Disaster Risk and Business Cycles

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Abstract

Motivated by the evidence that risk premia are large and countercyclical, this paper studies a tractable real business cycle model with a small, exogenously time-varying risk of disaster. Disaster risk affects both asset prices and macroeconomic quantities. An increase in disaster risk leads to a decline of output, investment, stock prices, and interest rates, and an increase in the expected return on risky assets. The model matches well data on quantities, asset prices, and the relations between quantities and prices. Empirically, shocks to disaster risk, or more generally shocks to risk premia, play a significant role in investment dynamics.

Keywords: business cycles, investment, production, equity premium, time-varying risk premium, disasters, rare events, jumps.

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1 Introduction

The empirical finance literature has provided substantial evidence that risk premia vary over time, and that they are countercyclical.[1] Yet, standard business cycle models such as the real business

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[1] See e.g. Campbell and Shiller (1988) and Fama and French (1989) for stocks, Cochrane and Piazzesi (2006) for Treasury bonds, Philippon (2008) and Gilchrist and Zakrajsek (2007) for corporate bonds, and Cochrane (2007) or Backus, Routledge and Zin (2008) for recent overviews.

cycle model, or the dynamic stochastic general equilibrium (DSGE) models used for monetary policy analysis, largely fail to replicate the level, the volatility, and the countercyclicality of risk premia. In these models, the variation in expected returns is entirely driven by variation in the risk-free interest rate. Is this a significant limitation of macroeconomic models? Do risk premia matter for macroeconomic dynamics?

To attack this question, I introduce a tractable real business cycle model with a small, stochastically time-varying risk of economic "disaster", following the work of Rietz (1988), Barro (2006), and Gabaix (2007). In my model, risk premia vary because the real quantity of risk varies, leading to a reaction of both asset prices and macroeconomic aggregates. Existing work has so far been confined to endowment economies, and hence does not consider the feedback from time-varying risk to macroeconomic aggregates. An increase in the probability of disaster creates a collapse of investment and a recession, as risk premia rise, increasing the cost of capital. Demand for precautionary savings increase, leading the yield on less risky assets to fall, while spreads on risky securities increase. These business cycle dynamics occur with no change in total factor productivity.[2]

Before turning to a quantitative analysis, I prove two theoretical results, which hold under the assumption that a disaster reduces total factor productivity (TFP) and the capital stock by the same amount. First, when the probability of disaster is constant, the path for macroeconomic quantities implied by the model is the same as that implied by a model with no disasters, but a different discount factor β. This "observational equivalence" (in a sample without disasters) is reminiscent of the numerical analysis of Tallarini (2000), who found that macroeconomic dynamics are essentially unaffected by the amount of risk or the degree of risk aversion. Second, when the probability of disaster is stochastic, an increase in probability of disaster is observationally equivalent to a preference shock. This implies that these shocks have a significant effect on macroeconomic aggregates, and this provides an interpretation of the "equity premium shocks" introduced by Smets and Wouters (2003) and other authors in their estimations of DSGE models. Consistent with the literature, the paper argues that these shocks play a significant role in macroeconomic dynamics. However I arrive at this conclusion from a very different path, since these shocks are calibrated to replicate asset prices in my model.

Quantitatively, I find that this parsimonious model can match many asset pricing facts - the mean, volatility, and predictability of returns - while doing at least as well as the RBC model in

[2] Because disasters are rare, the risk is usually not realized in sample. However, my results are not driven by sample selection (peso problem); see sections 4.3 and 5.5.

accounting for quantities. This is important since many asset pricing models which are successful in endowment economies do not generalize well to production economies.[3] Most interestingly, the model matches well the relations between macroeconomic aggregates (such as investment or output) and asset prices (such as expected returns, the P-D ratio, or the VIX index). As is well known, this connection between prices and quantities is problematic for most macroeconomic models.

Empirical tests of the disaster model are notoriously difficult. Barro (2006) measured historical disasters in cross-country data. To measure the time-varying probability of disaster, I use the most natural restriction of the model - disaster risk affects powerfully asset prices. I infer the probability of disaster from the observed price-dividend ratio. I then feed into the model this estimated probability of disaster. The variation over time in this probability appears to account for a share of business cycle dynamics, and is especially important during the sharpest downturns such as the current recession.

This risk of an economic disaster may be a strictly rational expectation. For instance, during the recent financial crisis, many commentators, including well-known macroeconomists, have highlighted the possibility that the U.S. economy might fall into another Great Depression.[4] My results suggest that the probability of a disaster was indeed high in Fall 2008. More generally it could reflect a time-varying belief, which may differ from the objective probability - i.e., waves of optimism or pessimism (see e.g. Jouini and Napp (2008)). My model studies the macroeconomic effects of such time-varying beliefs. (Of course in reality beliefs may be an endogenous, but understanding the effects that they have is important.) This simple modeling device captures the idea that aggregate uncertainty is sometimes high, i.e. people sometimes worry about the possibility of a deep recession. It also captures the idea that there are some asset price changes which are not obviously related to current or future TFP, i.e. "bubbles", "animal spirits", and which in turn affect the macroeconomy.

Introducing time-varying risk requires solving a model using nonlinear methods, i.e. going beyond the first-order approximation and considering higher order terms in the Taylor expansion.

[3] As explained in Jermann (1998), Lettau and Uhlig (2000), Kaltenbrunner and Lochstoer (2008).

[4] Greg Mankiw (NYT, Oct 25, 2008): "*Looking back at [the great Depression], it's hard to avoid seeing parallels to the current situation. (...) Like Mr. Blanchard at the I.M.F., I am not predicting another Great Depression. But you should take that economic forecast, like all others, with more than a single grain of salt.*"

Robert Barro (WSJ, March 4, 2009): "*... there is ample reason to worry about slipping into a depression. There is a roughly one-in-five chance that U.S. GDP and consumption will fall by 10% or more, something not seen since the early 1930s.*"

Paul Krugman (NYT, Jan 4, 2009): "*This looks an awful lot like the beginning of a second Great Depression.*"

Researchers disagree on the importance of these higher order terms, and a fairly common view is that they are irrelevant for macroeconomic quantities. Lucas (2003), in his presidential address, summarizes: "*Tallarini uses preferences of the Epstein-Zin type, with an intertemporal substitution elasticity of one, to construct a real business cycle model of the U.S. economy. He finds an astonishing separation of quantity and asset price determination: The behavior of aggregate quantities depends hardly at all on attitudes toward risk, so the coefficient of risk aversion is left free to account for the equity premium perfectly.*"[5] My results show, however, that when risk varies over time, risk aversion affects macroeconomic dynamics in a significant way, and hence matching the equity premium or other asset pricing facts can lead to different business cycle implications.

Overall, the **contribution** of the paper is twofold. Substantively, the quantitative and empirical results of this paper suggest an important role for time-varying risk in accounting for business cycles and asset prices. This result obtains in the context of a model which matches well data on prices, quantities, and the relations between quantities and prices, which in itself is an important achievement. Besides this substantive contribution, the technical contribution of the paper is to provide a tractable framework which leads to volatile, countercyclical risk premia in a standard macroeconomic model. The tractability of the framework is such that extensions to include credit frictions, monetary policy, or several countries, are quite feasible.

The paper is organized as follows: the rest of the introduction reviews the literature. Section 2 studies a simple analytical example in an AK model which can be solved in closed form and yields the central intuition for the results. Section 3 gives the setup of the full model and presents the analytical results. Section 4 studies the quantitative implications of the model numerically. Section 5 considers some extensions of the baseline model. Section 6 presents an empirical evaluation of the model, backing out the probability of disaster from asset prices.

Related Literature

Gabaix (2007, 2009) independently obtained propositions 1 and 2. On top of that, he develops a specific model where variation in the probability of disaster has no macroeconomic effect. In contrast, my paper uses the standard real business cycle model, and shows that a shock to the probability of disaster is equivalent to a preference shock (proposition 3) and hence has a macroeconomic effect. Unlike Gabaix then, my model generates an empirically compelling correlation between asset prices and macroeconomic quantities. Moreover, my paper is more

[5] Note that Tallarini (2000) actually picks the risk aversion coefficient to match the Sharpe ratio of equity. Since return volatility is very low in his model (there are no capital adjustment costs), the equity premium is much smaller in his model than in the data.

quantitative and uses Epstein-Zin utility.

This paper is mostly related to four strands of literature. First, a large literature in finance builds and estimates models which attempt to match not only the equity premium and the risk-free rate, but also the variation of risk premia (i.e. the predictability of excess returns). Two prominent examples are Bansal and Yaron (2004) and Campbell and Cochrane (1999). However, this literature is limited to endowment economies, and hence is of limited use to analyze the business cycle or to study policy questions.

Second, my paper is closely related to a small literature which studies business cycle models (i.e. with endogenous consumption, investment and output), and attempts to match both business cycle statistics but also asset returns first and second moments.[6] Many of these studies consider only the implications of productivity shocks, and generally study only the mean and standard deviations of return and do not attempt to match the predictability of returns. My paper contributes to this literature by focusing on the variation of risk premia and the correlations between asset prices or returns, on the one hand, and macroeconomic quantities, on the other hand. In contrast to my paper, many of these studies also abstract from employment, which is a critical business cycle variable. Many of these studies have difficulty reconciling business cycle dynamics and asset returns, but my model does well in this dimension.

Third, the paper draws from the recent literature on "disasters" or rare events (Rietz (1988), Barro (2006), Barro and Ursua (2008), Gabaix (2007), Gourio (2008a,2008b), Julliard and Ghosh (2008), Martin (2008), Santa Clara and Yan (2008), Wachter (2008), Weitzmann (2007), Backus, Chernov and Martin (2009)). Disasters are a powerful way to generate large risk premia. Moreover, as we will see, disasters are relatively easy to embed into a standard macroeconomic model.

Last, my paper studies the real effects of a shock to uncertainty, a channel recently emphasized by Bloom (2009). Bloom (2009) considers a partial-equilibrium model with heterogeneous firms facing fixed and linear costs to adjusting capital or labor. In his model, the uncertainty shock is a temporary increase in the variance of aggregate and especially idiosyncratic productivity shocks. His model generates a recession and a decrease in endogenous aggregate TFP in response to an uncertainty shock. My model also generates a recession in response to higher uncertainty, but there are several differences: (1) risk is modeled differently since the higher uncertainty affects both productivity and the capital stock; (2) the mechanism is different since it relies on the

[6] A non-exhaustive list includes Jermann (1998), Tallarini (2000), Boldrin, Christiano and Fisher (2001), Lettau and Uhlig (2000), Kaltenbrunner and Lochstoer (2008), Campanele et al. (2008), Croce (2005), Papanikolaou (2008), Kuehn (2008), Uhlig (2006), Jaccard (2008), and Fernandez-Villaverde et al. (2008).

general equilibrium feedback, i.e. risk-averse consumers are less willing to invest in risky capital when uncertainty is high; (3) the model does not generate any change in TFP. Most importantly, my model focuses on the relations between asset prices and the macroeconomy. For instance, my model can replicate the empirical finding that shocks to VIX affect output negatively.[7]

2 A simple analytical example in an AK economy

To highlight the key mechanism of the paper, this section studies a streamlined model. Section 3 relaxes many of the simplifying assumptions, such as constant productivity, no adjustment costs, etc., which are made for clarity. Consider a simple economy with a representative consumer who has power utility:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma},$$

where C_t is consumption and γ is the risk aversion coefficient (and also the inverse of the the intertemporal elasticity of substitution of consumption). This consumer operates an AK technology:

$$Y_t = AK_t,$$

where Y_t is output, K_t is capital, and A is productivity, which is assumed to be constant. The resource constraint is:

$$C_t + I_t \leq AK_t.$$

The economy is randomly hit by disasters. A disaster destroys a share b_k of the capital stock.[8] This may be because of a war which physically destroys capital, but there are alternative interpretations. For instance, b_k could reflect expropriation of capital holders (if the capital is taken away and then not used as effectively), or it could be a "technological revolution" that makes a large share of the capital worthless. It could also be that even though physical capital is not literally destroyed, some intangible capital (such as matches between firms, employees, and customers) is lost. Finally, one can imagine a situation where the demand for some goods falls sharply, rendering worthless the factories which produce them.[9]

[7] Fernandez-Villaverde et al. (2009) also study the effect of shocks to risk, but they focus on a small open economy which faces exogenous time-varying interest rate risk.

[8] A disaster does not affect productivity A. I will relax this assumption in section 3. In an AK model, a permanent reduction in productivity would lead to a permanent reduction in the growth rate of the economy, since the *level* of A affect the *growth rate* of output.

[9] In a large downturn, the demand for some luxury goods such as boats, private airplanes, etc. would likely fall sharply. If this situation were to last, the boats-producing factories would never operate at capacity, and hence

Throughout the paper I denote by x_{t+1} an indicator which is one if there is a disaster at time $t+1$, and 0 if not.

The probability of a disaster varies over time. To maintain tractability I assume in this section that it is *i.i.d.*: p_t, the probability of a disaster at time $t+1$, is drawn at the beginning of time t from a cumulative distribution function F. The law of accumulation for capital is thus:

$$K_{t+1} = ((1-\delta)K_t + I_t)(1 - x_{t+1}b_k).$$

Finally, I assume that the two random variables p_{t+1}, and x_{t+1} are independent. I also discuss this assumption in more detail in section 3.

This model has one endogenous state variable, the capital stock K and one exogenous state p, and there is one control variable C. There are two shocks: the realization of disaster $x' \in \{0,1\}$, and the draw of a new probability of disaster p'. The Bellman equation for the representative consumer is:

$$\begin{aligned}
V(K,p) &= \max_{C,\,I} \left\{ \frac{C^{1-\gamma}}{1-\gamma} + \beta E_{p',x'}\left(V(K',p')\right) \right\} \\
s.t. &: \\
C + I &\leq AK, \\
K' &= ((1-\delta)K + I)(1 - x'b_k).
\end{aligned}$$

The assumptions made ensure that V is homogeneous, i.e. V is of the form $V(K,p) = \frac{K^{1-\gamma}}{1-\gamma} g(p)$, where g satisfies the Bellman equation:

$$g(p) = \max_i \left\{ \frac{(A-i)^{1-\gamma}}{1-\gamma} + \beta \frac{(1-\delta+i)^{1-\gamma}\left(1-p+p(1-b_k)^{1-\gamma}\right)}{1-\gamma}\left(E_{p'}g(p')\right) \right\}, \tag{1}$$

and $i = \frac{I}{K}$ is the investment rate. This implies that consumption and investment are both proportional to the current stock of capital, but they typically depend on the probability of disaster as well:

$$\begin{aligned}
C_t &= f(p_t)K_t, \\
I_t &= h(p_t)K_t.
\end{aligned}$$

the value would fall to zero.

As a result, when a disaster occurs and the capital stock falls by a factor b_k, both consumption and investment also fall by a factor b_k. Given that there are no adjustment costs, the value of capital is equal to the quantity of capital, and hence it falls also by a factor b_k in a disaster. Finally, the return on an all-equity financed firm is:

$$R^e_{t,t+1} = (1 - \delta + A)\,(1 - x_{t+1} b_k),$$

i.e. it is $1 - \delta + A$ if there is no disaster, and $(1 - \delta + A)\,(1 - b_k)$ if there is a disaster. Clearly, the equity premium will be high, since the equity return and consumption are both very low during disasters. Moreover, the equity premium is larger when the probability of disaster p_t is higher.

Let us finally turn to the effect of p on the consumption-savings decision, i.e. the function $f(p)$. Using equation (1), the first-order condition with respect to i yields, after rearranging:

$$\left(\frac{A - i}{1 - \delta + i}\right)^{-\gamma} = \beta\left(1 - p + p(1 - b_k)^{1-\gamma}\right)\left(E_{p'} g(p')\right). \tag{2}$$

Given that p is *i.i.d.*, the expectation of g on the right-hand side is independent of the current p. The term $(1 - b_k)^{1-\gamma}$ is greater than unity if and only if $\gamma > 1$. Hence, the right-hand side is increasing in p if and only if $\gamma > 1$. Since the left-hand side is an increasing function of i, we obtain that i is increasing in p if $\gamma > 1$, it is decreasing in p if $\gamma < 1$, and it is independent of p if $\gamma = 1$.

The intuition for this result is as follows: if p goes up, investment in physical capital becomes more risky and hence less attractive, i.e. the risk-adjusted physical return on capital goes down.[10] The effect of a change in the return on the consumption-savings choice depends on the value of the intertemporal elasticity of substitution (IES), because of offsetting wealth and substitution effects. If the IES is unity (i.e. utility is log), savings are unchanged and thus the investment rate does not respond to a change in the probability of disaster. But if the IES is larger than unity, i.e. $\gamma < 1$, the substitution effect dominates, and i is decreasing in p. Hence, an increase in the probability of disaster leads initially, in this model, to a decrease in investment, and an increase in consumption, since output is unchanged on impact. Next period, the decrease in investment leads to a decrease in the capital stock and hence in output. This simple analytical example thus shows that a change in the perceived probability of disaster can lead to a decline in investment and output. The key mechanism is the effect of *rate-of-return uncertainty* on the optimal savings

[10] Following Weil (1989), I define the risk-adjusted return as $E(R^{1-\gamma})^{\frac{1}{1-\gamma}}$, where R is the physical return on capital.

decision.[11]

Extension to Epstein-Zin preferences

To illuminate the respective role of risk aversion and the intertemporal elasticity of substitution, it is useful to extend the preceding example to the case of Epstein-Zin utility. Assume, then, that the utility V_t satisfies the recursion:

$$V_t = \left(C_t^{1-\gamma} + \beta E_t \left(V_{t+1}^{1-\theta} \right)^{\frac{1-\gamma}{1-\theta}} \right)^{\frac{1}{1-\gamma}}, \tag{3}$$

where θ measures risk aversion towards static gambles, γ is the inverse of the intertemporal elasticity of substitution (IES) and β reflects time preference.[12] It is straightforward to extend the results above; the first-order condition now reads

$$\left(\frac{A-i}{1-\delta+i} \right)^{-\gamma} = \beta \left(1-p+p(1-b_k)^{1-\theta} \right)^{\frac{1-\gamma}{1-\theta}} \left(E_{p'} g(p')^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}},$$

and we can apply the same argument as above, in the realistic case where risk aversion $\theta \geq 1$: the now *risk-adjusted* return on capital is $\left(1-p+p(1-b_k)^{1-\theta} \right)^{\frac{1}{1-\theta}}$; it falls as p rises; an increase in the probability of disaster will hence reduce investment if and only if the IES is larger than unity.[13] Hence, the parameter which determines the sign of the response is the IES, and the risk aversion coefficient (as long as it is greater than unity) determines the magnitude of the response only. While this example is revealing, it has a number of simplifying features, which lead us to turn now to a quantitative model.

