

# The Browser Wars – Econometric Analysis of Markov Perfect Equilibrium in Markets with Network Effects

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ABSTRACT: When demands for heterogeneous goods in a concentrated market shift over time due to network effects, forward-looking firms consider the strategic impact of investment, pricing, and other conduct. A Markov perfect equilibrium model captures this strategic behavior, and permits the comparison of “as is” market trajectories with “but for” trajectories under counterfactuals where some “bad acts” by firms are eliminated. We give conditions for econometric identification and estimation of a Markov perfect equilibrium model from observations on partial trajectories, and discuss estimation of the impacts of firm conduct on consumers and rival firms. Our analysis is applied to a stylized description of the browser war between Netscape and Microsoft.

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# **The Browser Wars – Econometric Analysis of Markov Perfect Equilibrium in Markets with Network Effects**

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*“In the long run, there is just another short run.”*

*Abba Lerner*

## **I. Introduction**

Concentrated markets for heterogeneous goods are often linked through time through demand or supply. Demand-side effects come from habit, inertia (switching costs), contagion (bandwagon) effects, or network (connectivity, interoperability) externalities. Supply-side effects come from investments in production capacity, product quality, and brand promotion. Forward-looking firms in such markets consider the strategic impact of their actions, and seek to maximize the expected present value of profits given the information they have and their beliefs about the behavior of other market participants. Quantitative analysis of firm conduct in such markets requires dynamic models that predict market trajectories. The theory of *Markov Perfect Equilibrium* (MPE) introduced by Maskin and Tirole (1987, 1988a, 1988b), and developed as an analytic tool for industrial organization applications by Ericson and Pakes (1995), Pakes and McGuire (1994), and others, captures strategic behavior in evolving markets, and provides a platform for econometric analysis of dynamic market structure. This theory can be used to quantify the impacts on market participants of “bad acts” by a firm. This paper discusses the application of MPE to damage estimation in anti-trust litigation, and addresses some of the theoretical, computational, and econometric

issues that arise. Our analysis is applied to the browser wars between Netscape and Microsoft, and gives a framework for estimating the damages to Netscape caused by Microsoft “bad acts”.

The paper is organized as follows. Section 2 discusses the relationship between anti-trust litigation and dynamic models. Section 3 reviews the properties of MPE models, and discusses theoretical, computational, and econometric issues. Section 4 gives a capsule summary of the browser war between Netscape and Microsoft. Section 5 describes a simple MPE model designed to capture the most salient aspects of the browser war, and provide estimates of the “but for” development of this market absent Microsoft “bad acts”. Section 6 discusses computation. Section 7 gives specifics for the estimation strategy. Section 8 gives the results of the application. Section 9 gives conclusions.

## **2. Market Dynamics and Anti-Trust Law**

The fundamental goal of anti-trust law, in its current interpretation by the courts, is to promote market efficiency and protect the welfare of market participants by limiting the acquisition and exercise of market power. This is accomplished through statutes that provide *per se* limits on firm conduct, and give consumers and rival firms standing to sue for relief and damages from a firm’s “bad acts”. The acquisition and exercise of market power is a dynamic process in which erecting entry barriers, or eliminating rival firms through acquisition or predation, play out through time. Then, anti-trust law, and assessments of liability and damages based upon it, must account consistently for the dynamic evolution of concentrated markets under alternative rules for firm conduct.

Anti-trust law is not rooted in a comprehensive economic theory of concentrated markets, but its evolution has been influenced by economic analysis. Historically, the economic theory of concentrated markets has focused on short-run, or long-run steady state, market outcomes without explicit dynamics or complete models of demand behavior. The classic workable competition (Areeda, 1975), Cournot, Bertrand,

Stackelberg, and Kreps-Shenckman (1982) models of market conduct fit this mold. Two major developments in industrial organization offer the prospect of putting the anti-trust law on a sound dynamic foundation. The first is the adaptation from market research of models of demand behavior that describe consumer welfare and behavior in response to new products and changing product attributes, and capture network externalities and other demand-side dynamics; see Anderson-de Palma-Thisse (1996), Werden-Froeb (1994, 2002). The second is the vigorous development of the theory of dynamic stochastic games, particularly the MPE solution concept, and an expanding literature on its application to concentrated markets. The first of these developments has begun to influence anti-trust law through adoption of merger models that rely on analysis of consumer outcomes rather than mechanical measures of concentration (Kovacic-Shapiro, 2000; Werden and Froeb, 2002).

A more general reconsideration of the anti-trust law in light of modern economic analysis of demand behavior and market dynamics is timely. Demand-side dynamics is an important factor in most product markets, driving firm behavior on the positioning and launch of new products, and influencing their management of the product cycle. In the presence of such effects, firms conduct can have large strategic consequences. In some market circumstances, conduct that is illegal under current anti-trust law may increase market efficiency. For example, consumers may benefit in net from the ubiquity of proprietary product standards, even when they are promulgated to exclude competition. On the other hand, apparently minor announcement effects, exclusions, or delays that the law currently treats as inconsequential may be sufficient to tip a market with strong network demand or scale effects, and result in its monopolization. Even if the market remains contestable in principle, placing some limits on monopoly rents, the barriers created by these dynamic effects may be high enough to exclude entry by more innovative or potentially lower cost rivals.

The interpretation of firm behavior in the “as is” world, and projection of “but for” behavior when some acts are disallowed, should take into account the incentives created by the market dynamics, and the strategic elements in firm response. For example, consider the anti-trust charge of predatory pricing. Static analysis leads to the

Areeda-Turner standard that pricing must be below short-run marginal cost (or, loosely, average variable cost) to be predatory. Further, the law treats predation lightly, on the grounds that consumers benefit from low prices in the short run, and that predatory pricing is implausible behavior for a firm because the entry inviting a predatory response is likely to recur, removing the long-run benefit of forcing rivals to exit. However, when demand-side network effects, or supply-side scale or learning-by-doing effects, link the markets through time, neither the Areeda-Turner standard or the argument that predation is strategically unattractive necessarily hold, and the impacts of predation may be more dramatic, and the consequences for consumers more subtle, than the traditional analysis would suggest.

Another reason for making market dynamics explicit in anti-trust litigation is that the traditional focus of anti-trust cases on contemporaneous damages is inadequate when illegal conduct has strategic implications for market outcomes. In such cases, estimation of damages requires quantitative estimation of market trajectories under “as is” and “but for” conditions.

For the anti-trust law to work effectively as a regulator of resource allocation, it must be able to discriminate between circumstances and conduct that in combination reduce market efficiency, and those that are neutral or beneficial. This point is not new, and is the motivation for the theory of contestable markets (Baumol-Panzer-Willig, 1982). However, it is particularly relevant for some issues of conduct and the dynamics of market structure. For example, suppose a firm with deep pockets uses predatory pricing and innovation to monopolize a market characterized by network effects. Consumers benefit from low prices in the short run, and in the longer run gain from the network benefit of ubiquity and lose from the higher prices and lower rates of innovation provided by the monopolist. To determine whether the predation was socially undesirable, one needs to obtain the “as is” and “but for” trajectories, and calculate the expected present value of the net welfare loss. MPE models of the type studied by Hall-Royer-Van Audenrode (2004) and by this paper provide templates for such analysis.

### 3. Markov Perfect Equilibrium Models of Concentrated Market Dynamics

A time-linked concentrated market for heterogeneous goods can be modeled as a dynamic stochastic game, whose elements are the market participants and their policy spaces, information sets, and payoff functions, the state of the system, and its equation of motion. We note that both dynamic stochastic games and their application to concentrated markets are the subject of an extensive and deep literature; see for example Beggs-Klemperer (1992), Doraszelski-Satterthwaite (2003), Duffie-Geanakoplos-Mas Colell-McLennan (1994), Erickson-Pakes (1995), Fudenberg-Tirole (2000), Haller-Lagunoff (2000), Maskin-Tirole (1988a,b), Mertens (2003), Mertens-Parthasarathy (2003), Nowak (2003), Pakes-McGuire (1994, 2001), Reider (1979), Rosenberg (1998), Sorin (2003). One of our purposes in this paper is to re-examine with an econometrician's eye the role of various assumptions on information and structure that give equilibria in pure strategies, with the objective of providing a specialization of the general stochastic game model that is suitable for econometric identification and analysis. To this end, we will provide several self-contained, simple results on existence of pure-strategy equilibria under assumptions that also make econometric analysis tractable. Readers will recognize that the theoretical, computational, and econometric tools we use are somewhat inside the frontier in each of these specializations; our contribution is intended to be the integration of these areas in a consistent model with satisfactory characteristics in all three dimensions.

Several basic theoretical, econometric, and practical considerations enter the model specification:

(1) The Markov Perfect Equilibrium modeling approach that we adopt restricts the information of firms to the “payoff-relevant” history of the market, summarized in state variables that follow a first-order Markov process, and restricts response to “closed-loop” functions of each firm's information. Important motivations for the MPE assumption are econometric and computational tractability. However, the MPE approach is also behaviorally plausible in its relatively limited requirements for

commitment, coordination, and nuanced levels of rationality. Within the MPE framework, it is possible to obtain a rich variety of models by varying assumptions on what firms know about the structure of the market and about exogenous shocks, what information is public or private, and what firms believe about their rivals' information and behavior. We will assume below that the structure of the market game is stationary after some date  $T_N$ , and justify this as a necessary condition for obtaining an identifiable model that is useful for applications. An implication of this restriction is that the Markov assumption, which ensures that a finite-dimensional state vector is a sufficient statistic for the relevant history of the game, is not a substantial additional restrictive.<sup>5</sup> However, the restriction of Markov states to "payoff-relevant" history, and the restricted dimensionality of states in any practical model, significantly limit the possible solutions. In principle, one can test a particular MPE specification against more general solutions within the GMM estimation framework discussed later, and if necessary augment the state variables in the Markov model to accommodate observed strategic behavior.

(2) In applications, one will have more or less complete observations on market state and on firms' policies over a specific time interval. This may be augmented with data from outside the market that is informative on structural features of the market, such as market research studies on consumer preferences and engineering studies of production technology, and external factors that influence the market and potentially provide natural experiments, such as changing levels of foreign trade and other demand shifters. A critical criterion for MPE model specification is that the model be sufficiently rich to explain these observations, and sufficiently lean so that these observations are sufficient to identify the model structure and enable estimation of "but for" market outcomes. The first of these requirements will be met by introducing exogenous shocks that correspond to the non-systematic statistical

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<sup>5</sup>Technically, our model is a *continuous-state hidden Markov process* when time is discrete, and a *hidden Markov random field* when time is continuous. Kunsch-Geman-Kehagias (1995) show that the class of hidden Markov random fields is dense in the space of all stationary stochastic processes.

variations present in market data. In this connection, it is necessary to detail assumptions on what firms know and when they know it, and what the econometrician knows from the observed historical record. Note that when policy choices are made, shocks may be public information to all firms, private information to a single firm, or unknown to all firms. Further, the econometrician may be unable to observe some information that was known to the firms during historical market operation, and in other cases may be able to observe retrospectively information that was unknown to the firms at the time decisions are made. Note that in general, shocks that are known to firms will enter their policy response functions, while shocks that are unknown are omitted from their policy functions but enter their outcome functions. Further, the form of policy functions will be influenced by beliefs about unknown shocks. A classic literature in econometrics provides a template for these considerations; see Mundlak and Hoch (1965), and Fuss-McFadden-Mundlak (1978). The second requirement, that the structure of the MPE model be identified from observations on the market over a specified time interval, places conditions on what firms can know that the econometrician does not. Note that events subsequent to a firm's policy choice can influence those choices only if they are predictable by the firm at the time of the choice. A successful econometric model of firm behavior must incorporate the predictions of events that do influence choice. Thus, it is critical that the econometrician have all the information that is available to firms and predictive of future events. Put the other way around, information that is available to the firm and not to the econometrician must be non-predictive, or in econometric terminology, Granger non-causal, given the information that is common to the firm and the econometrician. With this assumption, events beyond the observation period that are not predictable by the econometrician cannot influence observed behavior, and by Occam's razor should be excluded from the model. A natural assumption consistent with this conclusion is that the market becomes stationary before the end of the observation period, once all predictable effects are built into the state vector.

With these considerations in mind, we write down a dynamic concentrated market model in fairly general terms, and later specialize it for application. Let  $n = 1, \dots, N$  index firms, which may be incumbents or potential entrants. Let  $t = 0, 1, \dots, T_N, \dots, T_O, \dots$  index time periods, with  $T_N < T_O < +4$ , where  $T_N$  is a time after which the market fundamentals are time-invariant, and the econometrician observes the market in periods  $t = 1, \dots, T_O$ , which include some when the system is time-invariant. The market is described by the following notation and accompanying assumptions:

$s_t$  a *state* vector in a compact subset  $S_t$  of a finite-dimensional Euclidean space that is endowed with its Borel  $\Phi$ -field and relative Lebesgue measure

$z_t$  a vector of observable exogenous variables in a compact subset of a finite-dimensional Euclidean space  $Z_t$  that is endowed with its Borel  $\Phi$ -field and a probability measure  $g_{z_t}$ .

$\gamma_t, 0_t$  independent public exogenous shocks in a compact subset  $\Sigma$  of a finite-dimensional Euclidean space that is endowed with its Borel  $\Phi$ -field and probabilities  $g_\gamma$  and  $g_0$ .

