{i_1=i_g^{\min}+1}^{i_g^{\max}} \sum_{i_2=i_g^{\min}}^{i_1-1} \left(\sum_{m \in \mathcal{M}} T_m^{-1} \left(\sum_{t \in I_m} \varphi_1(z_{i_1,t}) \right) \left(\sum_{t \in I_m} \varphi_2(z_{i_2,t}) \right) \right) \right|^2 \right] \\
&= \sum_{g \in \mathcal{G}} n_g^{-2} \sum_{i_1=i_g^{\min}+1}^{i_g^{\max}} \sum_{i_2=i_g^{\min}}^{i_1-1} \mathbb{E} \left[\left| \sum_{m \in \mathcal{M}} T_m^{-1} \left(\sum_{t \in I_m} \varphi_1(z_{i_1,t}) \right) \left(\sum_{t \in I_m} \varphi_2(z_{i_2,t}) \right) \right|^2 \right]. \tag{S576}
\end{aligned}$$

For any $m_1, m_2 \in \mathcal{M}$,

$$\begin{aligned}
& \left| \mathbb{E} \left[(T_{m_1} T_{m_2})^{-1} \left(\sum_{t \in I_{m_1}} \varphi_1(z_{i_1,t}) \right) \left(\sum_{t \in I_{m_1}} \varphi_2(z_{i_2,t}) \right) \left(\sum_{t \in I_{m_2}} \varphi_1(z_{i_1,t}) \right) \left(\sum_{t \in I_{m_2}} \varphi_2(z_{i_2,t}) \right) \right] \right| \\
&= \left| T_{m_1}^{-1} \text{Cov} \left(\sum_{t \in I_{m_1}} \varphi_1(z_{i_1,t}), \sum_{t \in I_{m_2}} \varphi_1(z_{i_1,t}) \right) \right| \left| T_{m_2}^{-1} \text{Cov} \left(\sum_{t \in I_{m_1}} \varphi_2(z_{i_2,t}), \sum_{t \in I_{m_2}} \varphi_2(z_{i_2,t}) \right) \right| \\
&\leq K a^{2|m_2-m_1|\delta'/(2+\delta')} \max_{m \in \mathcal{M}} \left\| T_m^{-1/2} \sum_{t \in I_m} \varphi_1(z_{i_1,t}) \right\|_{2+\delta'}^2 \max_{m \in \mathcal{M}} \left\| T_m^{-1/2} \sum_{t \in I_m} \varphi_2(z_{i_2,t}) \right\|_{2+\delta'}^2 \\
&\leq K a^{2|m_2-m_1|\delta'/(2+\delta')} \tag{S577}
\end{aligned}$$

where the first inequality is by Corollary A.2. in Hall and Heyde (1980) and Assumption 1(iv), the second inequality is by (S575) and Hölder's inequality. Using (S577), we can show that

$$\mathbb{E} \left[\left| \sum_{m \in \mathcal{M}} T_m^{-1} \left(\sum_{t \in I_m} \varphi_1(z_{i_1,t}) \right) \left(\sum_{t \in I_m} \varphi_2(z_{i_2,t}) \right) \right|^2 \right] \leq KM. \tag{S578}$$

The assertion in (S572) follows from (S576) and (S578).

Q.E.D.

Lemma S51. Let $\hat{\varphi}_{j,g,m} \equiv (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \varphi_j(z_{i,t})$, where $\varphi_j(\cdot): \mathcal{Z} \rightarrow \mathbb{R}$ are functions on the support $\mathcal{Z}$ of $z_{i,t}$ for $j = 1, 2$. Then under the conditions of Lemma S50, we have

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} = \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} [\hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m}] + O_p((GM)^{1/2}) = O_p(GM). \tag{S579}$$

PROOF OF LEMMA S51. First note that by Assumption 1(iii),

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} [\hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m}] = n_g^{-1} \mathbb{E} \left[T_m^{-1} \sum_{t_1, t_2 \in I_m} \varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}) \right]$$

where $\sum_{t_1, t_2 \in I_m}$ denotes the double summation over $t_1, t_2 \in I_m$. Therefore, for any $g \in \mathcal{G}$ and $m \in \mathcal{M}$, we can express the equations

$$\begin{aligned}
& \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} - \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} [\hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m}] \\
&= \sum_{g \in \mathcal{G}} n_g^{-1} \sum_{i \in I_g} \sum_{m \in \mathcal{M}} T_m^{-1} \sum_{t_1, t_2 \in I_m} (\varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}))^* \\
&+ \sum_{g \in \mathcal{G}} n_g^{-1} \sum_{i_1 = i_g^{\min} + 1}^{i_g^{\max}} \sum_{i_2 = i_g^{\min}}^{i_1 - 1} \left(\sum_{m \in \mathcal{M}} T_m^{-1} \left(\sum_{t \in I_m} \varphi_1(z_{i_1,t}) \right) \left(\sum_{t \in I_m} \varphi_2(z_{i_2,t}) \right) \right) \\
&+ \sum_{g \in \mathcal{G}} n_g^{-1} \sum_{i_1 = i_g^{\min} + 1}^{i_g^{\max}} \sum_{i_2 = i_g^{\min}}^{i_1 - 1} \left(\sum_{m \in \mathcal{M}} T_m^{-1} \left(\sum_{t \in I_m} \varphi_2(z_{i_1,t}) \right) \left(\sum_{t \in I_m} \varphi_1(z_{i_2,t}) \right) \right) \quad (\text{S580})
\end{aligned}$$

where $(\varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}))^* \equiv \varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}) - \mathbb{E} [\varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2})]$. For any $m \in \mathcal{M}$, we can use the triangle inequality, Hölder's inequality, Corollary A.2. in Hall and Heyde (1980) and Assumption 1(iv) to deduce that

$$\begin{aligned}
& \left| \mathbb{E} \left[T_m^{-1} \sum_{t_1, t_2 \in I_m} \varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}) \right] \right| \\
&\leq T_m^{-1} \sum_{t \in I_m} |\mathbb{E} [\varphi_1(z_{i,t}) \varphi_2(z_{i,t})]| + T_m^{-1} \sum_{l=1}^{T_m-1} \sum_{t=t_m^{\min}}^{t_m^{\max}-l} |\mathbb{E} [\varphi_1(z_{i,t}) \varphi_2(z_{i,t+l})]| \\
&+ T_m^{-1} \sum_{l=1}^{T_m-1} \sum_{t=t_m^{\min}}^{t_m^{\max}-l} |\mathbb{E} [\varphi_2(z_{i,t}) \varphi_1(z_{i,t+l})]| \\
&\leq K \left(1 + \sum_{l=1}^{T_m-1} (1 - l T_m^{-1}) a^{l\delta/(2+\delta)} \right) \max_{j=1,2} \sup_{i,t} \|\varphi_j(z_{i,t})\|_{2+\delta}^2 \leq K, \quad (\text{S581})
\end{aligned}$$

where $t_m^{\max} \equiv \max(I_m)$ and $t_m^{\min} \equiv \min(I_m)$, and the last inequality is by condition (ii) of Lemma S50. This implies that

$$\left| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} [\hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m}] \right| \leq K(GM). \quad (\text{S582})$$

By Assumption 1(iii) and (S571) in Lemma S50,

$$\begin{aligned}
& \text{Var} \left(\sum_{g \in \mathcal{G}} n_g^{-1} \sum_{i \in I_g} \sum_{m \in \mathcal{M}} T_m^{-1} \sum_{t_1, t_2 \in I_m} \varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}) \right) \\
&= \sum_{g \in \mathcal{G}} n_g^{-2} \sum_{i \in I_g} \text{Var} \left(\sum_{m \in \mathcal{M}} T_m^{-1} \sum_{t_1, t_2 \in I_m} \varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}) \right) \leq K(GM) \max_{g \in \mathcal{G}} n_g^{-1}
\end{aligned}$$

which together with Markov's inequality and (S581) implies that

$$\sum_{g \in \mathcal{G}} n_g^{-1} \sum_{i \in I_g} \sum_{m \in \mathcal{M}} T_m^{-1} \sum_{t_1, t_2 \in I_m} (\varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}))^* = O_p((GM)^{1/2} \max_{g \in \mathcal{G}} n_g^{-1/2}). \quad (\text{S583})$$

By Markov's inequality and (S572) in Lemma S50, we have

$$(GM)^{-1/2} \sum_{g \in \mathcal{G}} n_g^{-1} \sum_{i_1 = i_g^{\min} + 1}^{i_g^{\max}} \sum_{i_2 = i_g^{\min}}^{i_1 - 1} \left(\sum_{m \in \mathcal{M}} T_m^{-1} \left(\sum_{t \in I_m} \varphi_1(z_{i_1,t}) \right) \left(\sum_{t \in I_m} \varphi_2(z_{i_2,t}) \right) \right) = O_p(1) \quad (\text{S584})$$

and

$$(GM)^{-1/2} \sum_{g \in \mathcal{G}} n_g^{-1} \sum_{i_1 = i_g^{\min} + 1}^{i_g^{\max}} \sum_{i_2 = i_g^{\min}}^{i_1 - 1} \left(\sum_{m \in \mathcal{M}} T_m^{-1} \left(\sum_{t \in I_m} \varphi_2(z_{i_1,t}) \right) \left(\sum_{t \in I_m} \varphi_1(z_{i_2,t}) \right) \right) = O_p(1), \quad (\text{S585})$$

which together with (S580) and (S583) shows the first equality in (S579). The second equality in (S579) holds by (S582). Q.E.D.

Lemma S52. *Let $\varphi_j(\cdot): \mathcal{Z} \rightarrow \mathbb{R}$ be functions on the support $\mathcal{Z}$ of $z_{i,t}$ for $j = 1, 2, 3$. Suppose that for $j = 1, 2, 3$: (i) $\mathbb{E}[\varphi_j(z_{i,t})] = 0$; (ii) $\sup_{i,t} \mathbb{E}[|\varphi_j(z_{i,t})|^{6+\delta}] < K$ for some $\delta > 0$. Then under Assumption 1, we have*

$$(GM)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} = O_p \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left((n_g T_m)^{-1/2} + (GM)^{1/2} (n_g T_m)^{-1} \right) \right)$$

where $\hat{\varphi}_{j,g,m} \equiv (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \varphi_j(z_{i,t})$ for $j = 1, 2, 3$.

PROOF OF LEMMA S52. To prove the lemma, it is sufficient to show that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} |(n_g T_m) \mathbb{E}[\hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m}]| \leq K \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \quad (\text{S586})$$

and

$$\text{Var} \left(\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} \right) \leq K(GM) \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1}. \quad (\text{S587})$$

Because by the triangle inequality,

$$\begin{aligned} & \left| (GM)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} \right| \\ & \leq \left| (GM)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} - \mathbb{E}[\hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m}]) \right| \\ & \quad + (GM)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} |(n_g T_m) \mathbb{E}[\hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m}]| \\ & = O_p \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left((n_g T_m)^{-1/2} + (GM)^{1/2} (n_g T_m)^{-1} \right) \right), \end{aligned} \quad (\text{S588})$$

where equality follows by (S586), (S587) and Markov's inequality. The assertion of the lemma follows directly from (S588).

We first show (S586). By the independence assumption in Assumption 1 and condition (i) in the lemma, we have

$$\mathbb{E}[(n_g T_m)^2 \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m}] = (n_g T_m)^{-1} \sum_{i \in I_g} \mathbb{E} \left[\sum_{t_1, t_2, t_3 \in I_m} \varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}) \varphi_3(z_{i,t_3}) \right], \quad (\text{S589})$$

where $\sum_{t_1, t_2, t_3 \in I_m}$ denotes the triple summation $\sum_{t_1 \in I_m} \sum_{t_2 \in I_m} \sum_{t_3 \in I_m}$. Next, we write

$$\begin{aligned} \sum_{t_1, t_2, t_3 \in I_m} \varphi_1(z_{i,t_1}) \varphi_2(z_{i,t_2}) \varphi_3(z_{i,t_3}) &= \sum_{t \in I_m} \varphi_1(z_{i,t}) \varphi_2(z_{i,t}) \varphi_3(z_{i,t}) \\ &+ \sum_{(a,b,c) \in S_{1,m}} \sum_{t_1 < t_2 < t_3} \varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_2}) \varphi_c(z_{i,t_3}) \\ &+ 2^{-1} \sum_{(a,b,c) \in S_{1,m}} \sum_{t_1 \neq t_2} \varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_1}) \varphi_c(z_{i,t_2}), \end{aligned} \quad (\text{S590})$$

where $S_{1,m} \equiv \{(a,b,c) : a, b, c \in \{1, 2, 3\} \text{ and } a \neq b \neq c\}$, $\sum_{t_1 < t_2 < t_3}$ denotes the summation over $t_1, t_2, t_3 \in I_m$ with $t_1 < t_2 < t_3$, and similar definition applies to $\sum_{t_1 \neq t_2}$. By condition (ii) of the lemma, the triangle inequality and Hölder's inequality

$$\left| \mathbb{E} \left[\sum_{t \in I_m} \varphi_1(z_{i,t}) \varphi_2(z_{i,t}) \varphi_3(z_{i,t}) \right] \right| \leq \sum_{t \in I_m} |\mathbb{E}[\varphi_1(z_{i,t}) \varphi_2(z_{i,t}) \varphi_3(z_{i,t})]| \leq K T_m. \quad (\text{S591})$$

Consider any $(a,b,c) \in S_{1,m}$, we next study the moment of $\varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_2}) \varphi_c(z_{i,t_3})$ for any $t_1 < t_2 < t_3$. For this purpose, let $l \equiv \max\{t_2 - t_1, t_3 - t_2\}$. By Corollary A.2. in Hall and Heyde (1980) and Assumption 1(iv), we have

$$|\mathbb{E}[\varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_2}) \varphi_c(z_{i,t_3})]| \leq K a^{l\delta/(4+\delta)} \|\varphi_a(z_{i,t_1})\|_{2+\delta/2} \|\varphi_b(z_{i,t_2}) \varphi_c(z_{i,t_3})\|_{2+\delta/2} \leq K a^{l\delta/(4+\delta)}$$

when $l = t_2 - t_1$. The same upper bound $K a^{l\delta/(4+\delta)}$ applies to $|\mathbb{E}[\varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_2}) \varphi_c(z_{i,t_3})]|$ when $l = t_3 - t_2$. Therefore for any $t_1 < t_2 < t_3$ with $l \equiv \max\{t_2 - t_1, t_3 - t_2\}$, we have

$$|\mathbb{E}[\varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_2}) \varphi_c(z_{i,t_3})]| \leq K a^{l\delta/(4+\delta)}. \quad (\text{S592})$$

Since $t_1 < t_2 < t_3$, it is clear that $1 \leq l \leq T_m - 1$. Let

$$S_{2,m,l} \equiv \{(t_1, t_2, t_3) : t_1, t_2, t_3 \in I_m, t_1 < t_2 < t_3 \text{ and } l = \max\{t_2 - t_1, t_3 - t_2\}\}.$$

Then $\#S_{2,m,l} \leq 2l(T_m - l)$. Therefore,

$$\begin{aligned}
\left| \mathbb{E} \left[\sum_{t_1 < t_2 < t_3} \varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_2}) \varphi_c(z_{i,t_3}) \right] \right| &\leq \sum_{t_1 < t_2 < t_3} \mathbb{E} [|\varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_2}) \varphi_c(z_{i,t_3})|] \\
&= \sum_{l=1}^{T_m-1} \sum_{(t_1, t_2, t_3) \in S_{2,m,l}} \mathbb{E} [|\varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_2}) \varphi_c(z_{i,t_3})|] \\
&\leq K \sum_{l=1}^{T_m-1} l(T_m - l) a^{l\delta/(4+\delta)} \leq KT_m
\end{aligned} \tag{S593}$$

where the second inequality is by (S592), and the last inequality is by $a \in (0, 1)$. By the same arguments for deriving (S592),

$$|\mathbb{E} [\varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_1}) \varphi_c(z_{i,t_2})]| \leq K a^{|t_1 - t_2|\delta/(4+\delta)}$$

which implies that

$$\begin{aligned}
\left| \mathbb{E} \left[\sum_{t_1 \neq t_2} \varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_1}) \varphi_c(z_{i,t_2}) \right] \right| &\leq \sum_{t_1 \neq t_2} \mathbb{E} [|\varphi_a(z_{i,t_1}) \varphi_b(z_{i,t_1}) \varphi_c(z_{i,t_2})|] \\
&\leq K \sum_{t_1 \neq t_2} a^{|t_1 - t_2|\delta/(4+\delta)} \\
&\leq K \sum_{l=1}^{T_m-1} (T_m - l) a^{l\delta/(4+\delta)} \leq KT_m.
\end{aligned} \tag{S594}$$

Since the size of $\#S_{1,m} = 6$, combining the results in (S590), (S591), (S593) and (S594) we get

$$\left| \mathbb{E} \left[T_m^{-1} \left(\sum_{t \in I_m} \varphi_1(z_{i,t}) \right) \left(\sum_{t \in I_m} \varphi_2(z_{i,t}) \right) \left(\sum_{t \in I_m} \varphi_3(z_{i,t}) \right) \right] \right| \leq K,$$

which together with (S589) and the triangle inequality shows that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} |(n_g T_m)^2 \mathbb{E} [\hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m}]| \leq K. \tag{S595}$$

The desired result in (S586) follows by (S595).

We next prove (S587). By the same arguments for showing (S573),

$$\begin{aligned}
&\text{Var} \left(M^{-1/2} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} \right) \\
&\leq K \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} \right\|_{2+\delta'}^2 \\
&\leq K \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| (n_g T_m)^{3/2} \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} \right\|_{2+\delta'}^2
\end{aligned} \tag{S596}$$

where $\delta' = \delta/3$. Using similar arguments for deriving (S558), we can show that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \mathbb{E} \left[\left| (n_g T_m)^{1/2} \hat{\varphi}_{j,g,m} \right|^{6+\delta} \right] \leq K$$

which together with (S596) and Hölder's inequality implies that

$$\max_{g \in \mathcal{G}} \text{Var} \left(M^{-1/2} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} \right) \leq K \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1}. \quad (\text{S597})$$

By Assumption 1(iii) and (S597)

$$\begin{aligned} \text{Var} \left(\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} \right) &= \sum_{g \in \mathcal{G}} \text{Var} \left(\sum_{m \in \mathcal{M}} (n_g T_m) \hat{\varphi}_{1,g,m} \hat{\varphi}_{2,g,m} \hat{\varphi}_{3,g,m} \right) \\ &\leq K(GM) \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1}, \end{aligned}$$

which shows (S587). Q.E.D.

Lemma S53. *Under Assumptions 1, 8, 9 and 10, we have*

$$\begin{aligned} (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \\ - (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} \left[(\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] = O_p((GM)^{1/2} (nT)^{-1}), \end{aligned} \quad (\text{S598})$$

where $\hat{\Psi}_{\gamma,g,m} \equiv (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \psi_{\gamma}(z_{i,t})$ for any $g \in \mathcal{G}$ and $m \in \mathcal{M}$.

The proof of (S598) of Lemma S53 is omitted since it directly follows from (S579) of Lemma S51.

Lemma S54. *Under Assumptions 1, 8, 9 and 10, we have*

$$\begin{aligned} (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta_{\gamma,g,m}} (\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) \hat{\Psi}_{\gamma,g,m} \\ + (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma\gamma,g,m}^{-1} \mathbb{E} \left[(\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] \\ = O_p \left((1 + (GM)^{1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2}) (GM)^{1/2} (nT)^{-1} \right). \end{aligned} \quad (\text{S599})$$

PROOF OF LEMMA S54. Since

$$\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1} = -\Psi_{\gamma\gamma,g,m}^{-1} (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} - (\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1},$$

we can derive the following equations:

$$\begin{aligned}
& \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} (\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) \hat{\Psi}_{\gamma,g,m} \\
&= - \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \\
&\quad - \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} (\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \quad (\text{S600})
\end{aligned}$$

where

$$\begin{aligned}
& \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \\
&\quad - \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \mathbb{E} \left[(\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] = O_p((GM)^{1/2}) \quad (\text{S601})
\end{aligned}$$

by Lemma S51. Next, we will examine the second term on the right-hand side (RHS) of equation (S600). By the triangle inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left\| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} (\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\| \\
&\leq \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| \Psi_{\theta\gamma,g,m} (\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \psi_\gamma(z_{i,t}) \right\| \\
&= \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| \Psi_{\theta\gamma,g,m} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \psi_\gamma(z_{i,t}) \right\| \\
&\leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}(\Psi_{\theta\gamma,g,m} \Psi_{\gamma\theta,g,m}) \lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} \lambda_{\max}((\Psi_{\gamma\gamma,g,m}^{-1})^2) \\
&\quad \times \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right\|^2 \left\| \hat{\Psi}_{\gamma,g,m} \right\|, \quad (\text{S602})
\end{aligned}$$

where $\max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}(\Psi_{\theta\gamma,g,m} \Psi_{\gamma\theta,g,m}) \lambda_{\max}((\Psi_{\gamma\gamma,g,m}^{-1})^2)) \leq K$ because Assumption 9, (S620) of Lemma S60, and $\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2) = O_p(1)$ implied by Lemma S60(ii).

Equations (S561) and (S562) in Lemma S48 together with Markov's inequality and Hölder's inequality imply that

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right\|^2 \left\| \hat{\Psi}_{\gamma,g,m} \right\| = O_p((GM) \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2}). \quad (\text{S603})$$

Collecting the results in (S602) and (S603) obtains

$$\begin{aligned}
& \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} (\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \\
&= O_p((GM) \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2}). \quad (\text{S604})
\end{aligned}$$

The assertion of the lemma now follows from (S600), (S601) and (S604). Q.E.D.

Lemma S55. *Under Assumptions 1, 8, 9 and 10, we have*

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} \left[(\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}}) \Psi_{\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] = O(GM) \quad (\text{S605})$$

and

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma,g,m}^{-1} \mathbb{E} \left[(\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] = O(GM). \quad (\text{S606})$$

The proofs of (S605) and (S606) of Lemma S55 are omitted since they directly follow from (S582) in the proof of Lemma S51.

Lemma S56. *Under Assumptions 1, 8, 9 and 10, we have*

$$\begin{aligned} & \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta_{\gamma,g,m}} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma\gamma,g,m}^{-1}) \hat{\Psi}_{\gamma,g,m} \\ &= \sum_{g \in \mathcal{G}} \sum_{i \in I_g} (n_g T_m) \mathbb{E} \left[(\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma\gamma,g,m}^{-1} (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m})) \Psi_{\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] \\ &+ O_p \left((1 + (GM)^{1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2}) (GM)^{1/2} \right) = O_p(GM). \end{aligned}$$

PROOF OF LEMMA S56. By the triangle inequality, the Cauchy-Schwarz inequality, and (S620) of Lemma S60, we have

$$\begin{aligned} & \left\| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}}) (\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) \hat{\Psi}_{\gamma,g,m} \right\| \\ & \leq \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| (n_g T_m) (\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}}) \hat{\Psi}_{\gamma\gamma,g,m}^{-1} (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \Psi_{\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\| \\ & \leq K \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} \\ & \quad \times \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| \hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}} \right\| \left\| \hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right\| \left\| (n_g T_m) \hat{\Psi}_{\gamma,g,m} \right\|. \quad (\text{S607}) \end{aligned}$$

By Assumption 8(ii), and (S553) in Lemma S47, we have

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \mathbb{E} \left[\left\| (n_g T_m)^{1/2} (\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}}) \right\|^4 \right] \leq K \quad (\text{S608})$$

which together with Lemma S48, Markov's inequality and Hölder's inequality implies that

$$\begin{aligned} & \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| \hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}} \right\| \left\| \hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right\| \left\| (n_g T_m) \hat{\Psi}_{\gamma,g,m} \right\| \\ &= O_p((GM) \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2}). \quad (\text{S609}) \end{aligned}$$

Therefore by Lemma S60(ii), (S607) and (S609)

$$\begin{aligned} & \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}}) (\hat{\Psi}_{\gamma_{\gamma,g,m}}^{-1} - \Psi_{\gamma_{\gamma,g,m}}^{-1}) \hat{\Psi}_{\gamma_{\gamma,g,m}} \\ &= O_p((GM) \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2}). \end{aligned} \quad (\text{S610})$$

The claim of the lemma follows from Lemma S53, Lemma S54, Lemma S55 and (S610). *Q.E.D.*

3.2 Intermediate Results

Lemma S57. *Under Assumptions 1, 8 and 10, we have*

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} (D^\nu \psi(z_{i,t}, \phi) - \mathbb{E}[D^\nu \psi(z_{i,t}, \phi)]) \right| = O_p((GM)^{1/p}), \quad (\text{S611})$$

and

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \left| \frac{(n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} (D^\nu \psi(z_{i,t}, \phi) - \mathbb{E}[D^\nu \psi(z_{i,t}, \phi^*)])}{(GM)^{1/p} (n_g T_m)^{-1/2} + \|\phi - \phi^*\|} \right| = O_p(1) \quad (\text{S612})$$

for any $\nu \equiv (\nu_1, \dots, \nu_k)^\top$ with $|\nu| = \sum_{j=1}^k \nu_j \leq 3$.