3 A Real Business Cycle model with Time-Varying Risk of Disaster

This section introduces a real business cycle model with time-varying risk of disaster and study its implications analytically. The next section considers the quantitative implications of the model

[11] The effect of rate-of-return uncertainty differs from that of labor-income uncertainty, as is well known at least since Levhari and Srinivasan (1969) and Sandmo (1970). The example of this secton is related to work by Epaulard and Pommeret (2003).

[12] Note that it is commonplace to have a $(1-\beta)$ factor in front of $u(C,N)$ in equation (3), but this is merely a normalization, which it is useful to forgo in this case.

[13] The disaster reduces the mean return itself, but this is actually not important. We could assume that there is a small probability of a "capital windfall" so that a change in p does not affect the mean return on capital. Crucially, what matters here is the risk-adjusted return on capital, $E(R^{1-\theta})^{\frac{1}{1-\theta}}$, and a higher risk reduces this return. See section 5.6 for more details.

using numerical methods. The model extends the simple example of the previous section in the following dimensions: (a) the probability of disaster is persistent instead of *i.i.d.*; (b) the production function is neoclassical and affected by standard TFP shocks; (c) labor is elastically supplied; (d) disasters may affect total factor productivity as well as capital; (e) there are capital adjustment costs.

3.1 Model Setup

The representative consumer has preferences of the Epstein-Zin form, and the utility index incorporates hours worked N_t as well as consumption C_t:

$$V_t = \left(u(C_t, N_t)^{1-\gamma} + \beta E_t \left(V_{t+1}^{1-\theta} \right)^{\frac{1-\gamma}{1-\theta}} \right)^{\frac{1}{1-\gamma}}, \tag{4}$$

where the per period felicity function $u(C, N)$ is assumed to have the following form:

$$u(C, N) = C^{\upsilon} (1 - N)^{1-\upsilon}.$$

Note that u is homogeneous of degree one, hence γ is the inverse of the intertemporal elasticity of substitution (IES) over the consumption-leisure bundle, and θ measures risk aversion towards static gambles over the bundle.

There is a representative firm, which produces output using a standard Cobb-Douglas production function:

$$Y_t = K_t^{\alpha} \left(z_t N_t \right)^{1-\alpha},$$

where z_t is total factor productivity (TFP), to be described below. The firm accumulates capital subject to adjustment costs:

$$K_{t+1} = \left((1-\delta) K_t + \phi \left(\frac{I_t}{K_t} \right) K_t \right) (1 - x_{t+1} b_k).$$

where ϕ is an increasing and concave function, which curvature captures adjustment costs, and x_{t+1} is 1 if there is a disaster at time $t+1$ (with probability p_t) and 0 otherwise (probability $1 - p_t$). At this stage b_k is a parameter, which may be zero - i.e., a disaster only affects TFP. We explore quantitatively the role of b_k in section 5.2.

The resource constraint is

$$C_t + I_t \leq Y_t.$$

Aggregate investment cannot be negative: $I_t \geq 0$. Depending on parameter values, this constraint may bind in the periods immediately following a disaster.

Finally, we describe the shock processes. Total factor productivity (TFP) is assumed to follow a unit root process, and is affected by standard normally distributed shocks ε_t as well as disasters. Mathematically,

$$\log z_{t+1} = \log z_t + \mu + \sigma\varepsilon_{t+1} + x_{t+1} \log(1 - b_{tfp}),$$

where μ is the drift of TFP, and σ is the standard deviation of normal shocks, and b_{tfp} is the reduction in TFP following a disaster. Here too, we will consider various values for b_{tfp}, including possibly zero - i.e., a disaster only destroys capital but does not actually affect TFP. Last, the probability of disaster p_t follows a stationary Markov process with transition function T. In the quantitative application, I will simply assume that the log of p_t follows an AR(1) process.

I assume that the three exogenous shocks $p_{t+1}, \varepsilon_{t+1}$, and x_{t+1} are all independent conditional on p_t. This assumption requires that the occurrence of a disaster today does not affect the probability of a disaster tomorrow. This assumption may be wrong either way: a disaster today may indicate that the economy is entering a phase of low growth or is less resilient than thought, leading agents to revise upward the probability of disaster, following the occurrence of a disaster. But on the other hand, if a disaster occurred today, and capital or TFP fell by a large amount, it is unlikely that they will fall again by a large amount next year. Rather, historical evidence suggests that the economy is likely to grow above trend for a while (Gourio (2008a), Barro et al. (2009)). In section 5.3, I extend the model to consider these different scenarios.

3.2 Bellman Equation

In this section I set up a recursive formulation of the problem, which is used to prove analytical results. The model has three state variables: capital K, technology z and probability of disaster p. There are two independent controls: consumption C and hours worked N; and three shocks: the realization of disaster $x' \in \{0, 1\}$, the draw of the new probability of disaster p', and the normal shock ε'. The first welfare theorem holds, hence the competitive equilibrium is equivalent to a social planner problem, which is easier to solve. Denote $V(K, z, p)$ the value function, and define $W(K,z,p) = V(K,z,p)^{1-\gamma}$. The social planning problem can be formulated as:

$$
\begin{aligned}
W(K,z,p) &= \max_{C,I,N} \left\{ \left(C^{\upsilon}(1-N)^{1-\upsilon}\right)^{1-\gamma} + \beta \left(E_{p',\varepsilon',x'} W(K',z',p')^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}} \right\}, \qquad (5)\\
s.t. &: \\
C + I &\leq z^{1-\alpha} K^{\alpha} N^{1-\alpha}, \\
K' &= \left((1-\delta)K + \phi\left(\frac{I}{K}\right) K \right)(1 - x' b_k), \\
\log z' &= \log z + \mu + \sigma\varepsilon' + x' \log(1 - b_{tfp}).
\end{aligned}
$$

(Because we take a power $1-\gamma$ of the value function, if $\gamma > 1$, the max operator must be transformed into a min.) A standard homogeneity argument implies that we can write $W(K,z,p) = z^{\upsilon(1-\gamma)} g(k,p)$, where $k = K/z$, and g satisfies the associated Bellman equation:

$$
\begin{aligned}
g(k,p) &= \max_{c,i,N} \left\{ \begin{array}{c} c^{\upsilon(1-\gamma)}(1-N)^{(1-\upsilon)(1-\gamma)} \\ +\beta e^{\mu\upsilon(1-\gamma)} \left(E_{p',\varepsilon',x'} e^{\sigma\varepsilon'\upsilon(1-\theta)} (1 - x' b_{tfp})^{\upsilon(1-\theta)} g(k',p')^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}} \end{array} \right\}, \qquad (6)\\
s.t. &: \\
c + i &= k^{\alpha} N^{1-\alpha}, \\
k' &= \frac{(1 - x' b_k)\left((1-\delta)k + \phi\left(\frac{i}{k}\right) k\right)}{e^{\mu + \sigma\varepsilon'}(1 - x' b_{tfp})}.
\end{aligned}
$$

Here $c = C/z$ and $i = I/z$ are consumption and investment detrended by the stochastic technology level z. This homogeneity argument simplifies the problem substantially. It delivers some analytical results, and makes the numerical analysis simpler: first, k is stationary; second, the dimension of the state space is reduced.

3.3 Asset Prices

It is straightforward to compute asset prices in this economy. The stochastic discount factor is given by the formula

$$
M_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{\upsilon(1-\gamma)-1} \left(\frac{1 - N_{t+1}}{1 - N_t} \right)^{(1-\upsilon)(1-\gamma)} \left(\frac{V_{t+1}}{E_t\left(V_{t+1}^{1-\theta}\right)^{\frac{1}{1-\theta}}} \right)^{\gamma-\theta}. \qquad (7)
$$

The price of a one-period risk-free bond is $E_t\left(M_{t,t+1}\right)$, but this risk-free asset may not have an observable counterpart. Following Barro (2006), I will assume that government bonds are not risk-free but are subject to default risk during disasters.[14] More precisely, if there is a disaster, the recovery rate on government bonds is r, i.e. the loss is $1-r$. The T-Bill price can then be easily computed as $Q_{1,t} = E_t\left(M_{t,t+1}(1 + x_{t+1}(r-1))\right)) \stackrel{def}{=} Q_1(k,p)$. The ex-dividend value of the firm assets P_t is defined through the value recursion:

$$P_t = E_t\left(M_{t,t+1}\left(D_{t+1} + P_{t+1}\right)\right),$$

where $D_t = F(K_t, z_t N_t) - w_t N_t - I_t$ is the payout of the representative firm, and w_t is the wage rate, given by the marginal rate of substitution of the representative consumer between consumption and leisure. The equity return is then $R_{t,t+1} = \frac{D_{t+1}+P_{t+1}}{P_t}$. If the positivity constraint on investment does not bind, the unlevered equity return can be rewritten, following a standard Q-theory argument (See Jermann (1998) or Kaltenbrunner and Lochstoer (2008)) as

$$R_{t,t+1} = (1 - x_{t+1}b_k)\,\phi'\left(\frac{I_t}{K_t}\right)\left(\frac{1-\delta+\phi\left(\frac{I_{t+1}}{K_{t+1}}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} + \frac{\alpha K_{t+1}^{\alpha} z_{t+1}^{1-\alpha} N_{t+1}^{1-\alpha} - I_{t+1}}{K_{t+1}}\right), \tag{8}$$

where the first term emphasize that if $b_k > 0$, capital holders make a loss in the event of a disaster.

The empirical counterpart to this unlevered equity return is not stock returns, because in the real world, firms have financial leverage and operating leverage (e.g. fixed costs and labor contracts). This is a substantial source of profit and dividend volatility, which is not present in the model. Under the Modigliani and Miller theorem, in the absence of financial friction or taxes, the only effect of leverage is to modify the payout process and subsequently the asset prices. Rather than model the leverage explicitly, I follow the asset pricing literature (e.g. Abel (1999)) and compute the price of a claim to $D_t^{\text{lev}} = Y_t^{\lambda}$, where $\lambda > 1$ is the leverage parameter. This formulation implies that $\Delta \log D_t^{\text{lev}} = \lambda \Delta \log Y_t$, making dividends more volatile than output, as in the data. I will use the price of this levered claim to output as the model counterpart to stock prices. In section 5.1, I show that this formulation of leverage gives nearly identical results to a formulation based on a constant debt-equity ratio.

3.4 Analytical results

This section proves some analytical results in the special case $b_k = b_{tfp}$, i.e. productivity and capital fall by the same amount if there is a disaster. Under this assumption, we first establish

[14] Empirically, default often takes the form of high rates of inflation which reduces the real value of nominal government debt.

the behavior of quantities and returns following a disaster. Then we establish the equivalence between disaster risk and a change in impatience (discount factor). All these results stem directly from equation (6).[15]

Proposition 1 *Assume that $b_k = b_{tfp}$. Then, a disaster leads consumption, investment, and output to drop by a factor $b_k = b_{tfp}$, while hours do not change. The return on capital is also reduced by the same factor, while the return on government bonds is reduced by a factor r. There is no further effect of the disaster on quantities or prices, i.e. all the effect is on impact.*

Proof. Equation (6) leads to policy functions $c(k,p), i(k,p), N(k,p)$ and $y(k,p) = k^\alpha N(k,p)^{1-\alpha}$ which express the solution as a function of the probability of disaster p (the exogenous state variable) and the detrended capital k (the endogenous state variable). The detrended capital evolves according to the shocks ε', x', p' through

$$k' = \frac{(1 - x'b_k)\left((1-\delta)k + \phi\left(\frac{i(k,p)}{k}\right)k\right)}{e^{\mu+\sigma\varepsilon'}\left(1 - x'b_{tfp}\right)}.$$

The key remark is that if $b_k = b_{tfp}$, then

$$k' = \frac{\left((1-\delta)k + \phi\left(\frac{i(k,p)}{k}\right)k\right)}{e^{\mu+\sigma\varepsilon'}}$$

is independent of the realization of disaster x'. As a result, the realization of a disaster does not affect c, i, N, y, since k is unchanged, and hence it leads consumption $C = cz$, investment $I = iz$, and output $Y = yz$ to drop by a factor $b_k = b_{tfp}$ on impact. Furthermore, once the disaster has hit, it has no further effect since all the endogenous dynamics are captured by k, which is unaffected. The statement regarding returns follows from the expression of the stock return (8): given that the investment-capital ratio and output-capital ratios are unaffected by the disaster, the only effect of the disaster is to multiply $R_{t,t+1}$ by the factor $(1 - b_k)$. ■

The intuition for proposition 1 stems directly from the condition for the steady-state of the neoclassical growth model, which is determined by the level of TFP according to the familiar formula $\frac{1}{\beta} - 1 + \delta = \alpha K^{\alpha-1}(Nz)^{1-\alpha}$. Given the preference specification, the steady-state hours are unaffected by the change in TFP. The decrease in z hence requires an equal decrease in K to reach a steady-state. When $b_k = b_{tfp}$, the amount of capital destruction is exactly what is

[15] An alternative derivation, using the Euler equation, is provided in the appendix.

required for the economy to jump from one steady-state to another steady-state, and there are no further transitional dynamics.

In contrast, when $b_k \neq b_{tfp}$, a disaster leads both to impact effects and to further transitional dynamics. For instance, a capital destruction without reduction in productivity leads to high investment and a recovery as the economy converges back to its initial steady-state. Inversely, a productivity decline without capital destruction leads to a persistently low level of investment as the economy adjusts gradually to reach its new steady-state.

We can now state the first main result.

Proposition 2 *Assume that the probability of disaster p is constant, and that $b_k = b_{tfp}$. Then the policy functions $c(k)$, $i(k)$, $N(k)$, and $y(k)$ are the same as in a model without disasters ($p = 0$), but with a different time discount factor $\beta^* = \beta(1 - p + p(1 - b_k)^{\upsilon(1-\theta)})^{\frac{1-\gamma}{1-\theta}}$. Assuming $\theta \geq 1$, we have $\beta^* \leq \beta$ if and only if $\gamma < 1$. Asset prices and expected returns, however, will be different under the two models.*

Proof. Following proposition 1, note that k' is independent of the realization of disaster x'. As a result, we can simplify the expectation in the Bellman equation (6):

$$g(k) = \max_{c,i,N} \left\{ \begin{array}{c} c^{\upsilon(1-\gamma)}(1-N)^{(1-\upsilon)(1-\gamma)} \\ +\beta e^{\mu\upsilon(1-\gamma)} \left(E_{x'} (1 - x' b_{tfp})^{\upsilon(1-\theta)} \times E_{\varepsilon'} e^{\sigma\varepsilon'\upsilon(1-\theta)} g(k')^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}} \end{array} \right\},$$

i.e.:

$$g(k) = \max_{c,i,N} \left\{ c^{\upsilon(1-\gamma)}(1-N)^{(1-\upsilon)(1-\gamma)} + \beta^* e^{\mu\upsilon(1-\gamma)} \left(E_{\varepsilon'} e^{\sigma\varepsilon'\upsilon(1-\theta)} g(k')^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}} \right\}.$$

We see that this is the same Bellman equation as the one in a standard neoclassical model, with discount rate β^*. As a result, the policy functions $c(k), N(k), i(k)$ and $y(k)$ are also the same as a standard neoclassical model.

Asset prices, on the other hand, are driven by the stochastic discount factor, which weights the possibility of disaster (see the expression of the SDF in the computational appendix). Both consumption and the return on capital are low in a disaster as show in Proposition 1, hence the equity premium will be larger than in a model without disaster risk. ■

This result has several implications. First, in a sample without disasters, the quantities implied by the model (consumption, investment, hours, output and capital) are exactly the same as those implied by the standard RBC model, provided that the discount factor is adjusted. In practice, this adjustment is small and hence has very little effect on quantity dynamics. For the benchmark calibration, we have $\beta = .994$, and $\beta^* \approx .9934$. As a particular implication, the response to a standard normal TFP shock ε will be exactly the same, hence the model will generate the standard patterns of higher investment, output, employment and consumption following an increase in TFP.

Second, this analytical result clarifies the numerical findings of Tallarini (2000). As discussed in the introduction, he found, in a model where the IES is unity, that higher risk aversion has little effect on business cycle quantity dynamics (a finding often interpreted as "fixing the asset pricing properties of a RBC model need not change the quantity dynamics"). In my model, if the IES is unity ($\gamma = 1$), β^* is exactly equal to β, hence no adjustment is required and the equivalence of dynamics is an exact result. The model nevertheless generates a large equity premium, since a disaster leads to a large decline in consumption and in the equity return. This proposition hence shows how to construct a model with large risk premia and reasonable business cycle dynamics, addressing the question studied by Jermann (1998) and Boldrin, Christiano and Fisher (2001).

Third, the result implies that the steady-state level of capital stock will be affected by the probability of disaster. If risk aversion θ is greater than unity, and the IES is above unity, then $\beta^* < \beta$, leading people to save less: the steady-state capital stock is lower than in a model without disasters. While higher risk to productivity leads to higher precautionary savings, rate-of-return risk can reduce savings.

While this first result is interesting, it is not fully satisfactory however, since the constant probability of disaster implies constant risk premia. As is well known, constant risk premia imply that price-dividend ratios and returns are not volatile enough. This motivates extending the result for a time-varying p.