$\cdot_{nt}$  a private firm  $n$  exogenous shock in a non-degenerate compact rectangle  $\gamma_n$  in a finite-dimensional Euclidean space with its Borel  $\Phi$ -field and a probability  $g_{\cdot,n}$  that is absolutely continuous with respect to Lebesgue measure. Let  $\cdot_t = (\cdot_{1t}, \dots, \cdot_{Nt})$  and  $\gamma = \gamma_1 H \dots H \gamma_N$ . Also define  $\cdot_{nt} = (\cdot_{1t}, \dots, \cdot_{n-1t}, \cdot_{n+1t}, \dots, \cdot_{Nt})$ .

$x_{nt}$  a policy for firm  $n$ , in a compact subset  $X_{nt}$  of a finite-dimensional Euclidean space, endowed with its Borel  $\Phi$ -field. Let  $x_t = (x_{1t}, \dots, x_{Nt})$  denote a *profile of policies* of the firms, and  $X_t = X_{1t} H \dots H X_{Nt}$ . Define the profile of rivals' policies,  $x_{-n,t} = (x_{1t}, \dots, x_{n-1,t}, x_{n+1,t}, \dots, x_{Nt})$   $0$   $X_{-n,t} = X_{1t} H \dots H X_{n-1,t} H X_{n+1,t} H \dots H X_{Nt}$ .

$A_{nt}$  a profit function for firm  $n$  in period  $t$ , a function on  $S_t H X_t H Z_t H \Sigma H \gamma_n$ .

$h_t$  an equation of motion, a measurable map from  $S_t H X_t H Z_t H \Sigma$  into  $S_{t+1}$ .

$\exists$  a common constant discount factor.

The state may include both discrete and continuous payoff-relevant information such as entry status, market size, cumulative investment in product quality, and firm market shares. The exogenous shocks  $\gamma_t$  and  $\theta_t$  will represent demand effects. The distinction is that  $\gamma_t$  is assumed to be public knowledge to the firms when policy decisions are made in period  $t$ , while  $\theta_t$  is assumed to be unknown to all firms. The exogenous shock  $\eta_{nt}$  will represent cost effects that are private knowledge to firm  $n$  when decisions are made in period  $t$ ; we assume that  $\eta_{nt}$  enters the profit function for firm  $n$ , but not the profit functions of rivals or the equation of motion. Policy variables may include investment, pricing, product attributes, entry and exit from product lines, and actions such as bundling, tying, refusals to deal, and exclusionary contracts that are potentially “bad acts” under anti-trust law. The observable exogenous variables  $z_t$  may include demand shifters, prices of firm inputs, and rules of conduct that are determined outside the model. In particular, we treat firms’ beliefs about the applicability and consequences of the anti-trust law, and engagement in “bad acts” that violate this law, as determined exogenously by the legal system and a “firm culture” that promotes or discourages compliance. We assume that  $z_t$  is public knowledge to the firms when period  $t$  decisions are made. Summarizing, the payoff-relevant information of firm  $n$  when it makes its policy decision in period  $t$  is a point  $(s_t, z_t, \gamma_t, \eta_{nt}) \in S_t \times Z_t \times \Gamma_t \times \Theta_{nt}$ . A *Markov mixed strategy* for the firm is a measurable mapping  $f_{nt}$  from this information into the space  $P(X_{nt})$  of probabilities on  $X_{nt}$ . Note that this object is not the same as a *best-response* or *reaction* function, which maps the information above *and* the strategies of rivals into  $P(X_{nt})$ ; the latter is the mapping for which the former is a fixed point in a MPE. We will assume a non-atomic population of consumers who are price-takers and do *not* respond strategically. In particular, we assume that network effects influence demand solely

through the previous period market state, and that consumers do not form expectations about future market states based on market history or firm conduct.<sup>6</sup>

The per-period profit function  $A_{nt}(s_t, x_t, z_t, \gamma_t, \cdot_{nt})$  will reflect current production costs and revenues associated with its technology and the structure of consumer demand with its embedded network effects, and investment and marketing expenses. Note that it is a function solely of information it knows and the realized policies of all the firms; it does not depend on unknown shocks such as  $\theta_t$ , and should be interpreted as the expectation of profit with respect to any such exogenous variables that are unknown to the firm. An important special case we will analyze will be profit functions in which the private information has the same dimension as the policy and appears solely in a linear biadditive form,  $A_{nt}(s_t, x_t, z_t, \gamma_t, \cdot_{nt}) / \nabla_{nt}(s_t, x_t, z_t, \gamma_t) - x_t \ni \cdot_{nt}$ . In this form, the  $\cdot_{nt}$  can be interpreted as random costs associated with implementing the firm's policy, such as a setup cost on entry or scrap cost on exit, or a random surcharge on investment expenditure reflecting administrative efficiency. We will not require that  $X_{nt}$  be a convex set or that the profit function be concave in  $x_{nt}$ . The objective of the firm will be to maximize the expected present value of its stream of per-period profits, taking into account the information it has. It may have beliefs about its rivals' information and behavior, but these beliefs must in the end also depend only on the information that it has..

The equation of motion  $s_{t+1} = h_t(s_t, x_t, z_t, \gamma_t, \theta_t)$  maps the current state, exogenous variables, shocks, and policy realizations into the state in the following period. We assume that this equation does not depend on the private information of the individual firms, which influences the market solely through their realized policies. In the general theory of stochastic games, the law of motion is usually described in terms of Markov transition probabilities. In our notation, these would be obtained, for Borel sets  $A$  in  $S_t$ , from the relationship  $P_t(A|s_t, x_t, z_t, \gamma_t) = g_0(\{0|h_t(s_t, x_t, z_t, \gamma_t, 0) \in A\})$ . For example, in the special case that the state spaces are finite, the law of motion is a (controlled) Markov

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<sup>6</sup>Some markets are oligoposonistic, with atomistic buyers who can be included as players in the game. We will not analyze the possibility that firm announcements could become "bad acts" to promote

chain, and  $h_t(s_{t+1}|s_t, x_t, z_t, \gamma_t)$  denotes the transition probabilities, which will be assumed continuous in  $(s_t, x_t, z_t, \gamma_t)$ . With this basic model setup and the earlier discussion as motivation, we make the following assumptions:

**Assumption 1.** The shocks  $(\gamma_t, \theta_t, \epsilon_{nt})$  are independent and identically distributed across time, and independent of each other, and their probabilities  $g_\gamma$ ,  $g_\theta$ , and  $g_{\epsilon_n}$  are common knowledge. The observable vectors  $z_t \in Z_t$  are also independent across time, with probabilities  $g_{z_t}$  that are common knowledge. At each time  $t$  when policies  $x_{nt}$  are chosen, all firms know  $z_t$  and  $\gamma_t$ , and each firm  $n$  knows its private shock  $\epsilon_{nt}$ . Contemporaneous private shocks of rivals,  $\theta_t$ , and all future exogenous vectors and shocks, are not predictable by a firm. The econometrician observes  $z_t$  for  $t = 1, \dots, T_0$ , but does not observe the shocks  $\gamma_t$ ,  $\theta_t$ , or  $\epsilon_{nt}$ .<sup>7</sup>

**Assumption 2.** The market structure is time-homogeneous beginning at time  $T_N$ . Specifically, the state space  $S_t$ , the exogenous vector space  $Z_t$ , the policy spaces  $X_{1t}, \dots, X_{Nt}$ , the payoff functions  $A_{1t}, \dots, A_{Nt}$ , and the equation of motion  $h_t$  are independent of  $t$  for  $t \geq T_N$ .

We will study the game under the following simplifying regularity assumption, chosen because it is easy to check and likely to be satisfied in applications.

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market power when consumers respond strategically, but note that this is a problem of interest.

<sup>7</sup>The exogenous variables  $z_t$  and shocks  $\gamma_t$  differ only in what the *econometrician* observes, and are treated as similar contemporaneously known public information by the firms. The assumption that the exogenous variables and shocks are not predictable by the firms can be softened by interpreting these variables as *innovations* in stochastic processes, and building the *predictable* parts of these processes into the state vectors and laws of motion.

**Assumption 3.** The profit functions  $A_{nt}(s_t, x_t, z_t, \gamma_t, \cdot_{nt})$  and the equation of motion  $h_t(s_t, x_t, z_t, \gamma_t, 0_t)$  are Lipschitz in their arguments.<sup>8</sup> The discount factor satisfies  $0 < \beta < 1$ .

A *Markov Perfect Equilibrium* (MPE) is a profile of Markov mixed strategies  $f_{nt}^*$  for all firms and periods with the subgame-perfect Nash property that for each period  $t$  and firm  $n$ , given the current strategies of rivals and all future strategies, firm  $n$  has no incentive to alter its assigned strategy  $f_{nt}^*$ . Let  $V_{nt}(s_t, z_t, \gamma_t, \cdot_{nt})$  denote the *valuation function* of firm  $n$ , the maximum expected present value of its stream of profits from period  $t$  forward, given its payoff-relevant information. The MPE and the valuation functions of the firms are characterized by Bellman's backward recursion. Let  $E_{-n,t}$  denote the expectation operator with respect to variables that are *unknown* to firm  $n$  at the time of its policy decision in period  $t$ :  $z_{tN}$  and the shocks  $\gamma_{tN, \cdot tN}$  for  $tN > t$  and the contemporaneous shocks  $0_t$  and the private information of rivals,  $\cdot_{-n,t} = (\cdot_{1t}, \dots, \cdot_{n-1,t}, \cdot_{n+1,t}, \dots, \cdot_{Nt})$ . The valuation functions then satisfy Bellman's equations,

$$(1) \quad V_{nt}(s_t, z_t, \gamma_t, \cdot_{nt}) = \max_{x_{nt}} \{ \Delta_{nt}(s_t, x_{nt}, z_t, \gamma_t, \cdot_{nt}) + \beta E_{-n,t} V_{nt}(s_t, x_{nt}, z_t, \gamma_t, \cdot_{nt}) \},$$

with the notation

$$(2) \quad \Delta_{nt}(s_t, x_{nt}, z_t, \gamma_t, \cdot_{nt}) / E_{-n,t} \int_{X_{-n,t}} f_{jt}^*(dx_{jt} | s_t, z_t, \gamma_t, \cdot_{jt}) A_{nt}(s_t, x_{-n,t}, x_{nt}, z_t, \gamma_t, \cdot_{nt})$$

for the expected value of current profit, and

$$(3) \quad \cdot_{nt}(s_t, x_{nt}, z_t, \gamma_t) / E_{-n,t}$$

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<sup>8</sup>If a state or exogenous vector has some discrete components, then the Lipschitz condition is

$$\int_{X_{-n,t}} f_{jt}^*(dx_{jt}|s_t, z_t, \gamma_t, \cdot_{jt}) V_{n,t+1}(h_t(s_t, x_{-n,t}, x_{nt}, z_t, \gamma_t, 0_t), z_{t+1}, \gamma_{t+1}, \cdot_{n,t+1})$$

for the expected value of future profit. Let  $X_{nt}^*(s_t, z_t, \gamma_t, \cdot_{nt})$  denote the maximizer correspondence, the set of points  $x_{nt}$  that achieve the maximum in (1). The Nash property of MPE requires that the support of  $f_{nt}^*(\cdot|s_t, z_t, \gamma_t, \cdot_{nt})$  be contained in  $X_{nt}^*(s_t, z_t, \gamma_t, \cdot_{nt})$ .

### *Existence of MPE*

We will show that under Assumptions 1-3 above, plus the substantial technical simplification that the spaces  $S_t$ ,  $Z_t$ , and  $\Sigma$  are finite, that a MPE exists. With an additional assumption, we will show that an MPE in pure strategies exists almost surely. Our results are a special case of the analysis of existence of subgame-perfect Nash equilibria in stochastic games, with our strong assumptions implying the existence of an exact MPE solution to the stationary game after time  $T_N$ . We do not deal with the coordination problem in the case of multiple MPE, or consider a supergame in which coordinating communication between players is possible. For related and more general existence results see Rieder (1979), Mertens and Parthasarathy (1987), Majumdar and Sundaram (1991), Duffie, Geanakoplos, Mas-Colell, McLennan (1994), and Solon (2000).

**Theorem 1.** Suppose Assumptions 1-3. Then there exist valuation functions  $V_{nt}$  and a profile of strategies  $f_{nt}^*$  for  $n = 1, \dots, N$  that are time-invariant for  $t \geq T_N$  and satisfy condition (1) for a MPE.

*Proof:* We first establish the existence of a time-invariant valuation function and strategy profile for the time-invariant subgame starting in period  $T_N$ . Let  $T$  denote any time greater than  $T_N$ . The functions  $A_{nT}(s_T, x_T, z_T, \gamma_T, \cdot_{nT})$  and  $h_t(s_t, x_t, z_t, \gamma_t, 0_t)$  are Lipschitz; let  $M$

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satisfied trivially in these components.

denote a uniform bound on these function and their Lipschitz constants, which exists since all domains are compact. We will use a fixed point argument that there exist valuation functions that are reproduced by, and strategies for the firms that are consistent with, the Bellman recursion (1). We begin by defining a series of sets and mappings, and obtaining their properties.

[1] Let  $W_n$  denote the set of real-valued functions  $V$  on  $S_T H Z_T H \Sigma H 7_n$  that are Lipschitz in  $7_n$ , with a bound  $M/(1-\beta)$  on the function and on the Lipschitz constant. Then,  $W_n$  is a convex compact subset of the Banach space of bounded measurable functions on  $S_T H Z_T H \Sigma H 7_n$ ; see Dunford and Schwartz (1965, Theorem IV.5.6).

