

Inventory control with positive fixed cost

Stokey and Lucas (1989), Exercise 5.14, 5.15

Kei Kawakami

UCLA Department of Economics

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1 Discrete time

Consider a manager who can sell up to one unit of a certain product each period, at a price p . If he has $x \geq 0$ units in stock, he can sell $\min\{x, 1\}$ units. He can also order any amount y of new goods, to be delivered at the beginning of next period, at a cost $c_0 + c_1y$, paid now. His discount factor is $\beta \in (0, 1)$. His objective is to choose an ordering policy that maximizes the present value of revenues less costs. Assume $0 < c_0$ and $0 < c_1 < \beta p$.

The value function v must satisfy the functional equation:

$$v(x) = \begin{cases} px + \max \left[\beta v(0), \sup_{y>0} \{ \beta v(y) - c_0 - c_1y \} \right] & (0 \leq x_t \leq 1), \\ p + \max \left[\beta v(x-1), \sup_{y>0} \{ \beta v(y+x-1) - c_0 - c_1y \} \right] & (1 < x_t). \end{cases}$$

Let C be the space of bounded continuous functions on R_+ .

1.1 Existence, uniqueness

Exercise 1 Show that (1) mapping defined above has a unique fixed point in C , (2) $0 \leq v(x) \leq \frac{p}{1-\beta}$, and (3) an optimal ordering policy exists. Is this policy unique?

Proof. Since $y_t = 0$ is always feasible, $0 \leq v(x)$. Since period return is bounded by p , $v(x) \leq \frac{p}{1-\beta}$. Denote the mapping defined by the functional equation by T . We need to show two things: 1. T satisfies Blackwell's sufficient conditions, and 2. $T : C \rightarrow C$. Given these two, we can apply the contraction mapping theorem on a Banach space C . ■

- Blackwell's sufficient conditions.

Proof. We need to check Discounting and Monotonicity. Discounting is easy to verify: $\forall x, T(v+a)(x) = Tv(x) + \beta a$. For monotonicity, take $f \leq g$ in C and show $Tf \leq Tg$. Obvious for $x \in [0, 1]$. For $x \in (1, \infty)$, since v is bounded while $(c_0 + c_1y) 1_{\{y>0\}}$ is not, we can write T as follows: $Tv(x) = \sup_{y \in [0, \bar{y}]} \{p - (c_0 + c_1y) 1_{\{y>0\}} + \beta v(y+x-1)\}$. We can always find maximizing $y^* \in [0, \bar{y}]$. Define $F(x, y) = p - (c_0 + c_1y) 1_{\{y>0\}} + \beta f(y+x-1)$ and $G(x, y) = p - (c_0 + c_1y) 1_{\{y>0\}} + \beta g(y+x-1)$. Suppose $\exists x$ s.t. $F(x, y^*(f)) > G(x, y^*(g))$, where $y^*(f)$ denotes the value of y which attains $\sup_{y \in [0, \bar{y}]} F(x, y)$. This results in $F(x, y^*(f)) > G(x, y^*(g)) \geq G(x, y^*(f))$. But $F(x, y^*(f)) > G(x, y^*(f))$ contradicts with $f(x) \leq g(x) \forall x$. ■

- T maps C into itself.

Proof. We can't use the standard result in SLP since return function is not continuous and theorem of the maximum does not hold. We need to check both boundedness and continuity manually. Boundedness is obvious. For continuity, it is obvious at $x \in [0, 1]$. At $x = 1$, from left, $Tv(\uparrow 1) = p + \max \left[\beta v(0), \sup_{y>0} \{\beta v(y) - c_0 - c_1y\} \right]$. From right, $Tv(\downarrow 1) = p + \max \left[\beta v(\downarrow 0), \sup_{y>0} \{\beta v(y) - c_0 - c_1y\} \right]$. By continuity of v , this shows Tv is continuous at $x = 1$. At $x \in (1, \infty)$, we use upper bound of v to bound choice set. Think about two continuous functions: $f(x) = \beta v(x-1)$ and $g(x) = \max_{y \in [0, \bar{y}]} \{\beta v(y+x-1) - c_0 - c_1y\} = \max_{y \in [0, \bar{y}]} G(x, y)$, where \bar{y} is some number satisfying $\frac{\beta p}{1-\beta} - c_0 - c_1\bar{y} \leq 0$. We can use theorem of the maximum to claim $g(x)$ is continuous. Also, $\forall x, f(x) > G(x, 0)$. Hence, $Tv(x) = p + \max \{f(x), g(x)\}$ in $x \in (1, \infty)$ and is continuous. ■

- The optimal policy.

