Allocations with Incomplete Information

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1 Introduction

Suppose the government wants to allocate a given amount of funds between two entities (e.g. municipalities, villages, or families) in a way that maximizes their weighted welfare for a given set of weights. It can achieve this by directly allocating the resources or by delegating the task to two independent organizations (e.g. NGOs, charities, or local politicians) that also allocate resources by maximizing the two entities' welfare. The two organizations can be of one of two types: (i) with some probability, they assign a higher weight to the first region than the government; (ii) with the remaining probability they assign a lower weight to that region. The organizations know their own type, but not the type of the other one, only the probability distribution. If each one of the two organizations were to allocate the entire budget on its own, the organizations' weights are such that, on average, they would allocate the funds in the same way as the government. But, what is the outcome in the presence of uncertainty about the other organization's type? On average, do the organizations allocate the funds in the same way as the government? Do they allocate systematically more or less to one of the entities?

The main contribution of the paper is to answer the previous questions. We show that for all welfare functions that belong to the Hyperbolic Absolute Risk Aversion (HARA) class, the two organizations always allocate resources differently from the government's allocation. We also show that the difference always go in the same direction: the region with the higher weight according to the government always receives a smaller share of resources than the government's allocation would entail. As the HARA class includes the most commonly used welfare functions – logarithmic, constant absolute risk aversion, constant relative risk aversion, and quadratic – the result is fairly general.

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There may be reasons for delegating the allocation of public funds to local organizations. Some examples are better knowledge of local conditions and lower costs of implementing the allocation. Moreover, if the local organizations are small in size, the government may decide to delegate the allocation to more than one of them. The paper's results indicate that delegating this task is costly as the entity more in need of resources will always receive less than it is optimal. Governments should therefore weigh the costs against the benefits of delegation.

Several papers have a connection to the paper's results. Finan and Mazzocco (2020) estimate a model in which politicians choose independently and simultaneously the allocation of a fixed budget across regions using data on decisions by federal politicians. They find that about one quarter of the transfers are missallocated. The two main reasons behind the missallocations are electoral incentives and the uncertainty about the other deputies' types, with the uncertainty accounting for about 50% of the missallocations. The main result of this paper indicates that the missallocations generated by type uncertainty is not a consequence of the parameter estimates that characterize their context, but a general result that applies to all HARA welfare functions.

There is a large literature that studies the efficient allocation of resources across individuals or households that started in the nineties with Altug and Miller (1990), Cochrane (1991), Mace (1991), Altonji, Hayashi, and Kotlikoff (1992), and Townsend (1994), and was studied more recently in Ogaki and Zhang (2001), Blundell, Pistaferri, and Preston (2008), Schulhofer-Wohl (2011), Mazzocco and Saini (2012), and Chiappori et al. (2014). This literature's main interests is mostly on the derivation of tests of efficient risk sharing under different settings, whereas the present paper studies the actual allocation of funds under incomplete information.

The literature on distributive politics focuses for the most part on the effect of elections on the distribution of resources (see for instance Myerson (1993), Lizzeri and Persico (2001), Atlas et al. (1995), and Rodden (2002)). Instead, this paper focuses on the impact of welfare considerations on the allocation of funds.

2 Optimal Allocation of Resources

Consider and economy with two regions and a number of organizations that are in charge of allocating Q funds across the regions. We will consider two cases. One in which a single organization, the social planner, is in charge of allocating Q. A second case in which two organizations are in charge of distributing half of the resources to the two regions. Region i has welfare function W_i. It depends on the amount of resources q^i received from the entities. Entity j assigns weight μ_j to Region 1 and $1-\mu_j$ to Region 2. The welfare function W_i is increasing and concave in the total amount of resources received.

We will assume that the two regions have identical welfare functions that belong to the Hyperbolic Absolute Risk Aversion (HARA) class, that is

$$W(q) = \frac{(\alpha q + \eta)^{\delta}}{\delta},$$

where δ determines the curvature of the welfare function and η the region's subsistence level. The most popular welfare functions belong to the HARA class: the Constant Absolute Risk Aversion (CARA), the Constant Relative Risk Aversion (CRRA) and, hence, the logarithmic, and the quadratic welfare functions. For the HARA welfare function to be well defined, δ must be different from 0. Moreover, $\alpha > 0$ and $\delta < 1$ are required for the HARA welfare function to be increasing and concave. In the CRRA case, the restriction $\delta < 1$ is equivalent to the standard assumption that the relative risk aversion parameter is positive.

Throughout the paper we will use the first, second, and third derivatives of the HARA welfare function. They take the following form:

$$W'(q) = \alpha (\alpha q + \eta)^{\delta - 1} > 0,$$

$$W''(q) = -\alpha^2 (1-\delta)(\alpha q + \eta)^{\delta - 2} < 0,$$

and

$$W'''(q) = \alpha^3 (1-\delta)(2-\delta)(\alpha q + \eta)^{\delta - 3} > 0,$$

where the inequality follows from $\delta < 1$.

3 Social Planner's Allocation

We consider first the optimal allocation of funds when only one entity, the social planner, is in charge of Q. Her or his objective is to maximize the total welfare in the economy by choosing the allocation of Q that solves the following problem:

$$\max_{q} \; \mu W(q) + (1-\mu) W(Q-q).$$

The optimal choice q_{sp}^{μ} is the solution to the following first order condition:

$$\mu W^{'}(q) - (1-\mu) W^{'}(Q-q) = 0.$$

For a HARA welfare function, the first order condition becomes

$$\mu\alpha(\alpha q+\eta)^{\delta-1} - (1-\mu)\alpha(\alpha(Q-q)+\eta)^{\delta-1} = 0. \tag{1}$$

or, equivalently,

$$\left(\frac{\alpha(Q-q)+\eta}{\alpha q+\eta}\right)^{1-\delta} = \frac{1-\mu}{\mu}. \tag{2}$$

By solving for q, we obtain the social planner allocation

$$q_{sp}^{\mu} = \frac{\mu^{\frac{1}{1-\delta}}}{\mu^{\frac{1}{1-\delta}} + (1-\mu)^{\frac{1}{1-\delta}}} Q + \frac{\mu^{\frac{1}{1-\delta}} - (1-\mu)^{\frac{1}{1-\delta}}}{\mu^{\frac{1}{1-\delta}} + (1-\mu)^{\frac{1}{1-\delta}}} \frac{\eta}{\alpha}.$$

The social planner's choice has intuitive features. Everything else equal, the region with larger weight receives more resources, but the importance of the weights declines with the curvature parameter δ. The subsistence level has also the expected impact of increasing the amount of allocated resources, with an effect that rises with the region's weight.