[2] Let  $F_n$  denote the set of cumulative distribution functions  $f_n$  on  $X_{nT}$  that are conditioned on and measurable with respect to  $S_T H Z_T H \Sigma H 7_n$ . This is a subset of the Banach space  $L_1(X_{nT} H S_T H Z_T H \Sigma H 7_n)$ . Further, for each rectangle  $A = A_x H A_s H A_z H A_\gamma H A$ , in this space,

$$I_A f_n(x|s, z, \gamma, \cdot) g(d) g_z(dz) g_\gamma(d\gamma) ds dx \# g(A) g_z(A_z) g_\gamma(A_\gamma) :_s(A_s) :_x(A_x),$$

where  $:_s$  and  $:_x$  are relative Lebesgue measures on  $ST$  and  $X_{nT}$ . Then,  $F_n$  is weakly sequentially compact and convex; see Dunford and Schwartz (1965, Theorem IV.8.9).

[3] For each  $(V_1, \dots, V_N, f_1, \dots, f_n) \in W_1 H \dots H W_N H F_1 H \dots H F_N$ , the function  $\Delta_{nT}(S_T, X_{nT}, Z_T, \gamma_T, \cdot, \cdot, \cdot)$  from (2) is an expectation (with respect to  $\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot$ , whose distribution is independent of  $x_{nT}, \cdot, \cdot, \cdot$ ) of the function  $A_{nT}$  which is uniformly Lipschitz in  $(x_{nT}, \cdot, \cdot, \cdot)$ , and is therefore again Lipschitz in these arguments with the same constant.

[4] For each  $(V_1, \dots, V_N, f_1, \dots, f_n) \in W_1 H \dots H W_N H F_1 H \dots H F_N$ , the function  $\cdot_{nT}(S_T, X_{nT}, Z_T, \gamma_T)$  from (3) is an expectation (with respect to  $\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot$ , and  $\gamma_{T+1}$  whose distributions do not depend on  $x_{nT}$ , and with respect to  $S_{T+1}$  whose distribution has a Lipschitz dependence on  $x_{nT}$  and is independent of  $\cdot, \cdot, \cdot, \cdot$ ) of the function  $V_{n, T+1}$  which is independent of  $x_{nT}$ . Consequently,  $\cdot_{nT}$  inherits the bound  $M/(1-\beta)$  on  $V_{n, T+1}$ , and is Lipschitz in  $x_{nT}$ .

[5] The function  $\Delta_{nT}(s_T, x_{nT}, z_T, \gamma_T, \cdot_{nT}) + \exists \cdot_{nT}(s_T, x_{nT}, z_T, \gamma_T)$  in (1) is uniformly bounded by  $M + \exists M/(1-\exists) = M/(1-\exists)$ , is Lipschitz in  $\cdot_{nT}$  with constant  $M$ , and Lipschitz in  $x_{nT}$  on its compact domain. One implication is that the maximand correspondence  $X_{nT}^*(s_T, z_T, \gamma_T, \cdot_{nT})$  is non-empty and upper hemicontinuous; see Hildenbrand (1974). A second implication is that  $V_{nN}(s_T, z_T, \gamma_T, \cdot_{nT}) = \max_{x_{nT}} \{\Delta_{nT}(s_T, x_{nT}, z_T, \gamma_T, \cdot_{nT}) + \exists \cdot_{nT}(s_T, x_{nT}, z_T, \gamma_T)\}$  is bounded with constant  $M/(1-\exists)$ . Also, let  $x_{nT}(\cdot_{nT}) \in X_{nT}^*(s_T, z_T, \gamma_T, \cdot_{nT})$  be any selection, and note that maximization implies

$$\begin{aligned}
 & -M|\cdot_{nT}N - \cdot_{nT}N| \\
 & \# \Delta_{nT}(s_T, x_{nT}(\cdot_{nT}O), z_T, \gamma_T, \cdot_{nT}O) - \Delta_{nT}(s_T, x_{nT}(\cdot_{nT}N), z_T, \gamma_T, \cdot_{nT}O) \\
 & \# V_{nN}(s_T, z_T, \gamma_T, \cdot_{nT}N) - V_{nN}(s_T, z_T, \gamma_T, \cdot_{nT}O) \\
 & \# \Delta_{nT}(s_T, x_{nT}(\cdot_{nT}N), z_T, \gamma_T, \cdot_{nT}N) - \Delta_{nT}(s_T, x_{nT}(\cdot_{nT}N), z_T, \gamma_T, \cdot_{nT}O) \# M|\cdot_{nT}N - \cdot_{nT}N|.
 \end{aligned}$$

Therefore,  $V_{nN}$  is Lipschitz in  $\cdot_{nT}$  with constant  $M$ . Therefore,  $V_{nN} \in W_n$ .

[6] Let  $P_n$  denote the mapping from  $S_T H Z_T H \Sigma H \gamma_n$  into subsets of  $F_n$  defined by  $P_n(s_T, z_T, \gamma_T, \cdot_{nT}) = P(X_{nT}^*(s_T, z_T, \gamma_T, \cdot_{nT}))$ ; i.e., the set of all cumulative distribution functions with support contained in the maximizer correspondence. The correspondence  $P_n$  is non-empty and convex-valued. The upper hemicontinuity of the maximizer correspondence implies that  $P_n$  is upper hemicontinuous.

The preceding results establish that  $W_1 H \dots H W_N H F_1 H \dots H F_N$  is convex and compact, and that the mapping given by [5] and [6] is a upper hemicontinuous convex-valued correspondence from this space into itself. Then, by the Glicksberg-Fan fixed point theorem (Saveliev, 1999), there exists  $(V_{1T}, \dots, V_{NT}, f_{1T}^*, \dots, f_{NT}^*) \in W_1 H \dots H W_N H F_1 H \dots H F_N$  that satisfies (1) and returns the same valuation functions. This completes the proof that a stationary MPE exists in the subgame starting in period  $TN$ .

Now proceed by backward recursion of (1) from period  $TN$ . Let  $T$  be a period satisfying  $T \neq TN$ , and suppose that strategies  $f_{nt}^*$  and valuation functions  $V_{nt}$  have been established satisfying (1) for  $t > T$ . Define  $F_n$  and  $P_n$  as in [2] and [6] above, and note

that again by the same construction  $F_n$  is convex and compact, and  $P_n$  is a upper hemicontinuous convex-valued correspondence from  $F_1 \times \dots \times F_N$  into itself. Therefore, the Glicksburg-Kakutani fixed point theorem establishes the existence of strategies  $(f_{1T}^*, \dots, f_{NT}^*) \in F_1 \times \dots \times F_N$  satisfying (1). This completes the proof that a MPE exists for the full game. ~

For some purposes, it may be useful to identify invariant distributions for the state of the model after time  $T_N$  when the model is time-invariant, and estimate these distributions from the empirical distribution of realized states. With the MPE strategies given by Theorem 1, the law of motion of the model is a time-invariant Markov process, and there are some straightforward (high-level) assumptions under which it will have a unique invariant distribution and the ergodic property that the empirical distribution from any initial state will converge weakly to the invariant distribution. Define a Markov transition kernel

$$P_t(A|s_t, x_t, z_t, \gamma_t) = g_0(\{0|h_t(s_t, x_t, z_t, \gamma_t, 0) \cap A\}).$$

For this discussion, assume that the state spaces are finite. Suppose for any pair of states  $s^0, s^N$ , there is a set of  $z, \gamma, .$  occurring with positive probability on which  $h_N(s^0|s^N, z, \gamma, .) < 1$ , and there exists a finite sequence of states  $s^k$  with  $s^0 = s^N$  and  $s^K = s^0$  with the property that there are sets of  $z, \gamma, .$  occurring with positive probability on which  $h_N(s^{k+1}|s^k, z, \gamma, .) > 0$ . Then, the process is irreducible and acyclic, has an invariant distribution which is unique, and has the strong ergodic property that for any continuous function on  $S_T \times H \times Z_T \times H \times \Sigma^T$ , a time average of this function on any realized trajectory converges to its expectation at an exponential rate; see Appendix 1. Related results that are much more general are given by Duffie, Geanakoplos, Mas-Colell, McLennan (1994).

In computational approximation of MPE, mixed strategies create difficulties, so that the analysis is greatly aided if pure strategy MPE exist. This has been accomplished in previous literature through strong assumptions on the profit functions, or through the introduction of private information that acts as a mixing device. The first approach requires that the firm's policy set  $X_{nt}$  be convex, and the profit functions be strictly concave on this set, or that it satisfy some strong separability and super-modularity conditions. It is difficult to use the first approach in our application in which network effects influence the determination of market shares. The second approach has been used in specific applications by Pakes and McGuire (1994) and Doraszelski, and Satterthwaite (2003). We adapt the second approach to our model, introducing a functional specification that gives the desired result.

**Assumption 4.** For  $n = 1, \dots, N$  and  $t = 1, \dots, TO$ , the profit function of the firm has an additive bilinear dependence on  $\cdot_{nt}$ ; i.e.,  $A_{nt}(S_t, X_t, Z_t, \gamma_t, \cdot_{nt}) / \forall_{nt}(S_t, X_t, Z_t, \gamma_t) - X_t \in \cdot_{nt}$ .

Example:  $\cdot_{nt}$  is a vector of unit costs, commensurate with  $x_{nt}$ , that gives the firm's private cost of policy satisfies  $x_{nt}$ . Components of  $\cdot_{nt}$  may include entry setup and exit shutdown costs that do not otherwise enter the technology or product demand.

**Theorem 2.** Suppose Assumptions 1-4. Then, with probability one, the MPE established in Theorem 1 will require only pure strategies for all firms in all time periods.

Proof: Consider the function  $V_{nt}(S_t, Z_t, \gamma_t, \cdot_{nt}) = \max_{x_{nt}} \{ \Delta_{nt}(S_t, X_{nt}, Z_t, \gamma_t, \cdot_{nt}) + \exists \cdot_{nt}(S_t, X_{nt}, Z_t, \gamma_t) \}$  from (1). We will argue that  $V_{nt}$  is a convex function of  $\cdot_{nt}$ , and that its derivative with respect to  $\cdot_{nt}$ , if it exists, equals a unique pure strategy for the firm. But a convex function is twice continuously differentiable almost everywhere; see Alexandroff (1939). This result, combined with the Assumption 3 condition that the probability  $g$  is absolutely continuous with respect to Lebesgue measure, ensures that the firm's strategy is almost surely pure and continuous in  $\cdot_{nt}$ . We

complete the proof by demonstrating that  $V_{nt}$  is convex. Suppose that  $\cdot_{nt}N$  and  $\cdot_{nt}O$  are distinct points, consider a convex combination  $2\cdot_{nt}N+(1-2)\cdot_{nt}O$ , and let  $x_{nt}^*$  be any maximizer that gives the function  $V_{nt}(s_t, z_t, \gamma_t, 2\cdot_{nt}N+(1-2)\cdot_{nt}O)$ . Then,

$$\begin{aligned}
V_{nt}(s_t, z_t, \gamma_t, 2\cdot_{nt}N+(1-2)\cdot_{nt}O) &= \Delta_{nt}(s_t, x_{nt}^*, z_t, \gamma_t, \cdot_{nt}) + \exists_{\cdot_{nt}}(s_t, x_{nt}^*, z_t, \gamma_t) \\
&= \Delta_{nt}(s_t, x_{nt}^*, z_t, \gamma_t, 0) - (2\cdot_{nt}N+(1-2)\cdot_{nt}O) \cong x_{nt}^* + \exists_{\cdot_{nt}}(s_t, x_{nt}^*, z_t, \gamma_t) \\
&= 2\{\Delta_{nt}(s_t, x_{nt}^*, z_t, \gamma_t, 0) - \cdot_{nt}N \cong x_{nt}^* + \exists_{\cdot_{nt}}(s_t, x_{nt}^*, z_t, \gamma_t)\} \\
&\quad + (1-2)\{\Delta_{nt}(s_t, x_{nt}^*, z_t, \gamma_t, 0) - \cdot_{nt}O \cong x_{nt}^* + \exists_{\cdot_{nt}}(s_t, x_{nt}^*, z_t, \gamma_t)\} \\
&\# 2 V_{nt}(s_t, z_t, \gamma_t, 2\cdot_{nt}N) + (1-2)V_{nt}(s_t, z_t, \gamma_t, \cdot_{nt}O).
\end{aligned}$$

This completes the proof of concavity. ~

The preceding results do not establish a unique MPE, nor address the coordination problem associated with multiple MPE. Duffie, Geanakoplos, Mas-Colell, McLennan (1994) discuss extending the market game to include public information on “sunspots” that can be used as a coordination device. Uniqueness results based on game structure have been obtained only under stringent sufficient conditions; see Milgrom and Roberts (1990). The model we introduce later for our application does not meet these conditions, and we are left with the possibility that computational solutions in applications may find only some of multiple MPE solutions.