Proposition 3 *Assume that $b_k = b_{tfp}$, and that p follows a stationary Markov process. Then the policy functions $c(k,p)$, $i(k,p)$, $N(k,p)$, and $y(k,p)$ are the same as in a model without disasters ($p = 0$), but with stochastic discounting (i.e. β follows a stationary Markov process). Assuming $\theta \geq 1$, β is inversely related to p if and only if $\gamma < 1$.*

Proof. The proof also uses the fact that k' is independent of x', to simplify the expectation inside the Bellman equation (6):

$$g(k,p) = \max_{c,i,N} \left\{ \begin{array}{c} c^{\upsilon(1-\gamma)}(1-N)^{(1-\upsilon)(1-\gamma)} \\ +\beta e^{\mu\upsilon(1-\gamma)} \left(E_{x'|p} \left(1 - x' b_{tfp}\right)^{\upsilon(1-\theta)} E_{\varepsilon',p'} e^{\sigma\varepsilon'\upsilon(1-\theta)} g\left(k',p'\right)^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}} \end{array} \right\}.$$

Define

$$\beta(p) = \beta \left(E_{x'|p} \left(1 - x' b_{tfp}\right)^{\upsilon(1-\theta)} \right)^{\frac{1-\gamma}{1-\theta}} = \beta \left(1 - p + p(1 - b_{tfp})^{\upsilon(1-\theta)}\right)^{\frac{1-\gamma}{1-\theta}} .$$

Since p is Markov, β is Markov too. Assuming $\theta \geq 1$, β is increasing in p if and only if $\gamma < 1$. We have:

$$g\left(k,p\right) = \max_{c,i,N} \left\{ c^{\upsilon(1-\gamma)}(1-N)^{(1-\upsilon)(1-\gamma)} + \beta(p) e^{\mu\upsilon(1-\gamma)} \left(E_{\varepsilon',p'} e^{\sigma\varepsilon'\upsilon(1-\theta)} g\left(k',p'\right)^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}} \right\},$$

i.e. the Bellman equation of a model with time-varying β, but no disasters. ■

This result shows that the time-varying risk of disaster has the same implications for quantities as a preference shock. It is well known that these shocks have a significant effect on macroeconomic quantities (a point that we will quantify later). In a sense, this version of the model breaks the "separation theorem" of Tallarini (2000): when risk varies over time, risk aversion has an effect on the quantities. Asset prices will respond as well, generating correlations of risk premia and quantities.

This result is interesting in light of the empirical literature which suggests that "preference shocks" or "equity premium shocks" may be important (e.g., Smets and Wouters (2003)). Chari, Kehoe and McGrattan (2009) criticize these shocks which lack microfoundations. My model provides a simple microfoundation, which allows to tie these shocks to asset prices precisely, and justifies the label "equity premium shock". Of course, my model is significantly simpler than the medium-scale models of Smets and Wouters (2003), but I conjecture that this equivalence can be generalized to a large class of models.

Interestingly, this result also shows that it is technically feasible to solve DSGE models with time-varying risk premia. A full non-linear solution of a medium-scale DSGE model is daunting. But under this result, we can solve the quantities of the model by solving a model with shocks to β and no disasters, i.e. a fairly standard model which we can approximate well using log-linear methods. Once quantities are found, we can solve for asset prices using nonlinear methods. The computational appendix details this solution method.

The three propositions require that $b_k = b_{tfp}$; analytical results are impossible without this assumption. As discussed above, proposition 1 does not hold if $b_k \neq b_{tfp}$. On the other hand, numerical experiments suggest that proposition 2 is robust to this assumption, in that the dynamic response to a TFP shock is largely unaffected by the presence or type of disasters (i.e. b_k vs. b_{tfp}). Proposition 3 is somewhat more fragile. For instance, if disasters affect only TFP, and there are no adjustment costs, then an increase in p will lead people to want to hold more capital, for standard precautionary savings reasons. This is true regardless of the IES. We discuss this further and relax the assumption $b_k = b_{tfp}$ in Section 5.2.

4 Quantitative Results

This section first presents the calibration. I then successively study the implications of the model for business cycle quantities, for asset prices, and finally for the relations between asset prices and quantities. In general, the model cannot be solved analytically, leading me to resort to a numerical approximation. A nonlinear method is crucial to analyze time-varying risk premia. I use a standard policy function iteration algorithm, which is described in detail in the computational appendix.

4.1 Calibration

Parameters are listed in Table 1. The period is one quarter. Many parameters follow the business cycle literature (Cooley and Prescott (1995)). The risk aversion parameter is picked to replicate the mean equity premium, and it is set at 6. However, this is risk aversion over the consumption-hours bundle. Since the share of consumption in the utility index is .3, the effective risk aversion to a consumption gamble is 1.8 (Swanson (2010)).

The intertemporal elasticity of substitution of consumption (IES) is set at 2. There is a large debate regarding the value of the IES. Most direct estimates using aggregate data find low numbers (e.g. Hall (1988)), but this view has been challenged by several authors (see among others Bansal and Yaron (2004), Guvenen (2006), Mulligan (2004), Vissing-Jorgensen (2002)). As emphasized by Bansal and Yaron (2004), a low IES has the counterintuitive effects that higher expected growth lowers asset prices, and higher uncertainty increases asset prices. The IES plays a key role for only part of my results, namely the response of macroeconomic quantities to an increase in the probability of disaster.

The functional form for the adjustment cost function follows Jermann (1998): $\Phi(x) = a_1 \frac{x^{1-\eta}}{1-\eta} + a_2$, where a_1 and a_2 are set such that the steady-state is independent of η and marginal Q is one. The unique parameter η is set to match approximately the volatility of investment, relative to output, leading to $\eta = .15$, a value well in the range of empirical estimates.[16]

[16] The volatility of investment is limited by general equilibrium feedbacks, as in the RBC model, hence only moderate adjustment costs are required to lower further a bit the volatility of investment.

One crucial element of the calibration is the probability and size of disaster. I assume that $b_k = b_{tfp} = .43$ and the probability is .017 per year on average. These numbers are motivated by the evidence in Barro (2006) who reports this unconditional probability, and the risk-adjusted size of disaster is on average 43%. (Barro actually uses the historical distribution of sizes of disaster. In his model, this distribution is equivalent to a single disaster with size 43%.) In my model, with $b_k = b_{tfp} = .43$, both consumption and output fall by 43% if there is a disaster. Note that since the Solow residual is $z^{1-\alpha}$, the actual drop in productivity is 30.2%.

Whether one should model a disaster as a capital destruction or a reduction in TFP is an important question. Clearly some disasters, e.g. in South America since 1945, or Russia 1917, affected TFP, perhaps by introducing an inefficient government and poor policies. On the other hand, World War II led in many countries to massive physical destructions and losses of human capital. It would be interesting to gather further evidence on disasters, and measure b_k and b_{tfp} directly. This is beyond the scope of this paper. I concentrate on the parsimonious benchmark case $b_k = b_{tfp}$.This has the advantage of clarity, since the analytical results of section 3 apply, and generates the same consumption dynamics during disasters as assumed in the literature that uses endowment economies (Barro (2006), Gabaix (2007), Wachter (2008), Gourio (2008)). In section 5.2, I discuss an alternative calibration with $b_k = 0$, which generates many of the same results, provided that there are capital adjustment costs. Hence the capital destruction is not necessary for the model to match the data well.

The second crucial element is the persistence and volatility of movements in this probability of disaster. I assume that the log of the probability follows an AR(1) process:

$$\log p_{t+1} = \rho_p \log p_t + (1 - \rho_p) \log \overline{p} + \sigma_p \varepsilon_{p,t+1},$$

where $\varepsilon_{p,t+1}$ is *i.i.d.* $N(0,1)$.[17] The parameter $\overline{p}$ is picked so that the average probability is .017 per year, and I set $\rho_p = .92$ and the unconditional standard deviation $\frac{\sigma_p}{\sqrt{1-\rho_p^2}} = 1.85$, which allows the model to fit reasonably well the volatility and predictability of equity returns. Regarding the default of government bonds during disasters, I follow the work of Barro (2006): conditional on a disaster, government bonds default with probability .6, and the default rate is the size of the disaster. The leverage parameters λ is set to 2 (Abel (1999)).

On top of this benchmark calibration, I will also present results from different calibrations

[17] This equation allows the probability to be greater than one, however I will approximate this process with a finite Markov chain, which ensures that $0 < p_t < 1$ for all $t \geq 0$.

(no disasters, constant probability of disaster, and in section 5 more extensions) to illustrate the sensitivity of the results.

Some may argue that this calibration of disasters is extreme. A few remarks are in order. First, a long historical view makes this calibration sound more reasonable, as shown by Barro (2006) and Barro and Ursua (2008). An example is the U.K., which sounded very safe in 1900, but experienced a variety of very large negative shocks during the XXth century. The recent crisis also illustrates some large declines in consumption or GDP: for instance, real consumption in Iceland is expected to drop by 7.1% in 2008 and 24.1% in 2009, according to the official government forecast (January 2009). According to the IMF World economic outlook (April 2009), output in Germany, Ireland, Ukraine, Japan, Latvia, Singapore, Taiwan, are expected to contract by respectively 5.6%, 8.0%, 8.0%, 6.2%, 12.0%, 10.0%, 7.5% in 2009 *alone*. Second, it is also possible to change the calibration, and increase risk aversion[18] while reducing the size or probability of disasters. One can also employ fairly standard devices to boost the equity premium, and reduce the probability of disaster further - e.g., the disasters may be concentrated on a limited set of agents, or some agents may have background risk (private businesses); or idiosyncratic risk might be countercyclical. These features could all be added to the model, at a cost in terms of complexity, and would likely reduce the magnitude of disasters required to make the model fit the data.

4.2 An increase in the probability of a disaster

We can now perform the key experiment of an increase in the probability of disaster, i.e. an increase in risk. Figure 1 plots the impulse response of quantities to a doubling of the probability of disaster at time $t = 6$, starting at its long-run average (.017% per year or 0.00425% per quarter).[19] Investment decreases, and consumption increases, as in the analytical example of section 2, since the elasticity of substitution is assumed to be greater than unity. Employment decreases too, through an intertemporal substitution effect: the return on savings is low and thus working today is less attractive. (This is in spite of a negative wealth effect which tends to push employment up; given the large IES the substitution effect overwhelms the wealth effect both for

[18] The risk aversion in my calibration less than two, and hence even lower than in Barro (2006), because the variation over time in the probability of disaster is an additional source of risk.

[19] For clarity, to compute this figure, I assume that there are no realized disaster. The simulation is started of after the economy has been at rest for a long time (i.e. no realized disasters, no normal shocks, and no change in the probability of disaster). I obtain this figure by averaging out over 100,000 simulations which start at $t = 6$ in the same position, but then have further shocks to ε or p.

consumption and for leisure.) Hence, output decreases because both employment and the capital stock decrease, even though there is no change in current or future total factor productivity. This is one of the main result of the paper: this shock to risk leads to a recession. After impact, consumption starts falling. These results are robust to changes in parameter values, except for the IES which crucially determines the sign of the responses, and the assumption that $b_k = b_{tfp}$ (as we discuss in section 5.2 below). The size of adjustment costs, and the level of risk aversion, affect only the magnitude of the response of investment and hours. This figure is consistent with proposition 3: the shock is equivalent, for quantities, to a preference shock to β. The model predicts some negative comovement between consumption and investment, which may seem undesirable.[20] I discuss this further in Section 5.4.

Regarding asset prices, figure 2 reveals that, following the shock, the risk premium on equity increases (the spread between the red–crosses line and the black-full line becomes larger), and the short rate decreases, as investors try to shift their portfolio towards safer assets - a "flight to quality". Hence, in the model, an increase in risk premia coincides with a recession. On impact (at $t = 6$), the increase in the risk premium lowers equity prices substantially, through a discount rate effect.

4.3 First and second moments of quantities and asset returns

Table 2 reports the standard business cycle moments obtained from model simulations. Results are reported both for a sample where no disaster actually takes place (i.e. agents fear a disaster but it does not occur in sample), and, in the starred rows, for a full sample that includes disaster realizations (i.e. population moments). The data row reports the standard U.S. post-WWII statistics. Given the lack of disasters in these data, one should compare the data to the model results in a sample without disasters.

Row 2 shows the results for the standard model (i.e. $b_k = b_{tfp} = 0$). The success of the basic RBC model is clear: consumption is less volatile than output, and investment is more volatile than output. The volatility of hours is on the low side, a standard defect of the basic RBC model driven by the specification of the utility function and adjustment costs.

Introducing a constant probability of disaster, in row 3, does not change the moments significantly. This is consistent with proposition 2. However, the presence of the risk shock - the change

[20] Despite the fact that consumption rises on impact, states of nature with high probability of disaster are still "bad states", i.e. high marginal utility states. This is because the stochastic discount factor also includes current hours and future utility, and the higher uncertainty reduces the future value due to risk aversion (i.e. volatility is a priced factor; see e.g. Bansal and Yaron (2004) for a related analysis).

in the probability of disaster - leads to additional dynamics, which are visible in row 5. Specifically, the correlation of consumption with output is reduced. Total volatility increases, since there is an additional shock, but this is especially true for investment and employment. Overall the model gets closer to the data for most moments, except the relative volatility of investment which is slightly too high.

Turning to returns, table 3 shows that the benchmark model (row 4) can generate a large equity premium: about 6% (=4*(1.93-0.42)) per year for a levered equity (the model counterpart to real stocks). The unlevered equity also has a significant risk premium of 1.8% per year. These risk premia are computed over short-term government bonds, which are not riskless in the model; they would be larger if computed over the risk-free asset. Whether these risk premia are calculated in a sample with disasters or without disasters does not matter much quantitatively - the risk premia are reduced by 15–25 basis point per quarter or 0.6-1% per year. Hence, sample selection is not a critical issue.

Table 3 shows that the volatility of the levered equity approximately matches that of the data (7.14% per quarter vs. 8.14% in the data). This is in sharp contrast with the RBC model (1.59%) or the model with constant probability of disaster (1.53%). Importantly, the model matches the low volatility of short-term interest rates (0.85% vs. 0.81% in the data), an improvement over the studies of Jermann (1998) and Boldrin, Christiano and Fisher (2001) which implied highly volatile interest rates.

For completeness, it is important to note that an unlevered equity does not have volatile returns, however (0.40% per quarter). The intuition is that, without adjustment costs, Tobin q is unity, and the return on capital is simply $1-\delta+\alpha K_{t+1}^{\alpha-1}(z_{t+1}N_{t+1})^{1-\alpha}$, which is very smooth. My calibration has only a small amount of adjustment costs, hence Tobin q varies little and the return on unlevered capital is smooth.[21]

[21] This foonote describes, for completeness, the implications of the model for the term structure of interest rates. Because the model does not incorporate inflation, it is difficult to estimate the extent to which the model fits the data. Moreover, these results for the yield curve are similar to those in the endowment economy model of Gabaix (2007). Assume that all bonds default by the same amount during disasters. The model then generates a negative term premium, consistent with the evidence for indexed bonds in the UK. This negative term premium is not due to what happens during disasters, since short-term bonds and long-term bonds are assumed to default by the same amount. As usual, TFP shocks generate very small risk premia. The term premium is thus driven by the third shock, i.e. the shock to the probability of disaster. An increase in the probability of disaster reduces interest rates, as the demand for precautionary savings rises. As a result, long-term bond prices rise. Hence, long term bonds hedge against the shock to the probability of disaster, they have lower average return than the short-term bonds, and the yield curve is on average downward sloping. Obviously, one possibility to make the yield curve upward-sloping is to assume that long-term bonds will default by a larger amount, should a disaster happen.

4.4 Relations between asset prices and macroeconomic quantities

This section evaluates the ability of the model to reproduce some relations between asset prices and macroeconomic quantities.

4.4.1 Countercyclicality of risk premia

An important feature of the data is that risk premia are countercyclical. This has been illustrated strikingly by the recent crisis, where the yield on risky assets such as corporate bonds went up while the yield on safe assets such as government bonds went down. This pattern is common to most U.S. recessions. To illustrate it simply, figure 3 reports the covariance between detrended output $\widetilde{y}_t$ and stock excess returns at different leads and lags, i.e. $Cov\left(\widetilde{y}_t, R^e_{t+k} - R^f_{t+k}\right)$,for $k = -12$ to $k = 12$ quarters. In the data (full line), this covariance is positive for $k < 0$, reflecting the well-known fact that excess returns lead GDP, but this covariance becomes negative for $k \geq 0$, implying that output negatively leads excess returns, i.e. *risk premia are lower when output is high*.[22] I concentrate on the covariance rather than the correlation because the size of the association is critical (correlations can look good even if there is only a tiny variation, provided it has the right sign). GDP is detrended using the one-sided version of the Baxter-King (1999) filter.

The fact that returns lead GDP, while interesting, might be rationalized by several models, e.g. a model of advance information and adjustment costs or time-to-build. More simply, as can be seen in the figure 3, even the basic RBC model generates this pattern, since high returns reflect positive TFP shocks, and positive TFP shocks lead to a period of above-trend output. More interesting, and more discriminating, is the right-side of this picture, i.e. high output is associated with low future excess returns. The model without shocks to p, i.e. the RBC model, does not generate any variation in risk premia, so the model-implied covariance is very close to zero.[23] In contrast, my model generates about the right comovement of output and risk premia. This is a validation of the model key mechanism: changes in risk drive both expected returns and output.

[22] I use this particular statistic because it has a natural model counterpart. There are other, more powerful ways to show in the data that risk premia are countercyclical. First one can use additional variables, not present in the model: e.g. the unemployment rate forecasts excess stock returns negatively. Second, one can use a standard return forecasting regression, i.e. running future returns on the current dividend yield, the short rate and the term spread, and observe that the fitted values from this regression are significantly negatively correlated with detrended GDP.