4 Allocation with Two Organizations and Incomplete Information

Now suppose that the decision to allocate the resources is made by two organizations, each endowed with half the total budget $Q/2$. Organizations i can be one of two types based on the weight assigned to region 1. It assigns with probability p a low weight μ^L to region 1 and with probability $1-p$ a high weight μ^H. An organization knows its type when choosing the allocation, but it does not know the weight of the other organization, only the probability p.

We characterize the optimal choices of the two organizations using a Bayesian Nash Equilibrium (BNE). Let an organization i's strategy $q_i(\mu) = \left[q_i\left(\mu^L\right), q_i\left(\mu^H\right)\right]$ be an organization's allocation for any possible type, and an organization's profile $q(\mu) = [q_1(\mu), q_2(\mu)]$ the set of organizations' strategies. Then, the Bayesian-Nash equilibrium that characterizes our model can be defined as follows.

Definition 1 *The strategy profile $q^*(\mu)$ is a Bayesian-Nash equilibrium if, given $q_j^*(\mu)$, $q_i^*\left(\mu^k\right)$ maximize organization i's expected welfare, for $i = 1,\ 2$ and $\mu^k = \mu^L,\ \mu^H$, i.e.*

$$q_i^*\left(\mu^k\right) = \arg\max E\left[\mu W\left(q_i + q_j^*\right) + (1-\mu) W\left(Q - \left(q_i + q_j^*\right)\right) | k\right] \quad \textit{for } i = 1,\ 2 \textit{ and } \mu^k = \mu^L,\ \mu^H.$$

To characterize the BNE, suppose first that organization 1 is a low type. Then, its optimal allocation must be a best response to organization 2's strategy $q_2(\mu)$. It therefore solves the following problem:

$$\begin{aligned} &\max_{q_1^L}\ p\left[\mu^L W\left(q_1^L + q_2\left(\mu^L\right)\right) + \left(1-\mu^L\right) W\left(Q - \left(q_1^L + q_2\left(\mu^L\right)\right)\right)\right] + \\ &+ (1-p)\left[\mu^L W\left(q_1^L + q_2\left(\mu^H\right)\right) + \left(1-\mu^L\right) W\left(Q - \left(q_1^L + q_2\left(\mu^H\right)\right)\right)\right], \end{aligned}$$

Under the assumption that $W(q)$ belongs to the HARA class, the organization's first order condition takes the following form:

$$\begin{aligned} &p\left[\mu^L\left(\alpha\left(q_1^L + q_2\left(\mu^L\right)\right) + \eta\right)^{\delta-1} - \left(1-\mu^L\right)\left(\alpha\left(Q - q_1^L - q_2\left(\mu^L\right)\right) + \eta\right)^{\delta-1}\right] + \\ &+ (1-p)\left[\mu^L\left(\alpha\left(q_1^L + q_2\left(\mu^H\right)\right) + \eta\right)^{\delta-1} - \left(1-\mu^L\right)\left(\alpha\left(Q - q_1^L - q_2\left(\mu^H\right)\right) + \eta\right)^{\delta-1}\right] = 0. \end{aligned}$$

If, instead, organization's i is a high-type, it solves the following problem:

$$\begin{aligned} &\max_{q_1^H}\ p\left[\mu^H W\left(q_1^H + q_2\left(\mu^L\right)\right) + \left(1-\mu^H\right) W\left(Q - \left(q_1^H + q_2\left(\mu^L\right)\right)\right)\right] + \\ &+ (1-p)\left[\mu^H W\left(q_1^H + q_2\left(\mu^H\right)\right) + \left(1-\mu^H\right) W\left(Q - \left(q_1^H + q_2\left(\mu^H\right)\right)\right)\right]. \end{aligned}$$

If the welfare function is of the HARA type, the corresponding first order condition is

$$p\left[\mu^H\left(\alpha\left(q_1^H+q_2\left(\mu^L\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^H\right)\left(\alpha\left(Q-q_1^H-q_2\left(\mu^L\right)\right)+\eta\right)^{\delta-1}\right]+$$
$$+\left(1-p\right)\left[\mu^H\left(\alpha\left(q_2^H+q\left(\mu^H\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^H\right)\left(\alpha\left(Q-q_1^H-q_2\left(\mu^H\right)\right)+\eta\right)^{\delta-1}\right]=0.$$

For the strategies $q_1\left(\mu^k\right)$ and $q_2\left(\mu^k\right)$, for $k=L,\ H$, to be an equilibrium, it must be that

$$q_1^L=q_1\left(\mu^L\right)\quad\text{and}\quad q_1^H=q_1\left(\mu^H\right).$$

By replacing in the first order conditions we have

$$p\left[\mu^L\left(\alpha\left(q_1\left(\mu^L\right)+q_2\left(\mu^L\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^L\right)\left(\alpha\left(Q-q_1\left(\mu^L\right)-q_2\left(\mu^L\right)\right)+\eta\right)^{\delta-1}\right]+$$
$$+\left(1-p\right)\left[\mu^L\left(\alpha\left(q_1\left(\mu^L\right)+q_2\left(\mu^H\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^L\right)\left(\alpha\left(Q-q_1\left(\mu^L\right)-q_2\left(\mu^H\right)\right)+\eta\right)^{\delta-1}\right]=0.$$

and

$$p\left[\mu^H\left(\alpha\left(q_1\left(\mu^H\right)+q_2\left(\mu^L\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^H\right)\left(\alpha\left(Q-q_1\left(\mu^H\right)-q_2\left(\mu^L\right)\right)+\eta\right)^{\delta-1}\right]+$$
$$+\left(1-p\right)\left[\mu^H\left(\alpha\left(q_1^H\left(\mu^H\right)+q_2\left(\mu^H\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^H\right)\left(\alpha\left(Q-q_1\left(\mu^H\right)-q_2\left(\mu^H\right)\right)+\eta\right)^{\delta-1}\right]=0.$$