### *Computation of MPE Solutions and Trajectories*

To compute MPE, we follow the program developed by Pakes and McGuire (1994,2001), utilizing the adaptation by Judd (1999) of classical functional approximation by orthogonal polynomials; see Jackson (1941), Lorentz (1966), Press et al (1986). Consider the long run, time-invariant structure of the model. As in the proof of existence, one can use the Bellman recursion as a mapping from the space of valuation functions and mappings and strategies into itself. A fixed point of this mapping will give the

valuation function for the model once it is time-invariant. Given this function, the Nash strategies satisfying a fixed point and the valuation functions can be obtained by backward recursion. The computational technique is to approximate the family of valuation functions using Chebyshev polynomials and approximate the state space using a finite grid, and then to solve the fixed point problem in the finite-dimensional space spanned by these approximations. The following paragraphs outline the basic properties of Chebyshev approximations, and go on to describe a method for augmenting the grid that is computationally straightforward and permits the approximation error to be controlled on the observed trajectory of the game.

Consider approximating a function  $V(x)$  on the  $d$ -dimensional cube  $[-1,1]^d$  by a linear combination

$$V(x) = \sum_{i_1, \dots, i_d} b_{i_1, \dots, i_d} T_{i_1}(x_1) \dots T_{i_d}(x_d) + \gamma(x),$$

where the  $T_i(z)$  are univariate Chebyshev orthogonal polynomials of degree  $j = 0, \dots, m$ , and  $\gamma(x)$  is a residual. These polynomials satisfy the recursion

$$T_0(x) = 1, T_1(x) = x, \text{ and } T_{t+1}(x) = 2xT_t(x) - T_{t-1}(x) \text{ for } t = 2, 3, \dots$$

They satisfy the orthogonality condition

$$\int_{-1}^1 T_i(x) T_j(x) (1-x^2)^{1/2} dx = \frac{1}{2} \delta_{ij} \frac{1}{1+i}, \text{ and the roots of } T_t(x) \text{ are } x_{ti} = \cos\left(\frac{(i-1/2)\pi}{t}\right)$$

for  $t > 0$  and  $i = 1, \dots, t$ . Further, for  $j, k \neq m$ , they satisfy the finite orthogonality condition

$$\sum_{i=1}^m T_i(x_{mi}) T_k(x_{mi}) = \frac{1}{2} \delta_{ik} \frac{1}{1+i}.$$

Suppose  $z_{mj}$ ,  $j = 1, \dots, m+1$ , are the roots of the polynomial  $T_{m+1}(z)$ , and are used to form a grid on the cube  $[-1,1]^d$  in which the coordinates in each dimension are these roots. Reindex the polynomials and coefficients with a single running index  $j$ , and grid points

with a single running index  $k$ . Then, the system above evaluated at the grid points can be written

$$(4) \quad Y_k = \sum_{j=0}^M A_{kj} B_j + \gamma_k, \quad k = 1, \dots, K$$

where  $A_{kj}$  is the value of the  $j^{\text{th}}$  polynomial at the  $k^{\text{th}}$  grid point,  $Y_k$  is the value of  $V$  at the  $k^{\text{th}}$  grid point, and  $\gamma_k$  is a residual. We allow the possibility that some polynomials are omitted, so that the number of grid points  $K = (m+1)^d$  exceeds the number of polynomials  $M$ . If  $K = M$ , the system can be solved exactly, while if  $K > M$ , we assume the solution that minimizes the sum of squared residuals. Written in matrix notation, (4) is  $Y = AB + \gamma$  and the least squares solution is

$$(5) \quad B = [ANA]^{-1}ANY.$$

The orthogonality property of Chebyshev polynomials implies that  $ANA$  is a diagonal matrix  $D$ , so that this solution can be obtained without matrix inversion. Note that  $D$  has a simple form, with diagonal elements  $m^d/2^p$ , where  $m$  is the order of the Chebyshev polynomials and  $p$  is the number of univariate polynomials in the product for a given term in the approximation that are of degree greater than zero.

Suppose now that we start from a Chebyshev system with  $K$  evaluation points and  $M$  polynomials, with  $K \geq M$ , and add  $k$  additional evaluation points and  $m$  additional polynomials, with  $k \geq m$ . (Note that the notation  $k$  and  $m$  for the augmentation dimensions are not the same as the running index for grid points and the order of the Chebyshev polynomials that appeared in (4).) The additional evaluation points will be data-driven rather than polynomial roots, chosen so that observed states are in the grid. Consequently, there is no advantage to taking the additional polynomials to be orthogonal or Chebyshev. For example, they may be simple powers higher than the maximum degree of the included Chebyshev polynomials.

The system we seek to solve is an augmented version of (4) which can be written in partitioned matrix form

(6)

$$\begin{bmatrix} Y \\ y \end{bmatrix} \sim = \sim \begin{bmatrix} A \\ F \\ G \\ H \end{bmatrix} \begin{bmatrix} B \\ b \end{bmatrix} \sim + \sim \begin{bmatrix} \gamma \\ \langle \end{bmatrix},$$

where  $Y$  is  $KH1$ ,  $y$  is  $kH1$ ,  $A$  is the  $KHM$  array of Chebyshev polynomials evaluated at the original grid points,  $F$  is the  $KHm$  array of augmented polynomials evaluated at the original grid points,  $G$  is the  $kHM$  array of Chebyshev polynomials evaluated at the augmented grid points,  $H$  is the  $kHm$  array of augmented polynomials evaluated at the augmented grid points,  $B$  is  $MH1$ , and  $b$  is  $mH1$ . The least squares solution solves

(7)

$$\begin{bmatrix} ANY+GNy \\ FNY+HNY \end{bmatrix} \sim = \sim \begin{bmatrix} ANA+GNG \\ ANF+GNH \\ FNA+HNG \\ FNF+HNN \end{bmatrix} \begin{bmatrix} B \\ b \end{bmatrix}.$$

Define the  $MHM$  matrix

$$(8) \quad Q = D^{-1} - D^{-1}GN[I_k+GD^{-1}GN]^{-1}GD^{-1}.$$

Note that  $Q$  is the inverse of  $ANA + GNG / D + GNG$ . Since  $D$  is diagonal, the computation of  $Q$  requires only the  $kHk$  inverse  $[I_k+GD^{-1}GN]^{-1}$ .

The formula for a partitioned inverse is

(9)

$$\begin{aligned} & \{ \text{LEFT} [ \text{MATRIX} \{ \{ C_{11} \} \& \{ C_{12} \} \# \{ C_{21} \} \& \{ C_{22} \} \} \text{RIGHT} ] \}^{-1} = \\ & \text{LEFT} [ \text{MATRIX} \{ \{ (C_{11})^{-1} + (C_{11})^{-1} C_{12} R^{-1} C_{21} (C_{11})^{-1} \} \& \{ \text{---} \\ & (C_{11})^{-1} C_{12} R^{-1} \} \# \{ -R^{-1} C_{21} (C_{11})^{-1} \text{---} \} \& \{ R^{-1} \} \} \text{RIGHT} ] , \end{aligned}$$

where  $R = C_{22} - C_{21}(C_{11})^{-1}C_{12}$ . Apply this formula to (7). Note that

$$C_{11} = ANA + GNG, \text{ implying } (C_{11})^{-1} = Q,$$

$$C_{12} = ANF + GNH$$

$$C_{21} = FNA + HNG$$

$$C_{22} = FNF + HNH$$

Then R is the mHm matrix

$$R = FNF + HNH - (FNA + HNG)Q(ANF + GNH).$$

Computation of the solution to (7) requires inversion of the mHm matrix R, as well as computation of the matrix Q which contains the kHk inverse  $[I_k + GD^{-1}GN]^{-1}$ . The matrices to be inverted are symmetric positive definite, so a fast Cholesky inversion method can be used. When  $m = k = 1$ , no matrix inversion is required.

In application, we will have valuation functions  $V_{jt}(x,z,2)$  for each player that depend on the vector 2 of parameters of the problem, and we will have to approximate these functions in steady state, and in each of the observation periods where exogenous forcing variables are operating. However, the right-hand-side array in (6) does not depend on 2, so the weights to be applied to the  $Y(2)$  in the augmented version of (5) to calculate  $B(2)$  can be calculated once, before iterative search for 2.

The MPE solution algorithm for a given 2 can be characterized as follows: We have observation periods  $t = 1, \dots, T$  during which states  $x_t$  (e.g., installed customer base,

accumulated capital in product development) and policies  $w_{jt}$  (e.g., investment levels) for players  $j = 1, \dots, J$  are observed (at least partially) and some observed exogenous drivers  $z_t$  (e.g., exclusionary arrangements) are operative. This is followed by periods  $t = T+1, T+2, \dots$  where one does not have observed data. In these periods, the exogenous drivers are assumed to be constant, so that the valuation functions are stationary after time  $T$ . With these assumptions, we have valuation functions  $V_{jt}(x, z_t, 2)$  for the players  $j = 1, \dots, J$  in the periods  $t = 1, \dots, T, T+1, T+2, \dots$ , with  $z_t = 0$  (by normalization) and  $V_{j,t+1}(x, z_t, 2) = V_{jt}(x, z_t, 2)$  for  $t > T$ . Player  $j$ 's policy function is denoted  $w_{jt}(x, z, 2)$ . For  $t > T$ , the policy function will not depend on  $t$ . Assume that the law of motion is deterministic, conditioned on exogenous shocks; this is a special case of the general setup with a Markov transition kernel. The equations of motion can then be written

$$(10) \quad x_{t+1} = h(x_t, w_{1t}(x_t, z_t, 2), \dots, w_{Jt}(x_t, z_t, 2), z_t, 2),$$

where  $h$  is fully specified, but may depend on some of the parameters. Let  $P_{jt}(x, w_{1t}(x, z, 2), \dots, w_{jt}(x, z, 2), \dots, w_{Jt}(x, z, 2), z, 2)$  denote the period  $t$  payoff to player  $j$ , and  $\exists$  the discount factor, assumed constant. Bellman's equations and the Markov Perfect Nash equilibrium conditions determine the policy functions,

$$(11) \quad \begin{aligned} V_{jt}(x, z, 2) = & \mathbf{E}\{P_{jt}(x, w_{1t}(x, z, 2), \dots, w_{jt}(x, z, 2), \dots, w_{Jt}(x, z, 2), z, 2) \\ & + \exists V_{j,t+1}(h(x, w_{1t}(x, z, 2), \dots, w_{jt}(x, z, 2), \dots, w_{Jt}(x, z, 2), z, 2), z, 2)\} \\ & \exists \mathbf{E}\{P_{jt}(x, w_{1t}(x, z, 2), \dots, w_{jt}, \dots, w_{Jt}(x, z, 2), z, 2) \\ & + * V_{j,t+1}(h(x, w_{1t}(x, z, 2), \dots, w_{jt}, \dots, w_{Jt}(x, z, 2), z, 2), z, 2)\}. \end{aligned}$$

In this notation, exogenous shocks known to the firms are included in  $z$ , the expectation is with respect to information unknown to the firms. The solution method is to substitute augmented polynomial approximations (1) for the valuation functions, first solve (11) for the steady state optimal policy functions in  $T+1$ , using the requirement that  $V_{j,T+2} / V_{j,T+1}$ ,

and then solve by backward recursion for the policy functions and approximate valuation functions for  $T, T-1, \dots, t$ . Finally, starting from the observed initial state, and given the policy and valuation functions, solve the model forward to predict policies and states. Do the same thing with  $z$ 's reset to "no bad acts" to predict "but for" policies and states, and from comparing the paths compute various measures of the impacts of the bad acts, including market shares and profits.

### *Estimation Strategies*

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In our discussion of econometric estimation of the structure of an MPE model, we will use the language of parametric estimation, treating the equation of motion and the profit functions of the firms as fully specified up to unknown parameter vectors. This is the leading case for applications, where limited observations will support only sparsely parameterized models. However, it is useful to interpret these parametric models as method of sieves approximations to the true structural equation which are nonparametric, and to consider questions of identification in this more general context.

We divide the econometric problem into three stages, as is done by Bajari, Benkard, and Levin (2003). The first stage is structural estimation of the equation of motion and profit functions. This is done prior to computation of the MPE, so that at this stage the firms' policy functions are not specified structurally. The second stage is computation of the MPE, incorporating estimation of parameters not identified in the first stage, using a minimum distance criterion to match observed and computed trajectories of states and policies. The third stage is a one-step iteration for both first and second stage estimates to obtain minimum distance estimators for the complete system.

The motivation for this estimation sequence is that the equation of motion and profit functions contain most of the parameters and disturbances of the system, and their estimation within the limits of identification determine these partially or fully. This substantially reduces the burden of estimation search within the computation of the MPE. Specifically, structure that is estimated consistently in the first stage can be held fixed during stage two while remaining parts of the system are estimated, and then all the

parameters of the system updated in a single stage three iteration that does not require extensive recomputation of the MPE. A one-step theorem (Newey-McFadden, 1994) will assure that the three-stage procedure is asymptotically equivalent to direct minimum distance estimation of the full system.

In the first stage estimation, if the equation of motion and profit functions include both exogenous shocks and current policies that the firms choose knowing these shocks, then the current policy variables are endogenous. To deal with this, reduced form estimates of the policy functions may be usable as instrumental variables, but several econometric issues are involved. First, the policy variable reduced form equations are in general not parametrically specified at this stage, even if the equation of motion and profit functions are fully parametric, since their form depends on the solution of the game. Second, natural specifications of the structural equations may involve non-additive disturbances, making it necessary to use econometric methods appropriate to this case. Third, identification will depend on functional restrictions or the availability of exogenous variables that influence the determination of policy choices, but are otherwise excluded from some of the structural relationships. Suitable instruments will be variables observed retrospectively by the econometrician that are related to the private information of the firms, but independent of market-level shocks. Note that leads and lags of observed exogenous variables, and nonlinear transformations of predetermined variables, are not in general potential instruments for this problem. Under the Markov assumptions, lagged exogenous variables should not be predictive given current values. Leading exogenous variables should also not be predictive, since otherwise the players would form these predictions and incorporate them into their behavior. Finally, nonlinear transformations of predetermined variables are valid instruments only if one can be confident of the functional specification and their exclusion within this specification.