Proof. The optimal policy obviously exists. Non-uniqueness of policy could arise only between $y = 0$ and some positive y . We use following **Lemma**: If $y = 0$ is optimal at some point in $x \in [0, 1]$, then it is optimal for all $x \in [0, \infty)$. Also, if $y = 0$ is optimal at some point in $x^* \in [0, \infty)$, then it is optimal for all $x \geq x^*$. Since "never order" situation is not interesting, we assume parameter values such that $y > 0$ is optimal in $x \in [0, 1]$. Then from lemma, it is enough to check the possibility of non-uniqueness in $x \in (1, \infty)$. By continuity of $Tv(x)$ at $x = 1$, for x close to 1, $f(x) < g(x)$ with positive y . Since v is bounded, for large enough x , $f(x) > g(x)$. Hence $\exists x^* > 1$ s.t. $f(x^*) = g(x^*)$ and this is indifferent point between ordering zero and some positive number. ■

- Proof of lemma.

Proof. True for $x \in [0, 1]$ since choice of y is independent of x in $x \in [0, 1]$. At $x \in (1, \infty)$, no incentive to order because marginal return of order becomes lower than in $x \in [0, 1]$, while cost is same. The second statement follows from the same reasoning. ■

1.2 Characterization

Now we conjecture policy and value function. Consider actions of the following form: No initial stock, order n units immediately, and then sell these units at the rate of one per period over the next n periods. The value of this action is:

$$r_n = \beta \frac{1-\beta^n}{1-\beta} p - c_0 - c_1 n. \quad (r_0 \equiv 0)$$

The value of repeating r_n every other n periods is:

$$Y_n = \frac{r_n}{1-\beta^n} = \frac{\beta}{1-\beta} p - \frac{c_0+c_1 n}{1-\beta^n}. \quad (Y_0 \equiv 0)$$

Exercise 2 Characterize Y_n . Show that $\exists N$ s.t. $Y_N \geq Y_n$ for all n . Characterize $N(\beta, c_0, c_1)$.

Proof. $\frac{\partial Y_n}{\partial n} = -\frac{c_1(1-\beta^n)+(c_0+c_1n)\beta^n \log \beta}{(1-\beta^n)^2} \geq 0 \iff \frac{-\log \beta}{\beta^n-1} \geq \frac{1}{n+\frac{c_0}{c_1}}$. Note that both LHS and RHS are continuous and decreasing in n , $\lim_{n \rightarrow 0} LHS > \lim_{n \rightarrow 0} RHS$ and $\exists n^* > 0$ s.t. $\forall n > n^*$, $LHS < RHS$. The equality defines $n^* \left(\beta, \frac{c_0}{c_1} \right)$. This is increasing in both arguments, since LHS is increasing in β and RHS is decreasing in $\frac{c_0}{c_1}$. $N \equiv \arg \max_{\{\nu(n^*), \nu(n^*)+1\}} (Y_{\nu(n^*)}, Y_{\nu(n^*)+1})$, where $\nu(x)$ is the largest integer not exceeding x . ■

Note that $Y_N \leq 0$ results in "never order" case. So we assume $Y_N > 0$. By using Y_N , we conjecture a following value function:

$$w(x) = \begin{cases} px + Y_N & x \in [0, 1] \\ p + Y_N - c_1(x-1) & x \in (1, 1+A) \\ p + \beta w(x-1) & x \in [1+A, 2] \\ p + \beta w(x-1) & x \in (2, \infty) \end{cases}$$

To justify the conjecture for $x \in (1, 2]$, think about two alternative actions:

- Sell 1 and order $y = N - (x - 1)$.

– since order is less than N by $x - 1$, the associated value is $p + Y_N - c_1(x - 1)$.

- Sell 1 and order $y = 0$.

– the associated value is $p + \beta w(x - 1)$.

Being close to 1, we choose the first action, while with big enough x , we choose the second. Also, we conjecture that a point of indifference lies in $(1, 2)$.

Exercise 3 Show that $A \in (0, 1)$.

Proof. From indifference condition, $p + Y_N - c_1 A = p + \beta w(A) = p + \beta(pA + Y_N)$. $\therefore A = \frac{(1-\beta)Y_N}{\beta p - c_1} > 0$. At $x = 2$, the action 1 yields $p + Y_N - c_1$ and the action 2 yields $p + \beta w(1) = (1 + \beta)p + \beta Y_N$. The latter minus the former equals $\beta p + c_1 - (1 - \beta)Y_N$. This is positive since $Y_N = \frac{\beta}{1-\beta}p - \frac{c_0+c_1N}{1-\beta^N} < \frac{\beta p + c_1}{1-\beta}$. ■

Exercise 4 Solve the difference equation above to obtain the non-recursive conjectured value function $w(x)$. Show that w is bounded, continuous, and piecewise linear.