Since the two organizations, if of the same type, are identical, we will consider a symmetric equilibrium with $q_1\left(\mu^K\right)=q_2\left(\mu^K\right)=q\left(\mu^K\right)$, for $k=L,\ H$. In this case, the two first order conditions can be rewritten as follows

$$p\left[\mu^L\left(\alpha\left(2q\left(\mu^L\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^L\right)\left(\alpha\left(Q-2q\left(\mu^L\right)\right)+\eta\right)^{\delta-1}\right]+$$
$$+\left(1-p\right)\left[\mu^L\left(\alpha\left(q\left(\mu^L\right)+q\left(\mu^H\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^L\right)\left(\alpha\left(Q-q\left(\mu^L\right)-q\left(\mu^H\right)\right)+\eta\right)^{\delta-1}\right]=0.$$

and

$$p\left[\mu^H\left(\alpha\left(q\left(\mu^H\right)+q\left(\mu^L\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^H\right)\left(\alpha\left(Q-q\left(\mu^H\right)-q\left(\mu^L\right)\right)+\eta\right)^{\delta-1}\right]+$$
$$+\left(1-p\right)\left[\mu^H\left(\alpha\left(2q^H\left(\mu^H\right)\right)+\eta\right)^{\delta-1}-\left(1-\mu^H\right)\left(\alpha\left(Q-2q\left(\mu^H\right)\right)+\eta\right)^{\delta-1}\right]=0.$$

The solution to these first order conditions establishes the Bayesian Nash Equilibrium (BNE).

To derive the main result of the paper, we will make use of the function inside the brackets of the first order conditions

$$f^{\mu}(q) = \mu(\alpha q + \eta)^{\delta-1} - (1-\mu)(\alpha(Q-q) + \eta)^{\delta-1}.$$

We will refer to the function $f^{\mu}(q)$ as the *efficiency function*, as the allocation of a social planner with weight μ, q_{sp}^{μ} makes the function equal to zero. The following Proposition establishes properties of the *efficiency function* that will be used later in the paper.

Proposition 1 *The efficiency function $f^{\mu}(q)$ satisfies the following properties:*

(i) It is strictly decreasing in q;

(ii) There is a quantity q_0^{μ} such that $f^{\mu}(q)$ is strictly convex for $q < q_0^{\mu}$ and strictly concave for $q > q_0^{\mu}$;

(iii) The quantity q_0^{μ} satisfies the following conditions:

$$q_0^{\mu} \begin{cases} > q_{sp}^{\mu}, & \text{if } \mu < \frac{1}{2} < 1-\mu \\ = q_{sp}^{\mu}, & \text{if } \mu = \frac{1}{2} = 1-\mu \\ < q_{sp}^{\mu}, & \text{if } \mu > \frac{1}{2} > 1-\mu. \end{cases}$$

(iv) The derivative of $f^{\mu}(q)$ evaluated at the left and right of q_0^{μ} satisfies the following conditions:

$$-\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}-\phi} \begin{cases} > -\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}+\phi}, & \text{if } \mu < \frac{1}{2} < 1-\mu \\ = -\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}+\phi}, & \text{if } \mu = \frac{1}{2} = 1-\mu \\ < -\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}+\phi}, & \text{if } \mu > \frac{1}{2} > 1-\mu. \end{cases}$$

Proof. In the Appendix. ■

Using the *efficiency function*, we can write the two equations characterizing the BNE as follows:

$$pf^{\mu_L}\left(2q\left(\mu^L\right)\right) + (1-p)f^{\mu_L}\left(q\left(\mu^L\right) + q\left(\mu^H\right)\right) = 0 \tag{3}$$

and

$$pf^{\mu_H}\left(q\left(\mu^L\right)+q\left(\mu^H\right)\right)+(1-p)\,f^{\mu_H}\left(2q\left(\mu^H\right)\right)=0. \tag{4}$$

We start by considering the case in which the two types have identical probability $p=\frac{1}{2}$. To make the comparison between the social planner's allocation and the agents' allocation meaningful, we assume that the weights μ^L and μ^H satisfy the following condition:

$$q_{sp}^{\mu}=pq_{sp}^{\mu^L}+(1-p)\,q_{sp}^{\mu^H},$$

i.e. if a social planner has weight μ^L with probability p and weight μ^H with probability $1-p$, on average she or he chooses the same allocation as a social planner that has weight μ with probability 1. If the welfare function is the logarithmic function, the weights takes the following intuitive form: $\mu^L=\mu-\frac{\Delta}{p}$ with probability p and $\mu^H=\mu+\frac{\Delta}{1-p}$ with probability $1-p$. The following Proposition uses equations (3) and (4) to establish the main result of the paper.

Proposition 2 *If the social planner and the two organizations assign less weight to region 1 ($\mu^L<\mu<\mu^H\leq\frac{1}{2}$), then, on average, region 1 receives more resources from the two organizations than the social planner would optimally allocate.*

Proof. In the Appendix. ■

The result described in Proposition 2 is explained by the interaction between the uncertainty about the type of the other organization and the curvature of the welfare function. Suppose the first organization is a low type. Then, its optimal allocation entails transfering fewer funds to region 1 than the social planner's allocation. To achieve this, it has to account for the possibility that the second organization is a high type and, hence, will transfer more resources to region one than the social planner's allocation. It does this by transferring fewer funds than it would have without the type uncertainty, i.e. fewer resources than a social planner with weight μ_L would have transferred. If instead the first organization is a high type, the optimal choice is to allocate more funds to region 1 than the social planner would choose to transfer. But, as for the low type, it has to account for the likelihood that the second organization is a low type and, hence, transfers too few resources. This is achieved by transferring more resources to region 1 than it would be optimal without type uncertainty. The under-transferring of the low type and over-transferring of the high types are not equivalent since they take place at points of the welfare function with different degrees of curvature. If the low type reduces its transfer relative to the social planner by the same amount that the high type increases its transfer, the reduction has larger effects because it is done at a point of the welfare function that has more curvature. The uncertainty in types has larger effects on the low-type organization. It therefore decreases the allocation by less relative to the social planner, which explains the result.