Consider in more detail the first-stage identification and estimation of the equation of motion,

$$(12) \quad s_{t+1} = h_t(s_t, x_t, z_t, \gamma_t, 0_t)$$

and the profit functions

$$(13) \quad B_{nt} = \nabla_{nt}(s_t, x_t, z_t, \gamma_t) - \cdot_{nt} \cong x_{nt} - >_n$$

under Assumptions 1-4. By Theorem 2, the MPE strategies of the firms are almost surely pure, and hence can be written as functions

$$(14) \quad x_{nt} = r_{nt}(s_t, z_t, \gamma_t, \cdot_{nt})$$

for  $n = 1, \dots, N$ . In the first stage of estimation, the structure of  $r_{nt}$  in (14) is not yet determined. Note however that  $h_t$  and the  $\nabla_{nt}$  and  $r_{nt}$  are time-invariant for  $t \in TN$ .

We consider identification and estimation of these equations in several cases. The first and most straightforward occurs when there are no shocks in the equation of motion that are known to the firms when they choose their policies; i.e.,  $\gamma_t$  is absent. Then, the observed policies  $x_{nt}$  in (12) are independent of the remaining shock  $\theta_t$ . If this shock is additive, then (12) can be estimated by nonlinear least squares. If the shock is not additive, then Matzkin (2003) shows that either a dimension-reducing functional restriction or normalization of  $r_{nt}$  (at a point) or the distribution of  $\theta_t$  is needed for identification, and gives applicable estimators when the equation is identified.<sup>9</sup> Estimation of the profit functions is not simple because even though  $\gamma_t$  is absent, the policy  $x_{nt}$  will depend on the shock  $\cdot_{nt}$ . Identification can be achieved if there are variables in  $z_t$  such as observed input prices of rivals that are excluded from  $\nabla_{nt}$  but influence  $x_{nt}$  via their impact on the equation of motion and consequently on the valuation function of firm  $n$ . Even when (13) is identified, estimation is complicated by the appearance of the disturbance in an additive bilinear form  $\cdot_{nt} \cong x_{nt}$  with the endogenous variables. However, orthogonality of functions of  $(s_t, z_t)$  and the additive disturbance  $\cdot_{nt} \cong x_{nt} + >_n$  allows GMM estimation, provided there enough excluded exogenous variables or functional restrictions to provide instruments with the rank of the parameter

vector. An easy bridge from this result to nonparametric estimation is through the method of sieves.

The MPE dynamic computation works off concrete numbers, and can be interpreted as being applied using the first stage parameter estimates plus a grid for all remaining parameters. The third stage then uses simulations from the MPE computations at each grid point in the parameter space, and uses a minimum distance criterion to pick the best grid point. The minimum distance criterion might be defined in terms of differences of observed and calculated state and policy variables over the observed partial trajectory, or in terms of the distance of the MPE first-order conditions from zero when evaluated at the observed policy variables. Nothing is essentially different if instead of using a fixed grid, one iterates between the MPE computation and minimum distance stages using some search algorithm in parameter space. One major difference between our formulation and that of Bajari *et al* (2003) is that they concentrate on models where the current payoff functions are linear in unknown parameters. That is not generally the case for our models.

In the stage 3 minimum distance estimation, one can embed the Chebyshev approximation procedure for the valuation functions in an iterative algorithm to estimate  $\beta$  to minimize a criterion defined as a sum of squared deviations of the predicted “as is” policies and states from their observed values for  $t = 1, \dots, T$ . The search is facilitated by the smooth behavior of the valuation functions and policy functions with small changes in  $\beta$ , so long as the solutions to Bellman’s equations are regular. Schematically, we have

$$(12) \quad V_{jt}(x, z, \beta) = A(x, z)B_{jt}(x, z, \beta),$$

where  $A(x, z)$  is the augmented polynomial approximation (6). Suppose one starts from a solution to the MPE problem at an initial  $\beta_1$ . Differentiate the first-order conditions for the Bellman equations and the equations of motion to determine the derivatives of the  $B$ ’s

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<sup>9</sup>An example of a functional restriction is an “index” model in which  $s_t, x_t, z_t$  enter  $h_t$  through a one-dimensional linear transformation with parametric weights.

and  $w$ 's with respect to 2, and plug these derivatives into the expression for the derivative of the estimation criterion with respect to 2. For convenience, the derivatives can be done numerically. Then, do a gradient search to minimize the criterion. In principle, it is not necessary to re-solve the original problem, but in practice, periodic resolving should be done to correct the cumulative drift in the approximation from the MPE solution.

A fourth, efficiency-improving stage in the estimation, not implemented in this version of the paper, would be to do a one-step GMM update of both the parameters in the equation of motion estimated in the first stage and the additional parameters estimated in the third stage, see Newey-McFadden (1984). A side benefit of this procedure is that it gives ready estimates of asymptotic standard errors of the parameter estimates.

Some studies using MPE models have found that it is computationally advantageous to use polynomial approximations for the policy functions as well as the valuation functions. We have not done so in this paper, but note that for problems where convergence of the numerical search to a fixed point is difficult, such approximations may facilitate solution. Bajari et al (2003) estimate policy functions as well as the equation of motion in the first stage, assuming that these functions are linear in parameters. Because these policy functions depend structurally on the valuation functions derived in the second stage, all one can hope to get at the first stage are (nonparametric) reduced forms. These may nonetheless be useful, as their estimated values may be proper instruments for the policy variables that appear endogenously in the equation of motion, and they may provide good starting values for second-stage computations.

### *Issues in Damage Estimation*

Once the MPE model has been estimated, so that it fits baseline “as is” trajectories over the observed history of the market, it can be used to estimate trajectories under alternative “but for” conditions where “bad acts” are excluded, and damages attributable to these “bad acts”. A few precautions, applicable to many damage studies, apply here.

First, prediction of trajectories in these models generally requires simulation of unobserved shocks. For example, the empirical distribution of the fitted shocks obtained in estimation of the model may be the basis for simulation draws. To reduce the variance of damage estimates, common draws should be used under baseline and “but for” conditions.

Second, if the natural experiment provided by the observed market trajectory is insufficient to identify reliably the demand response to changes in “bad acts”, then it may be necessary to use external studies, such as market research studies, to estimate these effects.

Third, the statistical reliability of damage estimates should be provided as part of the analysis. In principle, the delta-method can be applied to linearize the model in its parameters around their fitted values, and this linearization can be combined with the asymptotic covariance estimate for the parameter estimates from the estimation phase to produce standard errors for the damage estimates. In practice, a bootstrapping procedure, with a common resample used for both baseline and “but for” scenarios, is likely to do a better job of capturing the second-order non-linearities of model, and is likely to be more robust

#### **4. The Browser Wars: Netscape versus Microsoft**

In this paper we examine the market for web browsers. Web browsers are software programs that allow the user to view over the Internet a particular type of web content called Hyper Text Markup Language (HTML). In early 1993, Marc Andreessen and a group of fellow students developed the first graphical web browser, called Mosaic, while working for the National Center for Supercomputing Applications (NCSA) at the University of Illinois at Urbana-Champaign (UIUC).<sup>10</sup> Word about Mosaic spread rapidly

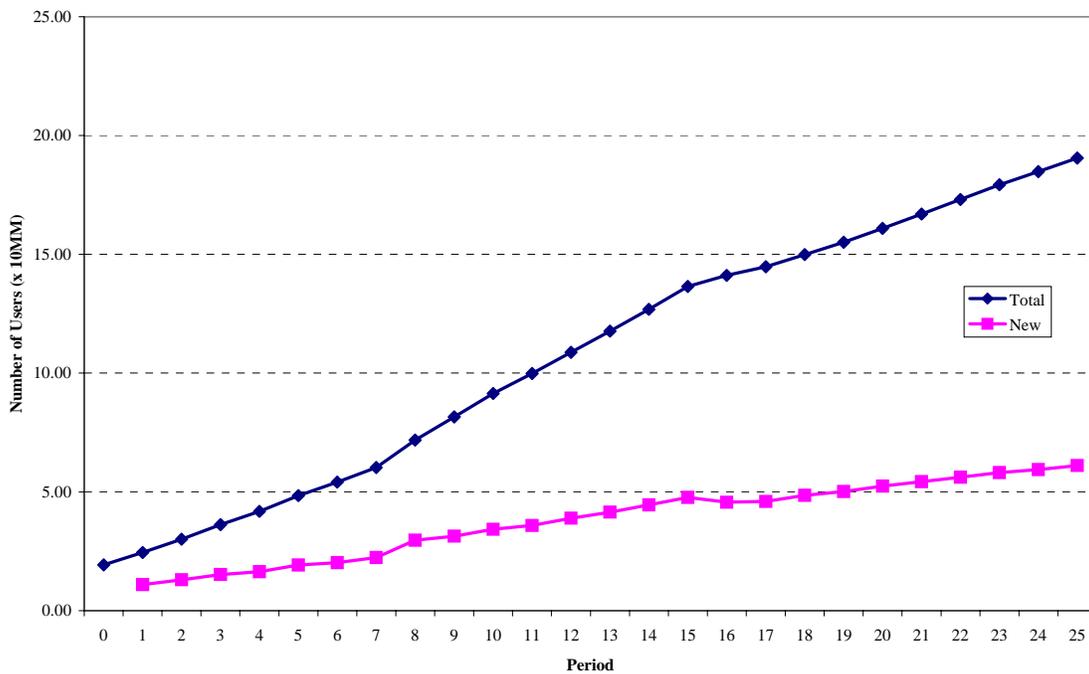
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<sup>10</sup> Cusumano and Yoffie, p. 3.

and by the end of 1994, Mosaic had been downloaded about two million times.<sup>11</sup> The introduction into the marketplace of a graphical browser was pivotal in spurring interest in the Internet; see Figure 1 below (the periods are quarters, with Period 0 representing the first quarter of 1996, and Period 25 representing the second quarter of 2002). By the second quarter of 2002 there were over 190 million Internet users as measured by IDC.

Figure 1

Number of Internet Users



Source: IDC.

<sup>11</sup> Cusumano and Yoffie, p. 95. The figures include other methods of obtaining Mosaic, but downloads were the dominant form of distribution.

In April 1994, Andreessen partnered with James Clark to form Netscape Communications.<sup>12</sup> The Netscape team developed a commercial Internet browser, called Netscape Navigator, which had a beta release in October 1994 and an official release in December 1994. Navigator was immediately successful. By February 1995, it had captured 60 percent of the fledgling market, and it was the leader of the “Internet Revolution”.<sup>13</sup> By January, 1996, its share of the browser market was 90 percent.<sup>14</sup> Navigator’s quality was a large factor in its success<sup>15</sup>; another contributing factor was Netscape’s innovative “free but not free” concept in which fully functional versions of Navigator could be downloaded for a free trial period and then users were asked but not required to pay \$39 per copy.<sup>16</sup>

Microsoft was late in entering the internet software business. Although it did plan to introduce a rudimentary browser in its Windows 95 operating system, it apparently did not initially perceive internet software as a significant market or threat to its Windows operating system core business.<sup>17</sup> Nonetheless, in June 1995, Microsoft and Netscape had a meeting at which Microsoft offered Netscape the option to either enter into a “special relationship” with Microsoft and agree to develop only non-Windows 95 browsers, or be regarded as a competitor and treated as such.<sup>18</sup> Netscape declined to enter into an agreement with Microsoft, as doing so would have foreclosed them from developing for Microsoft’s soon-to-be ubiquitous Windows platform. In December 1995, Microsoft CEO Bill Gates publicly declared that Netscape had “awakened a sleeping giant” and that Microsoft would reverse course and become “hard core” on “embracing

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<sup>12</sup> Cusumano and Yoffie, p. 7. The company was originally called Mosaic Communications.

<sup>13</sup> Cusumano and Yoffie, p. 7, 107.

<sup>14</sup> Cusumano and Yoffie, p. 9.

<sup>15</sup> Netscape was consistently rated as the best browser product then on the market. Cusumano and Yoffie, p. 96.

<sup>16</sup> Cusumano and Yoffie, p. 98.

<sup>17</sup> Cusumano and Yoffie, p. 9. Microsoft planned to incorporate a rudimentary web browser into the forthcoming release of Windows 95 but otherwise little apparent attention was paid to the Internet. Cusumano and Yoffie, pp. 107-108. Microsoft later changed its tune, realizing that the application programming interfaces (APIs) exposed by Netscape Navigator could potentially allow software developers to create programs that only required the presence of Navigator (and not Windows) to run. [Cite]

and extending” the Internet.<sup>19</sup> Thus began the “browser wars.”<sup>20</sup> Over the years that followed, Netscape and Microsoft would repeatedly one-up each other with new and improved versions of their respective browsers; see Table 1 below.