Proof. Nothing to do for $x \in [0, 1]$. For $x > 1$, define $\alpha(x) = x - \nu(x) \in [0, 1]$. Note that if $1 < x$ and $\alpha(x) \in [0, A)$, x will lie in $[1, 1 + A]$ in finite steps. If $1 < x$ and $\alpha(x) \in [A, 1)$, x will not reach $[1, 1 + A]$. Think about the first case. We earn p for $\nu(x) - 1$ times and are left with stock $1 + \alpha(x) < 1 + A$. The action suggested by $w(x)$ is then: sell 1 and order $N - \alpha(x)$. Hence the value associated with this is: $p \frac{1 - \beta^{\nu(x)}}{1 - \beta} + \beta^{\nu(x) - 1} (Y_N + c_1 \alpha(x))$. Similarly, for the second case, we earn p for $\nu(x) - 1$ times and are left with stock $1 + \alpha(x) > 1 + A$. Then we sell 1 and order 0 and are left with $\alpha(x) > A$. Finally we sell $\alpha(x)$ and order N . Hence, the value associated with this is: $p \frac{1 - \beta^{\nu(x)}}{1 - \beta} + \beta \beta^{\nu(x) - 1} (Y_N + p \alpha(x))$. Boundedness is obvious. For continuity, enough to check three points: $x \in \{1, 1 + A, 2\}$. At $x = 1$, we need to show: $p + Y_N = \lim_{x \searrow 1} \left\{ p \frac{1 - \beta^{\nu(x)}}{1 - \beta} + \beta^{\nu(x) - 1} (Y_N + c_1 \alpha(x)) \right\}$. This holds since $\lim_{x \searrow 1} \nu(x) = 1$ and $\lim_{x \searrow 1} \alpha(x) = 0$. At $x = 1 + A$, we need to show: $\lim_{x \nearrow 1 + A} p \frac{1 - \beta}{1 - \beta} + \beta^{1 - 1} (Y_N + c_1 \alpha(x)) = p \frac{1 - \beta}{1 - \beta} + \beta (Y_N + pA)$. Since $\lim_{x \nearrow 1 + A} \alpha(x) = A$, this is equivalent to $Y_N + c_1 A = \beta (Y_N + pA)$. This holds by definition of A . At $x = 2$, we need to show $p \frac{1 - \beta^2}{1 - \beta} + \beta^{2 - 1} (Y_N + c_1 \alpha(2)) = \lim_{x \nearrow 2} \left\{ p \frac{1 - \beta^{\nu(x)}}{1 - \beta} + \beta \beta^{\nu(x) - 1} (Y_N + p \alpha(x)) \right\}$. LHS = $p(1 + \beta) + \beta Y_N$ and this coincides with RHS since $\lim_{x \nearrow 2} \nu(x) = 1$ and $\lim_{x \nearrow 2} \alpha(x) = 1$. By construction, w is differentiable everywhere but countable points, i.e., $[0, \infty) \setminus \{I \cup \{I + A\}\}$, where I is a set of integers. Derivatives

$$\text{are given by } w'(x) = \begin{cases} p & x \in [0, 1) \\ \beta^{\nu(x) - 1} c_1 & x \in (1, \infty) \text{ and } \alpha(x) \in (0, A) \\ \beta^{\nu(x)} p & x \in (1, \infty) \text{ and } \alpha(x) \in (A, 1) \end{cases} \cdot \blacksquare$$

Finally, we need to show that conjectured $w(x)$ actually is a fixed point of T .

$$w(x) = \begin{cases} px + Y_N & x \in [0, 1] \\ p \frac{1 - \beta^{\nu(x)}}{1 - \beta} + \beta^{\nu(x) - 1} (Y_N + c_1 \alpha(x)) & 1 < x \text{ and } \alpha(x) \in [0, A) \\ p \frac{1 - \beta^{\nu(x)}}{1 - \beta} + \beta^{\nu(x)} (Y_N + p \alpha(x)) & 1 < x \text{ and } x \in [A, 1) \end{cases}$$

We use three preliminary results from following exercise.