5 Conclusions

This paper show that the delegation to local organizations of the allocation of public funds and resources to regions, villages, or families has a cost if the organizations are uncertain about how the other organizations will allocate their own budget. The cost is that the regions, villages, or families with to which the government assigns the higher weight receive a lower share of resources than the government would like them to receive. This result is of particular importance for developing countries where public resources are scarce and even a small missallocation can have significant effects on individual welfare.

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A Proof of Proposition 1

Property (i). It is satisfied by any concave welfare function. Indeed, the derivative of $f^{\mu}(q)$ with respect to q takes the following form:

$$\frac{\partial f^{\mu}(q)}{\partial q} = \mu W^{''}(q) + (1-\mu) W^{''}(Q-q) < 0,$$

as long as $W^{''}(q) < 0$ for any q.

Property (ii). For any welfare function,

$$\frac{\partial^2 f^{\mu}(q)}{\partial q^2} = \mu W^{'''}(q) - (1-\mu) W^{'''}(Q-q).$$

For a HARA function,

$$W^{'''}(q) = \alpha^3 (1-\delta)(2-\delta)(\alpha q + \eta)^{\delta-3} > 0,$$

with $\delta < 1$. Hence, $\frac{\partial^2 f^{\mu}(q)}{\partial q^2} f^{\mu}(q) \geq 0$ if

$$\mu (\alpha q + \eta)^{\delta-3} \geq (1-\mu)(\alpha (Q-q) + \eta)^{\delta-3},$$

or

$$\left(\frac{\alpha (Q-q) + \eta}{\alpha q + \eta}\right)^{3-\delta} \geq \frac{1-\mu}{\mu}.$$

There is only one q that satisfy the previous condition as an equality. It takes the following form:

$$q_0^{\mu} = \frac{\mu^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}} + (1-\mu)^{\frac{1}{3-\delta}}} Q + \frac{\mu^{\frac{1}{3-\delta}} - (1-\mu)^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}} + (1-\mu)^{\frac{1}{3-\delta}}} \frac{\eta}{\alpha},$$

with $\frac{\partial^2 f^{\mu}(q)}{\partial q^2} f^{\mu}(q) > 0$ for $q < q_0^{\mu}$ and $\frac{\partial^2 f^{\mu}(q)}{\partial q^2} f^{\mu}(q) < 0$ for $q > q_0^{\mu}$.

Property (iii). Consider first the case $\mu < \frac{1}{2}$. In this case, the social planner's allocation must satisfies $q_{sp}^{\mu} < Q - q_{sp}^{\mu}$, i.e. the region with lower weight receives less than half of total resources. We therefore have,

$$\frac{1-\mu}{\mu} = \left(\frac{\alpha (Q - q_{sp}^{\mu}) + \eta}{\alpha q_{sp}^{\mu} + \eta}\right)^{1-\delta} < \left(\frac{\alpha (Q - q_{sp}^{\mu}) + \eta}{\alpha q_{sp}^{\mu} + \eta}\right)^{3-\delta},$$

where the equality follows from the first order condition of the social planner and the inequality from

$\frac{\alpha\left(Q-q_{sp}^{\mu}\right)+\eta}{\alpha q_{sp}^{\mu}+\eta} > 1$ and $\delta < 1$. We also have,

$$\left(\frac{\alpha\left(Q-q_0^{\mu}\right)+\eta}{\alpha q_0^{\mu}+\eta}\right)^{3-\delta} = \frac{1-\mu}{\mu}.$$

Hence, since $\left(\frac{\alpha(Q-q)+\eta}{\alpha q+\eta}\right)^{3-\delta}$ decreases with q, we have $q_{sp}^{\mu} > q_0^{\mu}$. Applying the same steps for $\mu = \frac{1}{2}$ and $\mu > \frac{1}{2}$, we have $q_{sp}^{\mu} = q_0^{\mu}$ and $q_{sp}^{\mu} < q_0^{\mu}$, respectively.

Property (iv). Remember that, by properties (i)-(iii), $\frac{\partial f^{\mu}(q)}{\partial q} < 0$, $\frac{\partial^2 f^{\mu}(q)}{\partial q^2} > 0$ for $q < q_0^{\mu}$, and $\frac{\partial^2 f^{\mu}(q)}{\partial q^2} < 0$ for $q < q_0^{\mu}$. Since $\frac{\partial^2 f^{\mu}(q)}{\partial q^2}$ is continuous, we can apply the fundamental theorem of calculus to have

$$-\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}-\phi} = \int_{q_0^{\mu}-\phi}^{q_0^{\mu}} \frac{\partial^2 f^{\mu}(q)}{\partial q^2} dq - \left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}}.$$

and

$$-\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}+\phi} = -\int_{q_0^{\mu}}^{q_0^{\mu}+\phi} \frac{\partial^2 f^{\mu}(q)}{\partial q^2} dq - \left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}}.$$

Hence,

$$-\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}-\phi} \begin{cases} > -\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}+\phi}, & \text{if } \left.\frac{\partial^2 f^{\mu}(q)}{\partial q^2}\right|_{q=q_0^{\mu}-\psi} > -\left.\frac{\partial^2 f^{\mu}(q)}{\partial q^2}\right|_{q=q_0^{\mu}+\psi} \quad \text{for } 0 \le \psi \le \phi. \\ = -\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}+\phi}, & \text{if } \left.\frac{\partial^2 f^{\mu}(q)}{\partial q^2}\right|_{q=q_0^{\mu}-\psi} = -\left.\frac{\partial^2 f^{\mu}(q)}{\partial q^2}\right|_{q=q_0^{\mu}+\psi} \quad \text{for } 0 \le \psi \le \phi. \\ < -\left.\frac{\partial f^{\mu}(q)}{\partial q}\right|_{q=q_0^{\mu}+\phi}, & \text{if } \left.\frac{\partial^2 f^{\mu}(q)}{\partial q^2}\right|_{q=q_0^{\mu}-\psi} < -\left.\frac{\partial^2 f^{\mu}(q)}{\partial q^2}\right|_{q=q_0^{\mu}+\psi} \quad \text{for } 0 \le \psi \le \phi.. \end{cases}$$