The proofs of (S611) and (S612) of Lemma S57 are omitted since they directly follow from Lemma S49.

Lemma S58. *Under Assumptions 1, 7, 8 and 10(i), $\|\hat{\theta} - \theta^*\| + \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| = o_p(1)$.*

PROOF OF LEMMA S58. Let $\psi^*(z_{i,t}; \theta, \gamma) \equiv \psi(z_{i,t}; \theta, \gamma) - \mathbb{E}[\psi(z_{i,t}; \theta, \gamma)]$. Without loss of generality, let $\Phi = \Phi_\theta \times \Phi_\gamma$ where Φ_θ and Φ_γ include θ^* and $\gamma_{g,m}^*$, respectively. By the triangle inequality,

$$\begin{aligned} & \sup_{\theta, \{\gamma_{g,m}\}_{g \in \mathcal{G}, m \in \mathcal{M}}} \left| (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \sum_{i \in I_g} \sum_{t \in I_m} \psi^*(z_{i,t}; \theta, \gamma_{g,m}) \right| \\ & \leq (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m)^{1/2} \sup_{\phi \in \Phi} \left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \psi^*(z_{i,t}; \phi) \right| \\ & \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \psi^*(z_{i,t}; \phi) \right| (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m)^{1/2}, \end{aligned} \quad (\text{S613})$$

where $\sup_{\theta, \{\gamma_{g,m}\}_{g \in \mathcal{G}, m \in \mathcal{M}}}$ is taken over $\theta \in \Phi_\theta$ and $\gamma_{g,m} \in \Phi_\gamma$ for $g \in \mathcal{G}$ and $m \in \mathcal{M}$. Under Assumptions 1 and 8, we can apply (S611) of Lemma S57 to obtain

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\phi \in \Phi} \left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \psi^*(z_{i,t}; \phi) \right| = O_p((GM)^{1/p}). \quad (\text{S614})$$

Since $\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} n_g T_m = nT$, we have

$$(nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m)^{1/2} \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2},$$

which together with (S613) and (S614), and Assumption 10(i) implies that

$$\sup_{\theta, \{\gamma_{g,m}\}_{g \in \mathcal{G}, m \in \mathcal{M}}} \left| (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \sum_{i \in I_g} \sum_{t \in I_m} \psi^*(z_{i,t}; \theta, \gamma_{g,m}) \right| = o_p(1). \quad (\text{S615})$$

Recall that $\Psi_{g,m}(\theta, \gamma) \equiv (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \mathbb{E}[\psi(z_{i,t}; \theta, \gamma)]$. For any $\eta > 0$,

$$\begin{aligned} & \mathbb{P}(\|\hat{\theta} - \theta^*\| \geq \eta) \\ & \leq \mathbb{P}\left(\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \sum_{i \in I_g} \sum_{t \in I_m} \psi(z_{i,t}; \theta^*, \gamma_{g,m}^*) \leq \sup_{\|\theta - \theta^*\| \geq \eta, \{\gamma_{g,m}\}_{g,m}} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \sum_{i \in I_g} \sum_{t \in I_m} \psi(z_{i,t}; \theta, \gamma_{g,m})\right) \\ & \leq \mathbb{P}\left(o_p(1) \leq \sup_{\|\theta - \theta^*\| \geq \eta, \{\gamma_{g,m}\}_{g,m}} (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) [\Psi_{g,m}(\theta, \gamma_{g,m}) - \Psi_{g,m}(\theta^*, \gamma_{g,m}^*)]\right) \\ & \leq \mathbb{P}\left(o_p(1) \leq (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left[\sup_{\{\phi \in \Phi: \|\phi - \phi^*\| > \eta\}} \Psi_{g,m}(\phi) - \Psi_{g,m}(\phi^*) \right]\right) \\ & \leq \mathbb{P}\left(o_p(1) \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left[\sup_{\{\phi \in \Phi: \|\phi - \phi^*\| > \eta\}} \Psi_{g,m}(\phi) - \Psi_{g,m}(\phi^*) \right]\right) \rightarrow 0, \text{ as } n, T \rightarrow \infty, \end{aligned}$$

where $\sup_{\|\theta - \theta^*\| \geq \eta, \{\gamma_{g,m}\}_{g,m}}$ is taken over $\theta \in \Phi_\theta$ with $\|\theta - \theta^*\| \geq \eta$ and over $\gamma_{g,m} \in \Phi_\gamma$ for any $g \in \mathcal{G}$ and any $m \in \mathcal{M}$, the second inequality is by (S615), and the convergence is by Assumption 7. This shows the consistency of $\hat{\theta}$.

To show the consistency of $\hat{\gamma}_{g,m}$, we first note that by the definition of $\hat{\gamma}_{g,m}$

$$\hat{\gamma}_{g,m} = \arg \max_{\gamma_{g,m} \in \Phi_\gamma} \sum_{i \in I_g} \sum_{t \in I_m} \psi(z_{i,t}; \hat{\theta}, \gamma_{g,m})$$

for any $g \in \mathcal{G}$ and any $m \in \mathcal{M}$. Therefore for any $\eta > 0$, we have

$$\begin{aligned} & \mathbb{P}\left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| \geq \eta\right) \\ & \leq \mathbb{P}\left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\{\gamma_{g,m} \in \Phi_\gamma: \|\gamma_{g,m} - \gamma_{g,m}^*\| \geq \eta\}} \sum_{i \in I_g} \sum_{t \in I_m} [\psi(z_{i,t}; \hat{\theta}, \gamma_{g,m}) - \psi(z_{i,t}; \hat{\theta}, \gamma_{g,m}^*)] \geq 0\right) \quad (\text{S616}) \end{aligned}$$

By Assumption 8(i) and the triangle inequality,

$$\begin{aligned}
& \max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\gamma_{g,m} \in \Phi_\gamma} \left| (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \left[\psi(z_{i,t}; \hat{\theta}, \gamma_{g,m}) - \psi(z_{i,t}; \theta^*, \gamma_{g,m}) \right] \right| \\
& \leq \|\hat{\theta} - \theta^*\| \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} M(z_{i,t}) \\
& \leq \|\hat{\theta} - \theta^*\| \left(\sup_{i,t} \mathbb{E}[M(z_{i,t})] + \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left| (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} M^*(z_{i,t}) \right| \right). \tag{S617}
\end{aligned}$$

where $M^*(z_{i,t}) \equiv M(z_{i,t}) - \mathbb{E}[M(z_{i,t})]$. Applying (S611) of Lemma S57 yields

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left| (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} M^*(z_{i,t}) \right| = O_p \left((GM)^{1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right) = o_p(1) \tag{S618}$$

where the second equality is by Assumption 10(i). Since $\sup_{i,t} \mathbb{E}[M(z_{i,t})] \leq K$ by Assumption 8(ii), we can deduce from (S617), (S618) and the consistency of $\hat{\theta}$ that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\gamma_{g,m} \in \Phi_\gamma} \left| (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \left[\psi(z_{i,t}; \hat{\theta}, \gamma_{g,m}) - \psi(z_{i,t}; \theta^*, \gamma_{g,m}) \right] \right| = o_p(1). \tag{S619}$$

Invoking (S611) of Lemma S57 and Assumption 10(i) obtains

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\gamma_{g,m} \in \Phi_\gamma} \left| (n_g T_m)^{-1/2} \sum_{i \in I_g} \sum_{t \in I_m} \psi^*(z_{i,t}; \theta^*, \gamma_{g,m}) \right| = o_p(1)$$

which together with (S616), (S619) and Assumption 7 implies that

$$\begin{aligned}
& \mathbb{P} \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| \geq \eta \right) \\
& \leq \mathbb{P} \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\{\gamma_{g,m} \in \Phi_\gamma: \|\gamma_{g,m} - \gamma_{g,m}^*\| \geq \eta\}} (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} [\psi(z_{i,t}; \theta^*, \gamma_{g,m}) - \psi(z_{i,t}; \theta^*, \gamma_{g,m}^*)] \geq o_p(1) \right) \\
& \leq \mathbb{P} \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \sup_{\{\gamma_{g,m} \in \Phi_\gamma: \|\gamma_{g,m} - \gamma_{g,m}^*\| \geq \eta\}} [\Psi_{g,m}(\theta^*, \gamma) - \Psi_{g,m}(\theta^*, \gamma_{g,m}^*)] \geq o_p(1) \right) \\
& \leq \mathbb{P} \left(\min_{g \in \mathcal{G}, m \in \mathcal{M}} \left[\Psi_{g,m}(\phi^*) - \sup_{\{\phi \in \Phi: \|\phi - \phi^*\| > \eta\}} \Psi_{g,m}(\phi) \right] \leq o_p(1) \right) \rightarrow 0, \text{ as } n, T \rightarrow \infty,
\end{aligned}$$

which shows $\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| = o_p(1)$.

Q.E.D.

Lemma S59. *Under Assumptions 1, 8, 9 and 10, we have*

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\Psi}_{\theta\theta,g,m} - \Psi_{\theta\theta,g,m}\| + \|\hat{\Psi}_{\theta\gamma,g,m} - \Psi_{\theta\gamma,g,m}\| + \|\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}\|}{(GM)^{1/p} (n_g T_m)^{-1/2}} = O_p(1).$$

PROOF OF LEMMA S59. Follows from (S611) in Lemma S57.

Q.E.D.

Lemma S60. *Under Assumptions 1, 8, 9 and 10, we have*

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} (\|\Psi_{\theta\theta,g,m}\| + \|\Psi_{\theta\gamma,g,m}\| + \|\Psi_{\gamma\gamma,g,m}\|) \leq K. \quad (\text{S620})$$

We also have with probability approaching 1 (wpa1),

$$\begin{aligned} (i) \quad & \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\|\hat{\Psi}_{\theta\theta,g,m}\| + \|\hat{\Psi}_{\theta\gamma,g,m}\| + \|\hat{\Psi}_{\gamma\gamma,g,m}\|) \leq K; \\ (ii) \quad & \min_{g \in \mathcal{G}, m \in \mathcal{M}} \lambda_{\min}(-\hat{\Psi}_{\gamma\gamma,g,m}) > K^{-1}; \\ (iii) \quad & \min_{g \in \mathcal{G}, m \in \mathcal{M}} \lambda_{\min}(\hat{\Psi}_{\theta\gamma,g,m} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma\theta,g,m} - \hat{\Psi}_{\theta\theta,g,m}) > K^{-1}. \end{aligned}$$

PROOF OF LEMMA S60. (i) Equation (S620) follows from Assumption 8(iii). Lemma S59 and Assumption 10(i) imply that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \hat{\Psi}_{\theta\theta,g,m} - \Psi_{\theta\theta,g,m} \right\| + \left\| \hat{\Psi}_{\theta\gamma,g,m} - \Psi_{\theta\gamma,g,m} \right\| + \left\| \hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right\| = o_p(1). \quad (\text{S621})$$

The assertion in part (i) of the lemma now follows from (S621) and (S620).

(ii) Since $\min_{g \in \mathcal{G}, m \in \mathcal{M}} \lambda_{\min}(-\Psi_{\gamma\gamma,g,m}) > K^{-1}$ by Assumption 9, the assertion in this part of the lemma follows from (S621).

(iii) By the assertions in parts (i) and (ii) of the lemma, (S621) and (S620), we can conclude that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| (\hat{\Psi}_{\theta\gamma,g,m} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma\theta,g,m} - \hat{\Psi}_{\theta\theta,g,m}) - (\Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \Psi_{\gamma\theta,g,m} - \Psi_{\theta\theta,g,m}) \right\| = o_p(1).$$

This, combined with Assumption 9, shows the assertion in part (iii) of the lemma.

Q.E.D.

Lemma S61. *Let $\psi_\theta(z_{i,t}; \phi) \equiv \frac{\partial \psi(z_{i,t}; \phi)}{\partial \theta}$ and $\psi_\gamma(z_{i,t}; \phi) \equiv \frac{\partial \psi(z_{i,t}; \phi)}{\partial \gamma}$. The M-estimator $(\theta^\top, \gamma_{\mathcal{G},1}^\top, \dots, \gamma_{\mathcal{G},M}^\top)^\top$ satisfies the first-order condition*

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \sum_{i \in I_g} \sum_{t \in I_m} \psi_\theta(z_{i,t}; \hat{\phi}_{g,m}) = 0_{d_\theta \times 1}, \quad (\text{S622})$$

and hence,

$$0_{d_\gamma \times 1} = \hat{\Psi}_{\gamma,g,m} + \hat{\Psi}_{\gamma\phi,g,m}(\hat{\phi}_{g,m} - \phi_{g,m}^*) + 2^{-1} \left((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \tilde{\Psi}_{\gamma_j\phi\phi,g,m}(\hat{\phi}_{g,m} - \phi_{g,m}^*) \right)_{j \leq d_\gamma}, \quad (\text{S623})$$

where $\tilde{\Psi}_{\gamma_j\phi\phi,g,m} \equiv (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \frac{\partial^2}{\partial \phi \partial \phi^\top} \left(\frac{\partial \psi(z_{i,t}; \tilde{\phi}_{g,m})}{\partial \gamma_j} \right)$ for $j = 1, \dots, d_\gamma$ and $\tilde{\phi}_{g,m}$ lies between $\hat{\phi}_{g,m}$ and $\phi_{g,m}^*$. Under Assumptions 1, 7, 8, 9 and 10, we also have

$$\max_{j \leq d_\gamma} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \frac{\tilde{\Psi}_{\gamma_j\phi\phi,g,m} - \Psi_{\gamma_j\phi\phi,g,m}}{(GM)^{1/p} (n_g T_m)^{-1/2} + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|} \right\| = O_p(1). \quad (\text{S624})$$

PROOF OF LEMMA S62. The M-estimator also satisfies

$$\sum_{i \in I_g} \sum_{t \in I_m} \psi_\gamma(z_{i,t}; \hat{\phi}_{g,m}) = 0_{d_\gamma \times 1} \quad (\text{S625})$$

for any $g \in \mathcal{G}$ and $m \in \mathcal{M}$. Applying the Taylor expansion to (S625), we obtain (S623). By (S612) in Lemma S57, we obtain (S624). Q.E.D.

Lemma S62. *Under Assumptions 1, 7, 8, 9 and 10, we have*

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left\| \hat{\Psi}_{\gamma,g,m} + \hat{\Psi}_{\gamma\theta,g,m}(\hat{\theta} - \theta^*) + \hat{\Psi}_{\gamma\gamma,g,m}(\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \right\|}{(nT)^{-1} + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2} = O_p(1), \quad (\text{S626})$$

$$(nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 = O_p\left((GM)(nT)^{-1} + \|\hat{\theta} - \theta^*\|^2\right), \quad (\text{S627})$$

and

$$\hat{\theta} - \theta^* = O_p((nT)^{-1/2}).$$

PROOF OF LEMMA S62. By (S624) of Lemma S61, Assumptions 8(iii) and 10(i), and Lemma S58,

$$\max_{j \leq d_\gamma} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \tilde{\Psi}_{\gamma_j \phi \phi, g, m} \right\| = O_p(1) \quad (\text{S628})$$

which together with (S623) of Lemma S61 and the Cauchy-Schwarz inequality implies that (S626) holds.

Applying the Taylor expansion to (S622) obtains

$$\begin{aligned} 0_{d_\theta \times 1} &= \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left[\hat{\Psi}_{\theta,g,m} + \hat{\Psi}_{\theta\theta,g,m}(\hat{\theta} - \theta^*) + \hat{\Psi}_{\theta\gamma,g,m}(\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \right] \\ &\quad + 2^{-1} \left(\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \tilde{\Psi}_{\theta_j \phi \phi, g, m} (\hat{\phi}_{g,m} - \phi_{g,m}^*) \right)_{j=1, \dots, d_\theta} \\ &= \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left[\hat{\Psi}_{\theta,g,m} + \hat{\Psi}_{\theta\theta,g,m}(\hat{\theta} - \theta^*) + \hat{\Psi}_{\theta\gamma,g,m}(\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \right] \\ &\quad + O_p \left(\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2 \right), \end{aligned} \quad (\text{S629})$$

where $\tilde{\psi}_{\theta_j \phi \phi}(z_{i,t}) \equiv \psi_{\theta_j \phi \phi}(z_{i,t}; \tilde{\phi}_{g,m})$, $\tilde{\psi}_{\theta_j \phi \phi}(z_{i,t}; \phi) \equiv \frac{\partial^2}{\partial \phi \partial \phi^\top} \left(\frac{\partial \psi(z_{i,t}; \phi)}{\partial \theta_j} \right)$ for $j \leq d_\theta$ and $\tilde{\phi}_{g,m}$ lies between $\hat{\phi}_{g,m}$ and $\phi_{g,m}^*$, the second equality is by the Cauchy-Schwarz inequality and

$$\max_{j \leq d_\theta} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \tilde{\Psi}_{\theta_j \phi \phi, g, m} \right\| = O_p(1), \quad (\text{S630})$$

which can be established using the same arguments for showing (S628).

By (S626) of Lemma S62 and Lemma S60(i, ii),

$$\begin{aligned} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta\gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) &= - \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta\gamma, g, m} \hat{\Psi}_{\gamma\gamma, g, m}^{-1} (\hat{\Psi}_{\gamma, g, m} + \hat{\Psi}_{\gamma\theta, g, m} (\hat{\theta} - \theta^*)) \\ &\quad + O_p \left(1 + \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\phi}_{g, m} - \phi_{g, m}^*\|^2 \right). \end{aligned} \quad (\text{S631})$$

Applying the above representation in (S629), we obtain

$$\begin{aligned} 0_{d_\theta \times 1} &= (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta, g, m} - \hat{\Psi}_{\theta\gamma, g, m} \hat{\Psi}_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m}) \\ &\quad + (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta\theta, g, m} - \hat{\Psi}_{\theta\gamma, g, m} \hat{\Psi}_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma\theta, g, m}) (\hat{\theta} - \theta^*) \\ &\quad + O_p \left((nT)^{-1} + \|\hat{\theta} - \theta^*\|^2 + (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|^2 \right), \end{aligned} \quad (\text{S632})$$

where by Lemma S56, Assumptions 1 and 8(iii),

$$(nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta, g, m} - \hat{\Psi}_{\theta\gamma, g, m} \hat{\Psi}_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m}) = O_p \left((nT)^{-1/2} + (GM)(nT)^{-1} \right). \quad (\text{S633})$$

By (S626) of Lemma S62, Lemma S60(ii) and the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|^2 &\leq K \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| \hat{\Psi}_{\gamma\gamma, g, m}^{-1} (n_g T_m)^{1/2} \hat{\Psi}_{\gamma, g, m} \right\|^2 \\ &\quad + K \|\hat{\theta} - \theta^*\|^2 \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\Psi}_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma\theta, g, m} \right\|^2 \\ &\quad + O_p(1) \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left((nT)^{-2} + \|\hat{\phi}_{g, m} - \phi_{g, m}^*\|^4 \right). \end{aligned} \quad (\text{S634})$$

Note that

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\Psi}_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|^2 \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \lambda_{\max}((\hat{\Psi}_{\gamma\gamma, g, m}^{-1})^2) \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\Psi}_{\gamma, g, m} \right\|^2. \quad (\text{S635})$$

By (S562) in Lemma S48,

$$\mathbb{E} \left[\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| (n_g T_m)^{1/2} \hat{\Psi}_{\gamma, g, m} \right\|^2 \right] \leq K(GM)$$

which together with Markov's inequality implies that

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| (n_g T_m)^{1/2} \hat{\Psi}_{\gamma, g, m} \right\|^2 = O_p(GM). \quad (\text{S636})$$

Hence by Lemma S60(ii), (S635) and (S636),

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| \hat{\Psi}_{\gamma\gamma, g, m}^{-1} (n_g T_m)^{1/2} \hat{\Psi}_{\gamma, g, m} \right\|^2 = O_p(GM). \quad (\text{S637})$$

By Lemma S60(i, ii),

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\Psi}_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma\theta, g, m} \right\|^2 = O_p(nT). \quad (\text{S638})$$

Combining the results in Lemma S58, (S634), (S637) and (S638), we get (S627).

Lemma S60(iii) implies that wpa1,

$$\lambda_{\min} \left(-(nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta\theta, g, m} - \hat{\Psi}_{\theta\gamma, g, m} \hat{\Psi}_{\gamma\gamma, g, m}^{-1} \Psi_{\gamma\theta, g, m}) \right) > K^{-1}, \quad (\text{S639})$$

which together with (S632), (S633) and (S627) shows that

$$\|\hat{\theta} - \theta^*\| = O_p((nT)^{-1/2} + (GM)(nT)^{-1}). \quad (\text{S640})$$

The claim of the lemma follows from (S640) and Assumption 10(ii). Q.E.D.

Lemma S63. *Under Assumptions 1, 7, 8, 9 and 10, we have*

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\phi}_{g, m} - \phi_{g, m}^*)^\top \hat{\Psi}_{\theta_j \phi \phi, g, m} (\hat{\phi}_{g, m} - \phi_{g, m}^*) \\ &= (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} \left[\hat{\Psi}_{\gamma\gamma, g, m}^\top \Psi_{\gamma\gamma, g, m}^{-1} \Psi_{\theta_j \gamma \gamma, g, m} \Psi_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right] + o_p(1), \end{aligned}$$

where $\hat{\Psi}_{\theta_j \phi \phi, g, m} \equiv (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \frac{\partial^2}{\partial \phi \partial \phi^\top} \left(\frac{\partial \psi(z_{i, t}; \phi_{g, m}^*)}{\partial \theta_j} \right)$ for any $j \leq d_\theta$.