[23] It is important to use a one-sided filter for this purpose, since with a two-sided filter output is low when future output is high, i.e. in the RBC model when future TFP is high, i.e. when future returns are high: hence, the RBC model generates a negative covariance between two-sided filtered output and future excess returns.

4.4.2 VIX and GDP

The VIX index is a measure of the implied volatility of the SP500, constructed by the CBOE from option prices with different strikes. Mathematically, it is defined as $\sqrt{4var_t^Q\left(r_{t+1}^m\right)}$, where the variance is taken under the risk-neutral measure, and the factor 4 annualizes the variance. In an influential study, Bloom (2009) shows using a reduced-form VAR that shocks to the VIX index have a significant negative effect on output. Figure 4 reproduces this results by depicting the impulse response of output to a shock to VIX in the data (full blue line). This impulse response function is computed using a Cholesky decomposition, under the orthogonalization assumption that a shock to VIX has no instantaneous impact on GDP.[24]

Running the same VAR on the model-generated data yields a response that is fairly similar to the data (red crosses). In the model, VIX is largely driven by the fear of a disaster, i.e. VIX is nearly one-to-one with the state variable p. Increases in p lead to an increase in VIX and a decline in output. Hence, the model generates an impulse response consistent with the data. In contrast, in a real business cycle model without disaster risk, VIX is small and nearly constant, and the VAR finds actually a positive effect of VIX on output.

4.4.3 Investment and Asset Prices

One enduring puzzle in macroeconomics and finance is the relation between investment and the stock market. While the Q-theory correctly predicts a positive correlation, the level of adjustment costs required to match the investment and the stock market is widely considered excessive (see e.g. Philippon (2009) for a recent discussion). In contrast, I show here (see also section 6) that my model captures well the magnitude of the relation between the stock market and investment, even with small adjustment costs.

One way to measure this association is to compute the covariance between investment and asset prices in the RBC model. Figure 5 presents the cross-covariogram $\gamma_k = Cov(i_{t+k}, \log(P_t/D_t))$,where i_{t+k} is HP-filtered log investment, for $k = -12$ to $k = 12$ quarters.

The black (diamonds) line shows the data, reflecting the well-known pattern that investment and the stock market are positively correlated, with the stock market leading investment. The blue line (crosses) presents the covariogram for the model with only TFP shock, i.e. the basic RBC model. The model generates actually a small negative covariance between the price-dividend

[24] The orthogonalization assumption has little impact on these results. Following Bloom, both GDP and volatility are HP-filtered, but this is not critical either. Last, in the model, as in the data, it makes relatively little difference whether we use the implied volatility, based on the risk-neutral measure, or the physical volatility, $\sqrt{4var_t\left(r_{t+1}^m\right)}$.

ratio and investment, because TFP shocks have little effect on the stock market value - higher TFP increase future cash flows, but also increases interest rates, leading to offsetting effects on the levered equity.

The red line (diamonds) the covariogram for the benchmark model, i.e. including both TFP shocks and shocks to the probability of disaster. The key result is that the covariance is now of the right magnitude. Both models are, however, unable to replicate the exact timing of the association between the stock market and investment, i.e. to match the observed lag, but additional frictions such as time-to-plan may account for this.

4.4.4 Other asset pricing implications

The model has several other interesting implications for asset prices, which I describe briefly in this section. First, the model is consistent with the evidence that equity returns are predictable, but dividends are not. The standard regression

$$R^e_{t\to t+k} - R^f_{t\to t+k} = \alpha + \beta \frac{D_t}{P_t} + \varepsilon_{t+k},$$

yields a $R^2 = 25\%$ at the four-year horizon ($k = 4$) in the data (this figure, however, is sample-specific), and a $R^2 = 54\%$ in the model. In both data and model, the results are similar if the left-hand side variable is returns rather than excess returns. Second, in the model as in the data, dividends are much less forecastable than returns, i.e. in a regression

$$\frac{D_{t+k}}{D_t} = \alpha + \beta \frac{D_t}{P_t} + \varepsilon_{t+k},$$

the R^2 is 1% in the data at a four-year horizon, while it is 6% in the model. Hence, there is somewhat too much predictability of returns in the model, but the model is consistent with the finding that discount rate variation is the key driver of stock prices.

Third, the model generates an Euler equation error. A large literature has concentrated on the ability of models to generate a significant equity premium and volatile returns. Lettau and Ludvigson (2009) argue that a more challenging test is to generate a failure of the Euler equation, i.e. estimating the Euler equation with CRRA utility on data simulated from the model should lead, as in the data, to a rejection of the model. These author show that few models can pass this test, because in most models aggregate consumption is highly correlated with returns. In my model, the shock to the probability of disaster induces a negative comovement between asset returns and aggregate consumption, leading the CRRA model to be rejected.

Fourth, the model can generate qualitatively, but not quantitatively, results similar to those of Beaudry and Portier (2006, BP hereafter), which have stimulated a large recent literature on "news shocks". BP show empirically that shocks to the stock market lead to a gradual increase in TFP and GDP, suggestive of advance information about cash flows. An alternative interpretation of their findings is that the stock market movements are driven by changes in risk premia, which then feed back to GDP (and possibly in measured TFP through variation in utilization). To evaluate this possibility, I ran the same bivariate VAR with GDP growth and the stock market return in the data, in the RBC model, and in my model.[25] The impulse response are similar to BP, i.e. a "return shock" leads to a cumulative increase in GDP, however the magnitude of the response is much smaller in my model than in the data.

Fifth, it is possible to introduce corporate bonds, if one assumes (as in Philippon (2009)) that firms are run to maximize total value (not equity value). One can then calculate the bond prices implied by the firm value and an exogenous leverage policy. The credit spreads generated by the model are high and volatile, and the model generates a strong negative correlation between investment and credit spreads.

5 Robustness and Extensions

In this section, I discuss several extensions of the baseline model, and check that the results are robust to various changes in the calibration.

5.1 A calibration with financial leverage

The benchmark model follows a formulation of leverage which is standard in the asset pricing literature, i.e. the dividend process is $D_t = Y_t^\lambda$. One may worry that the nonlinearity is important. To check this, I computed the return on a levered equity, given an exogenous debt issuance policy.[26] Since the Modigliani-Miller theorem holds, the debt policy has no impact on the allocation. Assume that the firm each period adjusts its debt issues to keep the maturity equal to 5 years, and the book leverage ratio equal to 0.45 (Abel (1999), Barro (2006)). The expression for the

[25] Following BP, I use the orthogonalization assumption that the stock market shock does not affect GDP instantaneously. Obviously in my model this assumption is incorrect, hence the shocks picked by the VAR are combinations of the fundamental shocks, of opposite effect on output.

[26] I assume that the debt has the same default characteristics as government debt, i.e. it will default in a disaster, but by less than the capital stock. The results are stronger if the debt is truly risk-free. An interesting extension of the model is to make default endogenous.

levered firm return is

$$R_{t+1}^{\text{lev}} = \frac{P_{t+1} + D_{t+1} + \omega_t Q_{t+1}^{(n-1)} - \omega_{t+1} Q_{t+1}^{(n)}}{P_t - \omega_t Q_t^{(n)}},$$

where $Q_t^{(n)}$ is the price of a zero-coupon n-period bond, and ω_t is the number of bonds issued, e.g. for a constant book leverage policy $\omega_t Q_t^{(n)} = .45 K_t$. The mean return on the levered equity is then 2.25% per quarter, while the standard deviation of the return is 9.46%. This contrasts with 1.93% and 7.14% in my simple formulation of leverage. Moreover, in simulations, the two returns have a correlation above .95. Hence, the results are very similar if I use this formulation of leverage. Alternatively, one can assume that the firm keeps the market leverage ratio constant. In the benchmark model, the market value of the firm is only slightly more volatile than its capital stock, due to (rather small) adjustment costs, hence the results are nearly identical (2.23% for mean return and 9.37% for volatility of return).

5.2 A calibration without capital destruction

An interesting question is whether one should model a disaster as a reduction in TFP or a destruction of the existing capital stock.[27] Decreases in TFP arise for instance because of poor government policies or extreme misallocation, while destructions of the capital stock can be due to wars or expropriations (see the discussion in section 2). Tables 4 and 5 study the sensitivity of the key results to this assumption, and propose a different parametrization of the model without capital destruction which gives results similar to the benchmark. In tables 4 and 5, I keep the parameters as in the benchmark, except for those noted in the first column.

First, note that a calibration with only capital destruction and no TFP decline does not fit the data well (row 6). Business cycle statistics are, to a first order, similar to the benchmark model, but the equity premium is small and returns are not volatile. Intuitively, a disaster does not impact agents much in this economy, because capital share is only one-third, and hours increase following the disaster, thereby limiting the initial decline in output, and moreover the economy returns fairly quickly to its steady-state. Moreover, with recursive utility, agents take into account their future (high) consumption and hence do not mind disasters all that much.[28]

Second, when disasters affect solely TFP, and there are no adjustment costs (row 3), the model

[27] In the benchmark, I assumed that a disaster affects TFP and the capital stock equally. This generates a responses of consumption following a disaster which is the same as the endowment economy literature (e.g. Rietz (1988), Barro (2006)).

[28] Following Gourio (2008), a low IES would make the equity premium larger, but this would also make increases in p lead to booms, generating unrealistic correlations between asset prices and quantities.

generates a sizeable equity premium, and volatile returns. The business cycle statistics, however, imply too much volatility of investment and the correlation of consumption and investment and output is negative, contrary to the data. There is also a qualitative change: an increase in the probability of disaster now leads to an increase in the capital stock for precautionary savings reasons. As a result, in this case, and regardless of the IES, an increase in the probability of disaster leads to a boom in investment and output, i.e. the sign of the impulse responses depicted in figure 1 are reversed.

Adding adjustment costs can however undo this effect. Intuitively, with adjustment costs, the price of capital will fall significantly if a disaster occurs. Hence investing in capital is now more risky when the disaster probability rises, generating rate of return uncertainty (as discussed in section 2). In row 4, I use the benchmark level of adjustment costs ($\eta = .15$), and in row 5 I use a higher value ($\eta = .5$). These calibrations now imply that a rise in the probability of disaster leads to a recession. The equity premium is high (about 4% per year) and returns are volatile (over 6% per quarter). As the degree of adjustment costs is increased, the volatility of investment becomes close to the data, and the negative correlation of consumption with output or investment is overturned. Overall I conclude that a calibration without capital destruction can be successful, provided that adjustment costs are large enough.

5.3 Disaster Dynamics

As pointed out by Constantinides (2008) and Barro et al. (2009), disasters have more complicated dynamics in the real world than the pure jump typically assumed. First, disasters may last several years. Second, a recovery might then follow. This leads me to consider two variations on the model to study how these features affect my results. First, I consider disasters which last more than one period. Assume that a disaster leads only to a 20% drop in both productivity and the capital stock. However, a disaster also makes the probability of a disaster next period increase to 50%. Next period, either a disaster occurs, in which case the probability of a further disaster remains at 50%, or it doesn't, in which case this probability shifts back to a standard value. The last row of tables 4 and 5 shows the impact of this modification on the results. First, the business cycle dynamics are largely unaffected. Second, while the disaster is substantially smaller, the model still generates a high equity premium and volatile returns (though a bit less than in the data or benchmark). In this version of the model, a disaster initially leads to a large drop of investment, and a smaller drop in consumption, due to the very high risk of a further disaster which leads people to cut back on investment. Moreover, asset prices fall further during the disaster, since they are hit both by the realization of a disaster today and by the fear of another disaster tomorrow. The key lesson from this illustrative computation is that adding some fear of further disasters is a very powerful ingredient.

Second, I study how the results are affected by the presence of recoveries. More precisely, assume that following a disaster, there is a probability of recovery, i.e. an upward jump of capital and TFP which brings these quantities back to the initial trend. Following a disaster, one of three things can happen. Either there is a recovery right away (with probability 10%); there is no recovery (with probability 20%); or the recovery is uncertain (with probability 70%). When the recovery is uncertain, a new draw next period determines if there is a recovery, or not, or if it is still uncertain; and so on. Overall this leads to a recovery with probability 50%, with an uncertain timing. This is roughly in line with the estimates of Barro et al. (2009). The results of this model are shown in row 7. The business cycle statistics show somewhat less volatility than the benchmark. The equity premium and return volatility are also somewhat smaller than in the benchmark model, but they remain significant. Summarizing, the results of this section show that the model results are weakened, but not in a dramatic way,[29] when disasters are modeled in a more realistic way.

5.4 Comovement of consumption and investment

An implication of the model that may seem odd is that, when the probability of disaster rises, consumption initially increases, while output, employment and investment fall. Given the high IES, the wealth effect is overwhelmed by the substitution effect, hence hours go down and consumption goes up. More generally, given that productivity does not change, and the capital stock is predetermined, the labor demand schedule (marginal product of labor) is unchanged on impact, and, as explained by Barro and King (1984), this makes it impossible to generate on impact positive comovement between consumption and hours worked.[30]

It is not clear that this lack of impact comovement is necessarily a deficiency of the model.

[29] In these experiments I have kept all parameters fixed as I changed the process for disasters, to illustrate the effect of disaster dynamics. Needless to say, to put the models on an equal footing, I could have recalibrated all the parameters, which would allow me to improve significantly the match of the data, e.g. by having slightly higher risk aversion.

[30] This issue is shared by many other papers which incorporate either a shock to preferences (e.g. Smets and Wouters 2008) a shock to investment good prices (e.g. Justiniano, Primiceri and Tambalotti 2010), a shock to micro-uncertainty (e.g. Gilchrist, Ortiz and Zakrajsek (2009), or a shock to financial frictions (e.g. Hall (2009)). These models adopt the last proposed solution to the comovement puzzle, namely sticky prices (except Hall (2009) who assumes that markups are countercyclical).

Consumption, investment, and hours are far from perfectly correlated in the data (see Table 2), which means that any model needs a shock which pushes consumption and investment in opposite directions sometimes. Indeed, we see in Table 2 that adding the shock to the probability of disaster brings the model closer to the data regarding the correlations. Moreover, while the impact response displays negative comovement, consumption eventually falls, although after a long delay.

Alternatively, there are some extensions of the model which may overturn this result. Here I discuss intuitively some possibilities. The most simple extension is simply to assume that the shocks to the probability of disaster and the TFP shocks are correlated. Indeed, we observe that during recessions, investors and workers seem to fear terrible outcomes. This can also be generated endogenously through a learning mechanism as follows. Disasters are modeled for convenience as jumps, but in reality the contraction is not instantaneous. As a result, it is sometimes difficult for consumers and firms to determine if a decrease in productivity and output is a standard recession (a shock ε in our setup) or is the start of a large depression. A large decrease in productivity, due either to a disaster or to a large negative shock ε, will then lead agents to anticipate a further decrease next period. Because consumption reacts negatively to a standard TFP shock, this learning mechanism would attenuate the comovement puzzle.

Alternatively, it may be possible to employ non-standard preferences or adjustment cost formulations, as in Jaimovich and Rebelo (2009) for instance.

Finally and probably more interestingly, countercyclical markups may alter this result. Suppose one were to embed the model in a standard New Keynesian framework with sticky prices, which generates countercyclical markups endogenously. A perfect monetary policy could replicate the flexible price allocation, i.e. the results of this paper. In this case, an increase in the probability of disaster would require the central bank to decrease short-term interest rates. If, for some reason, monetary policy is not accommodative enough, or it is impossible to decrease interest rates because of the zero lower bound, then consumption would have to adjust. Since the real interest rate is too high, consumption would fall. This intuition suggests that this (very substantial) extension of the model may resolve the comovement puzzle.[31]

[31] I thank Emmanuel Farhi for this suggestion.

5.5 Government policy

In the model, the welfare theorems hold, implying no role for government policy. However, it is tempting to "offset" the time-varying wedge in the Euler equation created by the volatility in the probability of disaster. In the case where $b_k = b_{tfp}$, a simple policy can achieve this goal: the government commits to bail out capital holders. That is, the government provides a subsidy to capital holders, proportional to their holdings of capital, in the event of a disaster. If the government can finance this policy with lump-sum taxes, the agents' decisions are the same as in a model without disasters. The intuition for this result (proved in the appendix) can be easily seen in the model without adjustment costs and expected utility, where the return on capital is

$$R_{t+1}^K = (1 - x_{t+1} b_k)(1 - \delta + F_1(K_{t+1}, z_{t+1} N_{t+1})),$$

and the presence of disaster risk affects the equilibrium only through the Euler equation

$$U_1(C_t, N_t) = \beta E_t \left(R_{t+1}^K U_1(C_{t+1}, N_{t+1}) \right).$$

Suppose now that the government engineers a state-contingent subsidy, so that the return to capital holders is $R_{t+1}^K (1 + x_{t+1} \zeta)$, where ζ is the amount of subsidy, to be determined. The equilibrium is then characterized by the equation

$$U_1(C_t, N_t) = \beta E_t \left((1 - \delta + F_1(K_{t+1}, z_{t+1} N_{t+1}))(1 - x_{t+1} b_k)(1 + x_{t+1} \zeta) U_1(C_{t+1}, N_{t+1}) \right). \quad (9)$$

Assuming that the utility function takes the form $U(C, N) = \frac{C^{1-\gamma}}{1-\gamma} v(N)$, it is easy to see that if $1 + \zeta = (1 - b_k)^{\gamma - 1}$, the time-varying wedge disappears: changes in p have no effect, since investors anticipate that they will be bailed out should a disaster happen. Hence, this policy reduces overall volatility. But of course this is sub-optimal since investment *should* vary in response to changes in the probability of disaster. This policy thus reduce welfare. This policy is attractive only if agents have incorrect beliefs, and the government wants to maximize the expected utility of the agents under the correct beliefs.