The function $\frac{\partial^2 f^{\mu}(q)}{\partial q^2}$ takes the following form:

$$\frac{\partial^2 f^{\mu}(q)}{\partial q^2} = \alpha^3 (1-\delta)(2-\delta)\left(\mu(\alpha q+\eta)^{\delta-3} - (1-\mu)(\alpha(Q-q)+\eta)^{\delta-3}\right).$$

Since $\alpha^3(1-\delta)(2-\delta) > 0$ and independent of q, we will ignore this term for the rest of the proof.

Thus, we have

$$\left.\frac{\partial^2 f^\mu(q)}{\partial q^2}\right|_{q=q_0^\mu-\psi} = \mu\left(\alpha\left(q_0^\mu-\psi\right)+\eta\right)^{\delta-3} - (1-\mu)\left(\alpha\left(Q-q_0^\mu+\psi\right)+\eta\right)^{\delta-3}$$

$$\begin{aligned}
&= \mu\left(\alpha\left(\frac{\mu^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}Q+\frac{\mu^{\frac{1}{3-\delta}}-(1-\mu)^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}\frac{\eta}{\alpha}-\psi\right)+\eta\right)^{\delta-3}\\
&- (1-\mu)\left(\alpha\left(Q-\frac{\mu^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}Q-\frac{\mu^{\frac{1}{3-\delta}}-(1-\mu)^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}\frac{\eta}{\alpha}+\psi\right)+\eta\right)^{\delta-3}\\
&= \mu\left(\alpha\left(\frac{\mu^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}Q+\frac{2\mu^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}\frac{\eta}{\alpha}-\psi\right)\right)^{\delta-3}\\
&- (1-\mu)\left(\alpha\left(\frac{(1-\mu)^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}Q+\frac{2(1-\mu)^{\frac{1}{3-\delta}}}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}\frac{\eta}{\alpha}+\psi\right)\right)^{\delta-3}\\
&= \mu\left(\frac{\mu^{\frac{1}{3-\delta}}(\alpha Q+2\eta)}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}-\alpha\psi\right)^{\delta-3} - (1-\mu)\left(\frac{(1-\mu)^{\frac{1}{3-\delta}}(\alpha Q+2\eta)}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}+\alpha\psi\right)^{\delta-3}\\
&= \left(\frac{\mu^{-\frac{1}{3-\delta}}\mu^{\frac{1}{3-\delta}}(\alpha Q+2\eta)}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}-\frac{\alpha\psi}{\mu^{\frac{1}{3-\delta}}}\right)^{\delta-3} - \left(\frac{(1-\mu)^{-\frac{1}{3-\delta}}(1-\mu)^{\frac{1}{3-\delta}}(\alpha Q+2\eta)}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}+\frac{\alpha\psi}{(1-\mu)^{\frac{1}{3-\delta}}}\right)^{\delta-3}\\
&= \left(\frac{\alpha Q+2\eta}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}-\frac{\alpha\psi}{\mu^{\frac{1}{3-\delta}}}\right)^{\delta-3} - \left(\frac{\alpha Q+2\eta}{\mu^{\frac{1}{3-\delta}}+(1-\mu)^{\frac{1}{3-\delta}}}+\frac{\alpha\psi}{(1-\mu)^{\frac{1}{3-\delta}}}\right)^{\delta-3}\\
&= \left(g(\mu)-v(\mu)\right)^{\delta-3} - \left(g(\mu)+u(\mu)\right)^{\delta-3}.
\end{aligned}$$

Following the same steps we also have

$$\left.\frac{\partial^2 f^\mu(q)}{\partial q^2}\right|_{q=q_0^\mu+\psi} = \left(g(\mu)+v(\mu)\right)^{\delta-3} - \left(g(\mu)-u(\mu)\right)^{\delta-3}.$$

The easiest case to consider is $\mu = 1/2$, since in this case

$$v(\mu) = \frac{\alpha\psi}{\mu^{\frac{1}{3-\delta}}} = \frac{\alpha\psi}{(1-\mu)^{\frac{1}{3-\delta}}} = u(\mu)$$

As a consequence,

$$\left.\frac{\partial^2 f^\mu(q)}{\partial q^2}\right|_{q=q_0^\mu-\psi} = (g(\mu)-u(\mu))^{\delta-3}-(g(\mu)+u(\mu))^{\delta-3}$$
$$= -\left((g(\mu)+u(\mu))^{\delta-3}-(g(\mu)-u(\mu))^{\delta-3}\right) = -\left.\frac{\partial^2 f^\mu(q)}{\partial q^2}\right|_{q=q_0^\mu+\psi},$$

which implies

$$-\left.\frac{\partial f^\mu(q)}{\partial q}\right|_{q=q_0^\mu-\psi} = -\left.\frac{\partial f^\mu(q)}{\partial q}\right|_{q=q_0^\mu+\psi}.$$

Consider now the case $\mu < 1/2$. We have

$$\left.\frac{\partial^2 f^\mu(q)}{\partial q^2}\right|_{q=q_0^\mu-\psi} > -\left.\frac{\partial^2 f^\mu(q)}{\partial q^2}\right|_{q=q_0^\mu+\psi}.$$

if

$$(g(\mu)-v(\mu))^{\delta-3}-(g(\mu)+u(\mu))^{\delta-3} > -\left[(g(\mu)+v(\mu))^{\delta-3}-(g(\mu)-u(\mu))^{\delta-3}\right]$$

or, equivalently,

$$(g(\mu)-v(\mu))^{\delta-3}+(g(\mu)+v(\mu))^{\delta-3} > (g(\mu)-u(\mu))^{\delta-3}+(g(\mu)+u(\mu))^{\delta-3}. \tag{5}$$