Table 1

<b>Windows IE</b>		<b>Netscape</b>	
<u>Version</u>	<u>Final Release</u>	<u>Version</u>	<u>Final Release</u>
		1.0	Dec 1994
		1.1	Apr. 1995
1.0	Aug. 1995		
2.0	Nov. 1995		
		2.0	Mar. 1996
3.0	Aug. 1996	3.0	Aug. 1996
		4.0	Jun. 1997
4.0	Oct. 1997		
		4.5	Oct. 1998
5.0	Mar. 1999		
5.5	Jul. 2000		
		6.0	Nov. 2000
6.0	Oct. 2001		
		7.0	Aug. 2002

Source: <http://www.blooberry.com/indexdot/history/brc>

In bundling Internet Explorer with every copy of Windows, in making Internet Explorer separately available for free downloading, and in investing substantial resources into improving the quality of its browser (see Figure 2 below), Microsoft sought to “cut off Netscape’s air supply”<sup>21</sup> and undermine Netscape’s “free but not free” strategy.<sup>22</sup> An oft-cited figure in the DOJ’s antitrust case against Microsoft (discussed below) held that Microsoft invested approximately \$100 million a year.<sup>23</sup> Netscape’s *total* annual investment (including its investment in server software) was at most a third of that amount, with its

<sup>18</sup> Findings of Fact, ¶¶82-89.

<sup>19</sup> Cusumano and Yoffie, pp. 109-110.

<sup>20</sup> [Cite Cusumano and Yoffie]

<sup>21</sup> David A. Kaplan, *Silicon Boys*, p. 276. In January 1998, Netscape was finally forced to lower the price of its browser to nothing. Cusumano and Yoffie, p. 144.

<sup>22</sup> Cusumano and Yoffie, p. 141.

23[CITE]

browser investment amounting to perhaps no more than a quarter of the total, or approximately \$10 million.<sup>24</sup> The initial versions of Internet Explorer lacked the full functionality of Netscape Navigator, and it was common for consumers to download Navigator in preference to Internet Explorer even when IE came pre-installed on their new computers. However, with Version 3.0 of IE released in August 1996, the two browsers were generally viewed by reviewers and the public as comparable in features and quality. Nevertheless, Microsoft apparently felt that these actions might not be sufficient to dethrone Netscape from its dominant position in the browser market. As a result, Microsoft also attempted to cut off the primary distribution channels through which Netscape distributed its browsers.<sup>25</sup> First, Microsoft refused to unbundle Internet Explorer from Windows, making it unlikely that consumers and OEMs would seek alternative browsers.<sup>26</sup> Second, Microsoft entered into agreements with many Internet Service Providers that restricted their ability to distribute alternative browsers such as Netscape Navigator.<sup>27</sup> Third, Microsoft entered into agreements with Online Services, Internet Content Providers, and Independent Software Vendors that induced greater use of Internet Explorer. Following these actions by Microsoft, Netscape's market share slid steadily downward, while Microsoft's market share grew steadily upward. Figure 3 shows the two firm's market shares. Figure 4 shows that during this period, the total number of computers and the installed customer base for each browser grew rapidly, with Microsoft benefitting from its control of the distribution channels. In January 1998, Netscape was finally forced to lower the official price of its browser to zero.<sup>28</sup> By the time Microsoft's restrictions on OEMs and ISP agreements were lifted in 1998, the browser war was effectively over.<sup>29</sup> Even though Netscape still had 58 percent market share to Microsoft's 40 percent,<sup>30</sup> Microsoft dominated the distribution channels and had gained all the momentum. The actions by Microsoft against Netscape were the focus in the U.S.

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<sup>24</sup> Authors estimates, based on readings of Cusumano and Yoffie. The authors believe that while Netscape initially focused almost exclusively on browser development, later attempts to enter various server software markets led to sharp declines in browser investment as a percent of total investment.

<sup>25</sup> Findings of Fact, ¶143.

<sup>26</sup> Findings of Fact, ¶159. Microsoft also removed in Windows 98 the ability to uninstall Internet Explorer and prohibited modification of the Windows boot sequence to promote non-Microsoft software. Findings of Fact, ¶170, 204.

<sup>27</sup> Findings of Fact, ¶247.

<sup>28</sup> Cusumano and Yoffie, p. 144.

<sup>29</sup> [CITE source]

<sup>30</sup> Figures from <http://www.cen.uiuc.edu/bstats/latest.html>.

Department of Justice's antitrust action against Microsoft in 1997.<sup>31</sup> However, even though the District Court and the Court of Appeals found many of the Microsoft actions to be illegal,<sup>32</sup> the final settlement between the U.S. Department of Justice and Microsoft in November 2002 provided little relief to Netscape.<sup>33</sup> Netscape filed a separate civil antitrust suit against Microsoft in January 2002.<sup>34</sup> That suit was settled in May 2003 for \$750 million.<sup>35</sup> Today, Microsoft has 86 percent of the browser market and Netscape has only 9 percent.<sup>36</sup>

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<sup>31</sup> [Cite DOJ Complaint.]

<sup>32</sup> [Cite Findings of Fact, Appeal Court Opinion]

<sup>33</sup> See *Final Judgment*, viewed at [http://www.usdoj.gov/atr/cases/ms\\_index.htm](http://www.usdoj.gov/atr/cases/ms_index.htm) on March 4, 2004.

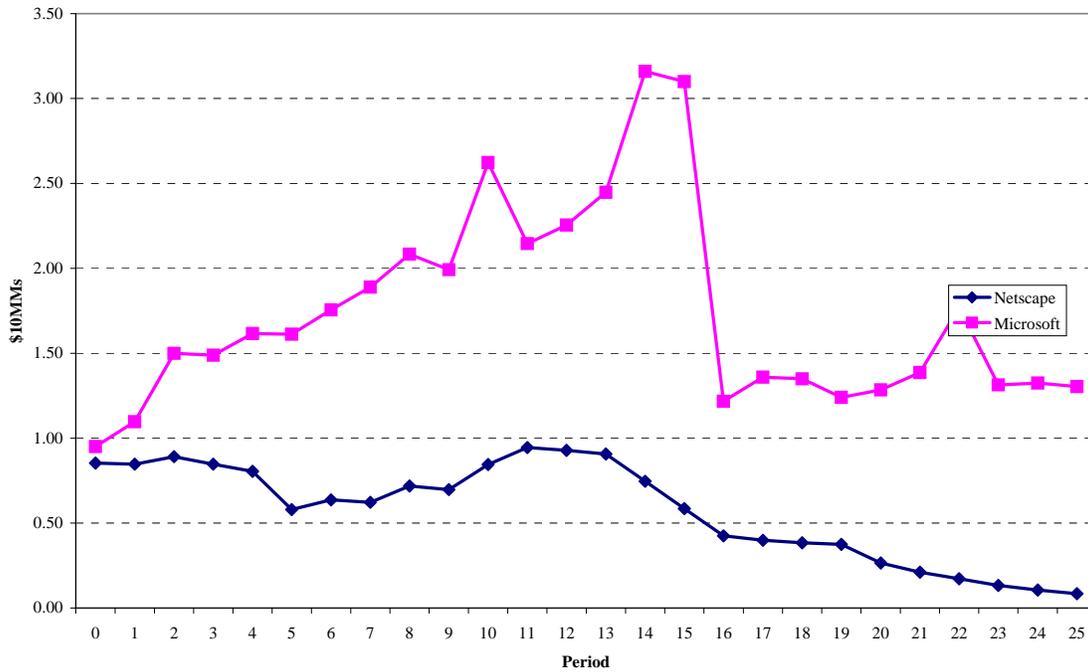
<sup>34</sup> See *Complaint* in *Netscape v. Microsoft*, viewed at <http://legal.web.aol.com/decisions/dlother/Netscape%20v.%20Microsoft.pdf> on March 4, 2004.

<sup>35</sup> See <http://money.cnn.com/2003/05/29/technology/microsoft/>.

<sup>36</sup> Figures from <http://www.cen.uiuc.edu/bstats/latest.html>.

Figure 2

Single-Period Browser Investment



Source: various, authors' guesstimates. This figure should not be construed as having been based in large part on actual data. Do not cite.

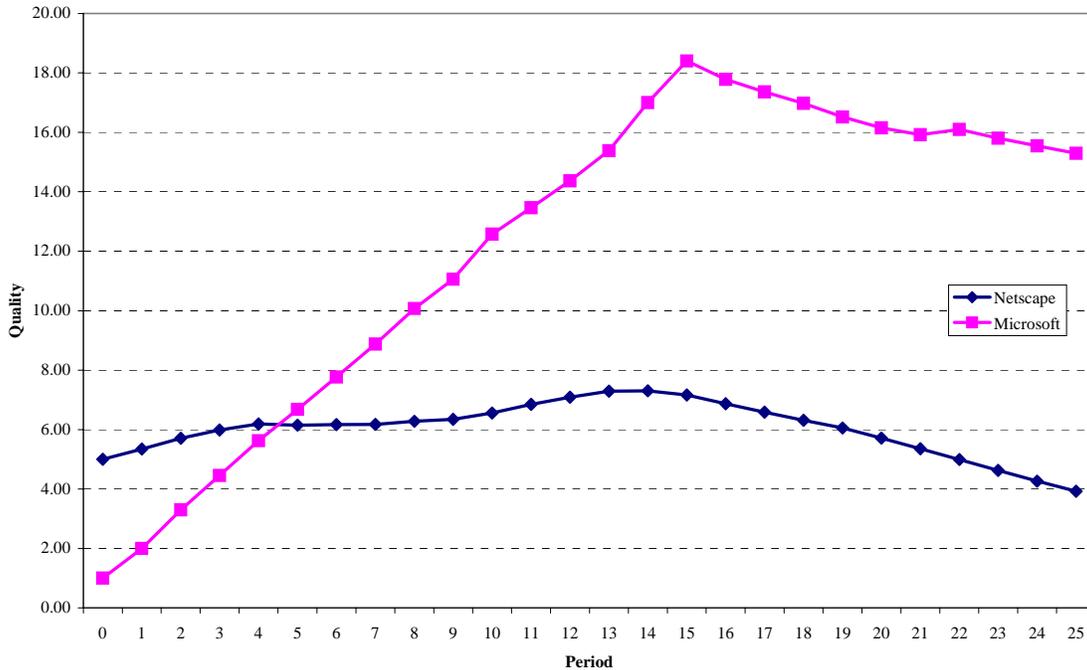
Gradually, Microsoft's substantial investment in browser technology led to an improved browser that was perhaps neck and neck with Netscape's browser in head-to-head comparisons in computer magazines.<sup>37</sup> While cumulative investment (accounting for depreciation) is a highly imperfect measure of product quality, by such a measure Microsoft was able to pull ahead in the browser quality race within four or five quarters. See Figure 3 below.

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37 [Cite Schmalensee report.]

Figure 3

Accumulated Browser Investment (Quality)



Source: single-period investment data, authors' depreciation estimates. Do not cite.

This figure should not be construed as having been based in large part on actual data.

Microsoft nevertheless apparently felt that these actions might not be sufficient to dethrone Netscape from its dominant position in the browser market before it began to be used extensively as an alternative to the Windows platform for applications software developers. As a result, Microsoft also attempted to cut off the primary distribution channels through which Netscape distributed its browsers.<sup>38</sup> First, Microsoft refused to unbundle Internet Explorer from Windows, making it unlikely that consumers and OEMs would seek alternative browsers.<sup>39</sup> Second, Microsoft entered into agreements with many Internet Service Providers that restricted their ability to distribute alternative browsers such as Netscape Navigator.<sup>40</sup> Third, Microsoft entered into agreements with Online Services, Internet Content Providers, and

<sup>38</sup> Findings of Fact, ¶143.

<sup>39</sup> Findings of Fact, ¶159. Microsoft also removed in Windows 98 the ability to uninstall Internet Explorer and prohibited modification of the Windows boot sequence to promote non-Microsoft software. Findings of Fact, ¶170, 204.

<sup>40</sup> Findings of Fact, ¶247.

Independent Software Vendors that induced greater use of Internet Explorer. Following these actions by Microsoft, Netscape's market share slid steadily downward, while Microsoft's market share grew steadily upward. By the time Microsoft's restrictions on OEMs and ISP agreements were lifted in 1998, the browser war was effectively over.<sup>41</sup> Even though Netscape still had 58 percent market share to Microsoft's 40 percent,<sup>42</sup> Microsoft dominated the distribution channels and had gained all the momentum. The actions by Microsoft against Netscape were the focus in the U.S. Department of Justice's antitrust action against Microsoft in 1997.<sup>43</sup> However, even though the District Court and the Court of Appeals found many of the Microsoft actions to be illegal,<sup>44</sup> the final settlement between the U.S. Department of Justice and Microsoft in November 2002 provided little relief to Netscape.<sup>45</sup> Netscape filed a separate civil antitrust suit against Microsoft in January 2002.<sup>46</sup> That suit was settled in May 2003 for \$750 million.<sup>47</sup> Today, Microsoft has 86 percent of the browser market and Netscape has only 9 percent.<sup>48</sup> See Figure 4 below for the browser market share history of these two firms.

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<sup>41</sup> Some sources placed the "end" in 1999 (see, e.g., Scott Bekker, "Zona Research: Browser War Over", 11/8/99, <http://www.entmag.com/news/article.asp?EditorialsID=1415>.)

<sup>42</sup> Figures from <http://www.cen.uiuc.edu/bstats/latest.html>.

<sup>43</sup> [Cite DOJ Complaint.]