5.6 Time-varying volatility vs. time-varying jump probability

The model concentrates on a specific model of risk, i.e. a jump with a fixed size. However, the results of the paper immediately generalize to a larger class of shock processes. More precisely, let X_{t+1} be a random variable with strictly positive support, which affects both the capital stock and productivity, i.e.

$$K_{t+1} = ((1-\delta)K_t + \Phi(I_t, K_t))\, X_{t+1},$$

and

$$\log z_{t+1} = \log z_t + \mu + \sigma\varepsilon_{t+1} + \log(X_{t+1}),$$

then propositions 2 and 3 of section 2.4 can be restated as: the decision rules are the same as in a model where the agent's discount factor between time t and time $t+1$, β_t is

$$\beta_t = E_t(X_{t+1}^{1-\theta})^{\frac{1-\gamma}{1-\theta}}, \tag{10}$$

with θ the risk aversion coefficient and γ the inverse of the IES. Hence, while I focused in the paper on the case where X_{t+1} is a binomial variable ($X_{t+1} = 1$ w/prob $1 - p_t$, and $X_{t+1} = 1 - b$ w/prob p_t), alternative stochastic processes generate similar effects. As an example, if X_{t+1} is log-normally distributed with unit mean, i.e. $\log(X_{t+1})$ is $N(-\sigma_t^2/2, \sigma_t^2)$, where σ_t follows an exogenous stochastic process, then

$$\beta_t = e^{-\theta(1-\gamma)\sigma_t^2},$$

and an increase in σ_t reduces β_t if and only if $\gamma < 1$ i.e. the IES is greater than unity, as in proposition 3. The key ingredient of my results is the fact that risk is time-varying, and not its specific distribution.[32] More generally, the expression (10) suggests that higher order moments may matter,[33] and the results of the paper hence readily apply to distributions with excess kurtosis, i.e. "fat tails".

6 The Empirical Importance of Time-Varying Risk

The previous sections show that a simple, parsimonious framework accounts for a variety of business cycle and asset pricing facts. However one may remain skeptical since the probability

[32] The empirical implications are, however, somewhat different since normally distributed shocks occur every period. The calibration would then be different, with a significantly higher risk aversion and smaller shocks, but the effects would be similar in the end.

[33] For instance, Weitzman (2007) shows that this expression may not even be finite for some "natural" distributions. Martin (2008) provides a decomposition using cumulants, and shows in some examples that the higher cumulants (higher moments) can play an important role in this conditional expectation. For my calibration, the second moment (time-varying risk) is the most important. Calibrations which emphasize the risk of even more extreme disasters, however, may imply that the time-varying skewness or kurtosis matters.

of disaster is difficult to measure.[34] This section tests the model directly by identifying the probability of disaster from asset prices – a natural approach, since asset prices are measured precisely and the probability of disaster affects them strongly. We then are able to back out the shocks to the probability of disasters, and to compare the data and the time series implied by two models: first, a model without disaster risk, where the only shock is a shock to TFP; second, a model with both TFP shocks and the shock to the probability of disaster.

Specifically, I pick the probability of disaster p_t to match the price-dividend ratio at each date. Following Campbell and Shiller (1988), we know that the stock market movements are largely due to variation in discount rates, which in my model come from variation in p. This motivates my choice of the P-D ratio as a reasonable source of information for p.[35] More precisely, in the model, given a vector of parameters Θ, the price-dividend ratio $\frac{P_t}{D_t}$ is a function of the two state variables, $k_t = \frac{K_t}{z_t}$ (where K_t is the capital stock and z_t is TFP) and of the probability of disaster p_t :

$$\frac{P_t}{D_t} = \psi\left(k_t, p_t; \Theta\right).$$

Standard data from the BEA lead to estimates of K_t and z_t, hence k_t, from 1948q1 to 2008q4. I then calculate, for each date, the value of the probability of disaster $\widehat{p_t}$ which allows to match exactly the observed price-dividend ratio in the data. Next, I feed this probability of disaster in the model, together with the measured TFP. For instance, the policy functions imply that aggregate investment is $I_t = z_t i(k_t, p_t)$. Finally, the implied series for investment and output are HP-filtered and compared to the data and to the baseline RBC model. The baseline RBC model is constructed using the same measurements of technology z_t and capital k_t, as $I_t = z_t i(k_t)$.[36]

Figure 6 depicts the probability of disaster obtained from this procedure. By construction, this time series is a nonlinear function of the price-dividend ratio $\frac{P_t}{D_t}$ and the detrended capital stock k_t. The short-run fluctuations hence mostly reflect changes in the stock market value. The probability of disaster - a measure of perceived risk - is highly volatile, consistent with the quantitative model. Interestingly, the highest probability of disaster is estimated to occur at the very end of the sample, in the last quarter of 2008.

[34] One indirect piece of evidence is that estimated DSGE models give a significant role to shocks to β, or to shocks to the relative price of investment goods, which have similar dynamic properties, in accounting for business cycle fluctuations. These estimation results are not based on asset prices data but on quantities alone.

[35] Alternatively, one could use option prices, e.g. the VIX index, to measure p. Since the model fits well the reduced-form relation between GDP and VIX (section 4.3), it is likely that if p is picked to match VIX in the data, the model would also match the GDP data well. Unfortunately, data for VIX are available only since 1986, limiting the power of this test.

[36] It makes little difference if the capital stock is not measured, i.e. the time series for I_t are calculated using only an initial estimate of the capital stock K_{1948} and only TFP is fed into the model to construct the path.

Figures 7 and 8 present the quantity implications. The first figure presents the data and the RBC model, with the NBER recessions marked as shaded areas. As is well known, the RBC model does a reasonable job at matching macroeconomic aggregates, given the observed path for TFP, until 1985, even if it does not generate quite enough volatility, especially for investment. The second figure plots the data and the model with disaster risk. Comparing the two figures side by side, the difference between the two models is small in "normal times". During recessions however, my model generates a sharper drop in output and especially in investment. For instance, the effect of the 1975 or 1981 recessions on investment are dramatically underpredicted by the RBC model, while my model generates about the right decline in investment. Another episode of special interest is the current recession. Figure 9 "zooms in" on the most recent data. Little happens to TFP in 2008, hence the RBC model does not predict a sharp recession. My model, however, generates a large drop in investment and output through the shock to p_t, i.e. higher perceived risk.

Table 6 summarizes the fit of the models by computing several statistics: first, the correlation and covariance between the data and each model; second, the mean absolute error, i.e. $E\left|\text{data}_t - \text{model}_t\right|$.[37] A close look at figure 8 suggests that there is a slight lag between the model and the data, hence I also report these statistics when the model series are lagged by 2 quarters. (There may well be delays to decisions and various adjustment costs which create such a delay, not captured in the simple version of the model.) Finally, I report the same statistics for the subsample of recessions.

The statistics of table 6 are consistent with the discussion of the figures in the previous paragraph. First, without the lag, adding the probability of disaster does not improve much the fit of the RBC model, if at all. Second, taking into account the lag, the model with shocks to p now improves a bit on the RBC model, especially for investment, but also for employment and output: the correlation and covariance of model and data becomes higher (Table 6, columns 3 and 4), and there is a reduction in the mean absolute error. The model does more poorly for consumption, however. Last, the improvement in fit for investment, output and employment is quite significant if one looks at the subsample of recessions. For instance, the correlation between the model investment and the data investment goes from 44.9 to 61.8, and the mean absolute error goes down from 187 to 130 (Table 6 rows 2 and 10, last two columns). The correlation of model and data employment similarly goes from 34.0 to 43.3.

[37] Here and table 7, I use the mean absolute error, because (unlike the variance) it is meaningful in a subsample with gaps, such as the subsample of recessions.

Table 7 shows a measure of volatility, $E\left|x_t\right|$, for $x =$data, RBC model, or RBC model with disaster risk. The table reveals that the model generates higher volatility: for instance, the investment statistic is only 3.74 in the RBC model vs. 9.55 in the data. The model with time-varying risk yields 6.51, a significant improvement. This is especially true in recessions: the volatility of investment or employment conditional on being in a recession almost doubles.

From an intuitive point of view, these result are not very surprising in light of the well-known empirical regularity that the stock market is correlated with GDP and investment. Section 4 shows that the model matches the relation between asset prices and investment. Hence, feeding in asset prices from the data (through p_t) allows the model to improve on quantities by using the empirical explanatory power of the stock market for investment or GDP. Overall, shocks to the probability of disaster, as *restricted* by asset prices data, appear to help the RBC model fit the data, especially during severe recessions, arguably the most interesting episodes.

7 Conclusion

This work shows how introducing disaster risk into a standard RBC model improves its fit of asset return data, preserves its success for quantities in response to a TFP shock, and creates some interesting new macroeconomic dynamics. The model can replicate not only the second moments of quantities and asset returns, but also a variety of empirical relationships between macroeconomic quantities and asset prices, which have so far largely eluded researchers.

This parsimonious setup is fairly tractable, which allows to derive some analytical results and makes it easy to embed into richer models. For instance, much of the research on financial frictions has taken place in models which largely abstract from macroeconomic risk premia - a potentially significant limitation which has been driven largely by technical considerations.

More broadly, the quantitative and empirical results of this paper suggest an important role for time-varying risk in macroeconomic models, and give some hope that we may be able to connect better asset prices and business cycles.

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Parameter	Symbol	Value	Target
Capital share	α	.34	Cooley and Prescott (1995)
Depreciation rate	δ	.02	Cooley and Prescott (1995)
Share of consumption in utility	υ	.3	Cooley and Prescott (1995)
Discount factor	β	.994	Mean short-term interest rate
Adjustment cost curvature	η	0.15	Investment volatility
Trend growth of TFP	μ	.0025	Measured TFP
Standard deviation of ordinary TFP shock	σ	.01	Measured TFP
IES	$1/\gamma$	2	A priori (see text)
Risk aversion over the consumption-leisure bundle	θ	6	Average equity premium
Size of disaster in TFP	b_{tfp}	.43	Barro (2006)
Size of disaster for capital	b_k	.43	Barro (2006)
Persistence of log(p)	ρ_p	.92	Predictability regressions
Unconditional std. dev. of log(p)	$\frac{\sigma_p}{\sqrt{1-\rho_p^2}}$	1.85	Volatility of returns
Leverage	λ	2	Abel (1999), Barro (2006)
Recovery rate for bonds during disasters	$1-r(1-b)$	.828	Barro (2006)

Table 1: **Parameter values for the benchmark model.** The time period is one quarter.

	$\frac{\sigma(\Delta \log C)}{\sigma(\Delta \log Y)}$	$\frac{\sigma(\Delta \log I)}{\sigma(\Delta \log Y)}$	$\frac{\sigma(\Delta \log N)}{\sigma(\Delta \log Y)}$	$\sigma(\Delta \log Y)$	$\rho_{C,Y}$	$\rho_{I,Y}$	$\rho_{N,Y}$	$\rho_{I,C}$
Data	0.57	2.68	0.92	0.98	0.45	0.68	0.71	0.49
No disaster	0.66	1.86	0.24	0.78	1.00	1.00	0.99	0.99
Constant p	0.67	1.87	0.24	0.78	1.00	1.00	0.99	0.99
Constant p*	0.96	1.12	0.06	3.10	1.00	1.00	0.52	0.99
Benchmark	0.73	3.03	0.54	0.83	0.66	0.85	0.72	0.21
Benchmark*	0.96	1.35	0.15	2.88	0.87	0.90	0.42	0.60

Table 2: **Business cycle statistics.** Second moments implied by the model, for different calibrations. Quarterly data. The statistics are computed in a sample without disasters, except for the rows marked with a star, which are computed in a full sample. rho(A,B) is the correlation of the growth rate of time series A and B. Data sources in appendix.

	$E(R_f)$	$E(R_b)$	$E(R_e)$	$E(R_{e,lev})$	$\sigma(R_f)$	$\sigma(R_b)$	$\sigma(R_e)$	$\sigma(R_{e,lev})$
Data	—	0.21	—	1.91	—	0.81	—	8.14
No disaster	0.71	0.71	0.71	0.74	0.04	0.04	0.24	1.59
Constant p	0.02	0.32	0.77	1.22	0.04	0.04	0.25	1.53
Constant p*	0.02	0.24	0.58	0.92	0.04	0.85	2.20	4.07
Benchmark	0.15	0.42	0.88	1.93	1.37	0.85	0.40	7.14
Benchmark*	0.17	0.36	0.69	1.60	1.29	1.28	2.06	7.94

Table 3: **Financial Statistics.** Mean and standard deviation of returns implied by the model for (a) a pure risk-free asset, (b) a one-quarter government bond, (c) a claim to dividends, (d) a claim on levered output. Quarterly data. The statistics are computed in a sample without disasters, except for the rows marked with a star, which are computed in a full sample. Data sources in appendix.

	$\frac{\sigma(\Delta \log C)}{\sigma(\Delta \log Y)}$	$\frac{\sigma(\Delta \log I)}{\sigma(\Delta \log Y)}$	$\frac{\sigma(\Delta \log N)}{\sigma(\Delta \log Y)}$	$\sigma(\Delta \log Y)$	$\rho_{C,Y}$	$\rho_{I,Y}$	$\rho_{N,Y}$	$\rho_{I,C}$
Data	0.57	2.68	0.92	0.98	0.45	0.68	0.71	0.49
Benchmark	0.73	3.03	0.54	0.83	0.66	0.85	0.72	0.21
$b_k = 0, \eta = 0$	0.77	9.45	1.07	1.34	-0.13	0.89	0.89	-0.41
$b_k = 0$	0.80	5.55	0.85	0.99	0.24	0.85	0.78	-0.21
$b_k = 0, \eta = .5$	0.90	3.71	0.65	0.80	0.56	0.77	0.57	-0.03
$b_{tfp} = 0, \eta = .15$	0.77	3.67	0.66	0.87	0.50	0.83	0.71	-0.03
Recoveries	0.71	2.56	0.44	0.81	0.79	0.88	0.74	0.43
Multiperiod disasters	0.73	2.90	0.52	0.82	0.69	0.86	0.73	0.26

Table 4: **Robustness and Extensions. Business cycle statistics.** Second moments implied by the model, for different calibrations. Quarterly data. The statistics are computed in a sample without disasters.

	$E(R_f)$	$E(R_b)$	$E(R_e)$	$E(R_{e,lev})$	$\sigma(R_f)$	$\sigma(R_b)$	$\sigma(R_e)$	$\sigma(R_{e,lev})$
Data	—	0.21	—	1.91	—	0.81	—	8.14
Benchmark	0.15	0.42	0.88	1.93	1.37	0.85	0.40	7.14
$b_k = 0, \eta = 0$	0.57	0.77	0.60	1.71	0.72	0.38	2.07	6.15
$b_k = 0$	0.51	0.72	0.64	1.71	0.69	0.29	1.68	6.25
$b_k = 0, \eta = .5$	0.48	0.68	0.72	1.71	0.74	0.33	1.11	6.22
$b_{tfp} = 0$	0.70	0.80	0.91	0.86	0.32	0.12	0.51	1.60
Recoveries	0.38	0.56	0.86	1.42	0.94	0.57	0.34	4.68
Multiperiod disasters	0.49	0.62	0.86	1.52	1.04	0.64	0.38	5.52

Table 5: **Robustness and Extensions. Financial statistics.** Mean and standard deviation of returns implied by the model for (a) a pure risk-free asset, (b) a one-quarter government bond, (c) a claim to dividends, (d) a claim on levered output. Quarterly data. The statistics are computed in a sample without disasters.

		Full sample		With two-quarter lag		Recessions	
		RBC	RBC+p	RBC	RBC+p	RBC	RBC+p
$Corr$(model,data)	C	43.26	24.96	31.20	11.74	-16.29	-31.72
	I	50.69	44.18	54.61	60.81	44.92	61.83
	N	16.07	11.02	48.90	48.90	33.96	43.36
	Y	67.71	64.33	56.56	60.80	28.88	39.27
Cov(model,data)	C	0.31	0.17	0.23	0.08	-0.10	-0.19
	I	5.15	8.00	5.56	10.85	3.08	8.79
	N	0.08	0.14	0.26	0.60	0.10	0.39
	Y	1.11	1.24	0.92	1.16	0.29	0.48
$E\,\lvert$data-model$\rvert$	C	194.51	210.35	206.65	227.82	43.51	50.99
	I	848.01	917.13	831.41	768.78	187.55	129.70
	N	369.59	384.13	345.39	326.76	90.44	72.00
	Y	243.94	253.55	255.29	247.30	65.31	56.20

Table 6: **Fit of the model.** The table reports three statistics of fit, for each time series (C,I,N,Y), and for the full sample, with a two-quarter lag, and for the subsample of recessions. The statistics are the correlation between model and data, the covariance between model and data, and the mean absolute error. See section 6 for the construction of the series.

	$x_t =$	Full sample			Recessions		
		Data	RBC	RBC+p	Data	RBC	RBC+p
$E\,\lvert x_t\rvert$	C	2.11	1.18	1.18	0.52	0.29	0.22
	I	9.55	3.74	6.51	2.72	1.00	1.92
	N	3.72	0.53	1.23	1.03	0.14	0.35
	Y	3.09	1.82	2.11	1.05	0.48	0.59

Table 7: **Volatility and conditional volatilities.** The table reports the mean absolute value of each time series (C,I,N,Y), for the data and for each model; results are reported for the full sample and for the subsample of recessions. This is a measure of the volatility implied by each model. See section 6 for the construction of the series.