The left- and right-hand sides of the previous inequality correspond to the function

$$(g(\mu)-x)^{\delta-3}+(g(\mu)+x)^{\delta-3}$$

with $x = v(\mu) > 0$ for the left-hand side and $x = u(\mu) > 0$ for the right-hand side. Then, since with $\mu < 1/2$ we have

$$v(\mu) = \frac{\alpha\psi}{\mu^{\frac{1}{3-\delta}}} > \frac{\alpha\psi}{(1-\mu)^{\frac{1}{3-\delta}}} = u(\mu),$$

inequality (5) is satisfied if

$$\frac{\partial}{\partial x}\left((g(\mu)-x)^{\delta-3}+(g(\mu)+x)^{\delta-3}\right) = -(\delta-3)\left((g(\mu)-x)^{\delta-4}-(g(\mu)+x)^{\delta-4}\right) > 0.$$

This is always the case, as $\delta - 3 < 0$, $\delta - 4 < 0$, and $x > 0$. Hence,

$$\left.\frac{\partial^2 f^\mu(q)}{\partial q^2}\right|_{q=q_0^\mu-\psi} > -\left.\frac{\partial^2 f^\mu(q)}{\partial q^2}\right|_{q=q_0^\mu+\psi},$$

and

$$-\left.\frac{\partial f^\mu(q)}{\partial q}\right|_{q=q_0^\mu-\psi} > -\left.\frac{\partial f^\mu(q)}{\partial q}\right|_{q=q_0^\mu+\psi}.$$

The case $\mu > 1/2$ can be dealt with by noticing that in this situation

$$v(\mu) = \frac{\alpha\psi}{\mu^{\frac{1}{3-\delta}}} < \frac{\alpha\psi}{(1-\mu)^{\frac{1}{3-\delta}}} = u(\mu)$$

and following the steps used for $\mu < 1/2$.

B Proof of Proposition 2

Let q_0^L and q_0^H be the quantities that satisfy property (ii) Proposition 1 for $\mu = \mu_L$ and $\mu = \mu_H$, respectively. We start with the simplest case to consider: a situation where $q_0^L > q_L^* + q_H^*$ and $q_0^H > 2q_H^*$, i.e. the function f^{μ_L} is convex from $2q_L^*$ to $q_L^* + q_H^*$, the lower and higher points in the low-type efficiency condition, and f^{μ_H} is convex from $q_L^* + q_H^*$ to $2q_H^*$.

Suppose the result established in the Proposition is not true and instead the two organizations allocate on average fewer resources to region 1 than the social planner, i.e.

$$2\left(\frac{1}{2}q_L^* + \frac{1}{2}q_H^*\right) = q_L^* + q_H^* \leq q_{sp}^\mu = \frac{1}{2}q_{sp}^L + \frac{1}{2}q_{sp}^H,$$

where the first term is multiplies by two to consider that in the model there are two organizations, each one endowed with half of total resources, and the last equality follows from the weights μ_L and μ_H being such that the social planner's allocation is equal to the average allocation of the organizations in case of complete information. The efficiency condition for the high type (4) for $p = \frac{1}{2}$ implies that

$$f^{\mu_H}(q_L^* + q_H^*) = -f^{\mu_H}(2q_H^*).$$

Since f^{μ_H} is continuous and $f^{\mu_H}\left(q_{sp}^H\right)=0$, the fundamental theorem of calculus implies that

$$f^{\mu_H}\left(q_L^*+q_H^*\right)=f^{\mu^H}\left(q_{sp}^H\right)-\int_{q_L^*+q_H^*}^{q_{sp}^H}\frac{\partial f^{\mu_H}(q)}{\partial q}dq=\int_{q_L^*+q_H^*}^{q_{sp}^H}-\frac{\partial f^{\mu_H}(q)}{\partial q}dq.$$

Also,

$$-f^{\mu_H}\left(2q_H^*\right)=-\int_{q_{sp}^H}^{2q_H^*}\frac{\partial f^{\mu_H}(q)}{\partial q}dq-f^{\mu_H}\left(q_{sp}^H\right)=\int_{q_{sp}^H}^{2q_H^*}-\frac{\partial f^{\mu_H}(q)}{\partial q}dq.$$

Property (ii) and (iii) in Proposition 1 imply that $-\frac{\partial f^{\mu_H}(q)}{\partial q}$ is strictly decreasing for $q<2q_H^*<q_0^H$ due to the convexity of f^{μ_H} for $q<q_0^H$. Then, it must be that

$$q_{sp}^H-\left(q_L^*+q_H^*\right)<2q_H^*-q_{sp}^H.$$

To see why, suppose that $q_{sp}^H-\left(q_L^*+q_H^*\right)\geq 2q_H^*-q_{sp}^H$. Then, there is a $q_L^*+q_H^*\leq\bar{q}<q_{sp}^H$ such that $2q_H^*-q_{sp}^H=q_{sp}^H-\bar{q}\leq q_{sp}^H-\left(q_L^*+q_H^*\right)$. Then,

$$-f^{\mu_H}\left(2q_H^*\right)=\int_{q_{sp}^H}^{2q_H^*}-\frac{\partial f^{\mu_H}(q)}{\partial q}dq<\int_{\bar{q}}^{q_{sp}^H}-\frac{\partial f^{\mu_H}(q)}{\partial q}dq\leq\int_{q_L^*+q_H^*}^{q_{sp}^H}-\frac{\partial f^{\mu_H}(q)}{\partial q}dq=f^{\mu_H}\left(q_L^*+q_H^*\right),$$

where the first inequality follows from $-\frac{\partial f^{\mu_H}(q)}{\partial q}$ being strictly decreasing and $2q_H^*-q_{sp}^H=q_{sp}^H-\bar{q}$, and the second from $q_{sp}^H-\bar{q}\leq q_{sp}^H-\left(q_L^*+q_H^*\right)$. Hence, we have

$$-f^{\mu_H}\left(2q_H^*\right)<f^{\mu_H}\left(q_L^*+q_H^*\right),$$

which contradict the efficiency condition for a high type (4). It must therefore be $q_{sp}^H-\left(q_L^*+q_H^*\right)<2q_H^*-q_{sp}^H$.