PROOF OF LEMMA S63. By (S611) in Lemma S57,

$$\max_{j=1, \dots, d_\theta} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| (n_g T_m)^{1/2} (\hat{\Psi}_{\theta_j \phi \phi, g, m} - \Psi_{\theta_j \phi \phi, g, m}) \right\| = O_p((GM)^{1/p}). \quad (\text{S641})$$

Since

$$\max_{j \leq d_\theta} \left\| \mathbb{E} [\psi_{\theta_j \phi \phi}(z_{i, t})] \right\| \leq K \quad (\text{S642})$$

by Assumption 8(iii), from (S641) and Assumption 10(i), we have

$$\max_{j \leq d_\theta} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \hat{\Psi}_{\theta_j \phi \phi, g, m} \right\| \leq K \quad (\text{S643})$$

wpa1. By the definition of $\hat{\Psi}_{\theta_j \phi \phi, g, m}$, we obtain the following expression:

$$\begin{aligned} (\hat{\phi}_{g, m} - \phi_{g, m}^*)^\top \hat{\Psi}_{\theta_j \phi \phi, g, m} (\hat{\phi}_{g, m} - \phi_{g, m}^*) &= (\hat{\theta} - \theta^*)^\top \hat{\Psi}_{\theta_j \theta \theta, g, m} (\hat{\theta} - \theta^*) \\ &\quad + (\hat{\gamma}_{g, m} - \gamma_{g, m}^*)^\top \hat{\Psi}_{\theta_j \gamma \gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \\ &\quad + 2(\hat{\theta} - \theta^*)^\top \hat{\Psi}_{\theta_j \theta \gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*). \end{aligned} \quad (\text{S644})$$

Applying Lemma S62 and (S643) yields

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\theta} - \theta^*)^\top \hat{\Psi}_{\theta_j \theta, g, m} (\hat{\theta} - \theta^*) = O_p(1). \quad (\text{S645})$$

By the similar arguments for deriving (S631), we can show that

$$\begin{aligned} & \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta_j \theta, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \\ &= - \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta_j \theta, g, m} \hat{\Psi}_{\gamma, g, m}^{-1} \left(\hat{\Psi}_{\gamma, g, m} + \hat{\Psi}_{\gamma \theta, g, m} (\hat{\theta} - \theta^*) \right) \\ & \quad + O_p \left(1 + \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\phi}_{g, m} - \phi_{g, m}^*\|^2 \right) \\ &= - \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta_j \theta, g, m} \hat{\Psi}_{\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} + O_p((nT)^{1/2}), \end{aligned} \quad (\text{S646})$$

where the second equality is by Assumption 10(ii), Lemma S62, Lemma S60(i), (S627) of Lemma S62 and (S643). By the similar arguments for showing Lemma S56, we have

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta_j \theta, g, m} \hat{\Psi}_{\gamma, g, m}^{-1} - \Psi_{\theta_j \theta, g, m} \Psi_{\gamma, g, m}^{-1}) \hat{\Psi}_{\gamma, g, m} = O_p(GM),$$

which together with $\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \Psi_{\theta_j \theta, g, m} \Psi_{\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} = O_p((nT)^{1/2})$ and Assumption 10(ii) implies that

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta_j \theta, g, m} \hat{\Psi}_{\gamma, g, m} \hat{\Psi}_{\gamma, g, m} = O_p((nT)^{1/2}). \quad (\text{S647})$$

Therefore by Lemma S62, (S646) and (S647),

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\theta} - \theta^*)^\top \hat{\Psi}_{\theta_j \theta, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) = O_p(1). \quad (\text{S648})$$

Next note that

$$\begin{aligned} & \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g, m} - \gamma_{g, m}^*)^\top \hat{\Psi}_{\theta_j \gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \\ & \leq \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g, m} - \gamma_{g, m}^*)^\top \Psi_{\theta_j \gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \\ & \quad + \max_{j=1, \dots, d_\theta} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \hat{\Psi}_{\theta_j \gamma, g, m} - \Psi_{\theta_j \gamma, g, m} \right\| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|^2 \\ & = \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g, m} - \gamma_{g, m}^*)^\top \Psi_{\theta_j \gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \\ & \quad + O_p \left((GM)^{1+1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right) \end{aligned} \quad (\text{S649})$$

where the equality is by Lemma S62, (S627) of Lemma S62 and

$$\max_{j=1,\dots,d_\theta} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \hat{\Psi}_{\theta_j \gamma \gamma, g, m} - \Psi_{\theta_j \gamma \gamma, g, m} \right\| = O_p \left((GM)^{1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right) \quad (\text{S650})$$

which holds by the similar arguments for showing Lemma S59. By the Cauchy-Schwarz inequality, (S642) and Lemma S1

$$\begin{aligned} & \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m})^\top \Psi_{\theta_j \gamma \gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m}) \\ & \leq K \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|^2 = O_p \left((GM)^{1+4/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \right). \end{aligned} \quad (\text{S651})$$

Similarly by the triangle inequality, Markov's inequality, Assumption 9, Lemma S1, Lemma S48 and (S642)

$$\begin{aligned} & \left\| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m})^\top \Psi_{\theta_j \gamma \gamma, g, m} \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\| \\ & \leq K \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\| \left\| \hat{\Psi}_{\gamma, g, m} \right\| \\ & \leq O_p((GM)^{2/p}) \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left\| \hat{\Psi}_{\gamma, g, m} \right\| = O_p \left((GM)^{1+2/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right) \end{aligned} \quad (\text{S652})$$

By the similar arguments for showing Lemma S53, we have

$$\begin{aligned} & \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma \gamma, g, m}^{-1} \Psi_{\theta_j \gamma \gamma, g, m} \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \\ & = \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} \left[\hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma \gamma, g, m}^{-1} \Psi_{\theta_j \gamma \gamma, g, m} \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right] + O_p((GM)^{1/2}). \end{aligned} \quad (\text{S653})$$

Combining the results in (S651), (S652) and (S653), and applying Assumption 10, we deduce that

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g, m} - \gamma_{g, m}^*)^\top \Psi_{\theta_j \gamma \gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \\ & = (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} \left[\hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma \gamma, g, m}^{-1} \Psi_{\theta_j \gamma \gamma, g, m} \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right] + O_p \left((GM)^{1/2} (nT)^{-1/2} \right) \\ & \quad + O_p \left(((GM)^{4/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} + (GM)^{2/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2}) (GM) (nT)^{-1/2} \right) \\ & = (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \mathbb{E} \left[\hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma \gamma, g, m}^{-1} \Psi_{\theta_j \gamma \gamma, g, m} \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right] + o_p(1) \end{aligned} \quad (\text{S654})$$

where the second equality is by Assumption 10. The assertion of the lemma now follows from (S644), (S645), (S648) and (S654). Q.E.D.

Lemma S64. *Under Assumptions 1, 7, 8, 9 and 10, we have*

$$\begin{aligned}
& (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta_{\gamma,g,m}} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \\
&= -(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left[\Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m} + \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} \Psi_{\gamma_{\theta,g,m}} (\hat{\theta} - \theta^*) \right] \\
&\quad - (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{i \in I_g} (n_g T_m) \mathbb{E} \left[\hat{U}_{\gamma,g,m} \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m} \right] \\
&\quad - \frac{(nT)^{-1/2}}{2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} \left(\mathbb{E} \left[\hat{\Psi}_{\gamma,g,m}^\top \Psi_{\gamma_{\gamma,g,m}}^{-1} \Psi_{\gamma_j \gamma_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m} \right] \right)_{j \leq d_\gamma} + o_p(1)
\end{aligned}$$

where $\hat{U}_{\gamma,g,m} \equiv \hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m}$.

PROOF OF LEMMA S64. By the definition of $\Psi_{\gamma_j \phi_{\gamma,g,m}}$ and the triangle inequality, we obtain:

$$\begin{aligned}
& \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left| \frac{(\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_j \phi_{\gamma,g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*) - (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \Psi_{\gamma_j \gamma_{\gamma,g,m}} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)}{(nT)^{-1} + (nT)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|} \right| \\
& \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\left| (\hat{\theta} - \theta^*)^\top \Psi_{\gamma_j \theta_{\gamma,g,m}} (\hat{\theta} - \theta^*) \right| + 2 \left| (\hat{\theta} - \theta^*)^\top \Psi_{\gamma_j \theta_{\gamma,g,m}} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \right|}{(nT)^{-1} + (nT)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|} = O_p(1) \quad (\text{S655})
\end{aligned}$$

where the last equality is by Assumption 8(iii), the Cauchy-Schwarz inequality and Lemma S62. By (S623) and (S624) of Lemma S61, we have

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \frac{\hat{\Psi}_{\gamma,g,m} + \hat{\Psi}_{\gamma \phi_{\gamma,g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*) + \frac{((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_j \phi_{\gamma,g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\gamma}}{2}}{(nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2 + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^3} \right\| = O_p(1). \quad (\text{S656})$$

Applying the triangle inequality, Assumption 10(i), Lemma S1 obtains

$$\begin{aligned}
\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|(n_g T_m)^{1/2} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)\| & \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| (n_g T_m)^{1/2} \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m} \right\| \\
& \quad + \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| (n_g T_m)^{1/2} (\hat{\gamma}_{g,m} - \gamma_{g,m}^* + \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m}) \right\| \\
& = \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| (n_g T_m)^{1/2} \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m} \right\| + o_p(1). \quad (\text{S657})
\end{aligned}$$

By the maximum inequality under L_p -norm, Markov's inequality, Assumption 8(iii) and (S553) in Lemma S47,

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| (n_g T_m)^{1/2} \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m} \right\| = O_p((GM)^{1/p}) \quad (\text{S658})$$

which together with (S657) implies that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|(n_g T_m)^{1/2} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)\| = O_p((GM)^{1/p}). \quad (\text{S659})$$

By Lemma S62 and (S659),

$$\begin{aligned} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|(n_g T_m)^{1/2}(\hat{\phi}_{g,m} - \phi_{g,m}^*)\| &\leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|(n_g T_m)^{1/2}(\hat{\theta} - \theta^*)\| \\ &+ \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|(n_g T_m)^{1/2}(\hat{\gamma}_{g,m} - \gamma_{g,m}^*)\| \\ &= O_p((GM)^{1/p}) \end{aligned}$$

which implies that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{(nT)^{-1} + (GM)^{1/p}(n_g T_m)^{-1/2} \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2 + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^3}{(nT)^{-1} + (GM)^{1/p}(n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 + (nT)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|} = O_p(1). \quad (\text{S660})$$

Collecting the results in (S655), (S656) and (S660), we deduce that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \frac{\hat{\Psi}_{\gamma,g,m} + \hat{\Psi}_{\phi,g,m}(\hat{\phi}_{g,m} - \phi_{g,m}^*) + \frac{((\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \Psi_{\gamma_j \gamma \gamma, g, m}(\hat{\gamma}_{g,m} - \gamma_{g,m}^*))_{j \leq d_\gamma}}{2}}{(nT)^{-1} + (GM)^{1/p}(n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 + (nT)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|} \right\| = O_p(1). \quad (\text{S661})$$

Next note that by Lemma S62

$$(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 = O_p\left((GM)(nT)^{-1/2}\right)$$

which together with Assumption 10 and (S659) implies that

$$\begin{aligned} &(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left((nT)^{-1} + (GM)^{1/p}(n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 + (nT)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| \right) \\ &\leq (nT)^{-1/2} + \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| \\ &+ (GM)^{1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 \\ &= O_p\left((nT)^{-1/2} + (1 + (GM)(nT)^{-1/2})(GM)^{1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2}\right) = o_p(1). \quad (\text{S662}) \end{aligned}$$

Therefore, by (S661) and (S662), we have

$$\begin{aligned} &(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta \gamma, g, m} \hat{\Psi}_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma \gamma, g, m} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \\ &= -(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta \gamma, g, m} \hat{\Psi}_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \\ &\quad - (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta \gamma, g, m} \hat{\Psi}_{\gamma \gamma, g, m}^{-1} \Psi_{\gamma \theta, g, m} (\hat{\theta} - \theta^*) \\ &\quad - \frac{(nT)^{-1/2}}{2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta \gamma, g, m} \hat{\Psi}_{\gamma \gamma, g, m}^{-1} ((\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \hat{\Psi}_{\gamma_j \gamma \gamma, g, m} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*))_j + o_p(1). \quad (\text{S663}) \end{aligned}$$

Since

$$\begin{aligned}\hat{\Psi}_{\theta\gamma,g,m}\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\theta\gamma,g,m}\Psi_{\gamma\gamma,g,m}^{-1} &= (\hat{\Psi}_{\theta\gamma,g,m} - \Psi_{\theta\gamma,g,m})\Psi_{\gamma\gamma,g,m}^{-1} + \Psi_{\theta\gamma,g,m}(\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) \\ &\quad + (\hat{\Psi}_{\theta\gamma,g,m} - \Psi_{\theta\gamma,g,m})(\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}),\end{aligned}$$

and

$$\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1} = -\hat{\Psi}_{\gamma\gamma,g,m}^{-1} \left(\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right) \Psi_{\gamma\gamma,g,m}^{-1},$$

by the triangle inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned}&\left\| (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta\gamma,g,m} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1}) \right\| \\ &\leq K \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\Psi}_{\theta\gamma,g,m} - \Psi_{\theta\gamma,g,m} \right\| \left\| \hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right\| \\ &\quad + K (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\Psi}_{\theta\gamma,g,m} - \Psi_{\theta\gamma,g,m} \right\| \\ &\quad + K \max_{g \in \mathcal{G}, m \in \mathcal{M}} (\lambda_{\max}((\hat{\Psi}_{\gamma\gamma,g,m}^{-1})^2))^{1/2} (nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m} \right\|. \tag{S664}\end{aligned}$$

By (S553) in Lemma S47,

$$\mathbb{E} \left[\left\| (n_g T_m)^{1/2} (\hat{\Psi}_{\theta\gamma,g,m} - \Psi_{\theta\gamma,g,m}) \right\|^2 \right] \leq K \text{ and } \mathbb{E} \left[\left\| (n_g T_m)^{1/2} (\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}) \right\|^2 \right] \leq K, \tag{S665}$$

which together with Markov's inequality, Assumption 10(i), Lemma S60(ii) and (S664) implies that

$$(nT)^{-1} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta\gamma,g,m} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1}) = o_p(1). \tag{S666}$$

Therefore by Lemma S62 and (S666), we have

$$\begin{aligned}&(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta\gamma,g,m} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} \Psi_{\gamma\theta,g,m} (\hat{\theta} - \theta^*) \\ &= (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \Psi_{\gamma\theta,g,m} (\hat{\theta} - \theta^*) + o_p(1).\end{aligned} \tag{S667}$$

By Lemma S56,

$$\begin{aligned}&(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta\gamma,g,m} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \\ &= (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \\ &\quad + (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{i \in I_g} (n_g T_m) \mathbb{E} \left[\hat{U}_{\gamma,g,m} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right] + o_p(1).\end{aligned} \tag{S668}$$

By Lemma S60(i, ii), Lemma S62, Lemma S59 and (S627) of Lemma S62,

$$\begin{aligned}
& \left\| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta_{\gamma,g,m}} \hat{\Psi}_{\gamma_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1}) ((\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \hat{\Psi}_{\gamma_j \gamma_{\gamma,g,m}} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*))_{j \leq d_\gamma} \right\| \\
& \leq K \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left(\left\| \hat{\Psi}_{\theta_{\gamma,g,m}} \hat{\Psi}_{\gamma_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} \right\| \max_{j \leq d_\gamma} \left\| \hat{\Psi}_{\gamma_j \gamma_{\gamma,g,m}} \right\| \right) \\
& \quad \times \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 \\
& = O_p((GM)^{1+2/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1}) = o_p((nT)^{1/2}). \tag{S669}
\end{aligned}$$

Similarly, we can show that

$$(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} ((\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \hat{\Psi}_{\gamma_j \gamma_{\gamma,g,m}}^* (\hat{\gamma}_{g,m} - \gamma_{g,m}^*))_{j \leq d_\gamma} = o_p(1) \tag{S670}$$

where $\hat{\Psi}_{\gamma_j \gamma_{\gamma,g,m}}^* \equiv \hat{\Psi}_{\gamma_j \gamma_{\gamma,g,m}} - \Psi_{\gamma_j \gamma_{\gamma,g,m}}$. By the similar arguments for deriving (S654), we can show that

$$\begin{aligned}
& (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} ((\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \Psi_{\gamma_j \gamma_{\gamma,g,m}} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*))_{j \leq d_\gamma} \\
& = (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} (\mathbb{E}[\hat{\Psi}_{\gamma,g,m}^\top \Psi_{\gamma_{\gamma,g,m}}^{-1} \Psi_{\gamma_j \gamma_{\gamma,g,m}} \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m}])_{j \leq d_\gamma} + o_p(1). \tag{S671}
\end{aligned}$$

The assertion of the lemma follows from (S663), (S667) and (S668)-(S671). Q.E.D.

Lemma S65. *Under Assumptions 1, 7, 8, 9 and 10, we have*

$$(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|(n_g T_m)^{1/2} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)\|^s = O_p\left((GM)(nT)^{-1/2}\right)$$

for any $s \leq p$.

PROOF OF LEMMA S65. By the triangle inequality and the c_r -inequality,

$$\begin{aligned}
\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|(n_g T_m)^{1/2} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)\|^s & \leq K \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|(n_g T_m)^{1/2} (\hat{\gamma}_{g,m} - \gamma_{g,m}^* + \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m})\|^s \\
& \quad + K \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|(n_g T_m)^{1/2} \Psi_{\gamma_{\gamma,g,m}}^{-1} \hat{\Psi}_{\gamma,g,m}\|^s. \tag{S672}
\end{aligned}$$

By Lemma S1,

$$\begin{aligned}
& (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|(n_g T_m)^{1/2} (\hat{\gamma}_{g,m} - \gamma_{g,m}^* + \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m})\|^s \\
& \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{\|\hat{\gamma}_{g,m} - \gamma_{g,m}^* + \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m}\|^s}{(GM)^{2s/p} (n_g T_m)^{-s}} (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (GM)^{2s/p} (n_g T_m)^{-s/2} \\
& = O_p \left((GM)^{1+2s/p} (nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-s/2} \right) = o_p \left((GM)(nT)^{-1/2} \right) \quad (\text{S673})
\end{aligned}$$

where the second equality is by Assumption 10(ii). By Assumption 9 and (S553) in Lemma S47,

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \mathbb{E} \left[\|(n_g T_m)^{1/2} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m}\|^s \right] \leq K \max_{g \in \mathcal{G}, m \in \mathcal{M}} \mathbb{E} \left[\|(n_g T_m)^{1/2} \hat{\Psi}_{\gamma,g,m}\|^s \right] \leq K \quad (\text{S674})$$

which together with Markov's inequality implies that

$$(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|(n_g T_m)^{1/2} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m}\|^s = O_p((GM)(nT)^{-1/2}). \quad (\text{S675})$$

The assertion of the lemma now follows from (S673) and (S675). $Q.E.D.$

Lemma S66. *Under Assumptions 1, 7, 8, 9 and 10, we have*

$$\begin{aligned}
& (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left[\hat{\Psi}_{g,m}(\hat{\phi}_{g,m}) - \hat{\Psi}_{g,m}(\phi_{g,m}^*) \right] \\
& = (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\phi,g,m}(\hat{\phi}_{g,m} - \phi_{g,m}^*) \\
& \quad + \frac{(nT)^{-1/2}}{2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \hat{\Psi}_{\phi\phi,g,m}(\hat{\phi}_{g,m} - \phi_{g,m}^*) \\
& \quad + O_p \left((GM)^{1/2} (nT)^{-1/2} \left((GM)^{1/2+2/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} + \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right) \right).
\end{aligned}$$

PROOF OF LEMMA S66. Applying the Taylor expansion obtains

$$\begin{aligned}
& (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left[\hat{\Psi}_{g,m}(\hat{\phi}_{g,m}) - \hat{\Psi}_{g,m}(\phi_{g,m}^*) \right] \\
& = (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\phi,g,m}(\hat{\phi}_{g,m} - \phi_{g,m}^*) \\
& \quad + \frac{(nT)^{-1/2}}{2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \hat{\Psi}_{\phi\phi,g,m}(\hat{\phi}_{g,m} - \phi_{g,m}^*) \\
& \quad + \frac{(nT)^{-1/2}}{6} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) ((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \tilde{\Psi}_{\phi\phi\phi,g,m}(\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\phi}^\top (\hat{\phi}_{g,m} - \phi_{g,m}^*), \quad (\text{S676})
\end{aligned}$$

where

$$\tilde{\Psi}_{\phi\phi\phi_j,g,m} \equiv (n_g T_m)^{-1} \sum_{i \in I_g} \sum_{t \in I_m} \frac{\partial}{\partial \phi_j} \left(\frac{\partial^2 \psi(z_{i,t}; \tilde{\phi}_{g,m})}{\partial \phi \partial \phi^\top} \right)$$

for any $j \leq k$, and $\tilde{\phi}_{g,m}$ is between $\hat{\phi}_{g,m}$ and $\phi_{g,m}^*$. By (S564) in Lemma S49, we can show that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \frac{\tilde{\Psi}_{\phi\phi\phi_j,g,m} - \Psi_{\phi\phi\phi_j,g,m}}{(GM)^{1/p} (n_g T_m)^{-1/2} + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|} \right\| = O_p(1), \quad (\text{S677})$$

where $\Psi_{\phi\phi\phi_j,g,m} \equiv \mathbb{E} \left[\frac{\partial}{\partial \phi_j} \left(\frac{\partial^2 \psi(z_{i,t}; \phi_{g,m}^*)}{\partial \phi \partial \phi^\top} \right) \right]$. By Assumptions 8(iii), 10(i) and Lemma S58

$$\begin{aligned} \max_{j \leq k} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \tilde{\Psi}_{\phi\phi\phi_j,g,m} \right\| &\leq \max_{j \leq k} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \Psi_{\phi\phi\phi_j,g,m} \right\| \\ &\quad + O_p \left(((GM)^{1/p} + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|) \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right) \\ &= O_p(1). \end{aligned} \quad (\text{S678})$$

By the triangle inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) ((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \tilde{\Psi}_{\phi\phi\theta_j,g,m} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\theta}^\top (\hat{\theta} - \theta^*) \right| \\ &\leq \left(\max_{j \leq d_\theta} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \tilde{\Psi}_{\phi\phi\theta_j,g,m} \right\| \right) \|\hat{\theta} - \theta^*\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2. \end{aligned} \quad (\text{S679})$$

From Lemma S62 and Lemma S66,

$$\begin{aligned} &(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2 \\ &= (nT)^{1/2} \|\hat{\theta} - \theta^*\|^2 + (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 \\ &= O_p((nT)^{-1/2} + (GM)(nT)^{-1/2}) = O_p((GM)(nT)^{-1/2}), \end{aligned}$$

which together with (S679) and Lemma S62 implies that

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) ((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \tilde{\Psi}_{\phi\phi\theta_j,g,m} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\theta}^\top (\hat{\theta} - \theta^*) = O_p((GM)(nT)^{-1/2}). \quad (\text{S680})$$

Similarly, we can show that

$$\begin{aligned} &\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) ((\hat{\theta} - \theta^*)^\top \tilde{\Psi}_{\theta\theta\gamma_j,g,m} (\hat{\theta} - \theta^*))_{j \leq d_\gamma}^\top (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \\ &= O_p \left((GM)(nT)^{-1} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{1/2} \right) \end{aligned} \quad (\text{S681})$$

and

$$\begin{aligned} & \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) ((\hat{\theta} - \theta^*)^\top \tilde{\Psi}_{\theta \gamma \gamma_j, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*))_{j \leq d_\gamma}^\top (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \\ &= O_p((GM)(nT)^{-1/2}). \end{aligned} \quad (\text{S682})$$

By the triangle inequality, Cauchy-Schwarz inequality, Lemma S58 and (S677),

$$\begin{aligned} & \left| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) ((\hat{\gamma}_{g, m} - \gamma_{g, m}^*)^\top (\tilde{\Psi}_{\gamma \gamma \gamma_j, g, m} - \Psi_{\gamma \gamma \gamma_j, g, m}) (\hat{\gamma}_{g, m} - \gamma_{g, m}^*))_{j \leq d_\gamma}^\top (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \right| \\ & \leq (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|^3 \left\| \tilde{\Psi}_{\gamma \gamma \gamma_j, g, m} - \Psi_{\gamma \gamma \gamma_j, g, m} \right\| \\ & \leq O_p(1)(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m)^{1/2} \|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|^3 ((GM)^{1/p} + \|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\| + \|\hat{\theta} - \theta^*\|) \\ & = O_p((GM)^{1/p})(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m)^{1/2} \|\hat{\gamma}_{g, m} - \gamma_{g, m}^*\|^3 \\ & = O_p \left((GM)^{1+1/p} (nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \right), \end{aligned} \quad (\text{S683})$$

where the last equality is by Lemma S65. By Assumption 8(iii), Lemma S1 and Lemma S48

$$\begin{aligned} & \left| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) ((\hat{\gamma}_{g, m} - \gamma_{g, m}^*)^\top \Psi_{\gamma \gamma \gamma_j, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*))_{j \leq d_\gamma}^\top (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \right. \\ & \quad \left. - \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma \gamma, g, m}^{-1} \Psi_{\gamma \gamma \gamma_j, g, m} \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m})_{j \leq d_\gamma}^\top \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right| \\ & \leq \max_{j \leq d_\gamma} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \Psi_{\gamma \gamma \gamma_j, g, m} \right\| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\| \left\| \hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|^2 \\ & \quad + \max_{j \leq d_\gamma} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \Psi_{\gamma \gamma \gamma_j, g, m} \right\| \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left\| \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|^2 \left\| \hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\| \\ & = O_p(1)(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \left((GM)^{4/p} (n_g T_m)^{-1} \left\| \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\| + (GM)^{2/p} \left\| \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\|^2 \right) \\ & = O_p \left((GM)^{1+2/p} (nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \right), \end{aligned} \quad (\text{S684})$$

where the last equality is by Assumption 10(i). Applying Lemma S52, we get

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma \gamma, g, m}^{-1} \Psi_{\gamma \gamma \gamma_j, g, m} \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m})_{j \leq d_\gamma}^\top \Psi_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \\ & = O_p \left((GM)^{1/2} (nT)^{-1/2} \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} + (GM)^{1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \right) \right). \end{aligned} \quad (\text{S685})$$

Collecting the results in (S680)-(S685), we deduce that

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left((\hat{\phi}_{g, m} - \phi_{g, m}^*)^\top \tilde{\Psi}_{\phi \phi \phi_j, g, m} (\hat{\phi}_{g, m} - \phi_{g, m}^*) \right)_{j \leq k}^\top (\hat{\phi}_{g, m} - \phi_{g, m}^*) \\ & = O_p \left((GM)^{1/2} (nT)^{-1/2} \left((GM)^{1/2+2/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} + \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right) \right) \end{aligned} \quad (\text{S686})$$

which together with (S676) shows the assertion of the lemma.

Q.E.D.