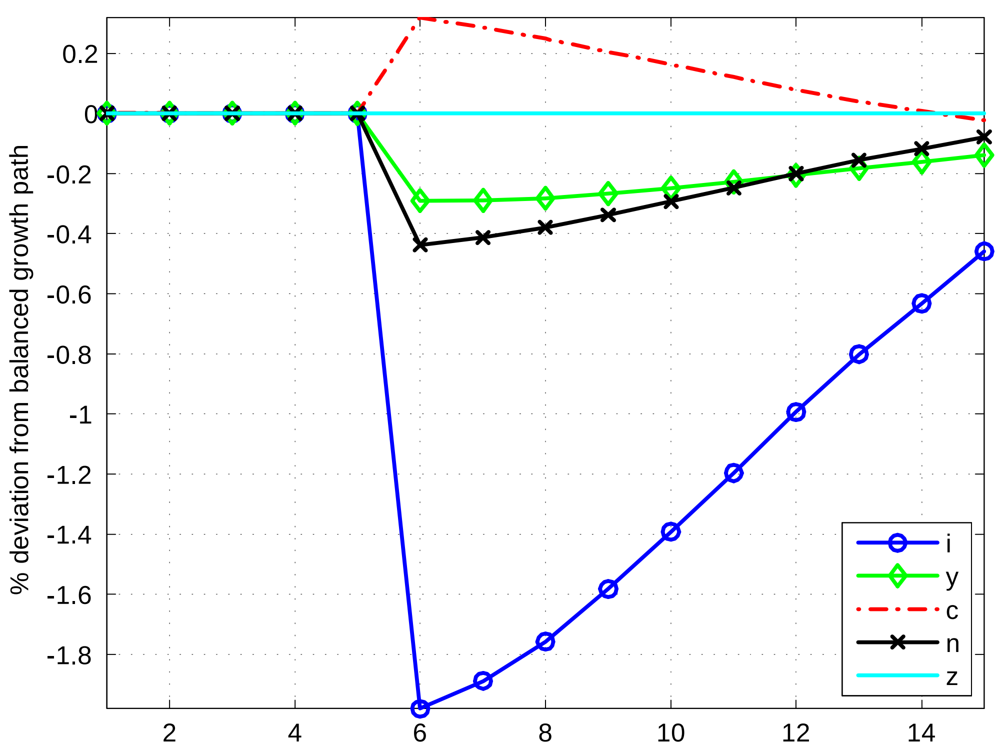

Figure 1: **The effect of an increase in the probability of disaster on macroeconomic quantities**. Impulse response of (I,Y,C,N,Z) to a shock to the probability of disaster at $t = 6$. Time (x-axis) is in quarters. The probability of disaster goes from its long-run average (0.425% per quarter) to twice its long-run average then mean-reverts according to its AR(1) law of motion. For clarity, this figure assumes that there is no shock to TFP, and no disaster realized.

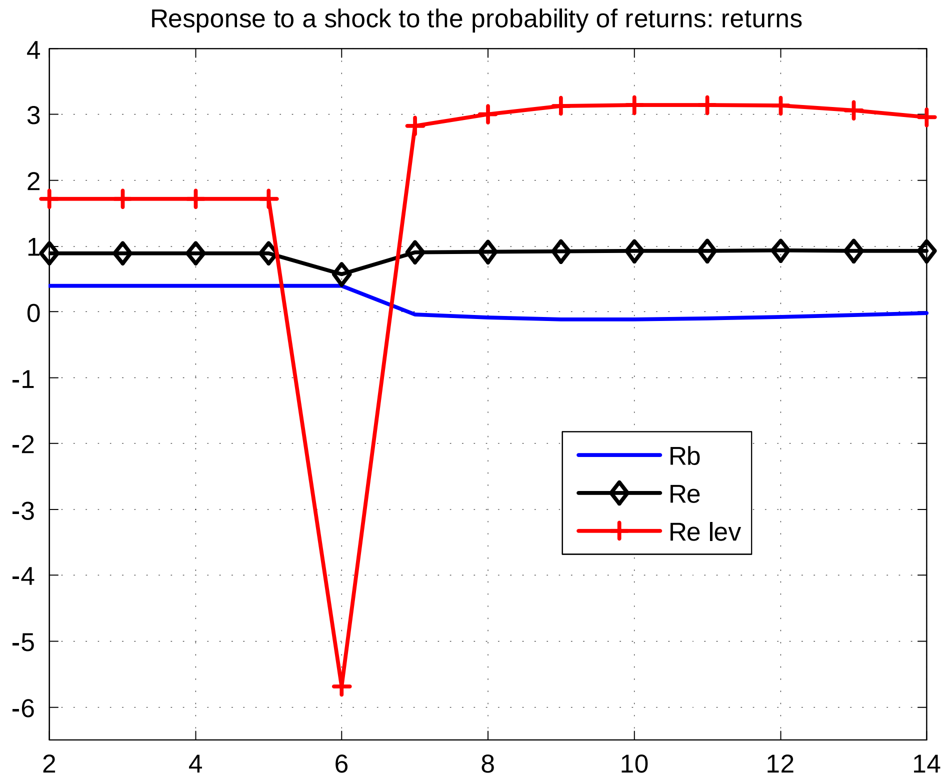

Figure 2: **The effect of an increase in the probability of disaster on asset returns and spreads**. Impulse response of asset returns to a shock to the probability of disaster at $t = 6$. Time (x-axis) is in quarters. The probability of disaster doubles at $t = 6$, starting from its long-run average. The figure plots the short-term government bond return, the equity return, and the levered equity return.

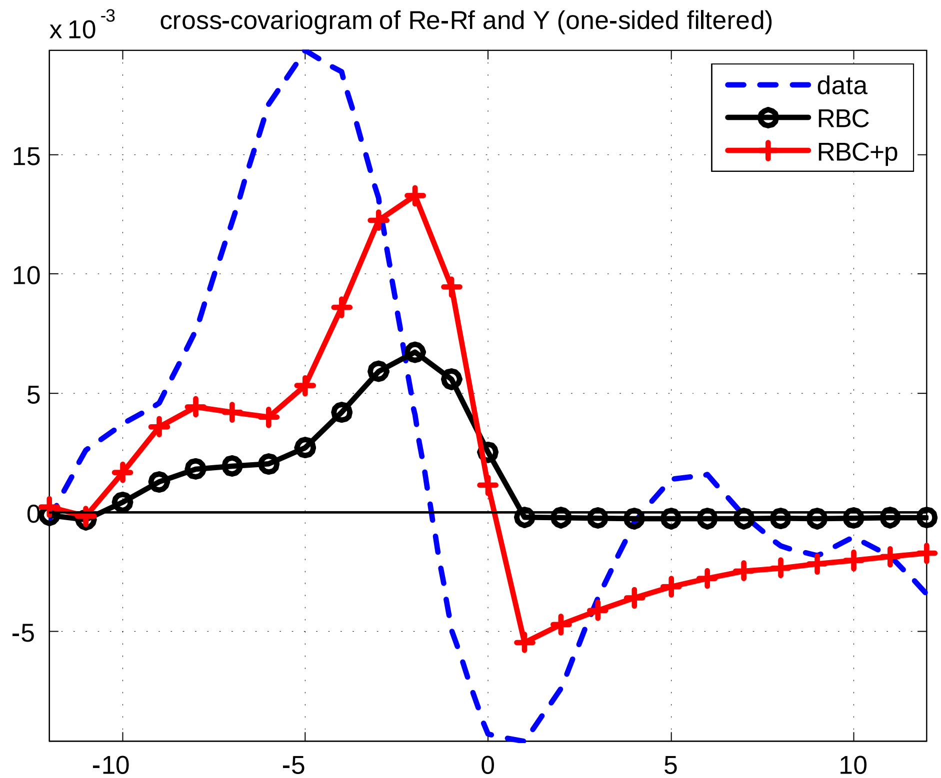

Figure 3: **Cross-covariogram of GDP and excess stock returns in the model and in the data.** Cross-covariogram of the (one-sided Baxter-King filtered) log GDP, and excess stock returns, in the data (blue dashed line), the RBC model, i.e. the model with only TFP shocks, (black circles) and the benchmark model with both p-shocks and TFP shocks (red crosses). The lag/lead (x-axis) is in quarters. The model covariograms are obtained by running 1000 simulations of length 200 each, and averaging.

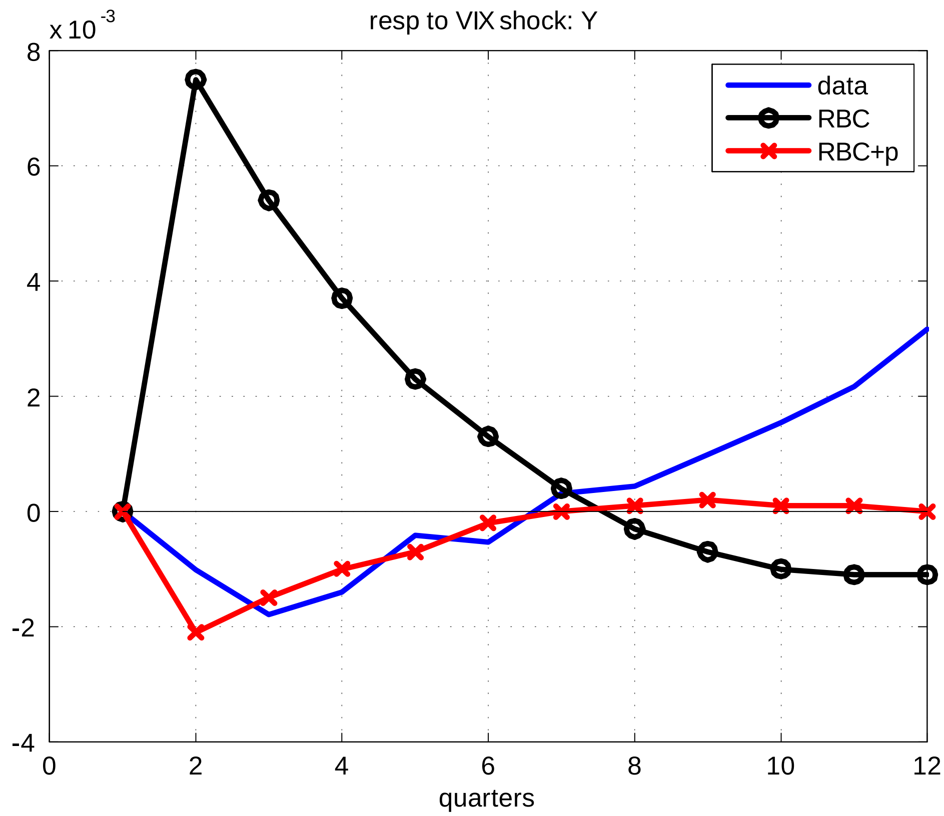

Figure 4: **Impulse response to a shock to VIX in a bivariate (VIX,GDP) VAR, in the model and in the data.** This figure gives the IRF to a one-standard deviation shock to VIX, in a bivariate VAR of HP-filtered log GDP, and HP-filtered VIX, in the data (blue full line), the RBC model, i.e. the model with only TFP shocks (black circles), and the benchmark model with both p-shocks and TFP shocks (red crosses). The model IRFs are obtained by running 1000 simulations of length 200 each, running the VAR on each simulation, and averaging. Orthogonalization assumption: GDP doesn't react to a VIX shock at $t = 0$.

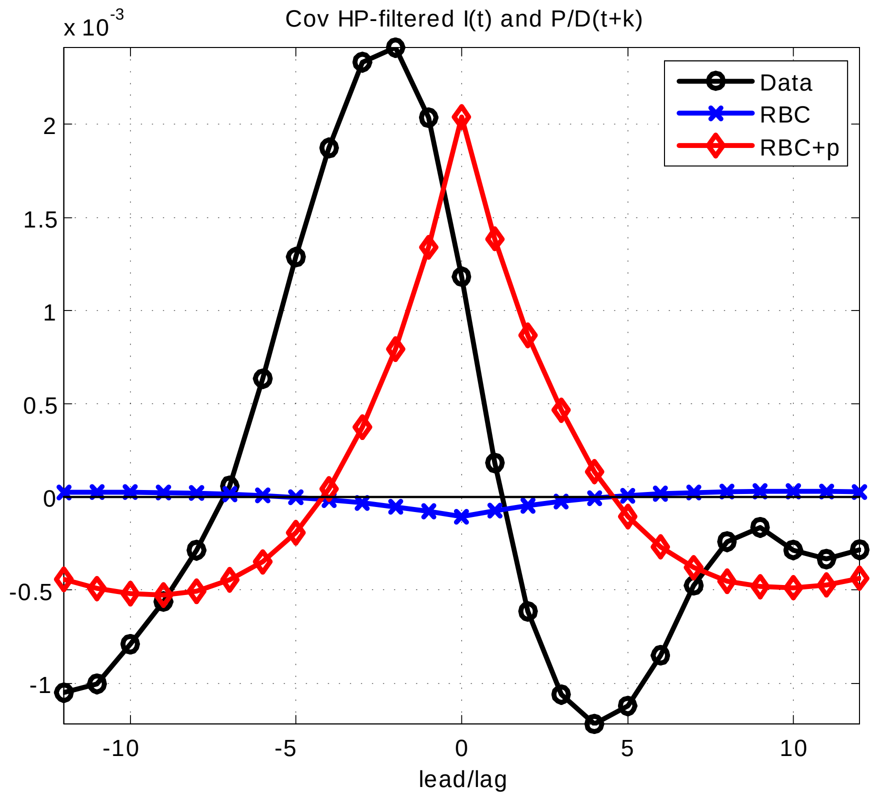

Figure 5: **Cross-covariogram of investment and P-D ratio in the model and in the data.** Cross-covariogram of the log HP filtered investment, and the HP filtered P-D ratio, in the data (black diamond line), in the RBC model, i.e. the model with only TFP shocks (blue crosses), and the benchmark model with both p-shocks and TFP shocks (red diamonds). The model covariograms are obtained by running 1000 simulations of length 200 each, and averaging.

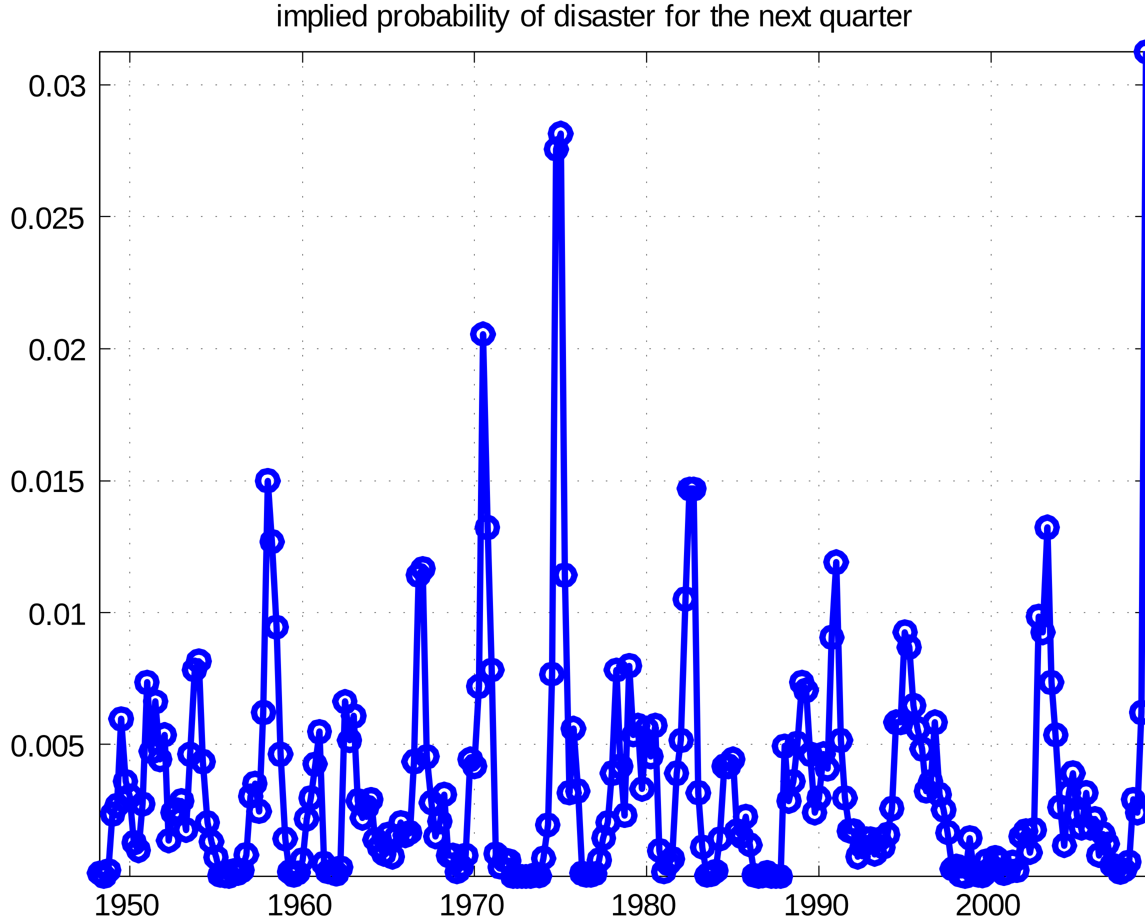

Figure 6: **Time-series for the quarterly probability of disaster (1948q1 to 2008q4).** This picture plots p_t, as implied by the model given the observed price-dividend ratio from CRSP and the measured capital stock and TFP (see section 6).