<sup>44</sup> [Cite Findings of Fact, Appeal Court Opinion]

<sup>45</sup> See *Final Judgment*, viewed at [http://www.usdoj.gov/atr/cases/ms\\_index.htm](http://www.usdoj.gov/atr/cases/ms_index.htm) on March 4, 2004.

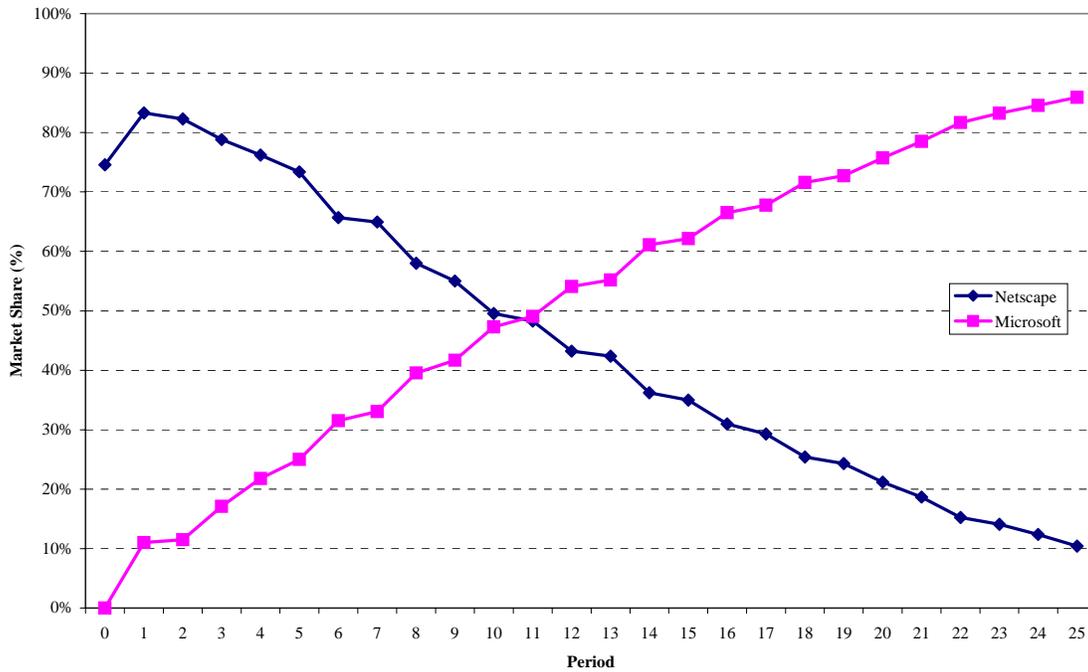
<sup>46</sup> See *Complaint in Netscape v. Microsoft*, viewed at <http://legal.web.aol.com/decisions/dlother/Netscape%20v.%20Microsoft.pdf> on March 4, 2004.

<sup>47</sup> See <http://money.cnn.com/2003/05/29/technology/microsoft/>.

<sup>48</sup> Figures from <http://www.cen.uiuc.edu/bstats/latest.html>.

Figure 4

Base Market Share



Source: UIUC data (<http://www.cen.uiuc.edu/bstats/latest.html>)

## 5. Motivating the Econometric Modeling

The question of how to estimate the impact of Microsoft's illegal actions raises the question as to what constitutes a proper model for analysing firm behavior in this market.<sup>49</sup> Traditional static industrial organization models fail to capture network effects and dynamic features of the market such as the importance of investment on future profits; as a result, a dynamic market model is likely to be necessary. In this paper, we have developed and estimated a dynamic Markov Perfect Equilibrium (MPE) model which is motivated by the following browser market characteristics:

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<sup>49</sup> We were engaged by Netscape to calculate damages in *Netscape v. Microsoft*.

1) **Firms.** The browser market consisted primarily of Netscape's Navigator and Microsoft's Internet Explorer for most of its history. While there have been other players in the market (e.g., Opera, Mozilla), these players tend to be minor. However, these minor players constituted a fringe that would have limited the ability of Netscape and Microsoft to raise prices substantially.

2) **Browser Acquisition and Use.** An important distinction in browsers is between acquisition by either pre-installation or initial download, or subsequent download of new versions, and use. This is potentially a rather complex consumer decision problem, as acquisition choices may be made with the option value of future choices and expectations of future environments in mind. However, it seems fairly realistic to simplify this drastically, and assume that consumers make a single initial browser choice at the time a new PC or new operating system is purchased, and then use this chosen browser for the life of the PC. We will assume further that the consumer makes the initial browser choice myopically on the basis of costs and qualities prevailing at the moment. (Note that this assumption is also consistent with an MPE model of the consumer, making the consumer's policy decision a function only of current state. However, we will not exploit this interpretation.) Assume that when both browsers are available pre-installed, the consumer simply chooses the one that will be used. Similarly, when neither are available pre-installed, the consumer simply downloads the preferred browser. Finally, when only one browser is available pre-installed, the consumer then decides whether to use this browser, or to abandon it and download the other browser (at an added cost).

3) **Demand Equation – Inclusion of Network Effect.** The browser market is characterized by strong network effects. As the Findings of Fact stated, “[Microsoft] believed that a comparable browser product offered at no charge would still not be compelling enough to consumers to detract substantially from Navigator’s existing share of browser usage. This belief was due, at least in part, to the fact that Navigator already

enjoyed a very large installed base and had become nearly synonymous with the Web in the public's consciousness."<sup>50</sup>

4) **Demand Equation – Exclusion of Price.** Price was not a primary consideration in choice of browser. Published browser price did not reflect the actual price typically paid by a user, as users mostly did not pay the voluntary license fee. Netscape received only a small revenue stream from browser licenses. Once Microsoft entered the market and priced its browser at zero, Netscape found it had to respond by doing similarly, thus extinguishing browser licenses as a revenue source. One aspect of browser pricing is that the effective price to the consumer exceeded the license price by the cost of downloading and installing the software (if required). This is potentially relevant in assessing consumer response to exclusion of Netscape from distribution on new computers. It would be useful to include prices in the model if predation were an issue and estimates of market prices under “but for” conditions were needed. However, that was not a focus of the legal case, and we do not include prices in the model analysed in this paper.

5) **Demand Equation – Inclusion of Investment.** Beyond the network effect, consumers cared primarily about the quality of their Internet experience. Newspaper and magazines regularly reviewed the then-current versions of Netscape Navigator and Internet Explorer in head-to-head matchups. Technologically-savvy computer users promoted the browser they viewed most favourably.<sup>51</sup> Quality is determined to a large extent on the amount of investment placed in upgrading each incremental version of the browser.

6) **Profit Equation – Inclusion of Derivative Revenues per Browser.** Browser revenues were always a relatively small portion of the business plan. Netscape intended to leverage its success in the browser business into other sources of revenue such as server software sales and advertising revenue from its Internet portal.<sup>52</sup> Microsoft could use Internet Explorer

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<sup>50</sup> Findings of Fact, ¶143.

<sup>51</sup> For example, XXXX

<sup>52</sup> Cusumano and Yoffie, p. 25, state “The key was to build market share and create the standard. Profits would eventually follow.” They also state, at p. 7, that “Initially, Netscape’s business model

to solidify its Windows monopoly and establish its own Internet portal. Thus Internet browsers can perhaps be viewed as a software with no direct revenues but with substantial potential revenues via linkages to other markets. These indirect revenues would likely be in direct proportion to the number of browsers, and could potentially be larger for Microsoft than for Netscape (to reflect the benefits to Microsoft's Windows monopoly).

## 6. The Model

The above discussion motivates a model in which

- 1) Market share for browsers is a function of investment and previous market share, reflecting quality and network effects.
- 2) Profits are a function of fixed revenues per user (with different values for Microsoft and Netscape) multiplied by market share for browsers multiplied by market size, less investment.
- 3) Each firm's choice variable is investment.

The following paragraphs give the details of the model specification. Table 2 summarizes the variables that appear.

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called for developing two sets of products—the browser, which would catapult Netscape to fame, and Web servers, which would pay the company's bills.”

Table 2. List of Variables		
Symbol	Type	Description
$j = 0,1$		firms
$t = 0,1,2,..$		time periods (months from IE launch)
$b_j(t)$	S	installed base of browsers for firm $j$ , $j=0,1$
$q_j(t)$	S	quality of browser for firm $j$ , $j=0,1$
$i_j(t)$	PO	current investment in product development for firm $j$ , $j=0,1$
$N(t)$	XO	number of installed browsers
$n(t)$	XO	number of consumers choosing new browser
$s_j(t)$	XO	share of consumers choosing firm $j$ 's browser, $j=0,1$
$r(t)$	XO	revenue per browser used
$a_k(t)$	XO	intensity of Microsoft's bad acts, $k=0,1,2$ ; 0=IAP, 1=OEM, 2=Tying
$\gamma(t)$	XU	preference-side, market-level demand shifter

Types: State (S), Policy (P), Exogenous (X), Observed (O), Unobserved (U)

This model has four state variables: installed base of browsers for each firm  $b_j(t)$  and browser quality for each firm  $q_j(t)$ . Quality is determined through an accounting relationship by the competing investments in development by the two firms, and installed base is determined through consumer choice. The policy variables determined within the model are the investments in development,  $i_0(t)$  and  $i_1(t)$ .

The model has observed exogenous variables,  $N(t)$ ,  $r(t)$ , and  $a_k(t)$ , and one unobserved policy variable  $\gamma(t)$ , that are known to the firms when they make their investment policy decisions in a period. The bad acts “intensity” variables,  $a_k(t)$ , are assumed to be equal to the percentage of consumers that are affected by each act. The intensity of the IAP act was calculated as the number of new subscribers to affected IAPs (i.e. IAPs that were involved in restrictive agreements with Microsoft) divided by the total number of Internet users. The intensity of the OEM act in a given period was calculated as the number of computers shipped by affected OEMs divided by the total installed base of computers. The intensity of the tying act was calculated as the number of Windows licenses sold through the finished goods channel (i.e. not pre-installed on an OEM computer) divided by the total number of Windows users.

Past the period of observation, all exogenous variables are assumed to be i.i.d. with distributions that reflect their characteristics during the period of observation. We assume that they are always treated by the firms as i.i.d. random variables over time, so that firms form expectations with respect to these variables when valuing the future.

Consumers’ decisions on acquisition and retirement of browsers are exogenous to the browser market. The number of consumers choosing a browser in each period is

$$(1) \quad n(t) = N(t) - (1-d_b)N(t-1),$$

where  $d_b$  denote the rate at which existing browser users reconsider their browser selection.

The share of new users choosing browser 0,  $s_0(t)$ , is determined by the logistic

$$(2) \quad s_0(t) = 1 / (1 + \exp(-x)),$$

where

$$x = \exp[2_0 + 2_1(b_0(t-1) - b_1(t-1)) + (2_2q_0(t-1) - 2_2q_1(t-1)) - 2_3a_1(t) - 2_4a_2(t) - 2_5a_3(t) + \gamma(t)].$$

The share of new users choosing browser 1,  $s_1(t)$ , is then

$$s_1(t) = 1 - s_0(t).$$

The installed base for each firm satisfies the equation of motion

$$(3) \quad b_j(t) = (1 - d_b)b_j(t-1) + n(t)s_j(t).$$

The quality variable, reflecting cumulative investment, satisfies the equation of motion

$$(4) \quad q_j(t) = (1 - d_q)q_j(t-1) + i_j(t),$$

The profit of each firm in period  $t$  is

$$(5) \quad B_j = r(t)b_j(t) - i_j(t) - \delta i_j(t)^2.$$

The first term in the profit function reflects the advertising and other revenue earned as a result of having an installed base of browser users, the second reflects the firm's dollar investment in browser development, and the third reflects the frictional costs associated with large scale R&D.

Because  $b_j(t)$  depends only on lagged or exogenous state and policy variables, current profits vary directly only with investment. The benefit of investment comes only through the impact of investment on future market shares.

The valuation functions of the firms are the expected (with respect to  $r, a, \gamma$ ) present value (with an observed discount factor  $\Xi$ ) of the stream of period profits. These expected present values at time  $t-1$  of profits from time  $t$  forward are functions of the state variables  $b_j(t-1)$ , and  $q_j(t-1)$ .

## 7. Data

Estimation of the model requires the following data:

1) **Number of Internet Users** - Data on number of Internet users were obtained from IDC. It is assumed that every Internet user utilizes exactly one browser, meaning the number of Internet users is equal to total installed base of browser users.

2) **Browser Market Shares** - Browser market shares were obtained from UIUC. UIUC measures market share by counting the number of hits registered by each browser on UIUC servers. For this model, it is assumed that these shares are reflective of total shares of Internet users using each browser.

3) **Browser Investment** - Browser investment for Netscape was estimated based on total firm-wide investment from analyst reports and on the authors' assumptions about the percentage of firm-wide investment devoted to browser development. These assumptions were based in part on anecdotal evidence from Cusumano and Yoffie. Browser investment for Microsoft was estimated based on anecdotal evidence from the Findings of Fact in the DOJ antitrust case.

4) **Revenue per browser** - Revenue per browser was based on Internet advertising revenue trends, and for Microsoft, a percentage of operating system revenues, which may have been at risk if another browser gained ubiquity.

5) **Illegal acts** - The bad act intensities were calculated as described in the previous section. For the IAP act, data on IAP subscriptions were obtained from AdKnowledge. For the OEM act, data on OEM shipments and installed base of computers were obtained from IDC. For the Tying act, number of Windows licenses sold as finished goods based on authors' estimates.