Lemma S67. *Under Assumptions 1, 7, 8, 9 and 10, we have*

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \hat{\Psi}_{\gamma_{g,m}} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \\ &= (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma_{g,m}}^\top \Psi_{\gamma_{g,m}}^{-1} \hat{\Psi}_{\gamma_{g,m}} + o_p \left((GM)^{1/2} (nT)^{-1/2} \right). \end{aligned}$$

PROOF OF LEMMA S67. By Lemma S62 and (S659),

$$\begin{aligned} & \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{(nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^2 + \|\hat{\phi}_{g,m} - \phi_{g,m}^*\|^3}{(nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2} \\ & \leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{(nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 + \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^3}{(nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2} \\ & + \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{(GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\theta} - \theta^*\|^2 + \|\hat{\theta} - \theta^*\|^3}{(nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2} \\ & \leq 1 + \max_{g \in \mathcal{G}, m \in \mathcal{M}} \frac{(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2}{(nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2} + o_p(1) = O_p(1). \end{aligned}$$

Therefore, we can use (S656) to obtain

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \left\| \frac{\hat{\Psi}_{\gamma_{g,m}} + \hat{\Psi}_{\gamma_{g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*) + \frac{((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_{g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\gamma}}{2}}{(nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2} \right\| = O_p(1). \quad (\text{S687})$$

To employ the above expansion, we write

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \hat{\Psi}_{\gamma_{g,m}} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \\ &= -(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top (\hat{\Psi}_{\gamma_{g,m}} + \hat{\Psi}_{\gamma_{g,m}} (\hat{\theta} - \theta^*)) \\ & \quad - 2^{-1} (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top ((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_{g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\gamma} + R_{1,n,T} \end{aligned} \quad (\text{S688})$$

where

$$R_{1,n,T} \equiv (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \left(\hat{\Psi}_{\gamma_{g,m}} + \hat{\Psi}_{\gamma_{g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*) + 2^{-1} ((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_{g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\gamma} \right).$$

We can apply the Cauchy-Schwarz inequality, Lemma S65, (S687) and (S659) to bound $R_{1,n,T}$

$$\begin{aligned}
|R_{1,n,T}| &\leq (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| ((nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2) \\
&\leq (nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| + (GM)^{1/p} (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m)^{1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^3 \\
&= O_p \left((GM)^{1/p} (nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} + (GM)^{1+1/p} (nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \right). \tag{S689}
\end{aligned}$$

Next, we bound the second term in the RHS of (S688). For this purpose, we write

$$\begin{aligned}
&(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top ((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_j \phi \phi, g, m} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\gamma} \\
&= (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top ((\hat{\theta} - \theta^*)^\top \Psi_{\gamma_j \theta \theta, g, m} (\hat{\theta} - \theta^*))_{j \leq d_\gamma} \\
&+ 2(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top ((\hat{\theta} - \theta^*)^\top \Psi_{\gamma_j \theta \gamma, g, m} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*))_{j \leq d_\gamma} \\
&+ (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top ((\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \Psi_{\gamma_j \gamma \gamma, g, m} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*))_{j \leq d_\gamma}. \tag{S690}
\end{aligned}$$

By the Cauchy-Schwarz inequality, Assumption 8(iii), Lemma S62 and (S659),

$$\begin{aligned}
&\left| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top ((\hat{\theta} - \theta^*)^\top \Psi_{\gamma_j \theta \theta, g, m} (\hat{\theta} - \theta^*))_{j \leq d_\gamma} \right| \\
&\leq \left(\max_{j \leq d_\gamma} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\Psi_{\gamma_j \theta \theta, g, m}\| \right) \|\hat{\theta} - \theta^*\|^2 (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| \\
&= O_p \left((nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\| \right) \\
&= O_p \left((GM)^{1/p} (nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right). \tag{S691}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left((\hat{\theta} - \theta^*)^\top \Psi_{\theta \gamma \gamma, g, m} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \right)_{j \leq d_\gamma} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \right| \\
&= \left| (\hat{\theta} - \theta^*)^\top (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \left((\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \Psi_{\theta_j \gamma \gamma, g, m} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \right)_{j \leq d_\gamma} \right| \\
&\leq K \|\hat{\theta} - \theta^*\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 = O_p((GM)(nT)^{-1}), \tag{S692}
\end{aligned}$$

where the equality is by Lemma S62 and Lemma S65. Applying (S684) and (S685), we obtain

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top ((\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \Psi_{\gamma_j \gamma_{j,g,m}} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*))_{j \leq d_\gamma} \\ &= O_p \left((GM)^{1/2} (nT)^{-1/2} \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} + (GM)^{1/2+2/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \right) \right). \end{aligned} \quad (\text{S693})$$

Collecting the results in (S690), (S691), (S692) and (S693) and applying Assumption 10 yield

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top ((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_j \phi_{j,g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\gamma} \\ &= o_p((GM)^{1/2} (nT)^{-1/2}). \end{aligned} \quad (\text{S694})$$

Next, by Lemma S1, Lemma S51 and Lemma S62

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \hat{\Psi}_{\gamma \theta, g, m} (\hat{\theta} - \theta^*) \\ &= (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma \gamma, g, m}^{-1} \Psi_{\gamma \theta, g, m} (\hat{\theta} - \theta^*) \\ &+ (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma \gamma, g, m}^{-1} (\hat{\Psi}_{\gamma \theta, g, m} - \Psi_{\gamma \theta, g, m}) (\hat{\theta} - \theta^*) + O_p \left((GM)^{1+2/p} (nT)^{-1} \right) \\ &= O_p \left((nT)^{-1/2} + (GM)^{1+2/p} (nT)^{-1} \right) = o_p \left((GM)^{1/2} (nT)^{-1/2} \right), \end{aligned}$$

which together with (S689) and (S694) implies that

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \hat{\Psi}_{\gamma \gamma, g, m} (\hat{\gamma}_{g,m} - \gamma_{g,m}^*) \\ &= -(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \hat{\Psi}_{\gamma, g, m} + o_p \left((GM)^{1/2} (nT)^{-1/2} \right). \end{aligned} \quad (\text{S695})$$

We shall apply (S687) to the first term in the RHS of (S695) to show the claim of the lemma. For this purpose, we write

$$\begin{aligned} & (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g,m} - \gamma_{g,m}^*)^\top \hat{\Psi}_{\gamma, g, m} \\ &= R_{2,n,T} - (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\gamma, g, m} + \hat{\Psi}_{\gamma \theta, g, m} (\hat{\theta} - \theta^*))^\top \hat{\Psi}_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \\ &- (4nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) ((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_j \phi_{j,g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\gamma}^\top \hat{\Psi}_{\gamma \gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m}, \end{aligned} \quad (\text{S696})$$

where

$$R_{2,n,T} \equiv (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top \hat{\Psi}_{\gamma \gamma, g, m}^{-1} \left(\hat{\Psi}_{\gamma, g, m} + \hat{\Psi}_{\gamma \phi, g, m} (\hat{\phi}_{g,m} - \phi_{g,m}^*) + 2^{-1} \left((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_j \phi_{j,g,m}} (\hat{\phi}_{g,m} - \phi_{g,m}^*) \right)_{j \leq d_\gamma} \right).$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}
|R_{2,n,T}| &\leq \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\Psi}_{\gamma,g,m}^{-1}\| \\
&\quad \times (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\Psi}_{\gamma,g,m}\| ((nT)^{-1} + (GM)^{1/p} (n_g T_m)^{-1/2} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2).
\end{aligned} \tag{S697}$$

Since $\max_{g \in \mathcal{G}, m \in \mathcal{M}} \mathbb{E} \left[\|(n_g T_m)^{1/2} \hat{\Psi}_{\gamma,g,m}\|^p \right] \leq K$ by (S553) in Lemma S553, we can use Holder's inequality Markov's inequality to obtain

$$\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|(n_g T_m)^{1/2} \hat{\Psi}_{\gamma,g,m}\|^s = O_p(GM) \tag{S698}$$

for any positive s less than or equal to p , which together with Lemma S60(ii) implies that

$$\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\Psi}_{\gamma,g,m}^{-1}\| (nT)^{-3/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\Psi}_{\gamma,g,m}\| = O_p \left((nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} \right). \tag{S699}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}
&\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|(n_g T_m)^{1/2} \hat{\Psi}_{\gamma,g,m}\| \times \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^2 \\
&\leq \left(\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|(n_g T_m)^{1/2} \hat{\Psi}_{\gamma,g,m}\|^2 \right)^{1/2} \left(\sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} \|\hat{\gamma}_{g,m} - \gamma_{g,m}^*\|^4 \right)^{1/2} \\
&= O_p \left((GM) \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \right),
\end{aligned} \tag{S700}$$

where the equality is by (S698) and Lemma S65. Using Lemma S60(ii), (S697), (S699) and (S700), we get

$$\begin{aligned}
R_{2,n,T} &= O_p \left(\left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1/2} + (GM)^{1+1/p} \max_{g \in \mathcal{G}, m \in \mathcal{M}} (n_g T_m)^{-1} \right) (nT)^{-1/2} \right) \\
&= o_p \left((GM)^{1/2} (nT)^{-1/2} \right),
\end{aligned} \tag{S701}$$

where the second equality is by Assumption 10. By the similar arguments for showing (S694),

$$\begin{aligned}
&(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) ((\hat{\phi}_{g,m} - \phi_{g,m}^*)^\top \Psi_{\gamma_j \phi \phi, g, m} (\hat{\phi}_{g,m} - \phi_{g,m}^*))_{j \leq d_\gamma}^\top \hat{\Psi}_{\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \\
&= o_p((GM)^{1/2} (nT)^{-1/2}).
\end{aligned} \tag{S702}$$

By the triangle inequality,

$$\begin{aligned}
& \left\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta_{\gamma,g,m}} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\| \\
& \leq \left\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\| \\
& + \left\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}}) \hat{\Psi}_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} \right\| \\
& + \left\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta_{\gamma,g,m}} (\hat{\Psi}_{\gamma\gamma,g,m}^{-1} - \Psi_{\gamma\gamma,g,m}^{-1}) \hat{\Psi}_{\gamma,g,m} \right\| \\
& \leq O_p(1) + \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\Psi}_{\gamma\gamma,g,m}^{-1}\| \right) (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\Psi}_{\theta_{\gamma,g,m}} - \Psi_{\theta_{\gamma,g,m}}\| \times \|\hat{\Psi}_{\gamma,g,m}\| \\
& + \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\Psi}_{\gamma\gamma,g,m}^{-1}\| \|\Psi_{\gamma\gamma,g,m}^{-1}\| \right) (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\Psi}_{\gamma\gamma,g,m} - \Psi_{\gamma\gamma,g,m}\| \times \|\hat{\Psi}_{\gamma,g,m}\| \\
& = O_p(1 + (GM)(nT)^{-1/2}) = O_p(1), \tag{S703}
\end{aligned}$$

where the second inequality is by the Cauchy-Schwarz inequality and

$$(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta_{\gamma,g,m}} \Psi_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} = O_p(1)$$

which holds by S553 in Lemma S47; the first equality is by Lemma S60(ii), (S698), (S665) and Markov's inequality; and the second equality is by Assumption 10. By the Cauchy-Schwarz inequality, Lemma S62 and (S703)

$$(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\theta} - \theta^*)^\top \hat{\Psi}_{\theta_{\gamma,g,m}} \hat{\Psi}_{\gamma\gamma,g,m}^{-1} \hat{\Psi}_{\gamma,g,m} = O_p((nT)^{-1/2}). \tag{S704}$$

Next note that

$$\begin{aligned}
& \left| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top (\hat{\Psi}_{\gamma\gamma, g, m}^{-1} - \Psi_{\gamma\gamma, g, m}^{-1}) \hat{\Psi}_{\gamma, g, m} \right| \\
& \leq \left| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma\gamma, g, m}^{-1} (\hat{\Psi}_{\gamma\gamma, g, m} - \Psi_{\gamma\gamma, g, m}) \Psi_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right| \\
& + \left| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top (\hat{\Psi}_{\gamma\gamma, g, m}^{-1} - \Psi_{\gamma\gamma, g, m}^{-1}) (\hat{\Psi}_{\gamma\gamma, g, m} - \Psi_{\gamma\gamma, g, m}) \Psi_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right| \\
& \leq \left(\max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\Psi}_{\gamma\gamma, g, m}^{-1}\| \|\Psi_{\gamma\gamma, g, m}^{-1}\| \|\hat{\Psi}_{\gamma\gamma, g, m} - \Psi_{\gamma\gamma, g, m}\|^2 \right) (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\Psi}_{\gamma, g, m}\|^2 \\
& + O_p \left((GM)^{1/2} (nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left((n_g T_m)^{-1/2} + (GM)^{1/2} (n_g T_m)^{-1} \right) \right) \\
& = O_p \left((GM)^{1/2} (nT)^{-1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \left((n_g T_m)^{-1/2} + (GM)^{1/2+2/p} (n_g T_m)^{-1} \right) \right) = o_p((GM)^{1/2} (nT)^{-1/2})
\end{aligned} \tag{S705}$$

where the second inequality is by Lemma S52 and the Cauchy-Schwarz inequality, the first equality is by Lemma S60(ii), Lemma S59 and (S698) and the second equality is by Assumption 10.

Collecting the results in (S696), (S697), (S702), (S704) and (S705), we have

$$\begin{aligned}
& (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\gamma}_{g, m} - \gamma_{g, m}^*)^\top \hat{\Psi}_{\gamma, g, m} \\
& = -(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top \hat{\Psi}_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} + o_p((GM)^{1/2} (nT)^{-1/2}) \\
& = -(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\gamma, g, m}^\top \Psi_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} + o_p((GM)^{1/2} (nT)^{-1/2})
\end{aligned} \tag{S706}$$

which together with (S695) shows the claim of the lemma.

Q.E.D.

Lemma S68. *Under Assumptions 1, 7, 8, 9 and 10, we have*

$$(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\theta} - \theta^*)^\top \hat{\Psi}_{\theta\theta, g, m} (\hat{\theta} - \theta^*) = O_p \left((nT)^{-1/2} \right) \tag{S707}$$

and

$$(nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\theta} - \theta^*)^\top \hat{\Psi}_{\theta\gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) = O_p \left((nT)^{-1/2} + (GM)^{1+2/p} (nT)^{-1} \right). \tag{S708}$$

PROOF OF LEMMA S68. By the triangle inequality and the Cauchy-Schwarz inequality

$$\begin{aligned}
& \left| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\theta} - \theta^*)^\top \hat{\Psi}_{\theta\theta, g, m} (\hat{\theta} - \theta^*) \right| \\
& \leq \left\| \hat{\theta} - \theta^* \right\|^2 (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\Psi}_{\theta\theta, g, m}\| \\
& \leq \left\| \hat{\theta} - \theta^* \right\|^2 (nT)^{1/2} \max_{g \in \mathcal{G}, m \in \mathcal{M}} \|\hat{\Psi}_{\theta\theta, g, m}\| = O_p((nT)^{-1/2})
\end{aligned} \tag{S709}$$

where the equality is by Lemma S62 and Lemma S60(i). This shows the assertion in (S707). To show (S708), we note that by the triangle inequality

$$\begin{aligned}
& \left\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta\gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^*) \right\| \\
& \leq \left\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \Psi_{\theta\gamma, g, m} \Psi_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\| \\
& + \left\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) (\hat{\Psi}_{\theta\gamma, g, m} - \Psi_{\theta\gamma, g, m}) \Psi_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\| \\
& + \left\| (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \hat{\Psi}_{\theta\gamma, g, m} (\hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m}) \right\| \\
& \leq (nT)^{-1/2} \sum_{g \in \mathcal{G}} \sum_{m \in \mathcal{M}} (n_g T_m) \|\hat{\Psi}_{\theta\gamma, g, m}\| \left\| \hat{\gamma}_{g, m} - \gamma_{g, m}^* + \Psi_{\gamma\gamma, g, m}^{-1} \hat{\Psi}_{\gamma, g, m} \right\| \\
& + O_p(1 + (GM)(nT)^{-1/2}) \\
& = O_p(1 + (GM)^{1+2/p}(nT)^{-1/2})
\end{aligned} \tag{S710}$$

where the second inequality is by (S553) in Lemma S47 and Lemma S51, the equality is by Lemma S60(i) and Lemma S1. The assertion in (S708) now follows from Lemma S62, (S710) and the Cauchy-Schwarz inequality. *Q.E.D.*

Lemma S69. *Under the conditions of Theorem A4, we have*

$$(n\omega_{n,T}^2)^{-1} \sum_{i \leq n} \mathbb{E} \left[(\tilde{\Psi}_i + \tilde{V}_i) \tilde{U}_{j,i} | \mathcal{F}_{nT,i-1} \right] = O_p \left(T^{-1/2} + G_j^{-1/2} \max_{g \in \mathcal{G}_j} n_g^{-1} \right). \tag{S711}$$

PROOF OF LEMMA S69. For any $i = 1, \dots, n$, by the independence of $\{z_{i,t}\}_{t \leq T}$ across i we have

$$\frac{n_{g_j(i)} T^{1/2} \mathbb{E} \left[(\tilde{\Psi}_i + \tilde{V}_i) \tilde{U}_{j,i} | \mathcal{F}_{nT,i-1} \right]}{\omega_{n,T}} = \mathbb{E} \left[\frac{(\tilde{\Psi}_i + \tilde{V}_i) A_{j,i}^\top}{\omega_{n,T}} \right] \sum_{\{i' \in I_{g_j(i)} : i' < i\}} A_{j,i'} = \sum_{\{i' \in I_{g_j(i)} : i' < i\}} c_{j,i}^\top A_{j,i'},$$

where $c_{j,i} \equiv \mathbb{E}[(\tilde{\Psi}_i + \tilde{V}_i)A_{j,i}/\omega_{n,T}]$. Let $C_{j,i} \equiv \sum_{\{i' \in I_{g_j(i)}: i' > i\}} c_{j,i'}$. Then the above expression implies that

$$\frac{\sum_{i \leq n} \mathbb{E} \left[(\tilde{\Psi}_i + \tilde{V}_i) \tilde{U}_{j,i} | \mathcal{F}_{nT,i-1} \right]}{n\omega_{n,T}^2} = \sum_{i \leq n} \frac{C_{j,i}^\top A_{j,i}}{n_{g_j(i)} T^{1/2} n\omega_{n,T}}. \quad (\text{S712})$$

Since $\mathbb{E}[A_{j,i}] = 0$ and $A_{j,i}$ are independent across i ,

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{i \leq n} \frac{C_{j,i}^\top A_{j,i}}{n_{g_j(i)} n\omega_{n,T} T^{1/2}} \right|^2 \right] &= \sum_{i \leq n} \frac{C_{j,i}^\top \mathbb{E}[A_{j,i} A_{j,i}^\top] C_{j,i}}{n_{g_j(i)}^2 n^2 \omega_{n,T}^2 T} \\ &\leq \frac{K \max_{i \leq n} \lambda_{\max}(\mathbb{E}[A_{j,i} A_{j,i}^\top])}{n^2 \omega_{n,T}^2 T} \sum_{i \leq n} n_{g_j(i)}^{-2} C_{j,i}^\top C_{j,i}. \end{aligned} \quad (\text{S713})$$

Applying Assumptions 1, 9 and 12(ii), and Rosenthal's inequality of strong mixing processes (e.g., (S556) with $\tilde{p} = 4 + \delta$ in the proof of Lemma S47), we obtain

$$\max_{i \leq n} \max_{m \in \mathcal{M}_j} \mathbb{E}[|\tilde{\Psi}_{j\gamma,im}|^{4+\delta}] = \max_{i \leq n} \max_{m \in \mathcal{M}_j} \mathbb{E} \left[\left(T_m^{-1/2} \sum_{t \in I_m} \tilde{\psi}_{j\gamma}(z_{i,t}; \phi_{j,g,m}^*) \right)^{4+\delta} \right] \leq K. \quad (\text{S714})$$

By Corollary A.2. in Hall and Heyde (1980),

$$\begin{aligned} &\max_{i \leq n} \max_{m' \in \mathcal{M}_j} \sum_{m \in \mathcal{M}_j} \left| \mathbb{E}[\tilde{\Psi}_{j\gamma,im} \tilde{\Psi}_{j\gamma,im'}] \right| \\ &\leq K \max_{i \leq n} \max_{m \in \mathcal{M}_j} (\mathbb{E}[|\tilde{\Psi}_{j\gamma,im}|^{2+\delta}])^2 \sum_{h=0}^{\infty} |\alpha_i(h)|^{\delta/(2+\delta)} \leq K, \end{aligned} \quad (\text{S715})$$

where the second inequality is by Assumption 1(iv) and (S714). In view of the upper bound in (S715), we can use Geršgorin's disc theorem (see, e.g., Theorem 6.1.1 in Horn and Johnson (2012)) to show that

$$\max_{j=1,2} \max_{i \leq n} \lambda_{\max}(\mathbb{E}[A_{j,i} A_{j,i}^\top]) \leq K. \quad (\text{S716})$$

Using Lemma S46 (with $X_1 = (\tilde{\Psi}_i^* + \tilde{V}_i^*)/\omega_{n,T}$ and $X_2 = A_{j,i}$) and the inequality above obtains

$$\begin{aligned} \max_{i \leq n} \|c_{j,i}\|^2 &\leq \max_{i \leq n} (\lambda_{\max}(\mathbb{E}[A_{j,i} A_{j,i}^\top])) \max_{i \leq n} \mathbb{E} \left[\frac{(\tilde{\Psi}_{1,i}^* - \tilde{\Psi}_{2,i}^* + \tilde{V}_{1,i}^* - \tilde{V}_{2,i}^*)^2}{\omega_{n,T}^2} \right] \\ &\leq K \max_{i \leq n} \mathbb{E} \left[\frac{(\tilde{\Psi}_{1,i}^* - \tilde{\Psi}_{2,i}^*)^2}{\omega_{n,T}^2} \right] + K \frac{\max_{j=1,2} \max_{i \leq n} \mathbb{E}[|\tilde{V}_{j,i}^*|^2]}{\omega_{n,T}^2}. \end{aligned} \quad (\text{S717})$$

Applying Assumptions 1 and 12(i), and Rosenthal's inequality of strong mixing processes, we can show that

$$\max_{i \leq n} \mathbb{E} \left[\left(\frac{\tilde{\Psi}_{1,i}^* - \tilde{\Psi}_{2,i}^*}{\omega_{n,T}} \right)^4 \right] = \max_{i \leq n} \mathbb{E} \left[\left(T^{-1/2} \sum_{t \leq T} \frac{\psi_1^*(z_{i,t}; \phi_{1,i,t}^*) - \psi_2^*(z_{i,t}; \phi_{2,i,t}^*)}{\omega_{n,T}} \right)^4 \right] \leq K, \quad (\text{S718})$$

where $\psi_j^*(z_{i,t}; \phi_{j,i,t}^*) \equiv \psi_j(z_{i,t}; \phi_{j,i,t}^*) - \mathbb{E}[\psi_j(z_{i,t}; \phi_{j,i,t}^*)]$. Similarly for any $\delta' \in [0, 2 + \delta/2]$,

$$\mathbb{E}[|\tilde{V}_{j,i}^*|^{2+\delta'}] = \frac{\mathbb{E}\left[\left(\sum_{m \in \mathcal{M}_j} (\tilde{\Psi}_{j\gamma,im}^2 - \mathbb{E}[\tilde{\Psi}_{j\gamma,im}^2])\right)^{2+\delta'}\right]}{(2n_{g_j(i)}T^{1/2})^{2+\delta'}} \leq \frac{KM_j^{1+\delta'/2}}{(n_{g_j(i)}T^{1/2})^{2+\delta'}}. \quad (\text{S719})$$

Therefore by (S717), (S718) and (S719) with $\delta' = 0$, and Assumptions 1(i) and 11(i), we can deduce that

$$\max_{i \leq n} \|c_{j,i}\|^2 \leq K \left(1 + M_j \max_{g \in \mathcal{G}_j} n_g^{-2}\right). \quad (\text{S720})$$

Using the Cauchy-Schwarz inequality and (S720) yields

$$\begin{aligned} \max_{i \leq n} \frac{C_{j,i}^\top C_{j,i}}{n_{g_j(i)}^2} &= \max_{i \leq n} \frac{\sum_{i_1, i_2 \in \{i' \in I_{g_j(i)}: i' > i\}} |c_{j,i_1}^\top c_{j,i_2}|}{n_{g_j(i)}^2} \\ &\leq \max_{i \leq n} \left(\frac{\sum_{i' \in I_{g_j(i)}: i' > i} \|c_{j,i'}\|}{n_{g_j(i)}} \right)^2 \leq K \left(1 + M_j \max_{g \in \mathcal{G}_j} n_g^{-2}\right). \end{aligned} \quad (\text{S721})$$

Collecting the results in (S713), (S716) and (S721), and applying Assumption 11(i) obtains

$$\mathbb{E} \left[\left| \sum_{i \leq n} \frac{C_{j,i}^\top A_{j,i}}{n_{g_j(i)} n \omega_{n,T} T^{1/2}} \right|^2 \right] \leq \frac{K}{T n^2 \omega_{n,T}^2} \sum_{i \leq n} \frac{C_{j,i}^\top C_{j,i}}{n_{g_j(i)}^2} \leq K T^{-1} \left(1 + M_j \max_{g \in \mathcal{G}_j} n_g^{-2}\right). \quad (\text{S722})$$

The desired result in (S711) now follows from (S712), (S722), Assumption 10(ii) and Markov's inequality. Q.E.D.