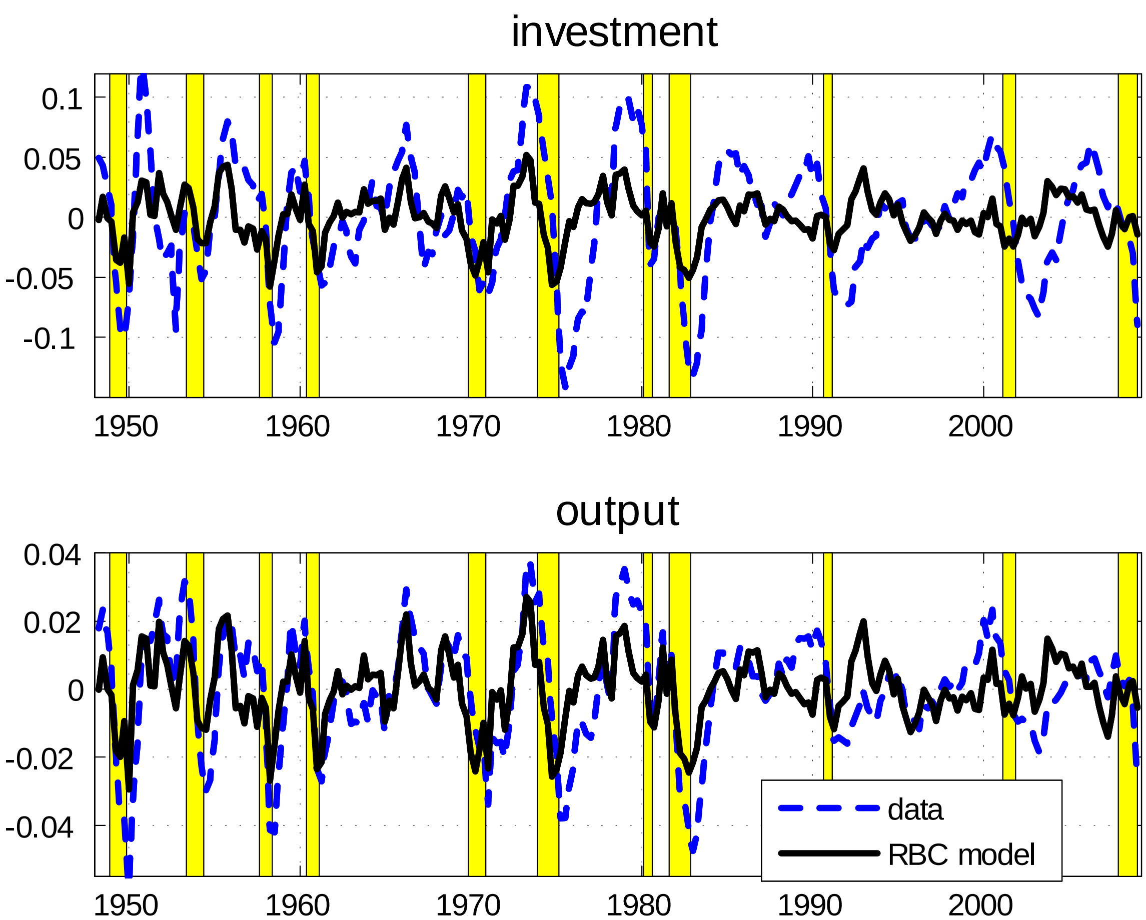

Figure 7: **Time-series of investment and output, in the data and in the RBC model.** This picture plots the data and the model-implied time series for macro aggregates for the RBC model (when TFP is fed into the model). All series are logged and HP-filtered, 1947q1-2008q4.

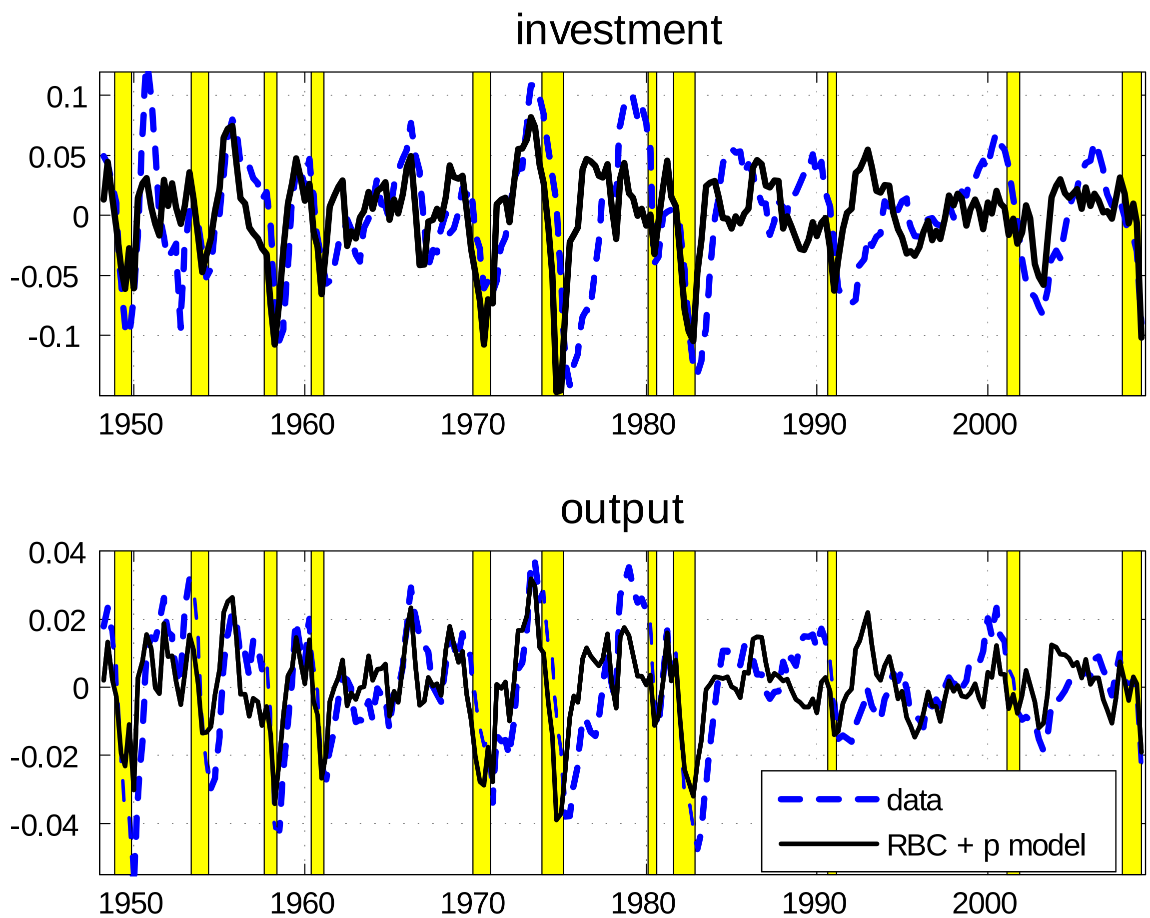

Figure 8: **Time-series of investment and output, in the data and in the benchmark model.** This picture plots the data and the model-implied time series for investment and output for the benchmark model (when both TFP and the disaster probability is fed into the model). All series are logged and HP-filtered, 1947q1-2008q4.

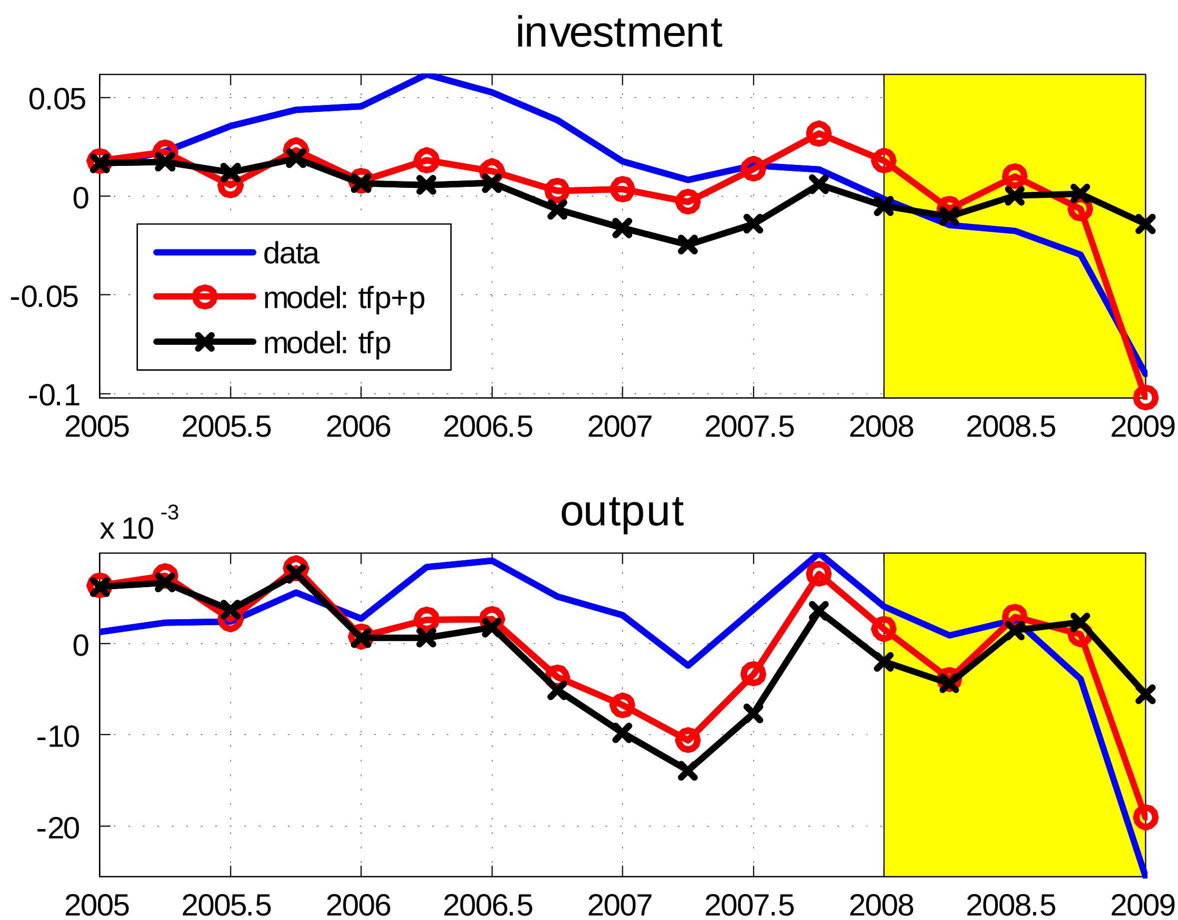

Figure 9: **Time-series of investment and output, in the data, in the RBC model, and in the benchmark model, from 2005q1 to 2008q4.** This picture is a zoomed-in version of the previous two figures, and displays the data, the model-implied time series for macro aggregates for the RBC model (when TFP is fed into the model) and for the benchmark model (when both TFP and the probability of disaster are fed into the model). See section 6 for details. All series are logged and HP-filtered, over 1947q1-2008q4, then cut from 2005q1 onwards.

Appendix to "Disaster Risk and Business Cycles"

NOT FOR PUBLICATION

March 2010

1 Euler equation approach

The proof of the results is based on the Bellman equation of the planner's problem. An alternative proof, which is also useful for intuition, uses Euler equations. For simplicity I show how this approach works in the special case of expected utility and no adjustment costs. In this case, the realized return on capital is

$$R_{t+1}^K = (1 - x_{t+1} b_k)\left(1 - \delta + F_1\left(K_{t+1}, z_{t+1} N_{t+1}\right)\right),$$

and the equilibrium of the economy is characterized by the four equations:

$$U_1(C_t, N_t) = \beta E_t\left(R_{t+1}^K U_1(C_{t+1}, N_{t+1})\right), \quad (1)$$

$$K_{t+1} = ((1-\delta) K_t + I_t)(1 - x_{t+1} b_k), \quad (2)$$

$$U_2(C_t, N_t) = z_t F_2\left(K_t, z_t N_t\right) U_1\left(C_t, N_t\right), \quad (3)$$

$$C_t + I_t = F(K_t, z_t N_t), \quad (4)$$

with $U(C, N) = \frac{C^{1-\sigma}}{1-\sigma} v(N)$, plus the equations characterizing the shock process $\{\varepsilon_{t+1}, x_{t+1}, p_{t+1}\}$:

$$\begin{aligned}
\log z_{t+1} &= \log z_t + \mu + \sigma \varepsilon_{t+1} + x_{t+1} \log(1 - b_{tfp}), \\
x_{t+1} &= 1 \text{ w/prob } p_t, \text{ and } 0 \text{ w/prob } 1 - p_t, \\
&\quad p_t \text{ Markov with transition } T.
\end{aligned}$$

One can then guess and verify that the solution can be rewritten as $C_t = c_t z_t$, $K_t = k_t z_t$, $I_t = i_t z_t$, $Y_t = y_t z_t$, which validates proposition 1. To obtain proposition 3, note that risk of disaster affects the equilibrium only through equation (1). Given proposition 1, we can decompose the expectation, between disaster states and non-disaster states:

$$\begin{aligned}
U_1(C_t, N_t) &= \beta(1 - p_t) E_{p_{t+1}, \varepsilon_{t+1}}^{\text{no disaster}}\left(1 - \delta + F_1\left(K_{t+1}, z_{t+1} N_{t+1}\right)\right) U_1(C_{t+1}, N_{t+1}) \\
&\quad + p_t\left(1 - b_k\right) E_{p_{t+1}, \varepsilon_{t+1}}^{\text{disaster}}\left(\left(1 - \delta + F_1\left(K_{t+1}, z_{t+1} N_{t+1}\right)\right) U_1(C_{t+1}, N_{t+1})\right) \\
&= \beta(1 - p_t + p_t\left(1 - b_k\right)^{1-\gamma}) E_{p_{t+1}, \varepsilon_{t+1}}^{\text{no disaster}}\left(1 - \delta + F_1\left(K_{t+1}, z_{t+1} N_{t+1}\right)\right) U_1(C_{t+1}, N_{t+1}), \quad (5)
\end{aligned}$$

where $E^{\text{no disaster}}_{p_{t+1},\varepsilon_{t+1}}$ is the expectation over $p_{t+1}, \varepsilon_{t+1}$, and conditional on $x_{t+1} = 0$ i.e. no disaster. This equation implies proposition 3: the model is equivalent to a model without disaster, but with a time-varying discount rate $\beta(p_t) = \beta(1 - p_t + p_t (1 - b_k)^{1-\gamma})$.

The risk-free rate is then $1/Q_t^{(1)}$, where

$$\begin{aligned} Q_t^{(1)} &= E_t \left(\frac{\beta U_1(C_{t+1}, N_{t+1})}{U_1(C_t, N_t)} \right) \\ &= (1 - p_t + p_t(1 - b_k)^{-\gamma}) E^{\text{no disaster}}_{p_{t+1},\varepsilon_{t+1}} \frac{\beta U_1(C_{t+1}, N_{t+1})}{U_1(C_t, N_t)}, \end{aligned}$$

which shows explicitely that the risk-free rate always falls when p_t rises, and illustrates the difference between the risk-free rate behavior and the expected equity return:

$$E_t \left(R^K_{t+1}\right) = (1 - p_t b_k) E^{\text{no disaster}}_{p_{t+1},\varepsilon_{t+1}} \left(1 - \delta + F_1 \left(K_{t+1}, z_{t+1} N_{t+1}\right)\right).$$

2 Government Policy

To prove the result, we follow the previous section of the appendix. Suppose that the government engineers a state-contingent subsidy ζ, then the return on capital is:

$$R^K_{t+1} = \left(1 + \zeta x_{t+1}\right) \left(1 - x_{t+1} b_k\right) \left(1 - \delta + F_1 \left(K_{t+1}, z_{t+1} N_{t+1}\right)\right),$$

and the equilibrium of the economy is characterized by the four equations:

$$\begin{aligned} U_1(C_t, N_t) &= \beta E_t \left(R^K_{t+1} U_1(C_{t+1}, N_{t+1})\right), \\ K_{t+1} &= \left((1 - \delta) K_t + I_t\right) \left(1 - x_{t+1} b_k\right), \\ U_2(C_t, N_t) &= z_t F_2 \left(K_t, z_t N_t\right) U_1 \left(C_t, N_t\right), \\ C_t + I_t &= F(K_t, z_t N_t). \end{aligned}$$

We can then rewrite the Euler equation:

$$\begin{aligned} U_1(C_t, N_t) &= \beta(1 - p_t) E^{\text{no disaster}}_{p_{t+1},\varepsilon_{t+1}} \left(1 - \delta + F_1 \left(K_{t+1}, z_{t+1} N_{t+1}\right)\right) U_1(C_{t+1}, N_{t+1}) \\ &\quad + p_t \left(1 - b_k\right) \left(1 + \zeta\right) E^{\text{disaster}}_{p_{t+1},\varepsilon_{t+1}} \left(\left(1 - \delta + F_1 \left(K_{t+1}, z_{t+1} N_{t+1}\right)\right) U_1(C_{t+1}, N_{t+1})\right) \\ &= \beta(1 - p_t + p_t \left(1 - b_k\right)^{1-\gamma} (1 + \zeta)) E^{\text{no disaster}}_{p_{t+1},\varepsilon_{t+1}} \left(1 - \delta + F_1 \left(K_{t+1}, z_{t+1} N_{t+1}\right)\right) U_1(C_{t+1}, N_{t+1}), \end{aligned}$$

This formula reveals that setting $1 + \zeta = (1 - b_k)^{\gamma - 1}$ leads to the disappearance of this wedge from the Euler equation, and the decision rules are then the same as in a model without disaster risk.

3 A simple method to solve models with time-varying risk premia

The model in this paper is simple enough that a full nonlinear solution is easy and relatively fast to implement. However, many interesting extensions require to add more state variables.[1] This makes a

[1] Also, many interesting extensions may violate the welfare theorems, making it impossible to use a social planner approach.

full non-linear solution difficult. Proposition 3 of the paper, however, suggest a simple two-step solution method: under the assumption that $b_k = b_{tfp}$, the model has the same implications for quantities than a model without disaster, but with a different, and time-varying discount factor β. As we know this model can be solved precisely using a standard first-order linearization. Hence, the proposed methodology is as follows:

(1) Define the average value of β, call it $\widetilde{\beta} = \int \beta(p) d\mu(p) = \int \beta (1 - p + p(1 - b_k)^{\upsilon(1-\theta)})^{\frac{1-\gamma}{1-\theta}} d\mu(p)$ where μ is the invariant distribution of p.

(2) Fit a linear time series process for $\beta(p)$. This is an approximation since the true process may not be linear.

(3) Solve for the nonstochastic steady-state, with discount factor $\widetilde{\beta}$.

(4) Linearize (or log-linearize) the model around this steady-state, using the approximate process for β found in (2). This solves for the quantities.