6) **Depreciation Rates** - The depreciation rate of installed base is assumed to be 0.30, the same as the depreciation rate of the experience base for MS Word, Excel and Windows as calculated in Hall (2004). The depreciation of accumulated investment is assumed to be 0.10 based on the relationship between incremental investment and quality ratings.

## 8. Model Results

The initial two stage least squares estimate of the equation of motion, instrumenting investment with third-order polynomials in base and quality generated the results in Table 2 below.

Table 2

Instrumental variables (2SLS) regression

Source	SS	df	MS			
Model	35.9385812	6	5.98976353	Number of obs =	25	
Residual	1.59403952	18	.088557751	F( 6, 18) =	68.41	
Total	37.5326207	24	1.5638592	Prob > F =	0.0000	
				R-squared =	0.9575	
				Adj R-squared =	0.9434	
				Root MSE =	.29759	

logss1	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
invest	.3493986	.2443484	1.43	0.170	-.1639584	.8627556
base	.1436096	.027166	5.29	0.000	.086536	.2006832
quality	.0369319	.0903374	0.41	0.687	-.1528599	.2267238
iap	12.78367	16.08325	0.79	0.437	-21.00598	46.57332
oem	-1.869162	4.938787	-0.38	0.710	-12.24517	8.506844
tie	-1520.734	2139.396	-0.71	0.486	-6015.439	2973.971
_cons	.2219995	1.231375	0.18	0.859	-2.365023	2.809022

Instrumented: invest

Instruments: base quality iap oem tie base1 base2 base3 qual1 qual2 qual3

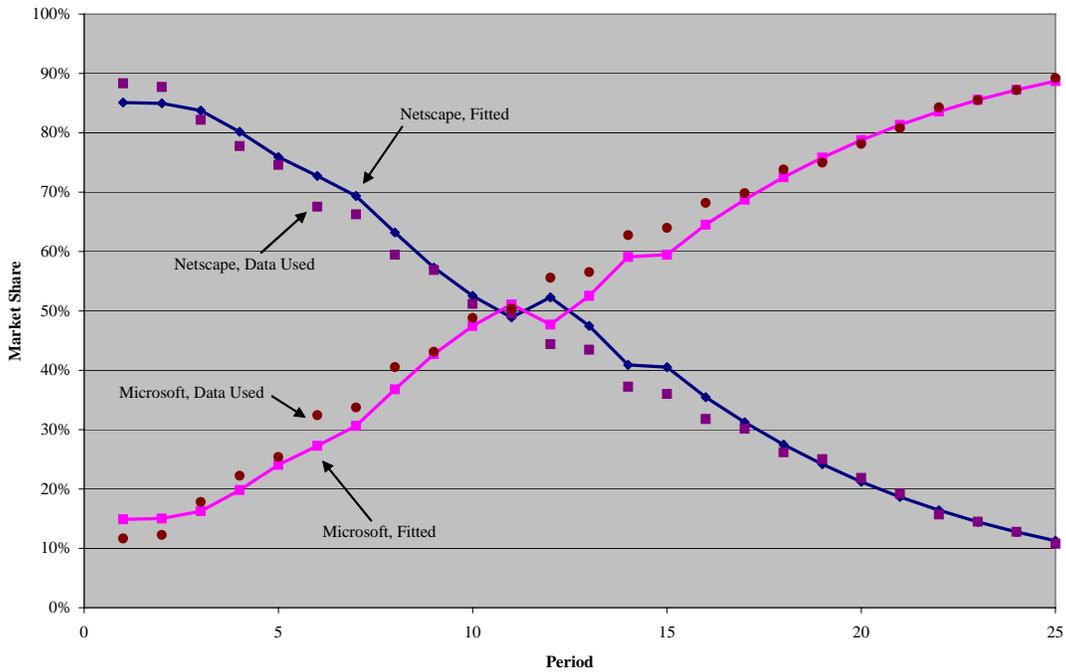
Note: Very preliminary result. Do not cite. This table should not be construed as having been based in large part on actual data.

Most of the coefficients in this estimation are not statistically significant. However, the coefficient for base is highly statistically significant, and of the expected sign: In other words, a greater base is likely to lead to a greater market share, *ceteris paribus*. In addition, the instrumented investment variable, while not statistically significant, has the expected sign that an increase in a firm's investment is predicted to lead to an expected increase in market share for that firm.

In Figure 5 below, we have plotted the predicted model market shares versus the data used for market share (these are the installed base market shares, not the incremental market shares) [NOTE: THIS FIGURE AND THE FIGURES THAT FOLLOW WERE CONSTRUCTED USING OLDER PARAMETER ESTIMATES AND WILL BE UPDATED IN THE NEXT DRAFT]. As one can see, the MPE model predicts the market shares reasonably well. At this time, we are in the process of refining our 2sls coefficient estimates by fitting the predictions of the model to the data we use.

Figure 5

Installed Base Market Share, Data Used vs. Fitted Data



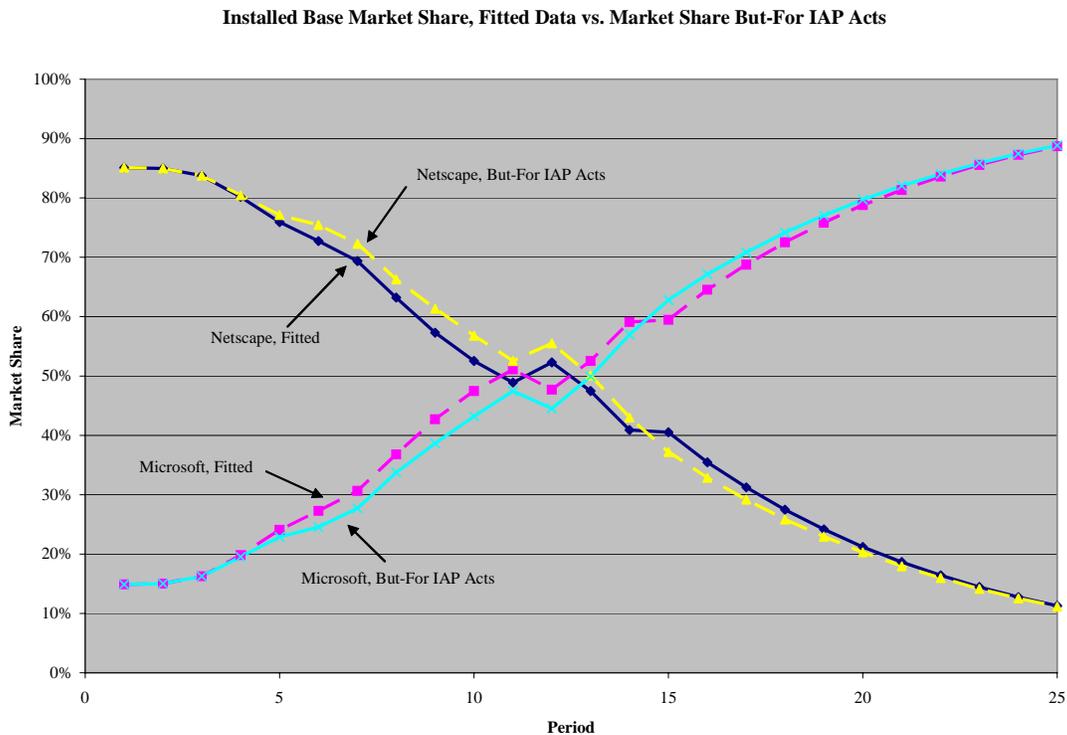
Note: Very preliminary result. Do not cite.

Of the “bad act” variables, the tying and oem acts do not have the expected positive sign (recall that the coefficient for the “bad acts” enters into Microsoft’s equation of motion: thus, a positive coefficient will in expectation *increase* Microsoft’s market share.) However, the iap bad acts do have the expected sign, albeit hardly significant. While these (highly preliminary) early results do not show a statistically significant effect of bad acts on the relative shares of Netscape and Microsoft, further work to more precisely quantify the “bad acts” could result in statistical significance. Our “bad acts” variables are currently crudely defined and may not be capturing the true impact of a properly measured “bad acts” variable.

Nevertheless, for purposes of illustrating the use of this model for predicting market share in the absence of bad acts, we have turned off the IAP variable, leaving the OEM

and tying variables intact (as well as the rest of the variables). Figure 6 below compares the predicted market shares with the IAP acts turned off to the predicted market shares with the IAP acts turned on. As one can see, the difference in the predicted market shares is not large. These results suggest that the IAP bad acts had relatively little impact on Netscape’s and Microsoft’s relative market shares, subject to all the caveats thus raised. [NOTE: THIS FIGURE WAS CONSTRUCTED USING OLDER PARAMETER ESTIMATES AND WILL BE UPDATED IN THE NEXT DRAFT].

Figure 6

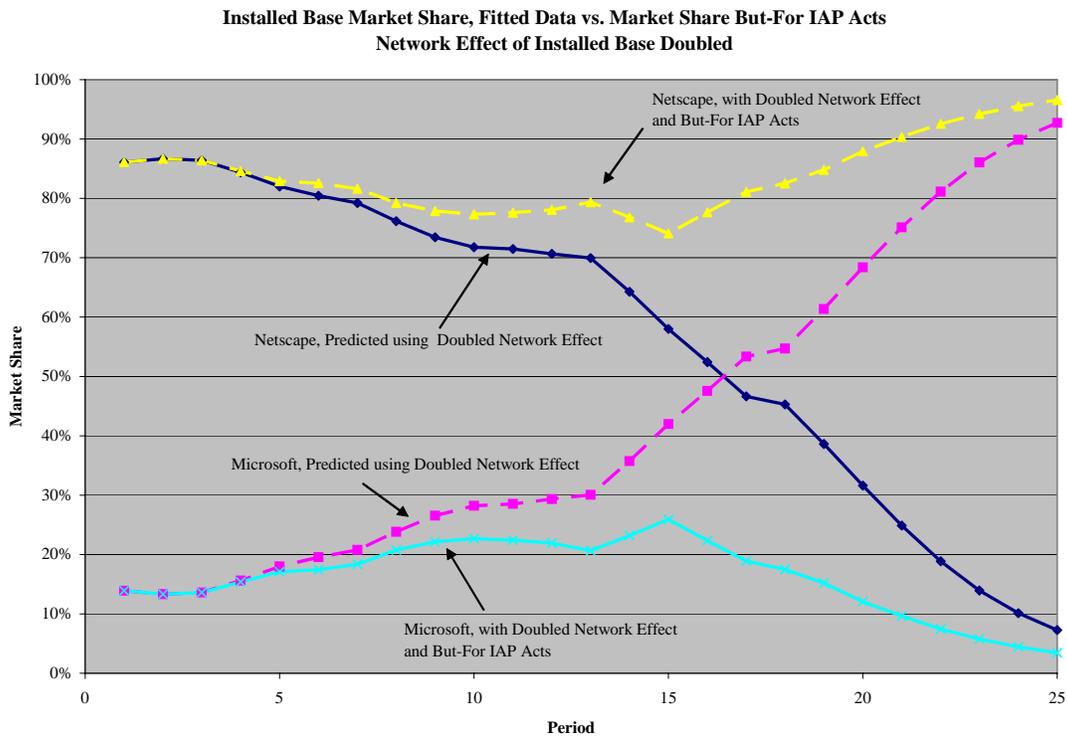


Note: Very preliminary result. Do not cite.

Though at this time we do not observe any substantial change in market share resulting from Microsoft bad acts (subject to the caveats raised), the model does not preclude the finding of a “market tipping.” To demonstrate this result, we only have to double the size of the “base” effect on market share in the equation of motion and then turning on and off

the IAP “bad act” generates the result in Figure 7 below. [NOTE: THIS FIGURE WAS CONSTRUCTED USING OLDER PARAMETER ESTIMATES AND WILL BE UPDATED IN THE NEXT DRAFT].

Figure 7



Note: Very preliminary result. Do not cite.

Had these coefficients been what we had found from the estimation, we would have found that the effect of the IAP “bad acts” was to tilt the market to Microsoft’s benefit. Without the IAP “bad act” in this scenario, Netscape would have been able to rely on the network effects deriving from its dominant installed base and would have been able to maintain its dominant market share through the time period observed. Thus with only somewhat different sets of estimated coefficients, we could find a substantial effect of Microsoft’s bad acts on the relative market shares of Netscape and Microsoft.

## 9. Conclusions

The results of this paper show that it is possible, and even feasible, to use the tools of Markov-perfect equilibrium models combined with data in order to arrive at a unique Markov perfect equilibrium. Using an MPE model that was motivated by observed characteristics of the browser market, and using data relating to Netscape and Microsoft, we successfully estimated an MPE model and used it to provide predictions of alternate hypotheses. In particular, we used the model to predict what Netscape's and Microsoft's market shares would have been "but for" the IAP acts that Microsoft committed. Despite the finding by the Appeals Court in *U.S. v. Microsoft* of illegal acts, our model predicts that the IAP acts had relatively little impact on Netscape's but-for market share. However, this is a highly preliminary result and subject to many caveats which are discussed in the paper; work on more precisely quantifying the illegal acts and the other variables is ongoing. We also show that with a network effect on installed base that is only double what we estimate, that in such a world, absent the IAP "bad acts", the model predicts that Netscape would have been able to maintain its initial dominant position. Thus the full span of possible outcomes is observed in this model, and results will depend critically on the quality of the data and the specific magnitudes of the parameter estimates.

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