Lemma S70. *Under the conditions of Theorem A4, we have*

$$(n\omega_{n,T}^2)^{-1} \sum_{i \leq n} \left(\mathbb{E}[\tilde{U}_i^2 | \mathcal{F}_{nT,i-1}] - \mathbb{E}[\tilde{U}_i^2] \right) = O_p \left(T^{-1/2} + \max_{j=1,2} \max_{g \in \mathcal{G}_j} (G_j n_g)^{-1/2} \right). \quad (\text{S723})$$

PROOF OF LEMMA S70. Since

$$\mathbb{E}[\tilde{U}_i^2 | \mathcal{F}_{nT,i-1}] = \mathbb{E}[\tilde{U}_{1,i}^2 | \mathcal{F}_{nT,i-1}] + \mathbb{E}[\tilde{U}_{2,i}^2 | \mathcal{F}_{nT,i-1}] - 2\mathbb{E}[\tilde{U}_{1,i}\tilde{U}_{2,i} | \mathcal{F}_{nT,i-1}],$$

the claim of the lemma follows if

$$\sum_{i \leq n} \left(\mathbb{E}[\tilde{U}_{j,i}^2 | \mathcal{F}_{nT,i-1}] - \mathbb{E}[\tilde{U}_{j,i}^2] \right) = O_p \left(T^{-1/2} + \max_{g \in \mathcal{G}_j} (G_j n_g)^{-1/2} \right), \quad \text{and} \quad (\text{S724})$$

$$\sum_{i \leq n} \left(\mathbb{E}[\tilde{U}_{1,i}\tilde{U}_{2,i} | \mathcal{F}_{nT,i-1}] - \mathbb{E}[\tilde{U}_{1,i}\tilde{U}_{2,i}] \right) = O_p \left(T^{-1/2} + \max_{j=1,2} \max_{g \in \mathcal{G}_j} (G_j n_g)^{-1/2} \right). \quad (\text{S725})$$

We next verify (S724) and (S725).

We first show (S724) for $j = 1$. The proof for $j = 2$ follows the same arguments. For simplicity of notations, we suppose that for any $g \in \mathcal{G}_1$, $I_g = \{N_g + 1, \dots, N_g + n_g\}$ where $N_g \equiv \sum_{g' < g} n_{g'}$. Following these notations, for any $i \in I_g$ and any $g \in \mathcal{G}_1$ we can write

$$\tilde{U}_{1,i} = (n_g T^{1/2})^{-1} \sum_{i'=N_g+1}^{i-1} A_{1,i}^\top A_{1,i'}.$$

Since $\sum_{i'=N_g+1}^{i-1} A_{1,i'}$ is non-random given $\mathcal{F}_{nT,i-1}$ and $A_{1,i}$ is independent of $\sum_{i'=N_g+1}^{i-1} A_{1,i'}$, for any $i \in I_g$ and any $g \in \mathcal{G}_1$

$$\begin{aligned} n_g^2 T \mathbb{E}[\tilde{U}_{1,i}^2 | \mathcal{F}_{nT,i-1}] &= \mathbb{E} \left[\left(\sum_{i'=N_g+1}^{i-1} A_{1,i}^\top A_{1,i'} \right)^2 \middle| \mathcal{F}_{nT,i-1} \right] \\ &= \sum_{i_1, i_2=N_g+1}^{i-1} A_{1,i_1}^\top (\mathbb{E}[A_{1,i} A_{1,i_1}^\top]) A_{1,i_2} \\ &= \sum_{i'=N_g+1}^{i-1} A_{1,i'}^\top (\mathbb{E}[A_{1,i} A_{1,i'}^\top]) A_{1,i'} + 2 \sum_{i_1=N_g+2}^{i-1} \sum_{i_2=N_g+1}^{i_1-1} A_{1,i_1}^\top (\mathbb{E}[A_{1,i} A_{1,i_1}^\top]) A_{1,i_2} \\ &= \text{Tr} \left(\mathbb{E}[A_{1,i} A_{1,i}^\top] \left(\sum_{i'=N_g+1}^{i-1} A_{1,i'} A_{1,i'}^\top + 2 \sum_{i_1=N_g+2}^{i-1} \sum_{i_2=N_g+1}^{i_1-1} A_{1,i_2} A_{1,i_1}^\top \right) \right), \end{aligned}$$

which implies that

$$\sum_{i \leq n} \mathbb{E}[\tilde{U}_{1,i}^2 | \mathcal{F}_{nT,i-1}] = \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+1}^{N_g+n_g-1} \frac{A_{1,i}^\top D_{1,i} A_{1,i}}{n_g^2 T} + 2 \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g-1} \sum_{i'=N_g+1}^{i-1} \frac{A_{1,i}^\top D_{1,i} A_{1,i'}}{n_g^2 T}, \quad (\text{S726})$$

where

$$D_{1,i} \equiv \sum_{i'=i+1}^{N_g+n_g} \mathbb{E}[A_{1,i'} A_{1,i'}^\top]$$

for any $i \in I_g$. Since $\mathbb{E}[A_{1,i}^\top D_{1,i} A_{1,i'}] = 0$ for any $i, i' \in I_g$ with $i \neq i'$, by (S726) and the law of iterated expectation,

$$\sum_{i \leq n} \mathbb{E}[\tilde{U}_{1,i}^2] = \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+1}^{N_g+n_g-1} \frac{\mathbb{E}[A_{1,i}^\top D_{1,i} A_{1,i}]}{n_g^2 T}. \quad (\text{S727})$$

By (S726) and (S727),

$$\begin{aligned} \sum_{i \leq n} \left(\mathbb{E}[\tilde{U}_{j,i}^2 | \mathcal{F}_{nT,i-1}] - \mathbb{E}[\tilde{U}_{j,i}^2] \right) &= \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+1}^{N_g+n_g-1} \frac{A_{1,i}^\top D_{1,i} A_{1,i} - \mathbb{E}[A_{1,i}^\top D_{1,i} A_{1,i}]}{n_g^2 T} \\ &\quad + 2 \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g-1} \sum_{i'=N_g+1}^{i-1} \frac{A_{1,i}^\top D_{1,i} A_{1,i'}}{n_g^2 T}. \end{aligned} \quad (\text{S728})$$

In view of the decomposition in (S728), it is clear that (S724) holds for $j = 1$ by Markov's inequality if

$$\text{Var} \left(\sum_{g \in \mathcal{G}_1} \sum_{i=N_g+1}^{N_g+n_g-1} \frac{A_{1,i}^\top D_{1,i} A_{1,i}}{n_g^2 T} \right) \leq K G_1^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1} \quad (\text{S729})$$

and

$$\text{Var} \left(\sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g-1} \sum_{i'=N_g+1}^{i-1} \frac{A_{1,i}^\top D_{1,i} A_{1,i'}}{n_g^2 T} \right) \leq K T^{-1}. \quad (\text{S730})$$

We next prove (S729) and (S730).

First note that the upper bound on the eigenvalues of $\mathbb{E}[A_{j,i} A_{j,i}^\top]$ established in (S716) implies that

$$\max_{i \leq n} \mathbb{E}[A_{j,i}^\top A_{j,i}] = \max_{i \leq n} \text{Tr}(\mathbb{E}[A_{j,i} A_{j,i}^\top]) \leq M_j \max_{i \leq n} \lambda_{\max}(\mathbb{E}[A_{j,i} A_{j,i}^\top]) \leq K M_j, \quad (\text{S731})$$

where the first inequality is because $\text{Tr}(\mathbb{E}[A_{j,i} A_{j,i}^\top])$ is equal to the sum of the eigenvalues of $\mathbb{E}[A_{j,i} A_{j,i}^\top]$, which is bounded by $M_j \lambda_{\max}(\mathbb{E}[A_{j,i} A_{j,i}^\top])$. Also in (S719), we have shown that

$$\max_{i \leq n} \mathbb{E} \left[\left(M_j^{-1/2} \sum_{m \in \mathcal{M}_j} (\tilde{\Psi}_{j\gamma,im}^2 - \mathbb{E}[\tilde{\Psi}_{j\gamma,im}^2]) \right)^2 \right] \leq K. \quad (\text{S732})$$

By (S731) and (S732), and the Cauchy-Schwarz inequality,

$$\begin{aligned} \max_{i \leq n} \mathbb{E}[(A_{j,i}^\top A_{j,i})^2] &= \max_{i \leq n} \mathbb{E} \left[\left(\sum_{m \in \mathcal{M}_j} \tilde{\Psi}_{j\gamma,im}^2 \right)^2 \right] \\ &\leq 2 \max_{i \leq n} \mathbb{E} \left[\left(\sum_{m \in \mathcal{M}_j} (\tilde{\Psi}_{j\gamma,im}^2 - \mathbb{E}[\tilde{\Psi}_{j\gamma,im}^2]) \right)^2 \right] + \left(\sum_{m \in \mathcal{M}_j} \mathbb{E}[\tilde{\Psi}_{j\gamma,im}^2] \right)^2 \\ &\leq K M_j + \max_{i \leq n} (\mathbb{E}[A_{j,i}^\top A_{j,i}])^2 \leq K M_j^2. \end{aligned} \quad (\text{S733})$$

By (S716), we also have

$$\max_{g \in \mathcal{G}_1} \max_{i \in I_g} \frac{\lambda_{\max}(D_{1,i})}{n_g} = \max_{g \in \mathcal{G}_1} \max_{i \in I_g} \frac{\sum_{i'=i+1}^{N_g+n_g} \lambda_{\max}(\mathbb{E}[A_{1,i'} A_{1,i'}^\top])}{n_g} \leq K. \quad (\text{S734})$$

Getting back to (S729), we note that since $A_{1,i}$ are independent across i ,

$$\begin{aligned}
\text{Var} \left(\sum_{g \in \mathcal{G}_1} \sum_{i=N_g+1}^{N_g+n_g-1} \frac{A_{1,i}^\top D_{1,i} A_{1,i}}{n_g^2 T} \right) &= \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+1}^{N_g+n_g-1} \frac{\text{Var}(A_{1,i}^\top D_{1,i} A_{1,i})}{(n_g^2 T)^2} \\
&\leq \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+1}^{N_g+n_g-1} \frac{\mathbb{E}[(A_{1,i}^\top D_{1,i} A_{1,i})^2]}{(n_g^2 T)^2} \\
&\leq \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+1}^{N_g+n_g-1} \frac{(\lambda_{\max}(D_{1,i}))^2 \mathbb{E}[(A_{1,i}^\top A_{1,i})^2]}{(n_g^2 T)^2} \\
&\leq K \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+1}^{N_g+n_g-1} (n_g T)^{-2} M_1^2 \leq K M_1^2 T^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \\
&\leq K M_1^2 G_1 T^{-2} \max_{g \in \mathcal{G}_1} n_g^{-1} \leq K G_1^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1} \quad (\text{S735})
\end{aligned}$$

where the third inequality is by (S733) and (S734), and the last inequality is by (S535). This establishes the desired result in (S729).

Similarly,

$$\begin{aligned}
&\text{Var} \left(\sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g-1} \sum_{i'=N_g+1}^{i-1} \frac{A_{1,i}^\top D_{1,i} A_{1,i'}}{n_g^2 T} \right) \\
&= \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g-1} \sum_{i'=N_g+1}^{i-1} \frac{\text{Var}(A_{1,i}^\top D_{1,i} A_{1,i'})}{(n_g^2 T)^2} = \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g-1} \sum_{i'=N_g+1}^{i-1} \frac{\mathbb{E}[(A_{1,i}^\top D_{1,i} A_{1,i'})^2]}{(n_g^2 T)^2} \\
&\leq \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g-1} \sum_{i'=N_g+1}^{i-1} \frac{(\lambda_{\max}(D_{1,i}))^2 \lambda_{\max}(\mathbb{E}[A_{1,i'} A_{1,i'}^\top]) \mathbb{E}[A_{1,i}^\top A_{1,i}]}{(n_g^2 T)^2} \\
&\leq K \sum_{g \in \mathcal{G}_1} \sum_{i=N_g+2}^{N_g+n_g-1} \frac{M_1}{n_g T^2} \leq K M_1 G_1 T^{-2} \leq K T^{-1}, \quad (\text{S736})
\end{aligned}$$

where the second inequality is by (S716) and (S734), and the fourth inequality is by Assumptions 1(i) and 10(ii). This shows (S730).

To derive (S725), we use the same notation for the structure in $\mathcal{G}_1$ employed in verifying (S724) for $j = 1$ above. That is for any $g \in \mathcal{G}_1$, $I_g = \{N_g + 1, \dots, N_g + n_g\}$ where $N_g \equiv \sum_{g' < g} n_{g'}$. Moreover, for each $g \in \mathcal{G}_1$, we order I_g according to the group identity $g_2(i)$ such that $g_2(i)$ is non-decreasing. Based on the values of $g_2(i)$, we can divide I_g to M_g subsets ($1 \leq M_g \leq G_2$): $I_{g,l} \equiv \{i_{g,l-1} + 1, \dots, i_{g,l}\}$ ($l = 1, \dots, M_g$) with $i_{g,0} \equiv N_g$, $i_{g,M_g} \equiv N_g + n_g$ and $i_{g,j} \in I_g$ for $j = 1, \dots, M_g$ such that $g_2(i) = g_2(i')$ for any $i, i' \in I_{g,l}$. For each i , let

$$Q_{1,i} \equiv \{i' \in I_{g_2(i)}: i' \leq N_{g_1(i)}\} \quad \text{and} \quad Q_{2,i} \equiv \{i' \in I_{g_2(i)}: N_{g_1(i)} + 1 \leq i' \leq i - 1\}.$$

By definition, $Q_{1,i}$ includes the individuals who share the same group identity with individual i under model 2 but are in groups $I_{g'}$ with $g' < g_1(i)$ under model 1. On the other hand, $Q_{2,i}$ includes the individuals who share the same group identity with individual i under model 2, and are in the same group $g_1(i)$ of individual i under model 1 but have their index strictly less than i . Since $N_{g_1(i)} \leq i - 1$, it is clear that $Q_{1,i}$ and $Q_{2,i}$ are mutually exclusive, and $\{i' \in I_{g_2(i)} : i' < i\} = Q_{1,i} \cup Q_{2,i}$.

Using the above notations, for any $i \in I_g$ and any $g \in \mathcal{G}_1$ we can write

$$\begin{aligned} \tilde{U}_{1,i} \tilde{U}_{2,i} &= (n_g n_{g_2(i)} T)^{-1} \left(\sum_{i'=N_g+1}^{i-1} A_{1,i}^\top A_{1,i'} \right) \left(\sum_{\{i' \in I_{g_2(i)} : i' < i\}} A_{2,i}^\top A_{2,i'} \right) \\ &= \sum_{i_1=N_g+1}^{i-1} \sum_{i_2 \in Q_{1,i}} \frac{A_{1,i_1}^\top A_{1,i} A_{2,i}^\top A_{2,i_2}}{n_g n_{g_2(i)} T} + \sum_{i_1=N_g+1}^{i-1} \sum_{i_2 \in Q_{2,i}} \frac{A_{2,i_2}^\top A_{2,i} A_{1,i}^\top A_{1,i_1}}{n_g n_{g_2(i)} T}. \end{aligned} \quad (S737)$$

Let $d_{12,i} \equiv \mathbb{E}[A_{1,i} A_{2,i}^\top]$ and $d_{21,i} \equiv d_{12,i}^\top$. Then by the independence of $A_{j,i}$ across i , we obtain

$$\begin{aligned} \sum_{i \leq n} \mathbb{E}[\tilde{U}_{1,i} \tilde{U}_{2,i} | \mathcal{F}_{nT,i-1}] &= \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{i_2 \in Q_{1,i}} \sum_{i_1=N_g+1}^{i-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i)} T} \\ &\quad + \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{i_2 \in Q_{2,i}} \sum_{i_1=N_g+1}^{i-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i)} T}. \end{aligned} \quad (S738)$$

To study the two terms in the RHS of (S738), we first investigate $d_{12,i}$. Applying Lemma S46 (with $X_1 = A_{1,i}$ and $X_2 = A_{2,i}$) yields

$$\max_{i \leq n} \lambda_{\max}(d_{12,i} d_{21,i}) \leq \max_{i \leq n} \lambda_{\max}(\mathbb{E}[A_{1,i} A_{1,i}^\top]) \lambda_{\max}(\mathbb{E}[A_{2,i} A_{2,i}^\top]) \leq K, \quad (S739)$$

where the second inequality is by (S716).

Consider any $g \in \mathcal{G}_1$. Since $I_g = \cup_{l \leq M_g} I_{g,l}$, and for any $i \in I_{g,l}$, $Q_{1,i} = Q_{1,i_{g,l}}$ and $n_{g_2(i)} = n_{g_2(i_{g,l})}$, we can write

$$\begin{aligned} \sum_{i \in I_g} \sum_{i_2 \in Q_{1,i}} \sum_{i_1=N_g+1}^{i-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i)} T} &= \sum_{l \leq M_g} \sum_{i \in I_{g,l}} \sum_{i_2 \in Q_{1,i_{g,l}}} \sum_{i_1=N_g+1}^{i-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T} \\ &= \sum_{l \leq M_g} \sum_{i_2 \in Q_{1,i_{g,l}}} \frac{A_{2,i_2}^\top}{n_g n_{g_2(i_{g,l})} T} \sum_{i \in I_{g,l}} \sum_{i_1=N_g+1}^{i-1} d_{21,i} A_{1,i_1}. \end{aligned} \quad (S740)$$

By the definition of $I_{g,l}$,

$$\sum_{i \in I_{g,l}} \sum_{i_1=N_g+1}^{i-1} d_{21,i} A_{1,i_1} = \sum_{i=i_{g,l-1}+1}^{i_{g,l}} d_{21,i} \sum_{i_1=N_g+1}^{i-1} A_{1,i_1} = D_{1,g}^l \sum_{i_1=N_g+1}^{i_{g,l}-1} A_{1,i_1} + \sum_{i_1=i_{g,l-1}+1}^{i_{g,l}-1} D_{2,g i_1}^l A_{1,i_1},$$

where $D_{1,g}^l \equiv \sum_{i=i_{g,l-1}+1}^{i_{g,l}} d_{21,i}$ and $D_{2,gi}^l \equiv \sum_{i'=i+1}^{i_{g,l}} d_{21,i'}$. Hence the summation in the RHS of the second equality of (S740) can be further decomposed into two terms which leads to

$$\begin{aligned} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{i_2 \in Q_{1,i}} \sum_{i_1=N_g+1}^{i_{g,l}} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i)} T} &= \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_2 \in Q_{1,i_{g,l}}} \sum_{i_1=N_g+1}^{i_{g,l-1}} \frac{A_{2,i_2}^\top D_{1,g}^l A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T} \\ &+ \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_2 \in Q_{1,i_{g,l}}} \sum_{i_1=i_{g,l-1}+1}^{i_{g,l}} \frac{A_{2,i_2}^\top D_{2,gi}^l A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T}. \end{aligned} \quad (\text{S741})$$

For any $T \times 1$ real vector a , (S739) and the Cauchy-Schwarz inequality,

$$\begin{aligned} a^\top (D_{1,g}^l)^\top D_{1,g}^l a &= \sum_{i'_1, i'_2=i_{g,l-1}+1}^{i_{g,l}} a^\top d_{12,i'_1} d_{21,i'_2} a \leq \left(\sum_{i'=i_{g,l-1}+1}^{i_{g,l}} \|a^\top d_{12,i'}\| \right)^2 \\ &\leq \max_{i \leq n} \lambda_{\max}(d_{12,i} d_{21,i}) \left(\sum_{i'=i_{g,l-1}+1}^{i_{g,l}} (a^\top a)^{1/2} \right)^2 \leq K(a^\top a)(i_{g,l} - i_{g,l-1})^2, \end{aligned}$$

which implies that

$$\max_{g \in \mathcal{G}_1} \max_{l \leq M_g} \max_{i \in I_{g,l}} \frac{\lambda_{\max}((D_{1,g}^l)^\top D_{1,g}^l)}{(i_{g,l} - i_{g,l-1})^2} \leq K. \quad (\text{S742})$$

Similarly, we can show that

$$\max_{g \in \mathcal{G}_1} \max_{l \leq M_g} \max_{i \in I_{g,l}} \frac{\lambda_{\max}((D_{2,gi}^l)^\top D_{2,gi}^l)}{(i_{g,l} - i_{g,l-1})^2} \leq K. \quad (\text{S743})$$

By the independence of $A_{1,i}$ and $A_{2,i}$ across i ,

$$\begin{aligned} &\text{Var} \left(\sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_2 \in Q_{1,i_{g,l}}} \sum_{i_1=N_g+1}^{i_{g,l-1}} \frac{A_{2,i_2}^\top D_{1,g}^l A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T} \right) \\ &= \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_2 \in Q_{1,i_{g,l}}} \sum_{i_1=N_g+1}^{i_{g,l-1}} \frac{\mathbb{E}[A_{1,i_1}^\top (D_{1,g}^l)^\top \mathbb{E}[A_{2,i_2} A_{2,i_2}^\top] D_{1,g}^l A_{1,i_1}]}{(n_g n_{g_2(i_{g,l})} T)^2} \\ &\leq \max_{i \leq n} (\lambda_{\max}(\mathbb{E}[A_{j,i} A_{j,i}^\top]))^2 \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_2 \in Q_{1,i_{g,l}}} \sum_{i_1=N_g+1}^{i_{g,l-1}} \frac{\lambda_{\max}((D_{1,g}^l)^\top D_{1,g}^l) M_1}{(n_g n_{g_2(i_{g,l})} T)^2} \\ &\leq K M_1 T^{-2} \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \frac{(i_{g,l} - i_{g,l-1})^2 (i_{g,l-1} - N_g) (\#Q_{1,i_{g,l}})}{(n_g n_{g_2(i_{g,l})})^2} \\ &\leq K M_1 T^{-2} \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \frac{i_{g,l} - i_{g,l-1}}{n_g} \leq K G_1 M_1 T^{-2} \leq K T^{-1} \end{aligned} \quad (\text{S744})$$

where the $\#Q_{1,i_{g,l}}$ denotes the cardinality of $Q_{1,i_{g,l}}$, the second inequality is by (S716) and (S742), and the third inequality is by $n_g \geq i_{g,l-1} - N_g$ and $n_{g_2(i_{g,l})} \geq \#Q_{1,i_{g,l}} + i_{g,l} - i_{g,l-1}$ such that

$(i_{g,l} - i_{g,l-1})(\#Q_{1,i_{g,l}})n_{g_2(i_{g,l})}^{-2} \leq 1$, and the fourth inequality is due to $\sum_{l \leq M_g} (i_{g,l} - i_{g,l-1}) = n_g$, and the last inequality is by $G_1 M_1 T^{-1} = G_1 M_1 (nT)^{-1/2} (n/T)^{1/2} \leq K$ where the inequality holds by Assumptions 1(i) and 10(ii). Similarly,

$$\begin{aligned}
& \text{Var} \left(\sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_2 \in Q_{1,i_{g,l}}} \sum_{i_1 = i_{g,l-1}+1}^{i_{g,l}-1} \frac{A_{2,i_2}^\top D_{2,g i_1}^l A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T} \right) \\
&= \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_2 \in Q_{1,i_{g,l}}} \sum_{i_1 = i_{g,l-1}+1}^{i_{g,l}-1} \frac{\mathbb{E}[A_{1,i_1}^\top (D_{2,g i_1}^l)^\top \mathbb{E}[A_{2,i_2} A_{2,i_2}^\top] D_{2,g i_1}^l A_{1,i_1}]}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&\leq \max_{i \leq n} (\lambda_{\max}(\mathbb{E}[A_{j,i} A_{j,i}^\top]))^2 \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_2 \in Q_{1,i_{g,l}}} \sum_{i_1 = i_{g,l-1}+1}^{i_{g,l}-1} \frac{\lambda_{\max}((D_{2,g i_1}^l)^\top D_{2,g i_1}^l) M_1}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&\leq K M_1 T^{-2} \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \frac{(i_{g,l} - i_{g,l-1})^2 (i_{g,l-1} - N_g) (\#Q_{1,i_{g,l}})}{(n_g n_{g_2(i_{g,l})})^2} \leq K G_1 M_1 T^{-2} \leq K T^{-1}. \quad (\text{S745})
\end{aligned}$$

Since $\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{i_2 \in Q_{1,i}} \sum_{i_1 = N_g+1}^{i-1} \frac{\mathbb{E}[A_{2,i_2}^\top d_{21,i} A_{1,i_1}]}{n_g n_{g_2(i)} T} = 0$, by (S741), (S744), (S745) and Markov's inequality, we have

$$\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{i_2 \in Q_{1,i}} \sum_{i_1 = N_g+1}^{i-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i)} T} = O_p(T^{-1/2}). \quad (\text{S746})$$