Exercise 5 Show that:

$$(1) \forall z \geq 0, \sup_{y>0} \{\beta w(y+z) - c_0 - c_1 y\} = \max \left[\beta w(z) - c_0, \max_{n>z, n \in I} \{\beta w(n) - c_0 - c_1(n-z)\} \right],$$

$$(2) \text{ We can replace the second maximum above by } c_1 z + \max_{n>z, n \in I} \{r_n + \beta^n Y_N\},$$

$$(3) \max_{n \geq 1, n \in I} \{r_n + \beta^n Y_N\} = Y_N.$$

Proof. (1) We need to show that stock at differentiable points are not optimal. This is true because $\beta w'(x) \neq c_1$ for all differentiable points. (2) $\beta w(n) - c_0 - c_1 n = \beta p \frac{1-\beta^n}{1-\beta} + \beta \beta^{n-1} (Y_N + c_1 0) - c_0 - c_1 n = r_n + \beta^n Y_N$. (3) Suppose $\exists n \neq N$ s.t. $r_n + \beta^n Y_N > Y_N$. Then, $Y_N < \frac{r_n}{1-\beta^n} = Y_n$. This contradicts with definition of Y_N . ■

By using three results above, we can rewrite T as follows:

$$Tf(x) = \begin{cases} px + \max \{\beta f(0), Y_N\} & x \in [0, 1) \\ p + \max \{\beta f(x-1), Y_N + c_1(x-1)\} & x \in [1, 2) \\ p + \max \left\{ \beta f(x-1), \max_{n \geq x-1, n \in I} (r_n + \beta^n Y_N) + c_1(x-1) \right\} & x \in [2, \infty). \end{cases}$$

Exercise 6 Show $(Tw)(x) = w(x)$.

Proof. For $x \in [0, 1]$, easy to check by plugging in $w(x) = px + Y_N$. For $x \in (1, 2)$ and $\alpha(x) \in (0, A)$, $w(x-1) = p(x-1) + Y_N$. Hence, $\beta w(x-1) = \beta Y_N + \beta p \alpha(x) < Y_N + c_1 \alpha(x) = w(x)$. For $x \in (1, 2)$ and $\alpha(x) \in [A, 1)$, again, $w(x-1) = p(x-1) + Y_N$. Hence, $Y_N + c_1(x-1) = Y_N + c_1 \alpha(x) < \beta w(x-1) = \beta Y_N + \beta p \alpha(x) = w(x)$. For $x \in [2, \infty)$, enough to show $p + \beta w(x-1) = w(x)$, since $Y_N \geq \max_{n \geq x-1, n \in I} (r_n + \beta^n Y_N)$. For $x \in [2, \infty)$ and $\alpha(x) \in (0, A)$, $w(x-1) = p \frac{1-\beta^{\nu(x)-1}}{1-\beta} + \beta^{\nu(x)-2} (Y_N + c_1 \alpha(x))$. Hence, $p + \beta w(x-1) = p + p \frac{\beta - \beta^{\nu(x)}}{1-\beta} + \beta^{\nu(x)-1} (Y_N + c_1 \alpha(x)) = p \frac{1-\beta^{\nu(x)}}{1-\beta} + \beta^{\nu(x)-1} (Y_N + c_1 \alpha(x)) = w(x)$. For $x \in [2, \infty)$ and $\alpha(x) \in [A, 1)$, $w(x-1) = p \frac{1-\beta^{\nu(x)-1}}{1-\beta} + \beta^{\nu(x)-1} (Y_N + p \alpha(x))$. Hence, $p + \beta w(x-1) = p + p \frac{\beta - \beta^{\nu(x)}}{1-\beta} + \beta^{\nu(x)} (Y_N + c_1 \alpha(x)) = p \frac{1-\beta^{\nu(x)}}{1-\beta} + \beta^{\nu(x)} (Y_N + p \alpha(x)) = w(x)$. ■

2 Continuous time

Let demand be a flow of α units of goods per unit time, and let the interest rate be $r > 0$. The decision with initial stock $A \geq 0$ at date $t = 0$ is to choose the date $\tau \geq 0$ for the next order, and size y of the order.

$$v(A) = \max \left[\begin{array}{c} \int_0^{\frac{A}{\alpha}} e^{-rt} \alpha p dt, \\ \sup_{\tau \geq 0, y > 0} \left\{ \int_0^{\tau} e^{-rt} p z(t) dt + e^{-r\tau} \{v(x(\tau) + y) - c_0 - c_1 y\} \right\} \end{array} \right],$$

$$\text{where } x(0) = A, \frac{dx(t)}{dt} = -z(t) = \begin{cases} -\alpha & \text{if } x(t) > 0 \\ 0 & \text{if } x(t) = 0 \end{cases}.$$

Exercise 7 Show that $Y^* \equiv \max \left\{ 0, \sup_{\tau > 0} (1 + e^{-r\tau} + e^{-2r\tau} \dots) \right\}$, where $r_\tau \equiv \int_0^{\tau} e^{-rt} \alpha p dt - c_0 - c_1 \alpha \tau$, is well defined and finite.