We now apply similar arguments to show that $2q_L^*-q_{sp}^L<\left(q_L^*+q_H^*\right)-q_{sp}^L$. The efficiency condition for the low type (3) for $p=\frac{1}{2}$ implies that

$$-f^{\mu_L}\left(q_L^*+q_H^*\right)=f^{\mu_L}\left(2q_L^*\right).$$

We also have

$$f^{\mu_L}(2q_L^*) = -\int_{2q_L^*}^{q_{sp}^L} \frac{\partial f^{\mu_L}(q)}{\partial q} dq + f^{\mu_L}\left(q_{sp}^L\right) = \int_{2q_L^*}^{q_{sp}^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq$$

and

$$-f^{\mu_L}(q_L^* + q_H^*) = -\int_{q_{sp}^L}^{q_L^*+q_H^*} \frac{\partial f^{\mu_L}(q)}{\partial q} dq - f^{\mu_L}\left(q_{sp}^L\right) = \int_{q_{sp}^L}^{q_L^*+q_H^*} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq.$$

Using the same argument used for the high type, it must then be that

$$q_L^* + q_H^* - q_{sp}^L > q_{sp}^L - 2q_L^*.$$

To see why, suppose that $q_L^* + q_H^* - q_{sp}^L \leq q_{sp}^L - 2q_L^*$. Then, there is a $2q_L^* \leq \bar{q} < q_{sp}^L$ such that $q_L^* + q_H^* - q_{sp}^L = q_{sp}^L - \bar{q} \leq q_{sp}^L - 2q_L^*$. Hence,

$$-f^{\mu_L}(q_L^* + q_H^*) = \int_{q_L^*+q_H^*}^{q_{sp}^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq < \int_{\bar{q}}^{q_{sp}^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq \leq \int_{2q_L^*}^{q_{sp}^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq = f^{\mu_L}(2q_L^*),$$

where the first inequality follows from $-\dfrac{\partial f^{\mu_L}(q)}{\partial q}$ being strictly decreasing and $q_L^* + q_H^* - q_{sp}^L = q_{sp}^L - \bar{q}$, and the second from $q_{sp}^L - \bar{q} \leq q_{sp}^L - 2q_L^*$. Hence, we have

$$-f^{\mu_L}(q_L^* + q_H^*) < f^{\mu_L}(2q_L^*),$$

which contradicts the efficiency condition for a low type (3). It must therefore be $q_L^* + q_H^* - q_{sp}^L > q_{sp}^L - 2q_L^*$.

To summarize, we have: (i) $q_{sp}^H - (q_L^* + q_H^*) < 2q_H^* - q_{sp}^H$; (ii) $q_L^* + q_H^* - q_{sp}^L > q_{sp}^L - 2q_L^*$; and, given the initial assumption, (iii) $q_L^* + q_H^* \leq q_{sp}^\mu = \frac{1}{2}q_{sp}^L + \frac{1}{2}q_{sp}^H$. Condition (iii) implies that

$$2q_H^* - q_{sp}^H \leq q_{sp}^L - 2q_L^*,$$

whereas conditions (i) and (ii) imply that

$$2q_H^* - q_{sp}^H > q_{sp}^L - 2q_L^*,$$

creating a contradiction. It must therefore be $q_L^* + q_H^* > q_{sp}^\mu = \frac{1}{2}q_{sp}^L + \frac{1}{2}q_{sp}^H$.

We now consider the case $q_{sp}^L < q_0^L < q_L^* + q_H^*$ and $q_{sp}^H < q_0^H < 2q_H^*$. The previous proof applies with a couple of modifications. Suppose the result established in the Proposition is not true and instead the two organizations allocate on average fewer resources to region 1 than the social planner, i.e.

$$2\left(\frac{1}{2}q_L^* + \frac{1}{2}q_H^*\right) = q_L^* + q_H^* \leq q_{sp}^{\mu} = \frac{1}{2}q_{sp}^L + \frac{1}{2}q_{sp}^H,$$

The efficiency condition for the high type (4) for $p = \frac{1}{2}$ implies that

$$f^{\mu_H}\left(q_L^* + q_H^*\right) = -f^{\mu_H}\left(2q_H^*\right).$$

Since f^{μ_H} is continuous and $f^{\mu_H}\left(q_{sp}^H\right) = 0$, the fundamental theorem of calculus implies that

$$f^{\mu_H}\left(q_L^* + q_H^*\right) = f^{\mu_H}\left(q_{sp}^H\right) - \int_{q_L^*+q_H^*}^{q_{sp}^H} \frac{\partial f^{\mu_H}(q)}{\partial q} dq = \int_{q_L^*+q_H^*}^{q_{sp}^H} -\frac{\partial f^{\mu_H}(q)}{\partial q} dq.$$

Also,

$$-f^{\mu_H}\left(2q_H^*\right) = -\int_{q_{sp}^H}^{2q_H^*} \frac{\partial f^{\mu_H}(q)}{\partial q} dq - f^{\mu_H}\left(q_{sp}^H\right) = \int_{q_{sp}^H}^{2q_H^*} -\frac{\partial f^{\mu_H}(q)}{\partial q} dq = \int_{q_{sp}^H}^{q_0^H} -\frac{\partial f^{\mu_H}(q)}{\partial q} dq + \int_{q_0^H}^{2q_H^*} -\frac{\partial f^{\mu_H}(q)}{\partial q} dq.$$

where the last equality follows from properties (ii) and (iii) in Proposition 1.