We next study the second term in the RHS of (S738). For any $i \in I_{g,l}$, by the arrangement in I_g , it is clear that $Q_{2,i} = \{i_{g,l-1} + 1, \dots, i - 1\}$ and hence

$$\sum_{i_2 \in Q_{2,i}} \sum_{i_1 = N_g+1}^{i-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i)} T} = \sum_{i_2 = i_{g,l-1}+1}^{i-1} \sum_{i_1 = N_g+1}^{i_{g,l}-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T} + \sum_{i_1, i_2 = i_{g,l-1}+1}^{i-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T}. \quad (\text{S747})$$

Since

$$\begin{aligned}
\sum_{i \in I_{g,l}} \sum_{i_2 = i_{g,l-1}+1}^{i-1} \sum_{i_1 = N_g+1}^{i_{g,l}-1} \frac{A_{1,i_1}^\top d_{12,i} A_{2,i_2}}{n_g n_{g_2(i_{g,l})} T} &= \sum_{i_1 = N_g+1}^{i_{g,l}-1} \frac{A_{1,i_1}^\top}{n_g n_{g_2(i_{g,l})} T} \sum_{i = i_{g,l-1}+2}^{i_{g,l}} d_{12,i} \sum_{i_2 = i_{g,l-1}+1}^{i-1} A_{2,i_2} \\
&= \sum_{i_1 = N_g+1}^{i_{g,l}-1} \sum_{i_2 = i_{g,l-1}+1}^{i_{g,l}-1} \frac{A_{1,i_1}^\top (D_{2,g i_2}^l)^\top A_{2,i_2}}{n_g n_{g_2(i_{g,l})} T}, \quad (\text{S748})
\end{aligned}$$

by the similar arguments for showing (S744), we get

$$\begin{aligned}
& \text{Var} \left(\sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i \in I_{g,l}} \sum_{i_2 = i_{g,l-1}+1}^{i-1} \sum_{i_1 = N_g+1}^{i_{g,l-1}} \frac{A_{1,i_1}^\top d_{12,i} A_{2,i_2}}{n_g n_{g_2(i_{g,l})} T} \right) \\
&= \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_1 = N_g+1}^{i_{g,l-1}} \sum_{i_2 = i_{g,l-1}+1}^{i_{g,l-1}} \frac{\mathbb{E}[A_{1,i_1}^\top (D_{2,g i_2}^l)^\top \mathbb{E}[A_{2,i_2} A_{2,i_2}^\top] D_{2,g i_2}^l A_{1,i_1}]}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&\leq \max_{i \leq n} (\lambda_{\max}(\mathbb{E}[A_{j,i} A_{j,i}^\top]))^2 \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_1 = N_g+1}^{i_{g,l-1}} \sum_{i_2 = i_{g,l-1}+1}^{i_{g,l-1}} \frac{\lambda_{\max}((D_{2,g i_2}^l)^\top D_{2,g i_2}^l) M_1}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&\leq K M_1 T^{-2} \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \frac{(i_{g,l} - i_{g,l-1})^3 (i_{g,l-1} - N_g)}{(n_g n_{g_2(i_{g,l})})^2} \\
&\leq K M_1 T^{-2} \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \frac{i_{g,l} - i_{g,l-1}}{n_g} \leq K G_1 M_1 T^{-2} \leq K T^{-1}. \tag{S749}
\end{aligned}$$

Next note that

$$\begin{aligned}
& \sum_{i \in I_{g,l}} \sum_{i_1, i_2 = i_{g,l-1}+1}^{i-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i)} T} \\
&= \sum_{i \in I_{g,l}} \sum_{i_1 = i_{g,l-1}+1}^{i-1} \frac{A_{2,i_1}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T} + 2 \sum_{i \in I_{g,l}} \sum_{i_1 = i_{g,l-1}+2}^{i-1} \sum_{i_2 = i_{g,l-1}+1}^{i_1-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T} \\
&= \sum_{i = i_{g,l-1}+1}^{i_{g,l-1}} \frac{A_{2,i}^\top D_{2,g i}^l A_{1,i}}{n_g n_{g_2(i_{g,l})} T} + 2 \sum_{i_1 = i_{g,l-1}+2}^{i_{g,l-1}} \sum_{i_2 = i_{g,l-1}+1}^{i_1-1} \frac{A_{2,i_2}^\top D_{2,g i_1}^l A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T}. \tag{S750}
\end{aligned}$$

By the independence of $A_{1\gamma,i}$ and $A_{2\gamma,i}$ across i ,

$$\begin{aligned}
& \text{Var} \left(\sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i = i_{g,l-1}+1}^{i_{g,l-1}} \frac{A_{2,i}^\top D_{2,g i}^l A_{1,i}}{n_g n_{g_2(i_{g,l})} T} \right) \\
&= \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i = i_{g,l-1}+1}^{i_{g,l-1}} \frac{\text{Var}(A_{2,i}^\top D_{2,g i}^l A_{1,i})}{(n_g n_{g_2(i_{g,l})} T)^2} \leq \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i = i_{g,l-1}+1}^{i_{g,l-1}} \frac{\mathbb{E}[(A_{2,i}^\top D_{2,g i}^l A_{1,i})^2]}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&\leq \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i = i_{g,l-1}+1}^{i_{g,l-1}} \frac{\mathbb{E}[(A_{1,i}^\top (D_{2,g i}^l)^\top D_{2,g i}^l A_{1,i}) (A_{2,i}^\top A_{2,i})]}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&\leq \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i = i_{g,l-1}+1}^{i_{g,l-1}} \frac{\lambda_{\max}((D_{2,g i}^l)^\top D_{2,g i}^l) (\mathbb{E}[|A_{1,i}|^4] \mathbb{E}[|A_{2,i}|^4])^{1/2}}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&\leq K \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \frac{(i_{g,l} - i_{g,l-1})^3 M_1 M_2}{(n_g n_{g_2(i_{g,l})} T)^2} \leq K M_1 M_2 T^{-2} \sum_{g \in \mathcal{G}_1} n_g^{-1} \leq K G_2^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1}, \tag{S751}
\end{aligned}$$

where the second inequality is by the Cauchy-Schwarz inequality, the third inequality is by Hölder's inequality, the fourth inequality is by (S733) and (S743), and the last inequality is by Assumptions 1(i) and 10(ii). By the similar arguments for showing (S744), we also have

$$\begin{aligned}
& \text{Var} \left(\sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_1=i_{g,l-1}+2}^{i_{g,l}-1} \sum_{i_2=i_{g,l-1}+1}^{i_1-1} \frac{A_{2,i_2}^\top D_{2,g i_1}^l A_{1,i_1}}{n_g n_{g_2(i_{g,l})} T} \right) \\
&= \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_1=i_{g,l-1}+2}^{i_{g,l}-1} \sum_{i_2=i_{g,l-1}+1}^{i_1-1} \frac{\text{Var}(A_{2,i_2}^\top D_{2,g i_1}^l A_{1,i_1})}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&= \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_1=i_{g,l-1}+2}^{i_{g,l}-1} \sum_{i_2=i_{g,l-1}+1}^{i_1-1} \frac{\mathbb{E}[A_{1,i_1}^\top (D_{2,g i_1}^l)^\top \mathbb{E}[A_{2,i_2} A_{2,i_2}^\top] D_{2,g i_1}^l A_{1,i_1}]}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&\leq \max_{i \leq n} (\lambda_{\max}(\mathbb{E}[A_{j,i} A_{j,i}^\top]))^2 \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i_1=i_{g,l-1}+2}^{i_{g,l}-1} \sum_{i_2=i_{g,l-1}+1}^{i_1-1} \frac{\lambda_{\max}((D_{2,g i_1}^l)^\top D_{2,g i_1}^l) M_1}{(n_g n_{g_2(i_{g,l})} T)^2} \\
&\leq K M_1 T^{-2} \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \frac{(i_{g,l} - i_{g,l-1})^4}{(n_g n_{g_2(i_{g,l})})^2} \leq K G_1 M_1 T^{-2} \leq K T^{-1}. \tag{S752}
\end{aligned}$$

Collecting the results in (S747)-(S752), we deduce that

$$\begin{aligned}
\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \sum_{i_2 \in Q_{2,i}} \sum_{i_1=N_g+1}^{i-1} \frac{A_{2,i_2}^\top d_{21,i} A_{1,i_1}}{n_g n_{g_2(i)} T} &= \sum_{g \in \mathcal{G}_1} \sum_{l \leq M_g} \sum_{i=i_{g,l-1}+1}^{i_{g,l}-1} \frac{\mathbb{E}[A_{2,i}^\top D_{2,g i}^l A_{1,i}]}{n_g n_{g_2(i_{g,l})} T} \\
&\quad + O_p(T^{-1/2} + G_2^{-1/2} \max_{g \in \mathcal{G}_1} n_g^{-1/2}),
\end{aligned}$$

which together with (S738) and (S746) implies

$$\sum_{i \leq n} (\mathbb{E}[\tilde{U}_{1,i} \tilde{U}_{2,i} | \mathcal{F}_{nT,i-1}] - \mathbb{E}[\tilde{U}_{1,i} \tilde{U}_{2,i}]) = O_p(T^{-1/2} + G_2^{-1/2} \max_{g \in \mathcal{G}_1} n_g^{-1/2}). \tag{S753}$$

We can switch the role of $\mathcal{G}_1$ and $\mathcal{G}_2$, and use the same argument above to show that

$$\sum_{i \leq n} (\mathbb{E}[\tilde{U}_{1,i} \tilde{U}_{2,i} | \mathcal{F}_{nT,i-1}] - \mathbb{E}[\tilde{U}_{1,i} \tilde{U}_{2,i}]) = O_p(T^{-1/2} + G_1^{-1/2} \max_{g \in \mathcal{G}_2} n_g^{-1/2}). \tag{S754}$$

Since

$$\begin{aligned}
\min \left\{ G_1^{-1} \max_{g \in \mathcal{G}_2} n_g^{-1}, G_2^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1} \right\} &\leq \left(G_1^{-1/2} \max_{g \in \mathcal{G}_2} n_g^{-1/2} \right) \left(G_2^{-1/2} \max_{g \in \mathcal{G}_1} n_g^{-1/2} \right) \\
&\leq \frac{G_1^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1} + G_2^{-1} \max_{g \in \mathcal{G}_2} n_g^{-1}}{2} \leq \max_{j=1,2} \left(G_j^{-1} \max_{g \in \mathcal{G}_j} n_g^{-1} \right), \tag{S755}
\end{aligned}$$

the desired result in (S725) follows from (S753), (S754) and (S755).

Q.E.D.

Lemma S71. *Under the conditions of Theorem A4, we have for $j = 1, 2$*

$$\max_{g \in \mathcal{G}_j} \max_{i \in I_g} \mathbb{E} \left[\left| M_j^{-1/2} (n_g T)^{1/2} \tilde{U}_{j,i} \right|^4 \right] \leq K. \quad (\text{S756})$$

PROOF OF LEMMA S71. We only prove (S756) for $j = 1$, since the proof for $j = 2$ follows the same arguments. We shall use the structure of $\mathcal{G}_1$ employed in verifying (S724) for $j = 1$ in the proof of Lemma S70. That is for any $g \in \mathcal{G}_1$, $I_g = \{N_g + 1, \dots, N_g + n_g\}$ where $N_g \equiv \sum_{g' < g} n_{g'}$. Following these notations, we can write

$$\tilde{U}_{1,i} = (n_g T^{1/2})^{-1} \sum_{i'=N_g+1}^{i-1} A_{1,i}^\top A_{1,i'}$$

for any $i \in I_g$ and any $g \in \mathcal{G}_1$. Since $\{A_{j,i}^\top A_{j,i'}\}_{i'=1}^{i-1}$ are independent conditioning on $A_{j,i}$, we can use Rosenthal's inequality of independent random variables to obtain

$$\begin{aligned} \mathbb{E}[(n_g T^{1/2} \tilde{U}_{1,i})^4 | A_{1,i}] &= \mathbb{E} \left[\left(\sum_{i'=N_g+1}^{i-1} A_{1,i}^\top A_{1,i'} \right)^4 \middle| A_{1,i} \right] \\ &\leq K \left(\sum_{i'=N_g+1}^{i-1} \mathbb{E} \left[(A_{1,i}^\top A_{1,i'})^2 \middle| A_{1,i} \right] \right)^2 + K \sum_{i'=N_g+1}^{i-1} \mathbb{E} \left[(A_{1,i}^\top A_{1,i'})^4 \middle| A_{1,i} \right]. \end{aligned} \quad (\text{S757})$$

By the independence of $A_{j,i}$ and $A_{j,i'}$ for any $i \neq i'$,

$$\begin{aligned} \sum_{i'=N_g+1}^{i-1} \mathbb{E} \left[(A_{1,i}^\top A_{1,i'})^2 \middle| A_{1,i} \right] &= A_{1,i}^\top \left(\sum_{i'=N_g+1}^{i-1} \mathbb{E}[A_{1,i'} A_{1,i'}^\top] \right) A_{1,i} \\ &\leq (A_{1,i}^\top A_{1,i}) \sum_{i'=N_g+1}^{i-1} \lambda_{\max}(\mathbb{E}[A_{1,i'} A_{1,i'}^\top]) \leq K n_g (A_{1,i}^\top A_{1,i}), \end{aligned} \quad (\text{S758})$$

where the second inequality is by (S716). Therefore,

$$\max_{g \in \mathcal{G}_1} \max_{i \in I_g} \mathbb{E} \left[\left(n_g^{-1} \sum_{i'=N_g+1}^{i-1} \mathbb{E} \left[(A_{1,i}^\top A_{1,i'})^2 \middle| A_{1,i} \right] \right)^2 \right] \leq K \max_{i \leq n} \mathbb{E}[(A_{1,i}^\top A_{1,i})^2] \leq K M_1^2, \quad (\text{S759})$$

where the second inequality is by (S733).

For any $i, i' = 1, \dots, n$ with $i \neq i'$, by the independence between $A_{j,i}$ and $A_{j,i'}$, and the triangle

inequality

$$\begin{aligned}
\mathbb{E} \left[(A_{1,i}^\top A_{1,i'})^4 \right] &= \mathbb{E} \left[\left| \sum_{m \in \mathcal{M}_1} \tilde{\Psi}_{1\gamma,im} \tilde{\Psi}_{1\gamma,i'm} \right|^4 \right] \\
&\leq \sum_{m_1, m_2, m_3, m_4} |\mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \cdots \tilde{\Psi}_{1\gamma,im_4} \tilde{\Psi}_{1\gamma,i'm_1} \cdots \tilde{\Psi}_{1\gamma,i'm_4}]| \\
&\leq \max_{m_1, m_2, m_3, m_4} |\mathbb{E}[\tilde{\Psi}_{1\gamma,i'm_1} \cdots \tilde{\Psi}_{1\gamma,i'm_4}]| \sum_{m_1, m_2, m_3, m_4} |\mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \cdots \tilde{\Psi}_{1\gamma,im_4}]| \\
&\leq \max_{i \leq n} \max_{m \in \mathcal{M}_1} \mathbb{E}[|\tilde{\Psi}_{1\gamma,im}|^4] \sum_{m_1, m_2, m_3, m_4} |\mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \tilde{\Psi}_{1\gamma,im_2} \tilde{\Psi}_{1\gamma,im_3} \tilde{\Psi}_{1\gamma,im_4}]| \\
&\leq K \sum_{m_1, m_2, m_3, m_4} |\mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \tilde{\Psi}_{1\gamma,im_2} \tilde{\Psi}_{1\gamma,im_3} \tilde{\Psi}_{1\gamma,im_4}]|, \tag{S760}
\end{aligned}$$

where the third inequality is by Hölder's inequality and the last inequality is by (S714). Consider any m_1, m_2, m_3, m_4 from $\mathcal{M}_1$ with $m_1 \leq \dots \leq m_4$. Let $r \equiv \max_{j=1,2,3} \{m_{j+1} - m_j\}$ and $j_r \equiv \min\{j = 1, 2, 3 : m_{j+1} - m_j = r\}$. It is clear that r ranges from 0 to $M_1 - 1$. When $r = 0$, we have $m_1 = \dots = m_4 = m$ for some $m \in \mathcal{M}_1$ and in this case

$$|\mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \tilde{\Psi}_{1\gamma,im_2} \tilde{\Psi}_{1\gamma,im_3} \tilde{\Psi}_{1\gamma,im_4}]| = \mathbb{E}[\tilde{\Psi}_{1\gamma,im}^4] \leq K, \tag{S761}$$

where the inequality is by (S714). We next consider the case that $r = 1, \dots, M_1 - 1$. By Hölder's inequality and Corollary A.2. in Hall and Heyde (1980),

$$\begin{aligned}
&\left| \mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \tilde{\Psi}_{1\gamma,im_2} \tilde{\Psi}_{1\gamma,im_3} \tilde{\Psi}_{1\gamma,im_4}] - \mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \cdots \tilde{\Psi}_{1\gamma,im_{j_r}}] \mathbb{E}[\tilde{\Psi}_{1\gamma,im_{j_r+1}} \cdots \tilde{\Psi}_{1\gamma,im_4}] \right| \\
&\leq (\alpha_i(r))^{\delta/(4+\delta)} \left\| \tilde{\Psi}_{1\gamma,im_1} \cdots \tilde{\Psi}_{1\gamma,im_{j_r}} \right\|_{(4+\delta)/j_r} \left\| \tilde{\Psi}_{1\gamma,im_{j_r+1}} \cdots \tilde{\Psi}_{1\gamma,im_4} \right\|_{(4+\delta)/(4-j_r)} \\
&\leq (\alpha_i(r))^{\delta/(4+\delta)} \max_{m \in \mathcal{M}_1} \left\| \tilde{\Psi}_{1\gamma,im} \right\|_{(4+\delta)}^4 \leq K(\alpha_i(r))^{\delta/(4+\delta)}, \tag{S762}
\end{aligned}$$

where the last inequality is by (S714). Since $\mathbb{E}[\tilde{\Psi}_{1\gamma,im}] = 0$, by (S762) we have

$$\left| \mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \tilde{\Psi}_{1\gamma,im_2} \tilde{\Psi}_{1\gamma,im_3} \tilde{\Psi}_{1\gamma,im_4}] \right| \leq K(\alpha_i(r))^{\delta/(4+\delta)} \tag{S763}$$

if $j_r = 1$ or 3. On the other hand if $j_r = 2$, then

$$\left| \mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \cdots \tilde{\Psi}_{1\gamma,im_4}] \right| \leq K(\alpha_i(r))^{\delta/(4+\delta)} + \left| \mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \tilde{\Psi}_{1\gamma,im_2}] \mathbb{E}[\tilde{\Psi}_{1\gamma,im_3} \tilde{\Psi}_{1\gamma,im_4}] \right|. \tag{S764}$$

Since for each $r_0 = 1, \dots, M_1 - 1$, there are at most $3(M_1 - r_0)(1 + r_0)^2$ of indices m_1, m_2, m_3, m_4 satisfying $1 \leq m_1 \leq \dots \leq m_4 \leq M_1$ such that $\max_{j=1,2,3} \{m_{j+1} - m_j\} = r_0$. From (S761), (S762)

and (S763) we deduce that

$$\begin{aligned}
& \sum_{m_1 \leq \dots \leq m_4} \left| \mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \tilde{\Psi}_{1\gamma,im_2} \tilde{\Psi}_{1\gamma,im_3} \tilde{\Psi}_{1\gamma,im_4}] \right| \\
& \leq \sum_{m \in \mathcal{M}_1} \mathbb{E}[\tilde{\Psi}_{1\gamma,im}^4] + K \sum_{r_0=1}^{M_1-1} (\alpha_i(r_0))^{\delta/(4+\delta)} (M_1 - r_0)(1 + r_0)^2 \\
& \quad + \sum_{m_1 \leq \dots \leq m_4} \left| \mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \tilde{\Psi}_{1\gamma,im_2}] \mathbb{E}[\tilde{\Psi}_{1\gamma,im_3} \tilde{\Psi}_{1\gamma,im_4}] \right| \\
& \leq KM_1 + KM_1 \sum_{r_0=1}^{M_1-1} (\alpha_i(r_0))^{\delta/(4+\delta)} r_0^2 + \left(\sum_{m_1, m_2 \in \mathcal{M}_1} \left| \mathbb{E}[\tilde{\Psi}_{1\gamma,im_1} \tilde{\Psi}_{1\gamma,im_2}] \right| \right)^2 \\
& \leq KM_1 + KM_1^2 \leq KM_1^2, \tag{S765}
\end{aligned}$$

where the third inequality is by Assumption 1(iv) and (S715). Since for any $m_1, m_2, m_3, m_4 \in \mathcal{M}_1$ there are at most 24 ways to order them from the smallest to the largest, by (S760) and (S765), we deduce that $\mathbb{E}[(A_{1,i}^\top A_{1,i'})^4] \leq KM_1^2$ for any $i \neq i'$, which implies that

$$\max_{i \leq n} n_g^{-1} \sum_{i'=N_g+1}^{i-1} \mathbb{E} \left[(M_1^{-1/2} A_{1,i}^\top A_{1,i'})^4 \right] \leq K. \tag{S766}$$

Collecting the results in (S757), (S759), (S766), we obtain

$$\begin{aligned}
\max_{i \in I_g} \mathbb{E}[|M_1^{-1/2} (n_g T)^{1/2} \tilde{U}_{1,i}|^4] &= \max_{g \in \mathcal{G}_1} \max_{i \in I_g} \mathbb{E} \left[n_g^{-2} \mathbb{E}[(M_1^{-1/2} n_g T^{1/2} \tilde{U}_{1,i})^4 | A_{1,i}] \right] \\
&\leq K \max_{g \in \mathcal{G}_1} \max_{i \in I_g} \mathbb{E} \left[\left(n_g^{-1} \sum_{i'=N_g+1}^{i-1} \mathbb{E} \left[(M_1^{-1/2} A_{1,i}^\top A_{1,i'})^2 | A_{1,i} \right] \right)^2 \right] \\
&\quad + K \max_{g \in \mathcal{G}_1} \max_{i \in I_g} \left(n_g^{-2} \sum_{i'=N_g+1}^{i-1} \mathbb{E} \left[(M_1^{-1/2} A_{1,i}^\top A_{1,i'})^4 \right] \right) \leq K,
\end{aligned}$$

which shows (S756) for $j = 1$.

Q.E.D.

4 Further Examples of Linear Models

In this section, we provide several examples from the linear panel model

$$y_{i,t} = x_{i,t}^\top \beta + u_{i,t} + \varepsilon_{i,t},$$

where $y_{i,t}$ is the dependent variable, $x_{i,t}$ are the observed regressors, $u_{i,t}$ is a unobservable term which may include individual and/or time fixed effects, and $\varepsilon_{i,t}$ is i.i.d. across i and t and normally distributed with mean zero and variance σ^2 . The data are $\{z_{i,t}\}_{i \leq n, t \leq T}$, where $z_{i,t} = (y_{i,t}, x_{i,t}^\top)^\top$. As illustrated in the examples below, $u_{i,t}$ takes different forms under various model specifications.

Example 1. (Time Invariant Fixed Effect) In this model, $u_{i,t}$ includes a time invariant fixed effect, i.e., $u_{i,t} = \gamma_i$. Since there is a group structure in t and no group structure in i , we have $\mathcal{G} = \{1, \dots, n\}$, $I_g = \{g\}$ for any $g \in \mathcal{G}$, $\mathcal{M} = \{1\}$ and $I_m = \{1, \dots, T\}$ for $m \in \mathcal{M}$. Therefore $\gamma_{g,1} = \gamma_g$ for any $g \in \mathcal{G}$, and the log-likelihood function for any $i = g$ and any $t = m$ is

$$\log \psi(z_{it}; \theta, \gamma_{g,m}) = -\frac{\log(\sigma^2)}{2} - \frac{1}{2\sigma^2} \left(y_{i,t} - x_{i,t}^\top \beta - \gamma_g \right)^2,$$

which leads to the joint-likelihood function

$$\sum_{g \in \mathcal{G}} \sum_{i \in I_g} \sum_{m \in \mathcal{M}} \sum_{t \in I_m} \psi(z_{it}; \theta, \gamma_{g,m}) = -\frac{nT \log(\sigma^2)}{2} - \frac{1}{2\sigma^2} \sum_{t \leq T} \sum_{i \leq n} \left(y_{i,t} - x_{i,t}^\top \theta - \gamma_i \right)^2,$$

where $\theta = (\beta^\top, \sigma^2)^\top$.