Proof. Inside sup, $\frac{1}{1-e^{-r\tau}} \left(\frac{\alpha p}{r} (1 - e^{-r\tau}) - c_0 - c_1 \alpha \tau \right) = \frac{\alpha p}{r} - \frac{c_0 + c_1 \alpha \tau}{1 - e^{-r\tau}} < \frac{\alpha p}{r}$. Hence we can think of $\inf_{\tau > 0} \frac{c_0 + c_1 \alpha \tau}{1 - e^{-r\tau}}$. $K(\tau) = \frac{c_0 + c_1 \alpha \tau}{1 - e^{-r\tau}}$ is continuous in $\tau \in (0, \infty)$. $\frac{\partial K}{\partial \tau} = \frac{c_1 \alpha - e^{-r\tau} (c_0 r + c_1 \alpha + c_1 \alpha r \tau)}{(1 - e^{-r\tau})^2}$ and $\frac{\partial K}{\partial \tau} \geq 0 \Leftrightarrow e^{r\tau} \geq 1 + \frac{c_0 r}{c_1 \alpha} + r\tau$. ■

Note that r_τ, Y^*, τ^* correspond to r_n, Y_N, N in discrete setting. Similarly as before, assume $Y^* > 0$.

Exercise 8 Show that $w(A) = \int_0^{\frac{A}{\alpha}} e^{-rt} \alpha p dt + e^{-r\frac{A}{\alpha}} Y^*$ satisfies the functional equation.

Proof. $w(x(\tau) + y) = \frac{\alpha p}{r} \left(1 - e^{-r\frac{x(\tau)+y}{\alpha}} \right) + e^{-r\frac{x(\tau)+y}{\alpha}} Y^*$. Since $x(\tau) = z(\tau) = 0$ for $\tau > \frac{A}{\alpha}$, there is no incentive to choose $\tau > \frac{A}{\alpha}$. Note that $x(\tau) = A - \alpha\tau$ and $z(\tau) = \alpha$ for $\tau \leq \frac{A}{\alpha}$.

Hence, problem is: $\sup_{\tau \in [0, \frac{A}{\alpha}], y > 0} \left\{ \frac{\alpha p}{r} (1 - e^{-r\tau}) + e^{-r\tau} \left\{ \frac{\alpha p}{r} \left(1 - e^{-r\frac{A-\alpha\tau+y}{\alpha}} \right) + e^{-r\frac{A-\alpha\tau+y}{\alpha}} Y^* - c_0 - c_1 y \right\} \right\}$.

Inside sup, $\frac{\alpha p}{r} + e^{-r\tau} \left\{ \frac{\alpha p}{r} \left(1 - e^{-r\frac{A+y}{\alpha}} e^{r\tau} \right) - \frac{\alpha p}{r} + e^{-r\frac{A+y}{\alpha}} e^{r\tau} Y^* - c_0 - c_1 y \right\} = \frac{\alpha p}{r} + e^{-r\frac{A+y}{\alpha}} (Y^* - \frac{\alpha p}{r}) - e^{-r\tau} (c_0 + c_1 y)$. Therefore, $\tau = \frac{A}{\alpha}$ is initially chosen. Next, think about choice of y .

$\sup_{y > 0} \{e^{-r\tau} \{w(x(\tau) + y) - c_0 - c_1 y\}\} = \sup_{n > x(\tau)} e^{-r\tau} \{w(n) - c_0 - c_1(n - x(\tau))\} = e^{-r\tau} x(\tau) + \sup_{n > x(\tau)} e^{-r\tau} \{w(n) - c_0 - c_1 n\} = e^{-r\frac{A}{\alpha}} \sup_{n > 0} \{w(n) - c_0 - c_1 n\}$. Notice: $\sup_{n > 0} \{w(n) - c_0 - c_1 n\} =$

$\sup_{n > 0} \left\{ \int_0^{\frac{n}{\alpha}} e^{-rt} \alpha p dt + e^{-r\frac{n}{\alpha}} Y^* - c_0 - c_1 n \right\} = \sup_{n > 0} \{r\frac{n}{\alpha} + e^{-r\frac{n}{\alpha}} Y^*\} = Y^*$. Hence, we showed

$Tw = w$. Moreover, the optimal policy is "initially wait until $\tau = \frac{A}{\alpha}$ and thereafter order $\alpha\tau^*$ every τ^* period". ■