(5) Solve for asset prices using the standard recursions. For simplicity I present them in the expected utility case:

(a) Equity paying out a process $D_t = z_t^\lambda d(k_t, p_t)$; the price-dividend ratio $f(p_t, k_t) = P_t / D_t$ satisfies:

$$\begin{aligned} f(p,k) &= (1 - p + p(1-b)^{\lambda + \upsilon(1-\gamma)-1}) \\ &\times \int_{\varepsilon'} \int_{p'} e^{(\lambda+1-\gamma)(\mu+\sigma\varepsilon')} \frac{\beta u_1(c(k',p'), N(k',p'))}{u_1(c(k,p), N(k,p))} \frac{d(k',p')}{d(k,p)} \left(f(p',k')+1\right) dT(p'|p) dH(\varepsilon'), \end{aligned}$$

where H is the c.d.f. of the normal shock ε, and T is the transition function of the process $\{p_t\}$.

(b) Yield curve: the price of a zero-coupon bond maturing in n periods satisfies:

$$\begin{aligned} q^{(n)}(k,p) &= \left(1 - p + rp(1-b)^{\upsilon(1-\gamma)-1}\right) \\ &\times \int_{\varepsilon'} \int_{p'} e^{(1-\gamma)(\mu+\sigma\varepsilon')} \frac{\beta u_1(c(k',p'), N(k',p'))}{u_1(c(k,p), N(k,p))} q^{(n-1)}(k',p') dH(\varepsilon') dT(p'|p), \end{aligned}$$

with $q^{(0)}(k,p) = 1$ and $u(C,N) = \frac{(C^\upsilon N^{1-\upsilon})^{1-\gamma}}{1-\gamma}$. This simplification relies on this form of preferences, which are consistent with balanced growth. These two recursions can be solved given the policy functions $c(k,p)$, $N(k,p)$, and $d(k,p)$ which obtained in step (4).

With recursive utility, the same procedure works, but one also needs to compute the value function, since it appears in the stochastic discount factor and hence in the asset price recursions. The value function can be computed easily once the policy functions are known, or it can be computed inexpensively using second-order perturbation methods as in Fernandez-Villaverde et al. (2008).

In general, for any given model, one should check that propositions 2 and 3 hold. However in several models they do hold, hence this opens the door to solving medium-scale macro model with time-varying risk premia.

4 Computational Method

This method used in the paper is presented for the case of a Cobb-Douglas production function, and a Cobb-Douglas utility function, but it can be used for arbitrary homogeneous of degree one production function and utility function. Also, this presentation does not allow for correlation between p and ε, and for recoveries, but it is straightforward to modify the method to allow for this.

The Bellman equation for the "rescaled" problem is:

$$
\begin{aligned}
g(k,p) &= \max_{c,i,N} \left\{ \begin{array}{c} c^{\upsilon(1-\gamma)}(1-N)^{(1-\upsilon)(1-\gamma)} \\ +\beta e^{\mu\upsilon(1-\gamma)} \left(E_{p',\varepsilon',x'} e^{\sigma\varepsilon'\upsilon(1-\theta)} (1-x'b_{tfp})^{\upsilon(1-\theta)} g(k',p')^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}} \end{array} \right\}, \\
s.t. &: \\
c+i &= k^{\alpha}N^{1-\alpha}, \\
k' &= \frac{(1-x'b_k)\left((1-\delta)k+\phi\left(\frac{i}{k}\right)k\right)}{e^{\mu+\sigma\varepsilon'}(1-x'b_{tfp})}.
\end{aligned}
$$

Because we take a power $\frac{1}{1-\gamma}$ of the value function, the max needs to be transformed in a min if $\gamma > 1$. To approximate numerically the solution of this problem, I proceed as follows:

(1) Pick a grid for k, and a grid for i, and approximate the process for p with a Markov chain with transition matrix T. Discretize the normal shock ε, with probabilities $\pi(\varepsilon)$. I used 120 points for the grid for k, 1200 points for the grid for i, and 5 points for the grid for ε. Finally T is picked as in Rouwenhorst (1995). That method is a better approximation for highly persistent processes. I used 9 points for the grid for p.

(2) Compute for any k, i in the grid the value $N(k,i)$ which solves

$$R(k,i) = \max_N \left(k^{\alpha}N^{1-\alpha} - i\right)^{\upsilon}(1-N)^{(1-\upsilon)}.$$

(3) The state space and action space are now discrete, so this is a standard discrete dynamic programing problem, which can be rewritten as follows, with one endogenous state, one exogenous state, and two additional shocks: a binomial variable x equal to one if a disaster occurs (with probability p) and the normal shock ε:

$$
\begin{aligned}
g(k,p) &= \max_{i} \left\{ \begin{array}{c} R(k,i) + \beta e^{\mu\upsilon(1-\gamma)} \times ... \\ \left(\sum_{p',\varepsilon',x'\in\{0,1\}} \pi(\varepsilon')T(p,p') e^{\sigma\varepsilon'\upsilon(1-\theta)} pr(x',p) \left(1-x'+x'(1-b_{tfp})^{\upsilon(1-\theta)}\right) g(k',p')^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}} \end{array} \right\}, \\
s.t. &: \\
k' &= \frac{(1-x'b_k)\left((1-\delta)k+\phi\left(\frac{i}{k}\right)k\right)}{e^{\mu+\sigma\varepsilon'}(1-x'b_{tfp})},
\end{aligned}
$$

$$
g(k,p) = \max_{i} \left\{ \begin{array}{c} R(k,i) + \beta e^{\mu\upsilon(1-\gamma)} \times ... \\ \left(\sum_{p',\varepsilon',x'\in\{0,1\}} \pi(\varepsilon')T(p,p') e^{\sigma\varepsilon'\upsilon(1-\theta)} pr(x',p) \left(1-x'+x'(1-b_{tfp})^{\upsilon(1-\theta)}\right) g(k',p')^{\frac{1-\theta}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\theta}} \end{array} \right\}
$$

where $pr(x',p) = p1_{x'=1} + (1-p)\,1_{x'=0} = px' + (1-p)(1-x')$. I solve this Bellman equation using modified policy iteration[2] (Judd (1998), p. 416), starting with a guess value close to zero. Recursive utility implies that the Blackwell sufficient conditions do not hold here, hence it is not obvious that the Bellman operator is a contraction. However, convergence occurs in practice as long as β and the probability of disasters are not too large. Note that to compute the expectation, we need the value

[2] This turns out to be significantly faster than value iteration for this application.

function outside the grid points. I use linear interpolation in the early steps of the iteration, then switch to spline interpolation. The motivation is that linear interpolation is more robust, hence it is easier to make the iterations converge; but spline interpolation is more precise.

(4) Given g, we have $V(K,z,p) = z^{\upsilon} g(k,p)^{\frac{1}{1-\gamma}}$. We also obtain the policy functions $C = zc(k,p)$, $I = zi(k,p)$, $N = N(k,p)$, and the output policy function $Y = zk^{\alpha}N(k,p)^{1-\alpha}$. Because these policy functions are defined on a discrete grid, I use interpolation in the simulations and impulse responses to obtain more accurate results. (Linear or spline interpolations yield nearly the same results.)

(5) To compute asset prices, we need the stochastic discount factor, which is given by the standard formula:

$$M_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{\upsilon(1-\gamma)-1} \left(\frac{1-N_{t+1}}{1-N_t}\right)^{(1-\upsilon)(1-\gamma)} \left(\frac{V_{t+1}}{E_t\left(V_{t+1}^{1-\theta}\right)^{\frac{1}{1-\theta}}}\right)^{\gamma-\theta}.$$

Using homogeneity, the SDF between two states $s = (k,p)$ and $s' = (k',p')$ is:

$$\begin{aligned} M(s,s',\varepsilon',x') &= \beta \left(\frac{z'c(k',p')}{zc(k,p)}\right)^{\upsilon(1-\gamma)-1} \left(\frac{1-N(k',p')}{1-N(k,p)}\right)^{(1-\upsilon)(1-\gamma)} \left(\frac{z'^{\upsilon} g(k',p')^{\frac{1}{1-\gamma}}}{E_{z',p',x'}\left(z'^{\upsilon(1-\theta)} g(k',p')^{\frac{1-\theta}{1-\gamma}}\right)^{\frac{1}{1-\theta}}}\right)^{\gamma-\theta} \\ &= \beta \left(\frac{z'}{z}\right)^{(\gamma-\theta)\upsilon+\upsilon(1-\gamma)-1} \left(\frac{c(k',p')}{c(k,p)}\right)^{\upsilon(1-\gamma)-1} \times ... \\ &\quad ... \left(\frac{1-N(k',p')}{1-N(k,p)}\right)^{(1-\upsilon)(1-\gamma)} \left(\frac{g(k',p')^{\frac{1}{1-\gamma}}}{E_{z',p',x'}\left(\left(\frac{z'}{z}\right)^{\upsilon(1-\theta)} g(k',p')^{\frac{1-\theta}{1-\gamma}}\right)^{\frac{1}{1-\theta}}}\right)^{\gamma-\theta}. \end{aligned}$$

Note that we first need to compute the conditional expectation which appears on the denominator of the last term. Denote $k' = j(k,p,\varepsilon',x')$ the detrended capital next period, wich depends on the detrended investment $i(k,p)$ and on the realization of the shocks next period ε' and x' (but not p'). The conditional expectation is obtained as:

$$E_{z',p',x'}\left(\left(\frac{z'}{z}\right)^{\upsilon(1-\theta)} g(k',p')^{\frac{1-\theta}{1-\gamma}}\right)^{\frac{1}{1-\theta}} =$$

$$= \sum_{p',\varepsilon'} T(p,p')\Pr(\varepsilon')e^{\upsilon(1-\theta)\mu+\upsilon(1-\theta)\sigma\varepsilon'}\left(p(1-b_{tfp})^{\upsilon(1-\theta)}g(j(k,p,\varepsilon',1),p')^{\frac{1-\theta}{1-\gamma}} + (1-p)g(j(k,p,\varepsilon',0),p')^{\frac{1-\theta}{1-\gamma}}\right).$$

(6) We can now obtain the price of a one-period asset, with payoff $d(k',z',p',x',\varepsilon')$. e.g. a pure risk-free asset $d = 1$, or a short-term government bond: $d = 1 - q(1-r)x'$, as

$$P(k,p) = E_{x',p',\varepsilon'}M(s,s',x',\varepsilon')d(k',z',p',x',\varepsilon').$$

For instance, for a pure risk-free asset, the formula is:

$$E_t M_{t,t+1} = \frac{\begin{array}{c} \beta\sum_{p'}\sum_{\varepsilon'} T(p,p')\Pr(\varepsilon')e^{((\gamma-\theta)\upsilon+\upsilon(1-\gamma)-1)\left(\mu+\sigma\varepsilon'\right)} \times ... \\ ... \left(\begin{array}{c} p(1-b_{tfp})^{(\gamma-\theta)\upsilon+\upsilon(1-\gamma)-1}c(j(k,p,\varepsilon',1),p')^{\upsilon(1-\gamma)-1}\times \\ (1-N(j(k,p,\varepsilon',1),p'))^{(1-\upsilon)(1-\gamma)}\, g(j(k,p,\varepsilon',1),p')^{\frac{\gamma-\theta}{1-\gamma}} \\ +(1-p)c(j(k,p,\varepsilon',0),p')^{\upsilon(1-\gamma)-1}\times \\ (1-N(j(k,p,\varepsilon',0),p'))^{(1-\upsilon)(1-\gamma)}\, g(j(k,p,\varepsilon',0),p')^{\frac{\gamma-\theta}{1-\gamma}} \end{array}\right) \end{array}}{c(k,p)^{\upsilon(1-\gamma)-1}\left(1-N(k,p)\right)^{(1-\upsilon)(1-\gamma)} E_{z',p',x'}\left(\left(\frac{z'}{z}\right)^{\upsilon(1-\theta)} g(k',p')^{\frac{1-\theta}{1-\gamma}}\right)^{\frac{\gamma-\theta}{1-\theta}}}.$$

(7) Next, we can obtain the term structure of interest rates on government bonds, using the recursion:

$$Q_n(k,p) = E_{x',p',\varepsilon'}\left(M(s,s',x',\varepsilon')\left((1+x'(r-1))\,Q_{n-1}(k',p')\right)\right).$$

(8), Finally, we can compute the price of equity claims. I consider three types of “equities”. Define the representative firm’s dividend as earnings less investment:

$$D_t = Y_t - w_t N_t - I_t.$$

The first equity is simply a claim to the stream $\{D_t\}$. Let P_t denote its price, which satisfies the standard recursion:

$$P_t = E_t\left(M_{t,t+1}\left(P_{t+1} + D_{t+1}\right)\right).$$

Note that D_t can be written as $D_t = z_t d(k_t, p_t)$, where $d(k,p) = \alpha k^{\alpha} N^{1-\alpha} - i(k,p)$. Hence, we can rewrite the firm value recursion as:

$$zf(k,p) = E_{s'|s}\left(M(s,s') \times \left(z'd(k',p') + z'f(k',p')\right)\right), \tag{6}$$

where $P_t = z_t f(k_t,p_t)$. The equity return is then

$$\begin{aligned} R^e_{t+1} &= \frac{P_{t+1}+D_{t+1}}{P_t} \\ &= \frac{z_{t+1}}{z_t}\frac{f(k_{t+1},p_{t+1})+d(k_{t+1},p_{t+1})}{f(k_t,p_t)}. \end{aligned} \tag{7}$$

To solve the recursion 6 in practice, I iterate starting with an initial guess $f(k,p) = 0$. The recursion can be rewritten as:

$$\begin{aligned} f(k,p) &= E_{s'|s}\left(M(s,s')\frac{z'}{z}\left(d(k',p')+f(k',p')\right)\right) \\ &= \beta E_{p',\varepsilon',x'}\left[\begin{array}{c} \left(\frac{z'}{z}\right)^{(\gamma-\theta)\upsilon+\upsilon(1-\gamma)} \left(\frac{c(k',p')}{c(k,p)}\right)^{\upsilon(1-\gamma)-1} \left(\frac{1-N(k',p')}{1-N(k,p)}\right)^{(1-\upsilon)(1-\gamma)} \times \ldots \\ \left(\frac{g(k',p')^{\frac{1}{1-\gamma}}}{E_{z',p',x'}\left(\left(\frac{z'}{z}\right)^{\upsilon(1-\theta)} g(k',p')^{\frac{1-\theta}{1-\gamma}}\right)^{\frac{1}{1-\theta}}}\right)^{\gamma-\theta} \left(d(k',p')+f(k',p')\right) \end{array}\right]. \end{aligned}$$

This conditional expectation can be written down, as

$$f(k,p) = \frac{\begin{array}{c} \beta\sum_{p'}\sum_{\varepsilon'} T(p,p')\Pr(\varepsilon')e^{((\gamma-\theta)\upsilon+\upsilon(1-\gamma))(\mu+\sigma\varepsilon')} \times \ldots \\ \left(\begin{array}{c} p(1-b_{tfp})^{(\gamma-\theta)\upsilon+\upsilon(1-\gamma)} c(j(k,p,\varepsilon',1),p')^{\upsilon(1-\gamma)-1}\left(1-N(j(k,p,\varepsilon',1),p')\right)^{(1-\upsilon)(1-\gamma)} \times \\ g(j(k,p,\varepsilon',1),p')^{\frac{\gamma-\theta}{1-\gamma}}\left(d(j(k,p,\varepsilon',1),p')+f(j(k,p,\varepsilon',1),p')\right) \\ +(1-p)c(j(k,p,\varepsilon',0),p')^{\upsilon(1-\gamma)-1}\left(1-N(j(k,p,\varepsilon',0),p')\right)^{(1-\upsilon)(1-\gamma)} \times \\ g(j(k,p,\varepsilon',0),p')^{\frac{\gamma-\theta}{1-\gamma}}\left(d(j(k,p,\varepsilon',0),p')+f(j(k,p,\varepsilon',0),p')\right) \end{array}\right) \end{array}}{c(k,p)^{-\gamma}E_{z',p',x'}\left(\left(\frac{z'}{z}\right)^{\upsilon(1-\theta)} g(k',p')^{\frac{1-\theta}{1-\gamma}}\right)^{\frac{\gamma-\theta}{1-\theta}}}.$$

Note that the return computed by solving this recursion for firm value can be compared to the expression obtained by the investment return. The two must give the same result (if the investment positivity constraint never binds). In practice it is a useful numerical check to perform.

Finally, The levered equity assumes that the payoff streams is $\{Y_t^\lambda\}$. It is easy to adapt the same method to price the claims to these assets. Finally, I obtain the model statistics by simulating 1000 samples of length 400, started at the nonstochastic steady-state, and cutting off the first 200 periods. The Matlab(c) programs are available on my web page.

5 Data Sources

Section 4: Business cycle and return moments of Tables 2-7: consumption is nondurable + services consumption, investment is fixed investment, and output is GDP, from the NIPA Table 1.1.3, quarterly data 1947q1-2008q4. Hours is nonfarm business hours from the BLS productivity program (through FRED: HOABNS). The return data is from Ken French's webpage, (benchmark factors, aggregated to quarterly frequency, and deflated by the CPI (CPIAUCSL through FRED)).

Predictability regressions: the data is taken from John Cochrane's webpage (Cochrane, 2008) and the regression results replicate exactly his specification (for the simple regression of return on dividend yield last year). This is also the CRSP value weighted return, and the risk-free rate is from Ibottson.

VIX data: he data on volatility starts in 1963, reducing the sample for the VARs using volatility or VIX.

Section 6: the price-dividend ratio is from CRSP. Similar results are obtained using the Shiller data. As is standard in the literature, I divided the price by the sum of the last four quarters of dividends, given the substantial seasonality of dividends. Because of changes over the sample period in the distribution of equity payout between dividends and share repurchases, there are lower-frequency movements in the price-dividend ratio. Hence, I first removed these low-frequency movements using a HP filter with $\lambda = 3000$ (The smoothing parameter does not affect the results significantly, provided it is above 1600).The capital stock is obtained from the annual fixed assets tables, and is interpolated. TFP is computed as output divided by labor to the power 2/3 and capital to the power 1/3.