Example 2. (Time Varying Fixed Effect) Consider the time variate fixed effect model where $u_{i,t} = \gamma_{i,1}$ for $t = 1, \dots, T_0$ and $u_{i,t} = \gamma_{i,2}$ for $t = T_0 + 1, \dots, T$. In this case, there is no group structure along i but a group structure on t . Hence we have $\mathcal{G} = \{1, \dots, n\}$, $I_g = \{g\}$ for any $g \in \mathcal{G}$, $\mathcal{M} = \{1, 2\}$ and

$$I_m = \begin{cases} \{1, \dots, T_0\}, & \text{if } m = 1 \\ \{T_0 + 1, \dots, T\}, & \text{if } m = 2 \end{cases}.$$

Therefore the log-likelihood function for any $i = g$ and any $t \in I_m$ is

$$\log f(z_{it}; \theta, \gamma_{g,m}) = -\frac{\log(\sigma^2)}{2} - \frac{1}{2\sigma^2} \left(y_{i,t} - x_{i,t}^\top \beta - \gamma_{g,m} \right)^2$$

which leads to the joint-likelihood function

$$\sum_{g \in \mathcal{G}} \sum_{i \in I_g} \sum_{m \in \mathcal{M}} \sum_{t \in I_m} \psi(z_{it}; \theta, \gamma_{g,m}) = -\frac{nT \log(\sigma^2)}{2} - \frac{1}{2\sigma^2} \sum_{i \leq n} \sum_{m \in \{1,2\}} \sum_{t \in I_m} \left(y_{i,t} - x_{i,t}^\top \beta - \gamma_{im} \right)^2,$$

where $\theta = (\beta^\top, \sigma^2)^\top$.

Example 3. (Time Fixed Effect) In this example, only the time fixed effect shows up in $u_{i,t}$, i.e., $u_{i,t} = \zeta_t$ for any i and any t . Since there is no grouping structure in t and a group structure in i , we have $\mathcal{G} = \{1\}$, $I_g = \{1, \dots, n\}$ for any $g \in \mathcal{G}$, $\mathcal{M} = \{1, \dots, T\}$ and $I_m = \{m\}$ for $m \in \mathcal{M}$. Therefore $\gamma_{1,m} = \zeta_m$ for any $m \in \mathcal{M}$, and the log-likelihood function for any $i = g$ and any $t = m$ is

$$\log \psi(z_{it}; \theta, \gamma_{g,m}) = -\frac{\log(\sigma^2)}{2} - \frac{1}{2\sigma^2} \left(y_{i,t} - x_{i,t}^\top \beta - \zeta_m \right)^2,$$

which leads to the joint-likelihood function

$$\sum_{g \in \mathcal{G}} \sum_{i \in I_g} \sum_{m \in \mathcal{M}} \sum_{t \in I_m} \psi(z_{it}; \theta, \gamma_{g,m}) = -\frac{nT \log(\sigma^2)}{2} - \frac{1}{2\sigma^2} \sum_{t \leq T} \sum_{i \leq n} \left(y_{i,t} - x_{i,t}^\top \theta - \zeta_t \right)^2,$$

where $\theta = (\beta^\top, \sigma^2)^\top$.

Example 4. (Grouped Time Fixed Effects) In this example, there is a group structure in i but no group structure in t , i.e., $u_{i,t} = \gamma_{g(i),t}$, where $g(\cdot)$ is a known function with range $\{1, \dots, G\}$. Hence in this case, $\mathcal{G} = \{1, \dots, G\}$, $I_g = \{g(i) = g : i = 1, \dots, n\}$ for any $g \in \mathcal{G}$, $\mathcal{M} = \{1, \dots, T\}$ and $I_m = \{m\}$ for $m \in \mathcal{M}$. Bonhomme and Manresa (2015) investigated this model where the function $g(\cdot)$ was considered unknown, and one of their goals was to estimate the latent group structure that is implied by $g(\cdot)$. The log-likelihood function for any $i \in I_g$ and any $t = m$ is

$$\log \psi(z_{it}; \theta, \gamma_{g,m}) = -\frac{\log(\sigma^2)}{2} - \frac{1}{2\sigma^2} \left(y_{i,t} - x_{i,t}^\top \beta - \gamma_{g,t} \right)^2$$

which leads to the joint-likelihood function

$$\sum_{g \in \mathcal{G}} \sum_{i \in I_g} \sum_{m \in \mathcal{M}} \sum_{t \in I_m} \psi(z_{it}; \theta, \gamma_{g,m}) = -\frac{nT \log(\sigma^2)}{2} - \frac{1}{2\sigma^2} \sum_{g \in \mathcal{G}} \sum_{i \in I_g} \sum_{t \leq T} \left(y_{i,t} - x_{i,t}^\top \beta - \gamma_{g,t} \right)^2,$$

where $\theta = (\beta^\top, \sigma^2)^\top$.

Below, we present an expansion of the likelihood function and discuss Assumption 10.

Example 1. (Time Invariant Fixed Effect ctd.). In this example, the expansion of the likelihood function can be written as

$$\sum_{i \leq n} \hat{\Psi}_i(\hat{\phi}_i) = \sum_{i \leq n} \hat{\Psi}_i(\phi_i^*) - 2^{-1} \sum_{i \leq n} \hat{\Psi}_{\gamma,i}^2 / \Psi_{\gamma\gamma,i} + o_p \left(n^{1/2} T^{-1} \right),$$

where $\phi_i \equiv (\theta^\top, \gamma_i)^\top$, $\hat{\Psi}_i(\phi_i) \equiv T^{-1} \sum_{t \leq T} \psi(z_{i,t}; \phi_i)$, $\hat{\Psi}_{\gamma,i} \equiv T^{-1} \sum_{t \leq T} \partial \psi(z_{i,t}; \phi_i^*) / \partial \gamma$ and $\Psi_{\gamma\gamma,i} \equiv T^{-1} \sum_{t \leq T} \mathbb{E}[\partial^2 \psi(z_{i,t}; \phi_i^*) / \partial \gamma^2]$. Moreover we have $G = n$, $M = 1$, $n_g = 1$, $T_m = T$ so Assumption 10 boils down to $n^{4/p}/T = o(1)$ and $n/(nT)^{1/2} = (n/T)^{1/2} = O(1)$.

Example 2. (Time Varying Fixed Effect ctd.). In this example, the expansion of the likelihood function can be written as

$$\begin{aligned} \sum_{i \leq n} \sum_{m \in \mathcal{M}} T_m \hat{\Psi}_{im}(\hat{\phi}_{im}) &= \sum_{i \leq n} \sum_{m \in \mathcal{M}} T_m \hat{\Psi}_{im}(\phi_{im}^*) \\ &\quad - 2^{-1} \sum_{i \leq n} \sum_{m \in \mathcal{M}} T_m \hat{\Psi}_{\gamma,im}^\top \Psi_{\gamma\gamma,im}^{-1} \hat{\Psi}_{\gamma,im} + o_p \left((nM)^{1/2} \right), \end{aligned}$$

where $\phi_{im} \equiv (\theta^\top, \gamma_{im})^\top$, $\hat{\Psi}_{im} \equiv T_m^{-1} \sum_{t \in I_m} \psi(z_{i,t}; \phi_{im})$, $\hat{\Psi}_{\gamma,im} \equiv T_m^{-1} \sum_{t \in I_m} \partial \psi(z_{i,t}; \phi_{im}^*) / \partial \gamma$ and $\Psi_{\gamma\gamma,im} \equiv T_m^{-1} \sum_{t \in I_m} \mathbb{E}[\partial^2 \psi(z_{i,t}; \phi_{im}^*) / \partial \gamma^2]$. Since $G = n$, $M = 2$, $n_g = 1$, $T_1 = T_0$ and $T_2 = T - T_0$, Assumption 10 holds in this example if $n^{2/p}(T_0^{-1/2} + (T - T_0)^{-1/2}) = o(1)$ and $n/(nT)^{1/2} = (n/T)^{1/2} = O(1)$.

Example 3. (Time Fixed Effect ctd.). In this example, the expansion of the likelihood function can be written as

$$\sum_{t \leq T} \hat{\Psi}_t(\hat{\phi}_t) = \sum_{t \leq T} \hat{\Psi}_t(\phi_t^*) - 2^{-1} \sum_{t \leq T} \hat{\Psi}_{\gamma,t}^2 / \Psi_{\gamma\gamma,t} + o_p \left(T^{1/2} n^{-1} \right),$$

where $\phi_t \equiv (\theta^\top, \gamma_t)^\top$, $\hat{\Psi}_t(\phi_t) \equiv n^{-1} \sum_{i \leq n} \psi(z_{i,t}; \phi_t)$, $\hat{\Psi}_{\gamma,t} \equiv n^{-1} \sum_{i \leq n} \partial \psi(z_{i,t}; \phi_t^*) / \partial \gamma$ and $\Psi_{\gamma\gamma,t} \equiv n^{-1} \sum_{i \leq n} \mathbb{E}[\partial^2 \psi(z_{i,t}; \phi_t^*) / \partial \gamma^2]$. In this case, $G = 1$, $M = T$, $n_g = n$, $T_m = 1$. Hence Assumption 10 becomes $T^{4/p}/n = o(1)$ and $T/(nT)^{1/2} = (T/n)^{1/2} = O(1)$.

Example 4. (Grouped Time Fixed Effects ctd.). In this example, the expansion of the likelihood function can be written as

$$\sum_{t \leq T} \sum_{g \in \mathcal{G}} n_g \hat{\Psi}_{gt}(\hat{\phi}_{gt}) = \sum_{t \leq T} \sum_{g \in \mathcal{G}} n_g \hat{\Psi}_{gt}(\phi_{gt}^*) - 2^{-1} \sum_{t \leq T} \sum_{g \in \mathcal{G}} n_g \hat{\Psi}_{\gamma,gt}^\top \Psi_{\gamma\gamma,gt}^{-1} \hat{\Psi}_{\gamma,gt} + o_p\left((GT)^{1/2}\right)$$

where $\phi_{gt} \equiv (\theta^\top, \gamma_{g(i)t})^\top$, $\hat{\Psi}_{gt} \equiv n_g^{-1} \sum_{i \in I_g} \psi(z_{i,t}; \phi_{gt}^*)$, $\hat{\Psi}_{\gamma,gt} \equiv n_g^{-1} \sum_{i \in I_g} \partial \psi(z_{i,t}; \phi_{gt}^*) / \partial \gamma$ and $\Psi_{\gamma\gamma,im} \equiv n_g^{-1} \sum_{i \in I_g} \mathbb{E}[\partial^2 \psi(z_{i,t}; \phi_{gt}^*) / \partial \gamma^2]$. Since $M = T$, $n_g = \#I_g$ where $\#I_g$ denotes the cardinality of set I_g , and $T_m = 1$, Assumption 10 holds in this example if $(GT)^{2/p} \max_{g \leq G} n_g^{-1/2} = o(1)$ and $(GT)(nT)^{-1/2} = GT^{1/2}n^{-1/2} = O(1)$.

4.1 Time Invariant vs. Time Varying Individual Effects

Suppose that model 1 specifies a time invariant individual fixed effect with joint likelihood function $\sum_{i \leq n} \sum_{t \leq T} \psi_1(z_{i,t}; \theta_1, \gamma_{1,i})$, and model 2 specifies a time varying individual fixed effect with joint likelihood function $\sum_{i \leq n} \sum_{m=1}^{M_2} \sum_{t \in I_m} \psi_2(z_{i,t}; \theta_2, \gamma_{2,im})$ where $M_2 > 1$ is a fixed integer. In this example, we have $\mathcal{G}_1 = \{1, \dots, n\}$ and $\mathcal{M}_1 = \{1\}$ for model 1, and $\mathcal{G}_2 = \{1, \dots, n\}$ and $\mathcal{M}_2 = \{1, \dots, M_2\}$ for model 2.¹

We first provide the specific forms of $\tilde{V}_{j,i}$ and $\tilde{U}_{j,i}$ in this example. It is clear that

$$\tilde{\Psi}_{1\gamma,i} = \frac{T^{-1/2} \sum_{t \leq T} \psi_{1\gamma}(z_{i,t}; \phi_{1,i}^*)}{\left(T^{-1} \sum_{t \leq T} \mathbb{E} \left[-\psi_{1\gamma\gamma}(z_{i,t}; \phi_{1,i}^*) \right] \right)^{1/2}}$$

and

$$\tilde{\Psi}_{2\gamma,im} = \frac{T_m^{-1/2} \sum_{t \in I_m} \psi_{2\gamma}(z_{i,t}; \phi_{2,im}^*)}{\left(T_m^{-1} \sum_{t \in I_m} \mathbb{E} \left[-\psi_{2\gamma\gamma}(z_{i,t}; \phi_{2,im}^*) \right] \right)^{1/2}},$$

and we suppress the dependence of $\tilde{\Psi}_{1\gamma,im}$ on m since $\mathcal{M}_1 = \{1\}$. Therefore,

$$\tilde{V}_{1,i} = (2T^{1/2})^{-1} \tilde{\Psi}_{1\gamma,i}^2, \quad \tilde{V}_{2,i} = (2T^{1/2})^{-1} \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2 \quad \text{and} \quad \tilde{U}_{j,i} = 0 \text{ for } j = 1, 2.$$

The expansion of the QLR statistic in (101) thus implies that

$$\widehat{QLR}_{n,T} = n^{-1/2} \sum_{i \leq n} \left(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \frac{\tilde{\Psi}_{1\gamma,i}^2 - \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2}{2T^{1/2}} \right) + o_p(n^{-1/2}). \quad (\text{S767})$$

¹Since $G_2 = n$ in this example, we let M_2 be a fixed integer such that Assumption 10(ii) is satisfied.

Applying Theorem A3 obtains the asymptotic normality for $\widehat{QLR}_{n,T}$ with variance

$$\omega_{n,T}^2 = n^{-1} \sum_{i \leq n} \text{Var} \left(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \frac{\tilde{\Psi}_{1\gamma,i}^2 - \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2}{2T^{1/2}} \right).$$

In the case that $\psi_1(\cdot)$ and $\psi_2(\cdot)$ take the same function form (say $\psi(\cdot)$), model 2 nests model 1 under the null, since $\phi_{1,i}^* = \phi_{2,im}^*$ by the definitions of $\phi_{1,i}^*$ and $\phi_{2,im}^*$ and their identification conditions. Therefore, the expansion of the QLR statistic in (S767) is reduced to

$$\widehat{QLR}_{n,T} = (4nT)^{-1/2} \sum_{i \leq n} \left(\tilde{\Psi}_{1\gamma,i}^2 - \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2 \right) + o_p(n^{-1/2}). \quad (\text{S768})$$

Therefore in this scenario

$$\omega_{n,T}^2 = (4nT)^{-1} \sum_{i \leq n} \text{Var} \left(\tilde{\Psi}_{1\gamma,i}^2 - \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2 \right), \quad (\text{S769})$$

and Assumption 11(i) holds as long as $n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_{1\gamma,i}^2 - \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2)$ is bounded away from zero. We next provides the formula of $\omega_{n,T}^2$ in the simplified case that $\mathbb{E}[\partial^2 \psi(z_{i,t}; \phi_{i,t}^*) / \partial \gamma^2]$ is constant for any i and any t , and $\partial \psi(z_{i,t}; \phi_{i,t}^*) / \partial \gamma$ is independent with constant variance across t .

Lemma S72. *Consider two models with $\mathcal{G}_1 = \{1, \dots, n\}$ and $\mathcal{M}_1 = \{1\}$ for model 1, and $\mathcal{G}_2 = \{1, \dots, n\}$ and $\mathcal{M}_2 = \{1, \dots, M_2\}$ for model 2 where M_2 is a fixed positive integer. Suppose that: (i) $\psi_1(\cdot)$ and $\psi_2(\cdot)$ take the same form $\psi(\cdot)$; (ii) $\psi_{\gamma,i,t} \equiv \partial \psi(z_{i,t}; \phi_{i,t}^*) / \partial \gamma$ is independent across t with $\mathbb{E}[\psi_{\gamma,i,t}^4] < K$; (iii) $\mathbb{E}[\partial^2 \psi(z_{i,t}; \phi_{i,t}^*) / \partial \gamma^2] = \Psi_{\gamma\gamma}$ where $\Psi_{\gamma\gamma}$ is a nonzero constant; and (iv) $\min_{m \in \mathcal{M}_2} (T_m/T) \geq K^{-1}$. Then*

$$\begin{aligned} \omega_{n,T}^2 = & \frac{n^{-1} \sum_{i \leq n} \bar{\sigma}_{\gamma,i}^4 + \sum_{m=1}^{M_2} (1 - 2w_m) n^{-1} \sum_{i \leq n} \bar{\sigma}_{\gamma,im}^4}{2T\Psi_{\gamma\gamma}^2} \\ & + O \left(T^{-1} \max_{m \in \mathcal{M}_2} T_m^{-1} \right), \end{aligned} \quad (\text{S770})$$

where $w_m \equiv T_m T^{-1}$, $\bar{\sigma}_{\gamma,i} \equiv \sqrt{T^{-1} \sum_{t \leq T} \sigma_{\gamma,i,t}^2}$, $\bar{\sigma}_{\gamma,im} \equiv \sqrt{T_m^{-1} \sum_{t \in I_m} \sigma_{\gamma,i,t}^2}$ and $\sigma_{\gamma,i,t}^2 \equiv \text{Var}(\psi_{\gamma,i,t})$.

When $\psi_{\gamma,i,t}$ has constant variance across t , i.e., $\sigma_{\gamma,i,t}^2 = \sigma_{\gamma,i}^2$ for some finite constant $\sigma_{\gamma,i}^2$, we have

$$n^{-1} \sum_{i \leq n} \bar{\sigma}_{\gamma,i}^4 = n^{-1} \sum_{i \leq n} \bar{\sigma}_{\gamma,im}^4 = \sigma_{\gamma,i}^4,$$

and the approximated expression of $\omega_{n,T}^2$ in (S770) becomes

$$\omega_{n,T}^2 = \frac{M_2 - 1}{2} \frac{n^{-1} \sum_{i \leq n} \sigma_{\gamma,i}^4}{T\Psi_{\gamma\gamma}^2} + O \left(T^{-1} \max_{m \in \mathcal{M}_2} T_m^{-1} \right), \quad (\text{S771})$$

where the second term in the RHS of (S771) is asymptotically negligible if $\max_{m \in \mathcal{M}_2} T_m^{-1} = o(1)$.

4.2 Homogeneous vs. Heterogeneous Time Effects

Suppose that model 1 specifies a group heterogeneous time effect with joint likelihood function $\sum_{g=1}^{G_1} \sum_{i \in I_g} \sum_{t \leq T} \psi_1(z_{i,t}; \theta_1, \gamma_{1,gt})$ where $G_1 > 1$ is a fixed integer, and model 2 specifies a homogeneous time fixed effect with joint likelihood function $\sum_{i \leq n} \sum_{t \leq T} \psi_2(z_{i,t}; \theta_2, \gamma_{2,t})$. In this example, we have $\mathcal{G}_1 = \{1, \dots, G_1\}$ and $\mathcal{M}_1 = \{1, \dots, T\}$ for model 1, and $\mathcal{G}_2 = \{1\}$ and $\mathcal{M}_2 = \{1, \dots, T\}$ for model 2.²

Under the specifications of model 1 and model 2,

$$\tilde{\Psi}_{1\gamma,i,t} = \frac{\psi_{1\gamma}(z_{i,t}; \phi_{1,i,t}^*)}{\left(n_g^{-1} \sum_{i \in I_g} \mathbb{E} \left[-\psi_{1\gamma}(z_{i,t}; \phi_{1,i,t}^*) \right] \right)^{1/2}}$$

and

$$\tilde{\Psi}_{2\gamma,i,t} = \frac{\psi_{2\gamma}(z_{i,t}; \phi_{2,t}^*)}{\left(n^{-1} \sum_{i \leq n} \mathbb{E} \left[-\psi_{2\gamma}(z_{i,t}; \phi_{2,t}^*) \right] \right)^{1/2}}$$

where we write $\tilde{\Psi}_{j\gamma,im}$ as $\tilde{\Psi}_{j\gamma,i,t}$ since $\mathcal{M}_j = \{1, \dots, T\}$. Without loss of generality, we suppose that for any $g \in \mathcal{G}_1$, $I_g = \{N_g + 1, \dots, N_g + n_g\}$ where $N_g \equiv \sum_{g' < g} n_{g'}$. Then for any $i \in I_g$ and any $g \in \mathcal{G}_1$

$$\begin{aligned} \tilde{V}_{1,i} &= \frac{A_{1,i}^\top A_{1,i}}{2n_g T^{1/2}}, \quad \tilde{U}_{1,i} = \frac{\sum_{i'=N_g+1}^{i-1} A_{1,i}^\top A_{1,i'}}{n_g T^{1/2}}, \\ \tilde{V}_{2,i} &= \frac{A_{2,i}^\top A_{2,i}}{2n T^{1/2}} \quad \text{and} \quad \tilde{U}_{2,i} = \frac{\sum_{i'=1}^{i-1} A_{2,i}^\top A_{2,i'}}{n T^{1/2}}, \end{aligned}$$

where $A_{j,i} = (\tilde{\Psi}_{j\gamma,i,t})_{t \leq T}$ for $j = 1, 2$.³ Applying Theorem A3, we obtain the asymptotic normality for $\widehat{QLR}_{n,T}$ with variance

$$\omega_{n,T} = \sqrt{n^{-1} \sum_{i \leq n} \text{Var} \left(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{V}_{1,i} - \tilde{V}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i} \right)}.$$

Moreover, the lemma below shows that $\tilde{V}_{j,i}$ ($j = 1, 2$) are asymptotically negligible, and hence the asymptotic normality of $\widehat{QLR}_{n,T}$ is driven by $n^{-1} \sum_{i \leq n} (\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i})$. Moreover, the variance of $\widehat{QLR}_{n,T}$ is given by

$$\omega_{n,T}^* = \sqrt{n^{-1} \sum_{i \leq n} \left(\text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i}) + \text{Var}(\tilde{U}_{1,i} - \tilde{U}_{2,i}) \right)}.$$

Hence Assumption 11(i) holds as long as $\sum_{i \leq n} \text{Var}(\tilde{U}_{1,i} - \tilde{U}_{2,i})$ is bounded away from zero.

²Since $M_1 = T$ in this example, we let G_1 be a fixed integer such that Assumption 10(ii) is satisfied.

³Following the tradition that the summaion over an empty set is zero, we have $\tilde{U}_{1,N_g+1} = 0$ and $\tilde{U}_{2,1} = 0$ in the above display.

Lemma S73. Consider two models with $\mathcal{G}_1 = \{1, \dots, G_1\}$ and $\mathcal{M}_1 = \{1, \dots, T\}$ for model 1 where G_1 is a fixed positive integer, and $\mathcal{G}_2 = \{1\}$ and $\mathcal{M}_2 = \{1, \dots, T\}$ for model 2. Suppose that Assumptions 1, 7, 8, 9, 10(i) and 12 hold for models 1 and 2, and $\sum_{i \leq n} \text{Var}(\tilde{U}_{1,i} - \tilde{U}_{2,i}) > K^{-1}$. Then

$$\frac{\widehat{QLR}_{n,T} - \mathbb{E}[S_{n,T}] - QLR_{n,T}^*}{\omega_{n,T}^*} \rightarrow_d N(0, 1), \quad (\text{S772})$$

where $\mathbb{E}[S_{n,T}] = (2nT)^{-1/2} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \left(n_g^{-1} \mathbb{E}[A_{1,i}^\top A_{1,i}] - n^{-1} \mathbb{E}[A_{2,i}^\top A_{2,i}] \right)$, and

$$\begin{aligned} (\omega_{n,T}^*)^2 &= n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i}) \\ &\quad + \frac{\sum_{g \in \mathcal{G}_1} (\|\Sigma_{1\gamma,g}\|^2 - 2w_g \|\Sigma_{12\gamma,g}\|^2) + \left\| \sum_{g \in \mathcal{G}_1} w_g \Sigma_{2\gamma,g} \right\|^2}{2nT} \\ &\quad + O\left(n^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1}\right). \end{aligned} \quad (\text{S773})$$

where $w_g \equiv n_g n^{-1}$, $\Sigma_{j\gamma,g} \equiv n_g^{-1} \sum_{i \in I_g} \mathbb{E}[A_{j\gamma,i} A_{j\gamma,i}^\top]$ and $\Sigma_{12\gamma,g} \equiv n_g^{-1} \sum_{i \in I_g} \mathbb{E}[A_{1,i} A_{2,i}^\top]$.