Property (ii) and (iii) in Proposition 1 imply that $-\frac{\partial f^{\mu_H}(q)}{\partial q}$ is strictly decreasing for $q < q_0^H < 2q_H^*$ due to the convexity of f^{μ_H} for $q < q_0^H$. Moreover, property (iv) of Proposition 1 implies that

$$-\left.\frac{\partial f^{\mu_H}(q)}{\partial q}\right|_{q=q_0^H-\psi} > -\left.\frac{\partial f^{\mu_H}(q)}{\partial q}\right|_{q=q_0^H+\psi} \quad \text{for } \psi > 0. \tag{6}$$

Then, it must be that

$$q_{sp}^H - \left(q_L^* + q_H^*\right) < 2q_H^* - q_{sp}^H.$$

To see why, suppose that $q_{sp}^H - \left(q_L^* + q_H^*\right) \geq 2q_H^* - q_{sp}^H$. Then, there is a $q_L^* + q_H^* < \bar{q} < q_{sp}^H$ such that

$2q_H^* - q_{sp}^H = q_{sp}^H - \bar{q} \leq q_{sp}^H - (q_L^* + q_H^*)$. Hence,

$$\begin{aligned} -f^{\mu_H}(2q_H^*) &= \int_{q_{sp}^H}^{q_0^H} -\frac{\partial f^{\mu_H}(q)}{\partial q} dq + \int_{q_0^H}^{2q_H^*} -\frac{\partial f^{\mu_H}(q)}{\partial q} dq < \int_{q_{sp}^H - q_0^H}^{q_{sp}^H} -\frac{\partial f^{\mu_H}(q)}{\partial q} dq + \int_{\bar{q}}^{q_{sp}^H - q_0^H} -\frac{\partial f^{\mu_H}(q)}{\partial q} dq \\ &\leq \int_{q_L^* + q_H^*}^{q_{sp}^H} -\frac{\partial f^{\mu_H}(q)}{\partial q} dq = f^{\mu_H}(q_L^* + q_H^*), \end{aligned}$$

where the first inequality follows from $-\frac{\partial f^{\mu_H}(q)}{\partial q}$ being strictly decreasing and $2q_H^* - q_{sp}^H = q_{sp}^H - \bar{q}$, and the second from $q_{sp}^H - \bar{q} \leq q_{sp}^H - (q_L^* + q_H^*)$. Hence, we have

$$-f^{\mu_H}(2q_H^*) < f^{\mu_H}(q_L^* + q_H^*),$$

which contradict the efficiency condition for a high type (4). It must therefore be $q_{sp}^H - (q_L^* + q_H^*) < 2q_H^* - q_{sp}^H$.

We now apply similar arguments to show that $2q_L^* - q_{sp}^L < (q_L^* + q_H^*) - q_{sp}^L$. The efficiency condition for the low type (3) for $p = \frac{1}{2}$ implies that

$$-f^{\mu_L}(q_L^* + q_H^*) = f^{\mu_L}(2q_L^*).$$

We also have

$$f^{\mu_L}(2q_L^*) = -\int_{2q_L^*}^{q_{sp}^L} \frac{\partial f^{\mu_L}(q)}{\partial q} dq + f^{\mu^L}\left(q_{sp}^L\right) = \int_{2q_L^*}^{q_{sp}^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq$$

and

$$\begin{aligned} -f^{\mu_L}(q_L^* + q_H^*) &= -\int_{q_{sp}^L}^{q_L^* + q_H^*} \frac{\partial f^{\mu_L}(q)}{\partial q} dq - f^{\mu^L}\left(q_{sp}^L\right) = \int_{q_{sp}^L}^{q_L^* + q_H^*} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq \\ &= \int_{q_{sp}^L}^{q_0^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq + \int_{q_0^L}^{q_L^* + q_H^*} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq. \end{aligned}$$

Using the same argument used for the high type, it must then be that

$$q_L^* + q_H^* - q_{sp}^L > q_{sp}^L - 2q_L^*.$$

To see why, suppose that $q_L^* + q_H^* - q_{sp}^L \leq q_{sp}^L - 2q_L^*$. Then, there is a $2q_L^* \leq \bar{q} < q_{sp}^L$ such that

$q_L^* + q_H^* - q_{sp}^L = q_{sp}^L - \bar{q} \leq q_{sp}^L - 2q_L^*$. Hence,

$$\begin{aligned} -f^{\mu_L}\left(q_L^* + q_H^*\right) &= \int_{q_{sp}^L}^{q_0^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq + \int_{q_0^L}^{q_L^*+q_H^*} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq < \int_{q_{sp}^L - q_0^L}^{q_{sp}^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq + \int_{\bar{q}}^{q_{sp}^L - q_0^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq \\ &\leq \int_{2q_L^*}^{q_{sp}^L} -\frac{\partial f^{\mu_L}(q)}{\partial q} dq = f^{\mu_L}\left(2q_L^*\right), \end{aligned}$$

where the first inequality follows from $-\frac{\partial f^{\mu_L}(q)}{\partial q}$ being strictly decreasing and $q_L^* + q_H^* - q_{sp}^L = q_{sp}^L - \bar{q}$, and the second from $q_{sp}^L - \bar{q} \leq q_{sp}^L - 2q_L^*$. Hence, we have

$$-f^{\mu_L}\left(q_L^* + q_H^*\right) < f^{\mu_L}\left(2q_L^*\right),$$

which contradict the efficiency condition for a low type (3). It must therefore be $q_L^* + q_H^* - q_{sp}^L > q_{sp}^L - 2q_L^*$.

Then, by applying the step used for the previous case, we have

$$2q_H^* - q_{sp}^H > q_{sp}^L - 2q_L^*,$$

which contradict the initial assumption. It must therefore be $q_L^* + q_H^* > q_{sp}^{\mu} = \frac{1}{2}q_{sp}^L + \frac{1}{2}q_{sp}^H$.

There are other two cases to consider: (i) $q_{sp}^L < q_0^L < q_L^* + q_H^*$ and $q_0^H > 2q_H^*$; and (ii) $q_0^L > q_L^* + q_H^*$ and $q_{sp}^H < q_0^H < 2q_H^*$. Their proof can be obtain by combining the previous two proof.