To get a more explicit form for $(\omega_{n,T}^*)^2$, we consider the scenario that the conditions of Lemma S72 hold. In this case, model 1 nests model 1 under the null hypothesis and $\text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i}) = 0$. Moreover, $\Sigma_{j\gamma,g} = \Sigma_{12\gamma,g}$ with

$$\|\Sigma_{1\gamma,g}\|^2 = \|\Sigma_{12\gamma,g}\|^2 = \Psi_{\gamma\gamma}^{-2} \sum_{t \leq T} \bar{\sigma}_{\gamma,gt}^4 \quad \text{and} \quad \left\| \sum_{g \in \mathcal{G}_1} w_g \Sigma_{2\gamma,g} \right\|^2 = \Psi_{\gamma\gamma}^{-2} \sum_{t \leq T} \bar{\sigma}_{\gamma,t}^4$$

where $\bar{\sigma}_{\gamma,gt} \equiv \sqrt{n_g^{-1} \sum_{i \in I_g} \sigma_{\gamma,i,t}^2}$ and $\bar{\sigma}_{\gamma,t}^2 \equiv \sqrt{n^{-1} \sum_{i \leq n} \sigma_{\gamma,i,t}^2}$. Hence the expression in (S773) leads to

$$(\omega_{n,T}^*)^2 = \frac{T^{-1} \sum_{t \leq T} \bar{\sigma}_{\gamma,t}^4 + \sum_{g \in \mathcal{G}_1} (1 - 2w_g) T^{-1} \sum_{t \leq T} \bar{\sigma}_{\gamma,gt}^4}{2n \Psi_{\gamma\gamma}^2} + O\left(n^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1}\right).$$

4.3 Individual vs. Time Fixed Effect Models

Suppose that model 1 specifies individual fixed effects with joint likelihood $\sum_{i \leq n} \sum_{t \leq T} \psi_1(z_{i,t}; \theta_1, \gamma_{1,i})$, while model 2 specifies time fixed effects with joint likelihood $\sum_{i \leq n} \sum_{t \leq T} \psi_2(z_{i,t}; \theta_1, \gamma_{2,t})$. In this case, we have $\mathcal{G}_1 = \{1, \dots, n\}$ and $\mathcal{M}_1 = \{1\}$ for model 1, and $\mathcal{G}_2 = \{1\}$ and $\mathcal{M}_2 = \{1, \dots, T\}$ for model 2.

In this example, it is clear that

$$\tilde{\Psi}_{1\gamma,i} = \frac{T^{-1/2} \sum_{t \leq T} \psi_{1\gamma}(z_{i,t}; \phi_{1,i}^*)}{\left(T^{-1} \sum_{t \leq T} \mathbb{E}[-\psi_{1\gamma}(z_{i,t}; \phi_{1,i}^*)]\right)^{1/2}} \quad \text{and} \quad \tilde{\Psi}_{2\gamma,i,t} = \frac{\psi_{2\gamma}(z_{i,t}; \phi_{2,t}^*)}{\left(n^{-1} \sum_{i \leq n} \mathbb{E}[-\psi_{2\gamma}(z_{i,t}; \phi_{2,t}^*)]\right)^{1/2}}$$

where we suppress the dependence of $\tilde{\Psi}_{1\gamma,im}$ on m since $\mathcal{M}_1 = \{1\}$, and write $\tilde{\Psi}_{2\gamma,im}$ as $\tilde{\Psi}_{2\gamma,i,t}$ since $\mathcal{M}_2 = \{1, \dots, T\}$. Therefore,

$$\tilde{V}_{1,i} = \frac{\tilde{\Psi}_{1\gamma,i}^2}{2T^{1/2}}, \quad \tilde{U}_{1,i} = 0, \quad \tilde{V}_{2,i} = \frac{A_{2,i}^\top A_{2,i}}{2nT^{1/2}} \quad \text{and} \quad \tilde{U}_{2,i} = \frac{\sum_{i'=1}^{i-1} A_{2,i}^\top A_{2,i'}}{nT^{1/2}},$$

where $A_{2,i} = (\tilde{\Psi}_{2\gamma,i,t})_{t \leq T}$.⁴ Applying Theorem A3, we obtain the asymptotic normality for $\widehat{QLR}_{n,T}$ with variance

$$\omega_{n,T} = \sqrt{n^{-1} \sum_{i \leq n} \text{Var} \left(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{V}_{1,i} - \tilde{V}_{2,i} - \tilde{U}_{2,i} \right)}.$$

Since $\tilde{V}_{2,i}$ is the same as the one in the Example 2, we can show that it is asymptotically negligible and the asymptotic normality of $\widehat{QLR}_{n,T}$ is driven by $n^{-1} \sum_{i \leq n} (\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{V}_{1,i} - \tilde{U}_{2,i})$. As we shall prove in the lemma below, the variance of $\widehat{QLR}_{n,T}$ in this example is given by

$$\omega_{n,T}^* = \sqrt{n^{-1} \sum_{i \leq n} \left(\text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{V}_{1,i}) + \text{Var}(\tilde{U}_{2,i}) \right)}.$$

Hence Assumption 11(i) holds as long as $\sum_{i \leq n} \text{Var}(\tilde{U}_{2,i})$ is bounded away from zero.

Lemma S74. *Consider two models with $\mathcal{G}_1 = \{1, \dots, n\}$ and $\mathcal{M}_1 = \{1\}$ for model 1, and $\mathcal{G}_2 = \{1\}$ and $\mathcal{M}_2 = \{1, \dots, T\}$ for model 2. Suppose that Assumptions 1, 7, 8, 9, 10(i) and 12 hold for models 1 and 2, and $\sum_{i \leq n} \text{Var}(\tilde{U}_{2,i}) > K^{-1}$. Then*

$$\frac{\widehat{QLR}_{n,T} - \mathbb{E}[S_{n,T}] - QLR_{n,T}^*}{\omega_{n,T}^*} \rightarrow_d N(0, 1), \quad (\text{S774})$$

where $\mathbb{E}[S_{n,T}] = (2nT)^{-1/2} \sum_{i \leq n} (\mathbb{E}[\tilde{\Psi}_{1\gamma,i}^2] - n^{-1} \mathbb{E}[A_{2,i}^\top A_{2,i}])$, and

$$(\omega_{n,T}^*)^2 = n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{V}_{1,i}) + \frac{\left\| n^{-1} \sum_{i \leq n} \mathbb{E}[A_{2,i} A_{2,i}^\top] \right\|^2}{2nT} + O(n^{-2}). \quad (\text{S775})$$

4.4 Proofs for Lemmas S72, S73 and S74

PROOF OF LEMMA S72. Recall that in the two models investigated in this lemma, we have $\mathcal{G}_1 = \{1, \dots, n\}$ and $\mathcal{M}_1 = \{1\}$ for model 1, and $\mathcal{G}_2 = \{1, \dots, n\}$ and $\mathcal{M}_2 = \{1, \dots, M_2\}$ for model 2 where M_2 is a fixed positive integer. Under Conditions (i, iii) of the Lemma, we can write

$$\tilde{\Psi}_{1\gamma,i}^2 = \frac{\left(\sum_{m \in \mathcal{M}_2} (T_m/T)^{1/2} \tilde{\Psi}_{\gamma,im} \right)^2}{-\Psi_{\gamma\gamma}} = \sum_{m=1}^{M_2} \frac{T_m \tilde{\Psi}_{\gamma,im}^2}{-T\Psi_{\gamma\gamma}} + 2 \sum_{m=2}^{M_2} \frac{\sum_{m'=1}^{m-1} T_m^{1/2} \tilde{\Psi}_{\gamma,im} T_{m'}^{1/2} \tilde{\Psi}_{\gamma,im'}}{-T\Psi_{\gamma\gamma}}$$

⁴Following the tradition that the summaion over an empty set is zero, we have $\tilde{U}_{2,1} = 0$ in the above display.

and $\tilde{\Psi}_{2\gamma,im}^2 = \tilde{\Psi}_{\gamma,im}^2 / (-\Psi_{\gamma\gamma})$ for any $m \in \mathcal{M}_2$ and any $i = 1, \dots, n$. Therefore,

$$\tilde{\Psi}_{1\gamma,i}^2 - \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2 = \sum_{m=1}^{M_2} (1 - T_m/T) \frac{\tilde{\Psi}_{\gamma,im}^2}{\Psi_{\gamma\gamma}} - 2 \sum_{m=2}^{M_2} \sum_{m'=1}^{m-1} \frac{T_m^{1/2} \tilde{\Psi}_{\gamma,im} T_{m'}^{1/2} \tilde{\Psi}_{\gamma,im'}}{T \Psi_{\gamma\gamma}}. \quad (\text{S776})$$

Under Condition (ii) of the lemma, $\tilde{\Psi}_{\gamma,im}$ are independent across m for any i , which together with the expression in (S776) implies that

$$\begin{aligned} \text{Var} \left(\tilde{\Psi}_{1\gamma,i}^2 - \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2 \right) &= \sum_{m=1}^{M_2} (1 - T_m/T)^2 \frac{\text{Var}(\tilde{\Psi}_{\gamma,im}^2)}{\Psi_{\gamma\gamma}^2} \\ &\quad + 4 \sum_{m=2}^{M_2} \sum_{m'=1}^{m-1} \frac{\text{Var}(T_m^{1/2} \tilde{\Psi}_{\gamma,im}) \text{Var}(T_{m'}^{1/2} \tilde{\Psi}_{\gamma,im'})}{T^2 \Psi_{\gamma\gamma}^2}. \end{aligned} \quad (\text{S777})$$

By Condition (ii) of the lemma, $\text{Var}(T_m^{1/2} \tilde{\Psi}_{\gamma,im}) = \sum_{t \in I_m} \sigma_{\gamma,i,t}^2$ for any m , and hence

$$\begin{aligned} &4T^{-2} \sum_{m=2}^{M_2} \sum_{m'=1}^{m-1} \text{Var}(T_m^{1/2} \tilde{\Psi}_{\gamma,im}) \text{Var}(T_{m'}^{1/2} \tilde{\Psi}_{\gamma,im'}) \\ &= 4T^{-2} \sum_{m=2}^{M_2} \sum_{m'=1}^{m-1} \left(\sum_{t \in I_m} \sigma_{\gamma,i,t}^2 \right) \left(\sum_{t \in I_{m'}} \sigma_{\gamma,i,t}^2 \right) \\ &= 2 \left(T^{-1} \sum_{t \leq T} \sigma_{\gamma,i,t}^2 \right)^2 - 2 \sum_{m=1}^{M_2} w_m^2 \left(T_m^{-1} \sum_{t \in I_m} \sigma_{\gamma,i,t}^2 \right)^2. \end{aligned} \quad (\text{S778})$$

Without loss of generality for any $m \in \mathcal{M}_2$, let $I_m = \{t_{m,1}, \dots, t_{m,T_m}\}$. Then

$$\tilde{\Psi}_{\gamma,im}^2 = T_m^{-1} \sum_{t \in I_m} \psi_{\gamma,i,t}^2 + 2T_m^{-1} \sum_{j=2}^{T_m} \sum_{j'=1}^{j-1} \psi_{\gamma,i,t_{m,j}} \psi_{\gamma,i,t_{m,j'}}.$$

Let $\kappa_{\gamma,i,t}^4 \equiv \text{Var}(\psi_{\gamma,i,t}^2)$. By Condition (ii) of the lemma

$$\begin{aligned} \text{Var}(\tilde{\Psi}_{\gamma,im}^2) &= T_m^{-2} \sum_{t \in I_m} \text{Var}(\psi_{\gamma,i,t}^2) + 4T_m^{-2} \sum_{j=2}^{T_m} \sum_{j'=1}^{j-1} \text{Var}(\psi_{\gamma,i,t_{m,j}}) \text{Var}(\psi_{\gamma,i,t_{m,j'}}) \\ &= T_m^{-2} \sum_{t \in I_m} \kappa_{\gamma,i,t}^4 + 2T_m^{-2} \left[\left(\sum_{t \in I_m} \sigma_{\gamma,i,t}^2 \right)^2 - \sum_{t \in I_m} \sigma_{\gamma,i,t}^4 \right], \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{m=1}^{M_2} (1 - T_m/T)^2 \text{Var}(\tilde{\Psi}_{\gamma,im}^2) &= \sum_{m=1}^{M_2} (1 - w_m)^2 T_m^{-2} \left(\sum_{t \in I_m} \kappa_{\gamma,i,t}^4 - 2 \sum_{t \in I_m} \sigma_{\gamma,i,t}^4 \right) \\ &\quad + 2 \sum_{m=1}^{M_2} (1 - w_m)^2 \left(T_m^{-1} \sum_{t \in I_m} \sigma_{\gamma,i,t}^2 \right)^2. \end{aligned} \quad (\text{S779})$$

Combining the results in (S777), (S778) and (S779), we have

$$\begin{aligned} \text{Var} \left(\tilde{\Psi}_{1\gamma,i}^2 - \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2 \right) &= \frac{2 \left[\left(T^{-1} \sum_{t \leq T} \sigma_{\gamma,i,t}^2 \right)^2 + \sum_{m=1}^{M_2} (1 - 2w_m) \left(T_m^{-1} \sum_{t \in I_m} \sigma_{\gamma,i,t}^2 \right)^2 \right]}{\Psi_{\gamma\gamma}^2} \\ &\quad + \frac{\sum_{m=1}^{M_2} (1 - w_m)^2 T_m^{-2} \left(\sum_{t \in I_m} \kappa_{\gamma,i,t}^4 - 2 \sum_{t \in I_m} \sigma_{\gamma,i,t}^4 \right)}{\Psi_{\gamma\gamma}^2}. \end{aligned} \quad (\text{S780})$$

Since M_2 is fixed, by $\mathbb{E}[\psi_{\gamma,i,t}^4] < K$,

$$\sum_{m=1}^{M_2} (1 - w_m)^2 T_m^{-2} \left(\sum_{t \in I_m} \kappa_{\gamma,i,t}^4 + 2 \sum_{t \in I_m} \sigma_{\gamma,i,t}^4 \right) \leq K M_2 \max_{m \in \mathcal{M}_2} T_m^{-1} = O \left(\max_{m \in \mathcal{M}_2} T_m^{-1} \right),$$

which together with (S780) and the definitions of $\bar{\sigma}_{\gamma,i}$ and $\bar{\sigma}_{\gamma,im}$ yields

$$\sum_{i \leq n} \text{Var} \left(\tilde{\Psi}_{1\gamma,i}^2 - \sum_{m \in \mathcal{M}_2} \tilde{\Psi}_{2\gamma,im}^2 \right) = \frac{2 \left(\sum_{i \leq n} \bar{\sigma}_{\gamma,i}^4 + \sum_{m=1}^{M_2} (1 - 2w_m) \sum_{i \leq n} \bar{\sigma}_{\gamma,im}^4 \right)}{\Psi_{\gamma\gamma}^2} + O \left(\max_{m \in \mathcal{M}_2} T_m^{-1} \right).$$

The claim in (S770) now follows from (S769) and the above equation. *Q.E.D.*

PROOF OF LEMMA S73. Recall that in the two models investigated in this lemma, we have $\mathcal{G}_1 = \{1, \dots, G_1\}$ and $\mathcal{M}_1 = \{1, \dots, T\}$ for model 1 where G_1 is a fixed positive integer, and $\mathcal{G}_2 = \{1\}$ and $\mathcal{M}_2 = \{1, \dots, T\}$ for model 2. First notice that by $G_1 < K$, $G_2 = 1$ and $M_1 = M_2 = T$, Assumption 10(ii) holds. By the independence of $\{z_{i,t}\}$ across i for all t , we $\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i}$ is uncorrelated with $\tilde{U}_{1,i} - \tilde{U}_{2,i}$ for any i . Therefore,

$$\text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i}) = \text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i}) + \text{Var}(\tilde{U}_{1,i} - \tilde{U}_{2,i}). \quad (\text{S781})$$

The condition $\sum_{i \leq n} \text{Var}(\tilde{U}_{1,i} - \tilde{U}_{2,i}) > K^{-1}$ maintained in the lemma thus implies that

$$n(\omega_{n,T}^*)^2 > K^{-1}. \quad (\text{S782})$$

We first show (S772). Since $\tilde{V}_{j,i}$ are independent across i ,

$$\begin{aligned} \text{Var} \left(n^{-1/2} \sum_{i \leq n} (\tilde{V}_{1,i} - \tilde{V}_{2,i}) \right) &= n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_{1,i} - \tilde{V}_{2,i}) \\ &\leq n^{-1} \sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} n_g^{-2} \text{Var} \left(T^{-1/2} A_{1,i}^\top A_{1,i} \right) + n^{-3} \sum_{i \leq n} \text{Var} \left(T^{-1/2} A_{2,i}^\top A_{2,i} \right) \\ &\leq K \left(n^{-2} + n^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \right) \leq K n^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1} \end{aligned} \quad (\text{S783})$$

where the second inequality is by (S733) (where we show that $\text{Var}(M_j^{-1/2} A_{j,i}^\top A_{j,i}) \leq K$ with $M_j = T$ here). Next note that

$$\begin{aligned} \omega_{n,T}^2 &= n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i}) + n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_{1,i} - \tilde{V}_{2,i}) \\ &\quad + 2n^{-1} \sum_{i \leq n} \text{Cov}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i}, \tilde{V}_{1,i} - \tilde{V}_{2,i}) \\ &= (\omega_{n,T}^*)^2 + 2n^{-1} \sum_{i \leq n} \text{Cov}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i}, \tilde{V}_{1,i} - \tilde{V}_{2,i}) + O\left(n^{-1} \max_{g \in \mathcal{G}_1} n_g^{-1}\right), \end{aligned} \quad (\text{S784})$$

where the second equality is by (S783). By Hölder's inequality,

$$\begin{aligned} &\left| n^{-1} \sum_{i \leq n} \text{Cov}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i}, \tilde{V}_{1,i} - \tilde{V}_{2,i}) \right| \\ &\leq n^{-1} \sum_{i \leq n} (\text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i}) \text{Var}(\tilde{V}_{1,i} - \tilde{V}_{2,i}))^{1/2} \\ &\leq \left(n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i}) \right)^{1/2} \left(n^{-1} \sum_{i \leq n} \text{Var}(\tilde{V}_{1,i} - \tilde{V}_{2,i}) \right)^{1/2}, \end{aligned}$$

which together with the definition of $\omega_{n,T}^*$, (S781) and (S783) implies that

$$\left| n^{-1} \sum_{i \leq n} \text{Cov}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{U}_{1,i} - \tilde{U}_{2,i}, \tilde{V}_{1,i} - \tilde{V}_{2,i}) \right| = O\left(\omega_{n,T}^* n^{-1/2} \max_{g \in \mathcal{G}_1} n_g^{-1/2}\right). \quad (\text{S785})$$

By (S782), (S784) and (S785), we can deduce that

$$\frac{\omega_{n,T}^2}{(\omega_{n,T}^*)^2} = 1 + O\left(\frac{\max_{g \in \mathcal{G}_1} n_g^{-1/2}}{n^{1/2} \omega_{n,T}^*}\right) + O\left(\frac{\max_{g \in \mathcal{G}_1} n_g^{-1}}{n(\omega_{n,T}^*)^2}\right) = 1 + O\left(\max_{g \in \mathcal{G}_1} n_g^{-1/2}\right). \quad (\text{S786})$$

The claim in (S772) thus follows from Theorem A3 and Theorem A4, Assumption 12(iii) and Slutsky's theorem.

We next turn to the proof of (S773). Since $\mathbb{E}[A_{j,i}] = 0$ and $A_{j,i}$ are independent across i , for any $i \in I_g$ and any $g \in \mathcal{G}_1$

$$\begin{aligned} \sum_{i \in I_g} \text{Var}(\tilde{U}_{1,i}) &= \sum_{i=N_g+2}^{N_g+n_g} \sum_{i'=N_g+1}^{i-1} \frac{\text{Var}(A_{1,i}^\top A_{1,i'})}{n_g^2 T} = \sum_{i=N_g+2}^{N_g+n_g} \sum_{i'=N_g+1}^{i-1} \frac{\text{Tr}(\mathbb{E}[A_{1,i} A_{1,i'}^\top] \mathbb{E}[A_{1,i'} A_{1,i}^\top])}{n_g^2 T} \\ &= (2T)^{-1} \text{Tr} \left(\left(n_g^{-1} \sum_{i \in I_g} \mathbb{E}[A_{1,i} A_{1,i}^\top] \right)^2 - n_g^{-2} \sum_{i \in I_g} (\mathbb{E}[A_{1,i} A_{1,i}^\top])^2 \right) \\ &= (2T)^{-1} \left(\|\Sigma_{1\gamma,g}\|^2 - n_g^{-2} \sum_{i \in I_g} \left\| \mathbb{E}[A_{1,i} A_{1,i}^\top] \right\|^2 \right), \end{aligned}$$

which together with (S716) in the proof of Lemma S69 implies that

$$\sum_{g \in \mathcal{G}_1} \sum_{i \in I_g} \text{Var}(\tilde{U}_{1,i}) = (2T)^{-1} \sum_{g \in \mathcal{G}_1} \|\Sigma_{1\gamma,g}\|^2 + O\left(\max_{g \in \mathcal{G}_1} n_g^{-1}\right). \quad (\text{S787})$$

Similarly,

$$\begin{aligned} \sum_{i \leq n} \text{Var}(\tilde{U}_{2,i}) &= (2n^2T)^{-1} \text{Tr} \left(\left(\sum_{i \leq n} \mathbb{E}[A_{2,i} A_{2,i}^\top] \right)^2 - \sum_{i \leq n} (\mathbb{E}[A_{2,i} A_{2,i}^\top])^2 \right) \\ &= (2T)^{-1} \left\| \sum_{g \in \mathcal{G}_1} w_g \Sigma_{2\gamma,g} \right\|^2 + O(n^{-1}), \end{aligned} \quad (\text{S788})$$

and

$$\begin{aligned} 2 \sum_{i \leq n} \text{Cov}(\tilde{U}_{1,i}, \tilde{U}_{2,i}) &= 2(nT)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \text{Cov} \left(\sum_{i'=N_g+1}^{i-1} A_{1,i}^\top A_{1,i'}, \sum_{i'=1}^{i-1} A_{2,i}^\top A_{2,i'} \right) \\ &= 2(nT)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i=N_g+2}^{N_g+n_g} \sum_{i'=N_g+1}^{i-1} \text{Tr}(\mathbb{E}[A_{1,i'} A_{2,i'}^\top] \mathbb{E}[A_{2,i} A_{1,i}^\top]) \\ &= (nT)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \text{Tr} \left(\sum_{i,i' \in I_g} \mathbb{E}[A_{1,i} A_{2,i}^\top] \mathbb{E}[A_{2,i'} A_{1,i'}^\top] - \sum_{i \in I_g} \mathbb{E}[A_{1,i} A_{2,i}^\top] \mathbb{E}[A_{2,i} A_{1,i}^\top] \right) \\ &= T^{-1} \sum_{g \in \mathcal{G}_1} w_g \|\Sigma_{12\gamma,g}\|^2 - (nT)^{-1} \sum_{g \in \mathcal{G}_1} n_g^{-1} \sum_{i \in I_g} \left\| \mathbb{E}[A_{1,i} A_{2,i}^\top] \right\|^2 \\ &= T^{-1} \sum_{g \in \mathcal{G}_1} w_g \|\Sigma_{12\gamma,g}\|^2 + O(n^{-1}), \end{aligned} \quad (\text{S789})$$

where the last equality is by (S739) in the proof of Lemma S70. Collecting the results in (S787), (S788) and (S789), we obtain

$$\sum_{i \leq n} \text{Var}(\tilde{U}_{1,i}^* - \tilde{U}_{2,i}^*) = \frac{\sum_{g \in \mathcal{G}_1} (\|\Sigma_{1\gamma,g}\|^2 - 2w_g \|\Sigma_{12\gamma,g}\|^2) + \left\| \sum_{g \in \mathcal{G}_1} w_g \Sigma_{2\gamma,g} \right\|^2}{2T} + O\left(\max_{g \in \mathcal{G}_1} n_g^{-1}\right)$$

which establishes (S773).

Q.E.D.

PROOF OF LEMMA S74. Recall that in the two models investigated in this lemma, we have $\mathcal{G}_1 = \{1, \dots, n\}$ and $\mathcal{M}_1 = \{1\}$ for model 1, and $\mathcal{G}_2 = \{1\}$ and $\mathcal{M}_2 = \{1, \dots, T\}$ for model 2. First notice that Assumption 10(ii) holds with $G_1 = n$, $M_1 = 1$, $G_2 = 1$ and $M_2 = T$. In the proof of Lemma S73, we have shown that

$$\text{Var} \left(n^{-1/2} \sum_{i \leq n} \tilde{V}_{2,i} \right) = n^{-3} \sum_{i \leq n} \text{Var} \left(T^{-1/2} A_{2,i}^\top A_{2,i} \right) \leq K n^{-2}. \quad (\text{S790})$$

Therefore, using the same arguments for deriving (S786) obtains

$$\frac{\omega_{n,T}^2}{(\omega_{n,T}^*)^2} = 1 + O(n^{-1}),$$

and the claim in (S774) thus follows from Theorem A3 and Theorem A4 and Slutsky's theorem.

By the independence of $z_{i,t}$ across i for all t

$$\text{Var} \left(n^{-1/2} \sum_{i \leq n} (\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{V}_{1,i} - \tilde{U}_{2,i}) \right) = n^{-1} \sum_{i \leq n} \text{Var}(\tilde{\Psi}_{1,i} - \tilde{\Psi}_{2,i} + \tilde{V}_{1,i}) + n^{-1} \sum_{i \leq n} \text{Var}(\tilde{U}_{2,i}). \quad (\text{S791})$$

The expression in (S775) is can be derived directly from (S791) and (S788) in the proof of Lemma S73. *Q.E.D.*

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